Modulus of continuity of the coefficients and loss of derivatives in the strictly hyperbolic Cauchy problem

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Abstract

We deal with the Cauchy problem for a strictly hyperbolic second-order operator with non-regular coefficients in the time variable. It is well-known that the problem is well-posed in $L^2$ in case of Lipschitz continuous coefficients and that the log-Lipschitz continuity is the natural threshold for the well-posedness in Sobolev spaces which, in this case, holds with a loss of derivatives. Here, we prove that any intermediate modulus of continuity between the Lipschitz and the log-Lipschitz one leads to an energy estimate with arbitrary small loss of derivatives. We also provide counterexamples to show that the following classification:

modulus of continuity $\rightarrow$ loss of derivatives
is sharp

\[
\text{Lipschitz} \rightarrow \text{no loss},
\]

\[
\text{intermediate} \rightarrow \text{arbitrary small loss},
\]

\[
\text{log-Lipschitz} \rightarrow \text{finite loss}.
\]

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1. Introduction

Let us consider the Cauchy problem

\[
\begin{cases}
\partial_t^2 u - \sum_{i,j=1}^n a_{ij}(t,x) \partial_{xx_{ij}} u = 0 & \text{in } Q := [0,T] \times \mathbb{R}^n, \\
u(0,x) = u_0, \quad \partial_t u(0,x) = u_1
\end{cases}
\]

(1.1)

with coefficients \(a_{ij} \in C([0,T]; B^{\infty}(\mathbb{R}^n))\), \(a_{ij} = a_{ji}\). We assume that the strict hyperbolicity condition

\[
\Lambda|\xi|^2 \geq \sum_{i,j=1}^n a_{ij}(t,x) \xi_i \xi_j \geq \lambda |\xi|^2, \quad \lambda > 0,
\]

(1.2)

is satisfied for every \(\xi \in \mathbb{R}^n\) and, for later use, we denote

\[
A = \max_{1 \leq |\alpha| \leq 2} \|\partial^\alpha_x a_{ij}\|_{L^\infty(Q)}.
\]

(1.3)

It is well-known that if the coefficients \(a_{ij}\) are Lipschitz continuous in the time variable, then the problem (1.1) is \(C^\infty\) well-posed. More precisely, in this case the Cauchy problem is well-posed in Sobolev spaces and one can prove that there is a unique solution

\[
u \in C \left([0,T]; H^s(\mathbb{R}^n)\right) \cap C^1 \left([0,T]; H^{s-1}(\mathbb{R}^n)\right)
\]

for any given Cauchy data \(u_0 \in H^s, u_1 \in H^{s-1}\), and that the solution satisfies the estimate

\[
\|\nu(t)\|_{H^s}^2 + \|\partial_t \nu(t)\|_{H^{s-1}}^2 \leq C_s \left(\|\nu(0)\|_{H^s}^2 + \|\partial_t \nu(0)\|_{H^{s-1}}^2\right),
\]

(1.4)

\(C_s > 0, \ t \in [0,T] \) (e.g. [6, Chapter 9] or [7, Chapter 6]).
Interesting results have been obtained in weakening the regularity assumptions, with respect to the \( t \) variable, on the coefficients \( a_{ij} \). To this purpose, let us recall the following:

**Definition 1.1.** A function \( f : I \to \mathbb{R} \), \( I \) a real interval or more generally a pluri-interval in \( \mathbb{R}^n \), is said log-Lipschitz continuous if it satisfies

\[
\| f \|_{LL(I)} := \sup_{0 < |t-s| < 1/2} \frac{|f(t) - f(s)|}{|t-s||\log|t-s||} < +\infty.
\]

In [3] it has been proved, in case of coefficients depending only on the \( t \) variable, that if \( a_{ij} \in LL([0,T]) \) then the Cauchy problem (1.1) is still \( C^\infty \) well-posed but the phenomenon of the *loss of derivatives* arises; precisely in [3] it is proved that there exists \( \delta > 0 \) depending on \( \|a_{ij}\|_{LL([0,T])} \) such that for all \( u_0 \in H^s \), \( u_1 \in H^{s-1} \) there is a unique solution

\[
u \in C \left( [0, T]; H^{s-\delta}(\mathbb{R}^n) \right) \cap C^1 \left( [0, T]; H^{s-1-\delta}(\mathbb{R}^n) \right)
\]

and that the solution satisfies

\[
\|u(t)\|_{H^{s-\delta}}^2 + \|\partial_t u(t)\|_{H^{s-1-\delta}}^2 \leq C_s \left( \|u(0)\|_{H^s}^2 + \|\partial_t u(0)\|_{H^{s-1}}^2 \right).
\]

