A Priori Fourier Analysis for 2.5D Finite Elements Simulations of Logging-While-Drilling (LWD) Resistivity Measurements

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Abstract
Triaxial induction measurements provided by LWD tools generate crucial petrophysical data to determine several quantities of interest around the drilled formation under exploration, such as a map of resistivities. However, the corresponding forward modeling requires the simulation of a large-scale three-dimensional computational problem for each tool position. When the material properties are assumed to be homogeneous in one spatial direction, the problem dimensionality can be reduced to a so-called 2.5 dimensional (2.5D) formulation.

In this paper, we propose an \textit{a priori} adaptive algorithm for properly selecting and interpolating Fourier modes in 2.5D simulations in order to speed up computer simulations. The proposed method first considers an adequate range of Fourier modes, and it then determines a subset of those which need to be estimated via solution of a Partial Differential Equation (PDE), while the remaining ones are simply interpolated in a logarithmic scale, without the need of solving any additional PDE. Numerical results validate our selection of Fourier modes, delivering superb results in real simulations when solving via PDE only for a very limited number of Fourier modes (below 50%).

\textit{Keywords:} LWD simulations, 2.5D simulations, Fourier analysis, \textit{a priori} adaptativity, Formation exploration, Quantities of interest, Electromagnetic simulations

1 Introduction
Triaxial induction logging plays nowadays an important role for characterizing Earth formation in the context of oil and gas geophysical exploration [6, 2]. Complex measured data provided by LWD tools help analysts to determine and study several formation quantities of interest such
as formation anisotropy [5]. To that end, these tools incorporate three mutually orthogonal transmitter coils located at a fixed position along the tool axis, and three receivers coils all placed at a fixed distance from the transmitters [7]. Combining every transmitter with every receiver, nine magnetic field components are measured, the so-called XX, XY, XZ, YX, YY, YZ, ZX, ZY and ZZ couplings [5]. Although all components play a significant role for inversion [8], we restrict our study to the ZZ component, since it provides prominent information [14].

Inversion techniques using numerical simulations involving LWD measurements require solving multiple times a computationally expensive forward problem described by Maxwell’s equations [5]. For every iteration of the inversion algorithm, several logging positions are considered along the tool trajectory of interest, say $N_p$ (typically several hundreds), each one requiring solving a full 3D problem. Thus, if the inversion procedure consists of $N_i$ iterations, $N_i N_p$ forward problems need to be solved.

Alleviating the computational cost of the forward problem impacts drastically in the total computational timespent in the inversion process. To that end, 2.5D simulations of this problem have been considered in the past [10, 9, 11, 1] by introducing a Fourier transform along one of the two horizontal axis (say y-axis), for which the material properties are constant, i.e., materials are assumed to be two-dimensional. The main advantage of this approach is that the full 3D solution is obtained by combining the partial solutions of a sequence of 2D uncoupled problems, one for each Fourier mode under consideration. The main issues that impinge upon the computational cost of a 2.5D algorithm are related to the adaptativity in (i) the $x$–$z$ plane (in the configuration space); and in (ii) the Fourier space. While in the configuration space much research effort has been devoted to electromagnetics [3, 4], adaptativity in Fourier space is less common. In [13], only applicable to finite difference schemes, authors proposed a criteria based on a finite difference node spacing to optimally select the range of Fourier modes, but did not consider the study of the optimal Fourier modes for interpolation, nor for specific quantities of interest related to LWD simulations.

Given a 2.5D problem, in this work we introduce an a priori algorithm for selecting a subset of Fourier modes to be computed by solving numerically the corresponding PDE, while the remaining modes are merely determined by interpolation. This algorithm can also be interpreted as an a priori integration rule in the Fourier domain. The initial design decision corresponds to the selection of a set of uniformly distributed homogeneous (0D) materials, including the materials that appears in the original 3D problem. Then, we build a criteria for selection of Fourier modes that provides accurate results for all possible 0D problems. The resulting algorithm is directly applied to 2.5D problems, assuming that for the higher dimensional case we will obtain a similar accuracy for the Fourier integration. As a result, it enables an a priori construction of a Fourier integration rule in 2.5D, and due to the observed low number of Fourier modes needed to be computed by solving the corresponding PDE, it is indeed cheaper than a posteriori adaptativity. In any case, it is also complementary to most a posteriori adaptive methods.

