Gallai–Milgram properties for infinite graphs

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Abstract


We discuss extensions of the Gallai–Milgram theorem to infinite graphs. We define a path to be a directed graph whose transitive closure is a linear ordering. We show that an undirected graph with no infinite independent set is covered by finitely many pairwise disjoint paths; moreover for a given integer k this graph is covered by \( \leq k \) (resp. \( \leq k \) pairwise disjoint) paths if each finite set of vertices is contained in the union of \( \leq k \) (resp. \( \leq k \) pairwise disjoint) paths.

Hence an undirected graph with no independent set of size \( k + 1 \) is covered by \( \leq k \) pairwise disjoint paths. We prove also that if to each edge \((x, y)\) of a countable path is associated an element \( \theta(x, y) \) of a finite group, then some edges can be deleted so that the new graph is still a path and the product \( \theta(x_0, x_1) \cdot \theta(x_1, x_2) \cdot \ldots \cdot \theta(x_{n-1}, x_n) \) of the elements of the group along any finite path \((x_0, x_1, \ldots, x_n)\) of the new graph depends only upon the extremities \(x_0, x_n\) of the path.

1. Introduction

If \( G = (V, E), (E \subseteq V \times V) \), is a finite directed graph with no independent set of size \( k + 1 \), \((k < \omega)\), the Gallai–Milgram theorem [6] says that \( V \) can be covered by \( \leq k \) pairwise disjoint paths, where a path is a finite sequence of distinct vertices \((x_0, \ldots, x_n)\) such that \((x_i, x_{i+1}) \in E \) for \( i < n \). If the directed graph is an ordered set \( P = (P, \leq) \) we get Dilworth's theorem [5]: If \( P \) has no antichain of size \( k + 1 \), \((k < \omega)\), then \( P \) can be covered by \( \leq k \) (pairwise disjoint) chains. Dilworth's theorem can be extended in this form to any infinite ordered set, but we need the finiteness assumption on \( k \). The proof, using a standard compactness argument, follows from the fact that a set is a chain if and only if all its finite subsets are chains. Hence the transitivity of the ordering is involved. For the Gallai–Milgram theorem the situation is different; elementary compactness

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arguments do not seem to yield any useful information about infinite directed graphs for the case when \( k \) is finite. In fact, we still do not know if the theorem is true or false in this case. Of course, in order that the question should make some sense, it is first necessary to extend the notion of a path in a directed graph which allows for the possibility of paths of arbitrary (countable or uncountable) order type. We adopt the following fairly natural definition: A path is a directed graph whose transitive closure is a linear (i.e. total) ordering; and a path in a directed graph is simply a subgraph (not necessarily induced) which is a path. The question is then whether the following extension of the Gallai–Milgram theorem to infinite graphs holds:

If \( 1 \leq k < \omega \) and \( \mathcal{G} = (V, E) \) is a directed graph with no independent set of size \( k + 1 \), then \( V \) is covered by \( \leq k \) pairwise disjoint paths. (*)

A necessary condition for the conclusion of (*) is that any finite subset of \( V \) is contained in \( \leq k \) pairwise disjoint finite paths. To conjecture that this condition is also sufficient is stronger than (*) since, by the Gallai–Milgram theorem, the hypothesis of (*) is equivalent to the fact that any finite subset of \( V \) is the union of \( \leq k \) pairwise disjoint paths. We show in Section 3 that this stronger conjecture, and also plausible weakenings of it, are false in general. In Section 4.2 we prove (cf. Theorem 2) that this conjecture holds for the undirected graphs with no infinite independent set, where an undirected graph is defined to be a directed graph \( \mathcal{G} = (V, E) \) such that \((x, y) \in E \) if and only if \((y, x) \in E \) for all \( x, y \in V \). We also show in Section 4.2 (cf. Theorem 1) that the vertex set of an undirected graph with no infinite independent set is covered by finitely many pairwise disjoint paths. Some similar properties concerning the undirected graphs with no independent set of size \( v \), where \( v \) is an arbitrary, finite or infinite, cardinal, are surveyed in Section 4.3 (cf. Theorems 2' and 2''). The notions of path and chain that we use are studied in Section 2. In Section 5, we deal with a different problem, still related to the spirit of the Gallai–Milgram theorem since it concerns the linkage of different finite sets of the same size along a countable path. We prove (cf. Theorem 4) that if \( \theta : E \to H \) is a function of the edge set of a countable path \( \mathcal{G} = (V, E) \) to a finite group \( \mathcal{H} = (H, \ast) \), then there is a subset of edges \( E' \subseteq E \) such that the graph \( \mathcal{G}' = (V, E') \) is still a path and the element \( \tilde{\theta}(p) = \theta(x_0, x_1) \ast \theta(x_1, x_2) \ast \cdots \ast \theta(x_{n-1}, x_n) \in H \) depends only upon \( x_0 \) and \( x_n \) if \( p = (x_0, x_1, \ldots, x_n) \) is a finite path of the graph \( \mathcal{G}' \). The interpretation in terms of linking finite sets of the same size \( n \) corresponds to the case where \( \mathcal{H} \) is the group of the permutations of a set of size \( n \).