This result has been extended in [4] to the case of coefficients \( a_{ij} \) depending also on the space variables, giving the following precise estimate of the loss of derivatives: denoting

\[
E_s(u)(t) := \|u(t)\|_{H^s}^2 + \|\partial_t u(t)\|_{H^{s-1}}^2,
\]

we have

\[
E_{s-\beta t}(u)(t) \leq C_s E_s(u)(0),
\]

where \( C_s \) is a positive constant depending only on \( s \), \( n \) and \( \Lambda \) (see (1.2)), and where the constant \( \beta \) is given by

\[
\beta = \frac{1}{\Lambda} C(n) \left( \sup_{x \in \mathbb{R}^n} \sup_{1 \leq i,j \leq n} \|a_{ij}(\cdot, x)\|_{LL([0,T])} + A \right)
\]

with \( C(n) \) depending only on the dimension \( n \), \( \lambda \) given in (1.2), \( A \) given in (1.3). We can see that in case of coefficients depending only on the \( t \) variable we have \( A = 0 \).
On the other hand in [3], with a coefficient $a$ depending only on the variable $t$ which is Hölder continuous of any exponent smaller than 1, an example with a $x \in \mathbb{R}$ shows that the log-Lip assumption cannot be weakened a lot. In [4], generalizing such an example, it is proved that under any weaker hypothesis with respect to the log-Lipschitz regularity, the Cauchy problem (1.1) is in general not well-posed in $C^\infty$ since one has an infinite loss of derivatives.

In [8,9], the Cauchy problem (1.1) is discussed with coefficients that satisfy

$$|a_{ij}(t) - a_{ij}(s)| \leq C|t - s|(|\log |t - s||)^{\gamma},$$

$$C > 0, \ t, s \in [0, T], \ 0 < |t - s| \leq \frac{1}{2}$$

and the following two problems are posed:

1. For $0 < \gamma < 1$, is the problem (1.1) well-posed with an arbitrary small but, in general, not vanishing loss of derivatives?
2. For $\gamma = 1$, does, in general, a loss of derivatives really occur?

The aim of this paper is to give a positive answer to both these questions. In Section 2 we consider the case of a one-dimension space variable and of a coefficient $a(t)$ depending only on $t$. By means of the Fourier transform we first give a simple proof of the fact that assuming any slightly better regularity of the coefficients than the log-Lipschitz continuity (in particular in the case (1)), the Cauchy problem (1.1) is well-posed with an arbitrary small loss of derivatives. Then we give two examples: by the first one we show that in the case (2) of log-Lip coefficients a loss of derivatives really occurs in general. This loss is proportional as well as to the log-Lip norm and the time. By the second example, we show that if the coefficients have a slightly worse regularity than Lip, in particular in the case (1), then the phenomenon of the (even arbitrary small) loss of derivatives arises in general.

In Section 3, we prove the arbitrary small loss of derivatives in the Cauchy problem for a strictly hyperbolic operator

$$P = \partial_t^2 + \sum_{i=1}^n b_i(t, x) \partial_{x_i} - \sum_{i,j=1}^n a_{ij}(t, x) \partial_{x_i}^2 \partial_{x_j}$$

with coefficients depending on all variables. With respect to the variable $t$, they have an intermediate modulus of continuity between Lip and log-Lip whereas they are in $B^\infty$ with respect to $x$. There we bring to the pseudodifferential operators level the arguments of the first part of Section 2.

### 2. Sharp moduli of continuity

In this section, we deal with operators with coefficients depending only on the time variable to give the main ideas about the classification

modulus of continuity $\longrightarrow$ loss of derivatives
and to construct counterexamples that show that such a classification is sharp. Here, we take a one-dimension space variable \( x \in \mathbb{R} \) only for simplicity’s sake, the case \( x \in \mathbb{R}^n, \ n > 1, \) requires only a heavier notation.