The rest of the article is organized as follows: the mathematical problem for LWD forward simulations is described in the next section. Then, the proposed a priori algorithm is described. We subsequently show the application of the algorithm via an example for a particular LWD tool configuration. Finally, section 5 summarizes and concludes the paper.
2 The mathematical problem

We consider the two time-harmonic curl Maxwell’s equations in a unbounded, piecewise-homogeneous and isotropic medium:

\[
\nabla \times \mathbf{E} = i \omega \mu \mathbf{H} - \mathbf{M},
\]

(1)

\[
\nabla \times \mathbf{H} = (\sigma - i\omega \varepsilon) \mathbf{E},
\]

(2)

satisfying the Sommerfeld radiation condition [12], where \( \mathbf{E} \) is the electric field, \( \mathbf{H} \) is the magnetic field, \( i \) is the imaginary unit, \( \mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \) is the free-space magnetic permeability, \( \sigma \) is the piecewise-homogeneous conductivity of the medium, \( \varepsilon = \varepsilon_0 \approx 8.85 \times 10^{-12} \text{ F/m} \) is the constant permittivity of the medium, \( k \) is the wave number, \( k^2 = \omega \mu (\omega \varepsilon + i\sigma) \), \( \omega = 2\pi f \) is the angular frequency and \( f \) is the constant frequency determined by a fixed mode (frequency) of operation of the LWD tool. We will also refer to the resistivity of the medium as \( \rho = \sigma^{-1} \).

The time-harmonic magnetic dipole sources \( \mathbf{M} \) induced by the tri-axial transmitter coils mounted on the LWD tool are mathematically modeled by a time-dependent factor \( e^{i\omega t} \) (where \( t \) is time), its magnetic moment \( \mathbf{M}_0 \) and the transmitter location \( \mathbf{x}_{tx} = (x_{tx}, y_{tx}, z_{tx}) \), so that \( \mathbf{M} = \mathbf{M}_0 \delta(\mathbf{x}_{tx}) \). In this work, materials are considered two-dimensional, only varying along \( x \)- and \( z \)-axis, as shown in Fig. 1 for an scenario with a geological fault and three layers with piecewise constant resistivities.

Figure 1: Illustration of the LWD tool whose borehole trajectory dip angle is assumed to be 80 degrees and crosses a geological fault. Materials are assumed to be two-dimensional varying only along the \( x \)-axis (horizontal direction) and the \( z \)-axis (vertical direction).

Despite the numerical solution of the magnetic field is to be computed everywhere in \( \Omega \) using a numerical method such as the Finite Element Method (FEM), we are only interested in having a good numerical accuracy at the two receivers of the LWD tool, namely \( R_{x_1} \) and \( R_{x_2} \). Also, we will restrict our study to the two commonly used frequencies of operation for LWD tools, namely \( f_1 = 2 \text{ MHz} \) and \( f_2 = 400 \text{ KHz} \), and to conductivities in the range of \( \mathcal{S}_\sigma := [\sigma_{\text{min}}, \sigma_{\text{max}}] = [1/\rho_{\text{max}}, 1/\rho_{\text{min}}] = [1/10^3, 1/10^{-1}] \), characteristic of the Earth’s formation under study. In our simulations, for \( x \) and \( z \) directions, we truncate our computational domain far enough from the source location, where we can safely impose a homogeneous Dirichlet boundary condition. In \( y \) direction, we choose a period \( L \) large enough so that the periodic Fourier series representation
of the non-periodic source does not induce significant numerical errors. For the two frequencies of operation defined above, we consider \( y \in [-L/2, L/2] \), \( L = 4\pi \).