2. Notions of path and chain in infinite graphs

2.1. A (directed) graph is an ordered pair of sets \( \mathcal{G} = (V, E) \) where \( E \subseteq V \times V \). We say that \( \mathcal{G} \) is an undirected graph if \( E = E^{-1} \), with \( E^{-1} = \{(y, x) \in E \mid (x, y) \in E \} \).
An undirected graph can be identified in an obvious way with an ordered pair of sets $G = (V, E)$ where $E \subseteq [V]^2 \cup \{V, E' : 1 \leq |e| \leq 2\}$. The degree of an element $x \in V$ in an undirected graph $G = (V, E)$ is the cardinal $|(y \in V : x \neq y, (x, y) \in E)|$. The undirected graph $G = (V, E)$ is bipartite with sides $X$ and $Y$ if $V = X \cup Y$, $X \cap Y = \emptyset$, $E \subseteq (X \times Y) \cup (Y \times X)$. The graph $G' = (V', E')$ is a subgraph of the graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. Hence an induced subgraph of a graph $G = (V, E)$ is entirely determined by its vertex set $V'$, and $G' = (V', V')$ will denote the induced subgraph of $G$ with vertex set $V'$. The size of a graph is the cardinality of its vertex set; in particular $G = (V, E)$ is finite if $V$ is. A subset $X \subseteq V$ is an independent set of a graph $G = (V, E)$ if $(x, y) \notin E$ for all distinct $x, y \in X$. A graph is reflexive if $(x, x) \in E$ for every $x \in V$, transitive if $(x, y) \in E$ and $(y, z) \in E$ imply $(x, z) \in E$ for every $x, y, z \in V$. $G$ is a preordering if it is reflexive and transitive; it is an ordering if moreover at most one of the two relations $(x, y) \in E$, $(y, x) \in E$, holds for all distinct $x, y \in V$. If at least one of these two relations holds for all distinct $x, y \in V$ the graph $G$ is complete; that means that $V$ is a clique of $G$. If exactly one of these two relations holds for all distinct $x, y \in V$, $G$ is a tournament. A linear ordering is an ordering which is also a tournament. If the graph is an ordering, we use the terms chain and antichain for clique and independent set. An ordering is well founded if each non-empty subset of the vertex set contains a minimal element. A well ordering is a linear ordering which is well founded. A linear ordering is scattered if it does not contain an isomorphic copy of the natural ordering on the rational numbers. Finally $R \subseteq V \times V$ is an equivalence relation on $V$ if the graph $(V, R)$ is reflexive, undirected, and transitive; an $R$-equivalence class is a subset $C$ of $V$ such that $C = \{y \in V : (x, y) \in R\}$ for some $x \in V$. The transitive closure of a graph $G = (V, E)$ is the graph $G = (V, E')$, where $E'$ is the smallest set of edges containing $E$ such that $G$ is a preordering. For $x, y \in V$, we have $(x, y) \in E'$ if there are an integer $n \geq 0$ and a sequence $(x_0, \ldots, x_n)$ of vertices such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E$ if $i < n$. A graph $G = (V, E)$ is acyclic if its transitive closure is an ordering, which means there is no cycle, i.e. no sequence $(x_0, \ldots, x_n)$ of distinct vertices, where $n$ is an integer $\geq 1$ such that $(x_n, x_0) \in E$ and $(x_i, x_{i+1}) \in E$ if $i < n$.
element of its transitive closure (when it exists) and the extremities of a path are its first and its last vertex (when they exist). $X \subseteq P$ is cofinal, (respectively coinitial), in the path $\mathcal{P} = (P, E)$ if $X$ is cofinal, (respectively coinitial), in its transitive closure $\mathcal{P} = (P, \leq)$, i.e. if there is some $y \in X$ such that $x \leq y$, (respectively $y < x$), for any $x \in P$. A path of a graph $\mathcal{G}$ is a subgraph of $\mathcal{G}$ which is a path. We shall say that $W \subseteq V$ is contained in, (respectively $W \subseteq V$ is the union of), some paths of the graph $\mathcal{G} = (V, E)$ if $W$ is contained in, (respectively $W$ is), the union of their vertex sets. The intersection of two paths will in fact denote the intersection of their vertex sets. In particular, we shall say that two paths are disjoint if their vertex sets are. A chain of a graph $\mathcal{G} = (V, E)$ is a subset $C$ of $V$ such that any finite subset $F$ of $C$ is contained in a (finite) path of $\mathcal{G}$.