Let us consider the Cauchy problem in \([0, T] \times \mathbb{R}\)

\[
\begin{cases}
\partial_t^2 u - a(t) \partial_x^2 u = 0, \\
u(0, x) = u_0, \ \partial_t u(0, x) = u_1
\end{cases}
\tag{2.1}
\]

with data \( u_0 \in H^s(\mathbb{R}), \ u_1 \in H^{s-1}(\mathbb{R}), \ s > 0, \) under the strict hyperbolicity assumption

\[
0 < \lambda \leq a(t) \leq \Lambda.
\tag{2.2}
\]

With the energy \( E_s(u) \) defined by (1.6), the first result of this section is the following:

**Theorem 2.1.** Let us assume

\[
|a(t + \tau) - a(t)| \leq M \tau \log \tau |\omega(\tau)|, \quad M > 0, \quad |\tau| \leq \frac{1}{\xi}
\tag{2.3}
\]

with a function \( \omega(\zeta), \ \zeta > 0, \) such that \( \omega(\zeta) \downarrow 0, \ \zeta \to 0^+ .\)

Then, for every \( \delta > 0 \) the solution of (2.1) fulfills

\[
E_{s-\delta}(u)(t) \leq C_\delta E_s(u)(0),
\]

\[
C_\delta > 0, \ t \in [0, T].
\tag{2.4}
\]

**Proof.** Let us introduce the following regularization of \( a(t) \)

\[
a_\varepsilon(t) = \int a(t + \tau) \varrho \left( \frac{\tau}{\varepsilon} \right) \frac{1}{\varepsilon} d\tau, \quad \varepsilon > 0,
\tag{2.5}
\]

where \( \varrho \in C_0^\infty([-1, 1]), \ 0 \leq \varrho \leq 1, \ \int \varrho(\tau) d\tau = 1, \ \int |\varrho'(\tau)| d\tau \leq 4, \) and we have set \( a(\tau) = a(T) \) for \( \tau > T \) and \( a(\tau) = a(0) \) for \( \tau < 0. \)

From (2.2) we have

\[
0 < \lambda \leq a_\varepsilon(t) \leq \Lambda,
\tag{2.6}
\]

whereas from (2.3) we get

\[
|a_\varepsilon(t) - a(t)| = \left| \int (a(t + \tau) - a(t)) \varrho \left( \frac{\tau}{\varepsilon} \right) \frac{1}{\varepsilon} d\tau \right|
\leq M \varepsilon |\log(\varepsilon)| \omega(\varepsilon)
\tag{2.7}
\]
and
\[
|a'_\varepsilon(t)| = \left| \int (a(t + \tau) - a(t)) q' \left( \frac{\tau}{\varepsilon} \right) \frac{1}{\varepsilon^2} \, d\tau \right| \leq 4M|\log(\varepsilon)|\omega(\varepsilon). \tag{2.8}
\]

Now, let us denote by \(v(t, \xi)\) the Fourier transform with respect to the space variable \(x\) of the solution \(u(t, x)\) of the Cauchy problem (2.1) and let us introduce the approximated energy
\[
E^{(\varepsilon)}(t, \xi) = |v'|^2 + \xi^2 a_\varepsilon(t)|v|^2. \tag{2.9}
\]
Since \(v\) satisfies
\[
v'' + a(t)\xi^2 v = 0,
\]
applying the Gronwall inequality to \(\frac{d}{dt}E^{(\varepsilon)}\), from (2.6) to (2.8) we obtain
\[
E^{(\varepsilon)}(t, \xi) \leq E^{(\varepsilon)}(0, \xi) \exp \left( \frac{4M}{\lambda} t|\log(\varepsilon)|\omega(\varepsilon) + \varepsilon |\xi| \frac{M}{\sqrt{\lambda}} t|\log(\varepsilon)|\omega(\varepsilon) \right). \tag{2.10}
\]
Thus, choosing \(\varepsilon = |\xi|^{-1}\), we have
\[
E^{(\varepsilon)}(t, \xi) \leq E^{(\varepsilon)}(0, \xi)^{|\xi|^{\hat{M}t\omega(1/|\xi|)}} \tag{2.11}
\]
with a suitable \(\hat{M} > 0\). For every \(\delta > 0\) there is \(R_\delta > 0\) such that
\[
\omega(1/|\xi|) \leq \frac{2\delta}{\hat{M}T}
\]
for \(|\xi| \geq R_\delta\), so from (2.11) we obtain
\[
E^{(\varepsilon)}(t, \xi) \leq E^{(\varepsilon)}(0, \xi)|\xi|^{2\delta}. \quad |\xi| \geq R_\delta \tag{2.12}
\]
which implies the desired inequality (2.4) taking (2.6) into account. \(\square\)

Choosing \(\omega(\xi) = |\log(\xi)|^{\gamma-1}, 0 < \gamma < 1\), in (2.3), we prove a conjecture stated in [8,9].
Corollary 2.2. Let us assume

\[ |a(t + \tau) - a(t)| \leq M|\tau| \log |\tau|, \]

\[ 0 < \gamma < 1, \ M > 0, \ |\tau| \leq \frac{1}{2}. \]  

(2.13)

Then, for every \( \delta > 0 \) there is \( C_\delta > 0 \) such that the solution of (2.1) fulfills the inequality (2.4).