## 3 The Fourier Modes Selection algorithm

Our algorithm is based on the assumption that the behavior of the 0D solution for different material properties, including those that appear in the original 3D problem, provides us a valuable information into the behavior of the solution at the position of the receivers for the original 3D problem with 2D materials. The 0D solution at the receivers of the tool depends solely on the distance from the source but not on its particular location. Considering that the source is located at the origin of the coordinate system, the \( H_{zz} \) component of the magnetic field solution (the 0D solution) is given by [15]:

\[
H_{zz}(x, y, z) = \frac{\pi M_0 e^{ikd}}{4\pi d} \left( k^2 + \frac{ik}{d} - \frac{k^2 z^2 + 1}{d^2} - \frac{3ik z^2}{d^3} + \frac{3z^2}{d^4} \right),
\]

where \( d = \sqrt{x^2 + y^2 + z^2} \) is the distance from the source. In Fig. 2, we show \(|H_{zz}|\) as a function of \( y \) for the two frequencies of interest. We distinguish two regions with different behavior: one near the source, around \( y = 0 \), where the solution varies strongly; and a second region where the solution decays with a factor dominated by the term \( \frac{e^{ikd}}{4\pi d} \).

![Figure 2: \(|H_{zz}|\) in logarithmic scale, as a function of \( y \), for \( \sigma = 10^{-1} \) S/m.](image)

We now represent the periodic solution along the \( y \)-axis as the following Fourier series

\[
H_{zz}(x, y, z) = \sum_{n=-\infty}^{+\infty} \tilde{H}_{zzn}(x, z)e^{iny2\pi/L},
\]

where for every Fourier mode \( n \), the Fourier coefficient \( \tilde{H}_{zzn}(x, z) \) is given by

\[
\tilde{H}_{zzn}(x, z) = \frac{1}{L} \int_{-L/2}^{L/2} H_{zz}(x, y, z)e^{-iny2\pi/L}.
\]

Fig. 3 shows how \(|\tilde{H}_{zzn}(x, z)|\) decays as a function of \( n \) at the position of the receiver \( \mathcal{R}_{x_1} \). The
smooth decay shown for the two tool frequencies of operation of interest are qualitatively very similar for all \( \sigma \in S_{\sigma} \). Small wrinkles formed in the region where \( n \in [70, 90] \) are related to the abrupt change in the spectrum of \( H_{zz} \) observed at the vicinity of \( y = \pm 0.8 \) m (Fig. 2). By inspection of Fig. 3, for this particular resistivity distribution, we can: (i) truncate the Fourier series according to an error tolerance criteria; and (ii) discern, in the non-truncated regime, which Fourier modes are to be computed and which ones are to be interpolated (using the computed ones).

In order to determine the criteria to truncate the Fourier series for a general 0D case, we now turn our attention into how we compute the first quantity of interest (QoI) in our simulations, which is the attenuation \( A \), given by

\[
A = \log \left| \frac{H_{zz}(R_{x1})}{H_{zz}(R_{x2})} \right| = \log |H_{zz}(R_{x1})| - \log |H_{zz}(R_{x2})|. \tag{6}
\]

Substituting the Fourier series expansion in (4) into (6) gives

\[
A = \log \left| \sum_{n=-\infty}^{+\infty} \tilde{H}_{zzn}(x, z)e^{iny\pi/L} \right|_{R_{x1}} - \log \left| \sum_{n=-\infty}^{+\infty} \tilde{H}_{zzn}(x, z)e^{iny\pi/L} \right|_{R_{x2}}
\]

\[
= \log \left| \sum_{n=-\infty}^{+\infty} \tilde{H}_{zzn}(R_{x1}) \right| - \log \left| \sum_{n=-\infty}^{+\infty} \tilde{H}_{zzn}(R_{x2}) \right|. \tag{7}
\]