2.4. Examples; basic observations. In Figs. 1, 2, 3, we give examples of paths which are respectively well ordered, scattered, nonscattered, (and which are not linear orderings); (note that two vertices $x$, $y$ are joined by an arrow $x \rightarrow y$ to indicate that $(x, y) \in E$). $\omega$, $\omega^*$, $\eta$ denote respectively the order types of the natural orderings on the positive integers, on the negative integers, and on the

![Fig. 1. A path of order type $\omega + 1$.](image-url)
Fig. 2. A path of order type $\omega + \omega^*$. 

Fig. 3. A path of order type $\omega + (\omega^* + \omega) \cdot \eta + \omega^*$. 

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rational numbers. If $\mathcal{P}_0 = (P_0, \preceq_0)$ and $\mathcal{P}_1 = (P_1, \preceq_1)$ are linear orderings of order types $\alpha_0$ and $\alpha_1$, then $\alpha_0 + \alpha_1$ and $\alpha_0 \cdot \alpha_1$ denote respectively the order types of the linear orderings $\mathcal{P} = (P, \preceq)$ and $\mathcal{P}' = (P', \preceq')$ defined in the following way:

If $P_0 \times \{0\} \cup (P_1 \times \{1\})$ and $(x, i) \preceq (y, j)$ if and only if $i < j$ or $i = j$ and $x \preceq_1 y$ in $\mathcal{P}_1$; $P' = P_0 \times P_1$ and $(x_0, x_1) \preceq' (y_0, y_1)$ if and only if $x_1 \preceq_1 y_1$, or $x_1 = y_1$ and $x_0 \preceq_0 y_0$.

A well ordered path of the graph $\mathcal{G} = (V, E)$ can be identified with a sequence $(x_\beta)_{\beta \in \alpha}$ of distinct vertices, where $\alpha$ is an ordinal $\geq 1$, such that $\{ \gamma < \beta : (x_\gamma, x_\beta) \in E \}$ is cofinal in $\beta$ whenever $\beta < \alpha$. In particular a finite path of $\mathcal{G}$ can be identified with a sequence $(x_0, \ldots, x_\alpha)$ of distinct vertices, where $\alpha$ is an integer $\geq 0$, such that $(x_i, x_{i+1}) \in E$ if $i < \alpha$.

The union of an up-directed family of chains is a chain, (a set $\mathcal{F}$ is up-directed if for every $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ such that $A \cup B \subseteq C$). Therefore, by Zorn's lemma, each chain is contained in a maximal one. We shall see in Example 1 of Section 3 that this does not generally hold for vertex sets of paths. A subset of a chain is a chain and if each finite subset of a set $C$ is a chain, then $C$ is a chain.

By a standard compactness argument, the following statements are equivalent:

(i) $C$ is a chain of $\mathcal{G}$.

(ii) There is a linear ordering $\mathcal{C}$ on $C$, such that for every finite subset $F$ of $C$ there is a finite path $\mathcal{P}_F$ of $\mathcal{G}$ whose vertex set contains $F$ and is such that the ordering induced by $\mathcal{P}_F$ on $F$ coincides with the induced ordering $\mathcal{C} \restriction F$.

Indeed, let us assume that $C$ is a chain of $\mathcal{G}$. For each finite subset $F$ of $C$, let $\mathcal{U}_F = \{ R \subseteq C \times C : (C, R) \restriction F$ is a linear ordering which is induced on $F$ by the linear ordering defined by some finite path of $\mathcal{G}$ containing $F$). The $\mathcal{U}_F$'s, ($F$ finite subset of $C$), are closed non-empty subsets of the compact space $2^{C \times C}$. They finitely intersect since $\mathcal{U}_{F_1} \supseteq \mathcal{U}_{F_2}$ for $F_1 \subseteq F_2$. Hence there is $R \in \bigcap \{ \mathcal{U}_F : F$ finite subset of $C \}$, and $\mathcal{C} = (C, R)$ is a linear ordering which has the desired properties.