By means of examples, we prove now that in general one has a loss of derivatives which is finite in the log-Lipschitz case (2), arbitrary small in the case (1) of the introduction.

In the following two theorems, we consider \( 2\pi \)-periodic Cauchy data and solutions in (1.1). In order to simplify the proofs, instead of the energy \( E_s(u)(t) \) in (1.6), here we use

\[ \dot{E}_s(u)(t) = \|u(t)\|_{\dot{H}^s} + \|\dot{u}(t)\|_{\dot{H}^{s-\rho}}, \]  

(2.14)

where \( \dot{H}^s \) denotes the homogeneous Sobolev space on the one-dimension torus \( T = \mathbb{R}/2\pi \mathbb{Z} \).

For \( M > 0 \), let us denote

\[ \mathcal{A}(M) = \{ a : \mathbb{R} \rightarrow \mathbb{R}; \ |a(t) - 1| \leq \frac{1}{2}, \ \|a\|_{LL(\mathbb{R})} \leq M \}. \]

We have then the following theorem concerning the log-Lip coefficients case:

Theorem 2.3. For every \( M \) there exists a sequence \( \{a_k\} \subset \mathcal{A}(M) \) and a sequence of functions \( u_k(t, x) \) satisfying

\[ \partial_t^2 u_k - a_k(t) \partial_x^2 u_k = 0, \quad (t, x) \in \mathbb{R} \times T, \]  

(2.15)

such that

\[ \dot{E}_1(u_k)(0) = 1 \]  

(2.16)

whereas, for every \( t > 0 \) and every \( s_0 < \frac{1}{16}Mt \), one has

\[ \sup_k \dot{E}_{1-s_0}(u_k)(t) = +\infty. \]  

(2.17)

Remark 2.4. Given a Cauchy problem with a coefficient \( a \in \mathcal{A}(M) \), the above statement means that in \([0, t]\) one has in general a loss of derivatives \( \bar{s} \) which is proportional to \( Mt \).
Proof. Let us define

\[ a_k(t) = x_{\varepsilon_k}(v_k t), \quad k = 1, 2, 3, \ldots, \]

where (see [3,5]) we have set

\[ x_{\varepsilon}(\tau) = 1 - 4\varepsilon \sin(2\tau) - \varepsilon^2 (1 - \cos(2\tau))^2 \]

and where the sequences \( \varepsilon_k \downarrow 0, v_k \uparrow +\infty \) will be chosen later. In particular \( \varepsilon_k \) will satisfy

\[ \varepsilon_k \leq \frac{1}{10}, \quad (2.18) \]

which implies

\[ |x_{\varepsilon}(\tau) - 1| \leq \frac{1}{2}, \quad |x'_{\varepsilon}(\tau)| \leq 9\varepsilon. \quad (2.19) \]

If we now impose

\[ 10\varepsilon_k v_k = M \log v_k \quad (2.20) \]

then we obtain \( a_k \in \mathcal{A}(M) \) for \( k \geq \tilde{k} \). In fact, since \( x_{\varepsilon} \) is a \( \pi \)-periodic function and taking (2.19) into account, we have

\[
\sup_{|\tau| \leq 1/2} \frac{|a_k(t + \tau) - a_k(t)|}{|\tau||\log|\tau||} = \sup_{|\tau| \leq \pi/v_k} \frac{|a_k(t + \tau) - a_k(t)|}{|\tau|} \cdot \frac{1}{|\log|\tau||} \\
\leq \frac{9\varepsilon_k v_k}{\log(v_k/\pi)} = \frac{9}{10} M \frac{\log v_k}{\log v_k - \log \pi} < M \quad (2.21)
\]

for \( k \) sufficiently large.