In the above, we have selected \( y = 0 \) for the location of both transmitters and receivers. This selection is justified because 0D solutions are invariant under rotations around the \( z \)-axis. For every \( \sigma \) in \( S_{\sigma} \), by truncating the Fourier series in (4) at fixed value \( N(\sigma) \), expressed as

\[
\tilde{H}_{zz}(x, y, z) = \sum_{n=-N(\sigma)}^{N(\sigma)} \tilde{H}_{zzn}(x, z)e^{iny\pi/L}, \tag{8}
\]

we introduce an error in the estimation of \( A \). Denoting \( \hat{A}(N(\sigma)) \) the estimation of \( A \), we now consider its corresponding relative error, defined as

\[
A_{err}(N(\sigma)) = \left| \frac{\hat{A}(N(\sigma)) - A}{A} \right| \cdot 100. \tag{9}
\]
We select the value of $N(\sigma) \in \mathbb{N}$ in Eq. (8) as the minimum value $n \in \mathbb{N}$ such that $A_{err}(n)$ is under the tolerance $\alpha = 0.1\%$. That is,

$$N(\sigma) := \min \{ m \in \mathbb{N} | A_{err}(m) < \alpha \}.$$  

(10)

Finally, the optimal value for $N$, denoted by $N_{opt}$, is selected according to the worst-case, that is

$$N_{opt} = \max_{\sigma \in \sigma_{\min} \ldots \sigma_{\max}} N(\sigma).$$  

(11)

This value is shown in Fig. 4 for the two frequencies of operation of interest. For each frequency of operation, we show in blue quantity $A_{err}$ as a function of $n$, for different conductivities given in $S_{\sigma}$. The worst-case crossing the red line at constant value $\alpha$ suggests selecting as a reasonable value $N_{opt}$ for 2 MHz, and $N_{opt}$ = 50 for 400 KHz. Once we select the range of Fourier modes to be calculated, we describe how to determine the subset of modes to be computed by the FEM and the complementary subset of modes to be interpolated. As mentioned before, our goal is to approximate the well log at the receivers. More precisely, the objective is to estimate accurately enough the quantities $\log |H_{zz}(R_{x_1})|$ and $\log |H_{zz}(R_{x_2})|$.

We first analyze this quantity at the first receiver, for which the Fourier series of $\hat{H}_{zz}$ takes the following expression

$$\hat{H}_{zz}(R_{x_1}) = \sum_{n=-N(\rho)}^{N(\rho)} \tilde{H}_{zzn}(R_{x_1}) = \sum_{n=-N(\rho)}^{N(\rho)} 10^{\log \tilde{H}_{zzn}(R_{x_1})}.$$  

(12)

We proceed now to define $p_p(n)$ as the polynomial interpolant of degree $p$ for the target function $f(n)$ we want to interpolate,

$$f(n) := \log \tilde{H}_{zzn}(R_{x_1}),$$  

(13)

that we can express by its polynomial interpolant $p_p(n)$ and its corresponding truncation error $e_p(n)$, as

$$f(n) = p_p(n) + \frac{f^{p+1}(\xi)}{(p+1)!} \prod_{i=0}^{p} (n - n_i) := p_p(n) + e_p(n),$$  

(14)
for some $\xi \in (n_{\text{min}}, n_{\text{max}})$, where $n_i$ is the $i$-th Fourier mode chosen for interpolation. Therefore, we can now rewrite $\hat{H}_{zz}(R_{x_1})$ as a function of its interpolant $p_p(n)$ and its error $e_p(n)$, as follows

$$\hat{H}_{zz}(R_{x_1}) = \sum_{n=-N(p)}^{N(p)} 10^{p_p(n)} \cdot 10^{\left(\frac{p+p+1}{(p+1)}\right) i \Pi_{i=0}^{(n-n_i)}} = \sum_{n=-N(p)}^{N(p)} 10^{p_p(n)} \cdot 10^{e_p(n)},$$ (15)

as well as its approximation in (12), as

$$\hat{H}_{zz}(R_{x_1}) \approx \sum_{n=-N(p)}^{N(p)} 10^{p_p(n)}.$$ (16)

With the above expressions and aiming to satisfy the criteria in (10), we can now control the error induced by interpolation for the following quantity

$$\left|\log \left(\frac{\sum_{n=-N(p)}^{N(p)} 10^{p_p(n)} \cdot (1 - 10^{e_p(n)})}{H_{zz}(R_{x_1})}\right) + 1\right|.$$ (17)

That is, we need

$$\left|\sum_{n=-N(p)}^{N(p)} \frac{\hat{H}_{zz}(R_{x_1})}{H_{zz}(R_{x_1})} \cdot (1/10^{e_p(n)} - 1)\right|.$$ (18)