Now we choose

\[ v_k = 2^k \]

and, consequently,

\[ \varepsilon_k = \frac{M k \log 2}{10} \frac{2^k}{2^k}. \]

Notice that we have \( \varepsilon_k < \frac{1}{10} \) for \( k \geq \tilde{k}_M \).
Let \( u_k \) be the solution of the Cauchy problem
\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u_k - a_k(t) \frac{\partial^2}{\partial x^2} u_k &= 0, \\
u_k(0, x) &= 0, \quad \frac{\partial}{\partial t} u_k(0, x) = (\pi)^{-1/2} \sin(v_k x).
\end{align*}
\]
(2.22)
Then we have
\[
u_k(t, x) = (\pi)^{-1/2} v_k(t) \sin(v_k x),
\]
where \( v_k(t) \) is the solution of
\[
\begin{align*}
\frac{d^2}{dt^2} v_k(t) + v_k^2 a_k(t) v_k(t) &= 0, \\
v_k(0) &= 0, \quad v_k'(0) = 1.
\end{align*}
\]
Hence, we obtain (see again [3,5])
\[
v_k(t) = \frac{1}{v_k} w_{\varepsilon}(v_k t),
\]
(2.23)
where
\[
w_{\varepsilon}(\tau) = \sin \tau \cdot \exp \left[ \varepsilon \left( \tau - \frac{1}{2} \sin 2\tau \right) \right]
\]
(2.24)
is the solution of
\[
\begin{align*}
\frac{d^2}{d\tau^2} w_{\varepsilon}(\tau) + \kappa_{\varepsilon}(\tau) w_{\varepsilon}(\tau) &= 0, \\
w_{\varepsilon}(0) &= 0, \quad w_{\varepsilon}'(0) = 1.
\end{align*}
\]
(2.25)
Thus we have
\[
\dot{E}_1(u_k)(0) = 1
\]
whereas, for any fixed \( s_0 \), from (2.18), (2.20), (2.23) and (2.24), we obtain
\[
\begin{align*}
\dot{E}_{1-s_0}(u_k)(t) &= v_k^{-2s_0} (\sin v_k t)^2 + v_k^{-2s_0} [\cos v_k t + \varepsilon_k \sin v_k (1 - \cos 2v_k t)]^2, \\
&\geq \exp \left[ 2\varepsilon_k (v_k t - \frac{1}{2} \sin 2v_k t) \right] \\
&\geq v_k^{-2s_0} (1 - 4\varepsilon_k) e^{-\varepsilon_k} e^{2\varepsilon_k v_k t} \\
&\geq \frac{1}{2} v_k^{-2s_0} v_k^{1/2} M_t.
\end{align*}
\]
(2.26)
Hence, \( \dot{E}_{1-s_0}(u_k)(t) \) is not bounded for \( s_0 < \frac{1}{10} M_t \). This means that at the time \( t \) we have a loss of derivatives \( s_0 \geq \frac{1}{10} M_t \). \( \square \)
Let now $\Omega(s)$ be a continuous decreasing function in $(0, 1/2)$ such that

$$
\lim_{s\to 0} \Omega(s) = +\infty, \quad \lim_{s\to 0} \frac{\Omega(s)}{\log(1/s)} = 0.
$$

As an example, one can consider

$$
\Omega(s) = (\log(1/s))^\gamma, \quad 0 < \gamma < 1.
$$

For such a function $\Omega$ and $M > 0$, let us denote

$$
\mathcal{A}(\Omega, M) = \left\{ a : \mathbb{R} \to \mathbb{R}; \ |a(t) - 1| \leq \frac{1}{2}, \sup_{0 < |\tau| \leq 1/2} \frac{|a(t + \tau) - a(t)|}{|\tau| \Omega(|\tau|)} \leq M \right\}.
$$

Then we have the following theorem:

**Theorem 2.5.** Let $\Omega$ satisfy (2.27). For every $M > 0$ there exists a sequence $\{a_k\} \subset \mathcal{A}(\Omega, M)$ and a sequence of functions $u_k$ which satisfies (2.15) and

$$
\dot{E}_1(u_k)(0) = 1
$$

but

$$
\sup_k \dot{E}_1(u_k)(t) = +\infty\quad (2.28)
$$

for every $t > 0$.

On the other hand, for every $s_0 > 0$ and every $T > 0$ we have

$$
\sup_k \sup_{0 \leq t \leq T} \dot{E}_1(u_k)(t) < +\infty.
$$

**Proof.** Let $a_k$ be as in the proof of Theorem 2.3 but with

$$
10\varepsilon_k v_k = M \Omega(\pi/v_k)
$$

in place of (2.20). Arguing as in (2.21), we obtain $\{a_k\} \subset \mathcal{A}(\Omega, M)$.

Still denoting $u_k$ the solution of (2.22),

$$
\dot{E}_1(u_k)(0) = 1
$$
holds, but now we have

\[ \dot{E}_1(u_k)(t) \geq \frac{1}{2} \exp \left( \frac{2\varepsilon_k v_k t}{\Omega} \right) \]

with a very similar to (2.26) calculation. This gives (2.28) taking the first property of \( \Omega \) in (2.27) into account.

On the other hand, using (2.23) and (2.24) and arguing as in (2.26), for fixed \( s_0 > 0 \), \( T > 0 \) and \( \sigma = 2s_0/5MT \) we obtain

\[
\sup_{0 \leq t \leq T} \dot{E}_1 - s_0 (u_k)(t) \leq 2v_k^{-2s_0} \left[ 1 + 2\varepsilon_k + 4\varepsilon_k^2 \right] e^{2\varepsilon_k v_k t} \leq 2v_k^{-2s_0} \exp \left[ \frac{5MT \log v_k}{\log(v_k/\pi)} \right] \leq 2v_k^{-2s_0} v_k^{5MT\sigma} = 2
\]

provided that \( k \geq \bar{k}(\sigma) \) taking the second property of \( \Omega \) in (2.27) into account. ◻

**Remark 2.6.** We can see that the second part of Theorem 2.5, that is (2.29), follows also directly from Theorem 2.1 taking there

\[ \omega(\tau) = \frac{\Omega(\tau)}{\log(1/\tau)} \]

and taking the second property of \( \Omega \) in (2.27) into account.

3. **Coefficients depending also on space variables**

Let us consider the operator \( P \) in \([0, T] \times \mathbb{R}^n_x\)

\[ P = \partial_t^2 + 2i B(t, x, D_x) \partial_t + A(t, x, D_x), \quad (3.1) \]

\( D_x = (1/i) \nabla_x, \; i = \sqrt{-1}, \) with real symbols

\[ A(t, x, \xi) = \sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k, \quad B(t, x, \xi) = \sum_{j=1}^n b_j(t, x) \xi_j \]

that satisfy the condition of strict hyperbolicity

\[ B^2(t, x, \xi) + A(t, x, \xi) \geq c_0 |\xi|^2, \]

\[ c_0 > 0, \; t \in [0, T], \; x, \xi \in \mathbb{R}^n. \]
Given a continuous function \( \omega(\zeta) \) such that \( \omega(\zeta) \downarrow 0, \zeta \to 0^+ \), we denote by \( M^{\omega}([0, T]) \) the space of all functions \( a(t) \) such that
\[
(t + \tau) - a(t) \leq C|\tau| \log |\tau| |\omega(\tau)|,
\]
\[ C > 0, \ t, t + \tau \in [0, T], 0 < \tau \leq \frac{1}{T}. \] (3.4)

The main result of this section is the following:

**Theorem 3.1.** Let the coefficients of \( P \) be such that
\[
a_{jk}, b_j \in M^{\omega}([0, T]; \mathcal{B}^\infty). \] (3.5)

Then, for every \( s, \delta > 0 \) there is a constant \( C_{s, \delta} > 0 \) such that
\[
\|u(t)\|^2_{H^{s+1-\delta}} + \|\partial_t u(t)\|^2_{H^{s-\delta}} \leq C_{s, \delta} \left( \|u(0)\|^2_{H^{s+1}} + \|\partial_t u(0)\|^2_{H^s} + \int_0^t \|P u(\tau)\|^2_{H^s} d\tau \right), \] (3.6)
\[ t \in [0, T], \text{ for all } u \in \bigcap_{j=0}^2 C^j([0, T]; H^{s+2-j}). \]

The remaining part of this section is devoted to the proof of Theorem 3.1. We bring to the pseudodifferential operators level the arguments of the proof of Theorem 2.1 with a refinement of the main ideas of Agliardi and Cicognani [1,2].