To be small, in particular, very small for every $n$ in the range of interpolation. Let us define the local error function $\text{locerr}(R_{x_1}, n) := \left|\frac{\hat{H}_{zz}(R_{x_1})}{H_{zz}(R_{x_1})} \cdot (1/10^{e_p(n)} - 1)\right|$. In figures 5 and 6 we show the behavior of the two factors of $\text{locerr}$ as a function of $n$, for different polynomials orders $p$ with the choice $(n-n_i) = p$, and Fig. 7 displays the overall error. These figures suggest that it is not advisable to interpolate between the first ten Fourier modes. However, from the mode $n = 10$ on, it is recommended and safe to interpolate with a polynomial of increasing degree $p$ (starting with $p = 2$) and computing by FEM only one Fourier mode every $p$ modes, saving therefore $(1 - 1/p) \cdot 100\%$ of the total computations.

4 Numerical examples

In this section we consider a numerical example to show the applicability of our algorithm. The formation under consideration is the one depicted in Fig. 1. The coordinate system is defined in meters (m). The geological fault occurs at the origin of the x-axis (horizontal axis), defined in the range $x \in [-50, 50]$, whereas the vertical z-axis is defined in $z \in [4950, 5050]$. The logging trajectory is fixed at 100 evenly distributed positions, starting at the position

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Figure 5: $|\tilde{H}_{zz}(R_x) / H_{zz}(R_x)|$, as a function of the Fourier mode, $n$.

Figure 6: $|1/10^p(n) - 1|$, as a function of the Fourier mode, $n$. Each line corresponds to a particular value of $p$, ranging from 1 to 6.

Figure 7: $loc_{err}(R_x, n)$, as a function of the Fourier mode, $n$. Each line corresponds to a particular value of $p$, ranging from 1 to 6.
\((x_{\text{init}}, z_{\text{init}}) = (-7.4, 4998.7)\) and finishing at \((x_{\text{end}}, z_{\text{end}}) = (7.45, 5001.32)\), for \(y = 0\) and a dip angle of 80 degrees. The tool operates at a frequency of 2 MHz. For this test, our algorithm suggests to compute a range of 26 Fourier modes, from the mode 0 to 25. We select two different subset of Fourier modes, \(F_{\text{interp}11} := \{7, 9, 11, 13, 15, 16, 18, 19, 21, 22, 24\}\) and \(F_{\text{interp}15} := \{7, 9, 11 - 14, 16 - 24\}\), to be estimated by interpolation, while the remaining ones are computed by the 2.5D FEM. In Fig. 8 we show the relative error in percentage when computing the QoI defined in (6) as a function of the tool trajectory for the two cases of interpolating the subset of Fourier modes defined in \(F_{\text{interp}11}\) and \(F_{\text{interp}15}\). In both cases, this error is always below 1%, as predicted theoretically by our method for all 0D cases. It is also remarkable that the number of interpolated modes for the case \(F_{\text{interp}15}\) represents more than 50% of the total modes in the range of selected Fourier modes.

![Figure 8: Relative error (in %) for the QoI defined in (6), shown for the two cases of interpolating the subsets of Fourier modes \(F_{\text{interp}11}\) and \(F_{\text{interp}15}\).](image)

5 Conclusions

We have studied the qualitative and quantitative behavior of the Fourier series over a direction along which materials are constant in a given LWD 3D problem. The analysis focused on the computations of the attenuation, a quantity of interest for inversion that involves the solution of the so-called \(ZZ\)–component of the magnetic field at the receivers of the LWD tool.

We first considered a 0D model. Then, for such cases, we designed an \(a\) \(p\)riori \ rule to replace computation of the solution corresponding to certain Fourier modes by the interpolation in the logarithmic scale when computing the attenuation in an LWD tool. We then selected the
interpolation rule that satisfies a given tolerance error for all 0D problems, and we applied it to 2.5D problems along the direction where materials are constant.

Numerical results showed that such a priori rule for computing/interpolating Fourier modes produced prominent savings from the computational point of view (over 50%) while the discretization error was maintained below 1%.

References