Let us denote \( \varphi(y) = \omega(1/y) \). Without loss of generality, we can assume \( \omega \in C^\infty \) and
\[
|\varphi^{(j)}(y)| \leq C_j \langle y \rangle^{-j} \varphi(y), \quad \langle y \rangle = (1 + |y|^2)^{1/2} \] (3.7)
(e.g. [10, formula (M.4) p. 172]). In particular, we can assume that the functions of \( \zeta \in \mathbb{R}^n \)
\[
\varphi((\zeta)), \quad \psi(\zeta) = \varphi((\zeta)) \log(1 + \langle \zeta \rangle) \] (3.8)
are symbols
\[
\varphi \in S^0, \quad \psi \in \bigcap_{\delta > 0} S^\delta, \] (3.9)
where \( S^m, m \in \mathbb{R} \), denotes the usual symbol class of all functions \( p(x, \zeta) \) in \( \mathbb{R}^n_x \times \mathbb{R}^n_\zeta \) such that
\[
|\partial_\zeta^\alpha \partial_x^\beta p(x, \zeta)| \leq C_{2\beta} \langle \zeta \rangle^{m-|\beta|}. \]
We use also the weighted classes $S^m_\psi$, $\psi$ the function in (3.8), defined as the space of all symbols $p(x, \xi)$ such that

$$|\partial_x^\alpha \xi^\beta p(x, \xi)| \leq C_{2\beta} \langle \xi \rangle^{m-|\alpha|} \psi(\xi). \quad (3.10)$$

Let us denote $\lambda_1(t, x, \xi), \lambda_2(t, x, \xi)$ the characteristic roots of $P$, that is the roots of the equation in the variable $\tau$

$$\tau^2 + 2B(t, x, \xi)\tau - A(t, x, \xi) = 0.$$

From (3.3), $\lambda_1$ and $\lambda_2$ are real and from (3.5) we may assume

$$\lambda_1, \lambda_2 \in M^\omega([0, T]; S^1) \quad (3.11)$$

after a modification in a neighborhood of $\xi = 0$.

The first step in the proof of Theorem 3.1 is to carry the algebraic factorization of the symbol of $P$ to the operators level.

Let us introduce the following regularization of $\lambda_k$, $k = 1, 2$, with respect to the variable $t$:

$$\tilde{\lambda}_k(t, x, \xi) = \int \lambda_k(s, x, \xi)q((t - s)\langle \xi \rangle)\langle \xi \rangle \, ds, \quad (3.12)$$

where $q \in C^\infty_0(\mathbb{R})$, $0 \leq q \leq 1$, $\int q(s) \, ds = 1$ and we have set $\lambda_k(s, x, \xi) = \lambda_k(T, x, \xi)$ for $s > T$ and $\lambda_k(s, x, \xi) = \lambda_k(0, x, \xi)$ for $s < 0$.

From (3.3) and (3.11), it is easy to see (cf. (2.7) and (2.8)) that

$$\begin{align*}
|\tilde{\lambda}_1 - \tilde{\lambda}_2| &\geq c_0(\xi), \quad c_0 > 0, \\
\tilde{\lambda}_k - \lambda_k &\in C([0, T]; S^0_\psi), \\
\tilde{\psi}^h \tilde{\lambda}_k &\in C([0, T]; S^h_\psi), \quad h \geq 1.
\end{align*} \quad (3.13)$$

Thus the operator $P$ can be factorized as follows:

$$\begin{align*}
P = \widehat{\partial_t - i\tilde{\lambda}_2(t, x, D_x)}(\partial_t - i\tilde{\lambda}_1(t, x, D_x)) \\
+ R_0(t, x, D_x) + R_1(t, x, D_x)\partial_t, \\
R_j &\in C([0, T]; S^{1-j}_\psi), \quad j = 0, 1.
\end{align*} \quad (3.14)$$

For any given function $f = f(t, x)$, now we want to reduce the scalar equation $Pu = f$ to a $2 \times 2$ system $LU = F$. Let us define $U =^t (u_0, u_1)$ by

$$u_0 = \langle D_x \rangle u, \quad u_1 = (\partial_t - i\tilde{\lambda}_1(t, x, D_x))u. \quad (3.15)$$
From (3.14), the scalar equation is equivalent to a \(2 \times 2\) system \(LU = F\) with \(F = f(0, f)\) and

\[
\begin{cases}
L = \partial_t - i\Lambda(t, x, D_x) + B(t, x, D_x), \\
\Lambda = \begin{bmatrix} \lambda_1 & \langle D_x \rangle \\
0 & \lambda_2 \end{bmatrix}, \\
B(t, x, \xi) \in C([0, T]; S^0_\psi).
\end{cases}
\]

(3.16)

The operator \(\Lambda\) can be diagonalized by means of

\[
Q = \begin{bmatrix} 1 & q \\
0 & 1 \end{bmatrix}, \quad q(t, x, \xi) = \langle \xi \rangle / (\tilde{\lambda}_2(t, x, \xi) - \tilde{\lambda}_1(t, x, \xi)) \quad \text{for large } |\xi|.
\]

From (3.13) we have \(Q, Q^{-1} \in C([0, T]; S^0), \partial_t Q, \partial_t Q^{-1} \in C([0, T]; S^0_\psi).\) Thus, after having performed the diagonalization of the principal part of \(L\), we obtain

\[
\begin{cases}
L_1 := Q^{-1}LQ = \partial_t - i\Delta(t, x, D_x) + B_1(t, x, D_x), \\
\Delta = \begin{bmatrix} \tilde{\lambda}_1 & 0 \\
0 & \tilde{\lambda}_2 \end{bmatrix}, \\
B_1(t, x, \xi) \in C([0, T]; S^0_\psi).
\end{cases}
\]

(3.17)

For the system \(L_1\) we have the following energy estimate:

**Proposition 3.2.** For every \(s, \delta > 0\) there is a constant \(C_{s, \delta} > 0\) such that

\[
\|U(t)\|_{H^{s+\delta}}^2 \leq C_{s, \delta} \left( \|U(0)\|_{H^s}^2 + \int_0^t \|L_1U(\tau)\|_{H^s}^2 d\tau \right), \quad t \in [0, T]
\]

(3.18)

for all \(U \in \bigcap_{j=0}^1 C^j([0, T]; H^{s+1-j}).\)

**Proof.** For a sufficiently large \(\lambda\), let us consider the symbol

\[
w(t, \xi) = e^{i\lambda\tau}\psi(\xi) = (1 + \langle \xi \rangle)^{i\lambda\tau}\phi(\langle \xi \rangle).
\]

(3.19)

Since \(\phi(y) \downarrow 0\) for \(y \to +\infty\), from (3.7) we have

\[
w(t, \cdot) \in \bigcap_{\delta > 0} S^\delta, \quad 1 \geq w^{-1}(t, \xi) \geq C_\delta \langle \xi \rangle^{-\delta}, \quad C_\delta > 0
\]

(3.20)

for every \(\delta > 0\), so we obtain the inequality (3.18) if we prove

\[
\|V(t)\|_{H^s}^2 \leq C_s \left( \|V(0)\|_{H^s}^2 + \int_0^t \|L_2V(\tau)\|_{H^s}^2 d\tau \right), \quad t \in [0, T]
\]

(3.21)
for all $V \in \bigcap_{j=0}^{1} C^j([0, T]; H^{s+1-j})$, $V = w(t, D_x U)$, with

$$L_2 := w^{-1}(t, D_x) L_1 w(t, D_x).$$  
(3.22)

From (3.17), we have

$$L_2 = \partial_t - i\Delta(t, x, D_x) + \lambda \psi(D_x) + B_1(t, x, D_x) + R(t, x, D_x)$$  
(3.23)

with $R \in C([0, T]; S^{-1+\delta})$ for every $\delta > 0$. In particular, it is sufficient to prove (3.21) for $s = 0$ since $\langle D_x \rangle^s L_2 \langle D_x \rangle^{-s}$ is of the same type of $L_2$.

A sufficiently large $\lambda$ guarantees that $\lambda \psi(D_x) + B_1(t, x, D_x)$ becomes a positive operator since $B_1 \in C([0, T]; S^0_\psi)$. With a new remainder $R \in C([0, T]; S^{-1+\delta})$, after application of sharp Gårding’s inequality we obtain (3.21) for $s = 0$ by the usual Gronwall method applied to $\frac{d}{dt} \|V(t)\|^2_{H^0}$. □

We get the desired inequality (3.6) from (3.18). This completes the proof of Theorem 3.1.

**Remark 3.3.** The same proof applies also to the limit cases $\omega(\zeta) = 1/|\log \zeta|$ (Lipschitz coefficients) and $\omega(\zeta) = 1$ (log-Lip coefficients). The weight $w(t, \zeta)$ in (3.19) becomes respectively $w(t, \zeta) = e^{\lambda t}$ and $w(t, \zeta) = \langle \zeta \rangle^{\lambda t}$, so we recover the respective results quoted in the introduction about well-posedness without loss of derivatives and with finite loss proportional to $t$.

**References**