The notion of chain in an acyclic graph reduces to the classical notion of chain, (i.e. clique), of the transitive closure. For a transitive graph $\mathcal{G}$, the notions of chain and vertex set of a path both reduce to the notion of clique, (therefore there is no ambiguity for orderings in the notions of chain). But in general a clique of the transitive closure of a graph $\mathcal{G}$ is not a chain of $\mathcal{G}$, as shown by Fig. 4. The graph $\mathcal{G} = (V, E)$ (Fig. 4) is a finite undirected graph whose transitive closure is complete, but $V$ is not a chain of $\mathcal{G}$.

The vertex set of a path is a chain. A finite chain is contained in a finite path, hence a maximal chain which is finite is a maximal vertex set of a path. Fig. 5 shows that a maximal chain of an acyclic graph is not necessarily (contained in) the vertex set of a path. The graph $\mathcal{G}$ (Fig. 5) is a countable acyclic graph with no independent set of size 3. $\{x_n : n < \omega \} \cup \{z\}$ is a maximal chain of $\mathcal{G}$ which is not the vertex set of a path.
It is shown in [1] that a maximal chain of an undirected graph with no infinite independent set is the vertex set of a path of this graph. However a maximal chain of an arbitrary undirected graph is not necessarily the vertex set of a path, as shown by Fig. 6. The graph \( \mathcal{G} \) (Fig. 6) is a countable undirected graph. \( C = \{a_n; n \leq \omega + 1\} \) is a maximal chain of \( \mathcal{G} \) which is not the vertex set of a path, (see [1] for the details), since it is not connected.

The two notions of maximal chain and maximal vertex set of a path are identical if and only if each chain is contained in a maximal vertex set of a path (this is the case for finite graphs), i.e. if each maximal chain is the vertex set of a path. This is for example the case if each finite vertex set of a path is contained in a maximal vertex set of a path (this condition always holds for acyclic graphs), and if the set of the maximal vertex sets of the paths of the graph \( \mathcal{G} = (V, E) \) is closed in \( 2^V \), i.e. compact, for the product topology: Indeed each finite subset of a chain is then contained in a maximal vertex set of a path and the result follows by compactness.
3. Conjectures and counter-examples

The main problem is whether the following extension of the Gallai–Milgram theorem to infinite graphs holds.

**Problem 1.** If $1 < k < \omega$ and $G = (V, E)$ is a graph with no independent set of size $k + 1$, then $V$ is the union of $\leq k$ pairwise disjoint paths of $G$.

The problem is still open even if $G$ is countable, and if we do not require the paths to be disjoint.

Problem 1 has a positive answer for $k = 1$. If $E'$ is a maximal subset of $E$ such that $G' = (V, E')$ is acyclic, then any independent set of the transitive closure of $G$ is an independent set of $G$; hence $G$ is a path if $G$ is complete. The following observation, which shows that one can say a little more, was suggested to us by Aharoni and Milner.

**Fact 1.** If $G = (V, E)$ is a complete directed graph, then there is $E' \subseteq E$ such that $G' = (V, E')$ is a path with the additional property: For any distinct $x, y \in V$ such that $x$ precedes $y$ on $G$, either $(x, y) \in E'$ or there is $z \in V$ such that $(x, z) \in E'$ and $(z, y) \in E'$.

**Proof.** Without loss of generality, we can assume that $G$ is a tournament. Let $V = \{x_\alpha : \alpha < \kappa\}$ where $\kappa = |V|$. We inductively define an increasing sequence $(E_\alpha : \alpha \leq \kappa)$ of subsets of $E$ such that for $\alpha \leq \kappa$ the transitive closure $G_\alpha = (V, E_\alpha)$ is an ordering for which two distinct elements $x, y$ are incomparable if and only if $x, y \notin V_\alpha = \{x_\beta : \beta < \alpha\}$ and $I_\alpha(x) = I_\alpha(y)$ where $I_\alpha(z) = \{t \in V_\alpha : t < z\}$ ($z \in V$). We let $E_\alpha = \bigcup \{E_\beta : \beta < \alpha\}$ if $\alpha = 0$ or $\alpha$ is limit, and

$$E_\alpha = E_\rho \cup \{(x, y) \in E : x \neq y, x_\rho \in \{x, y\}, x, y \notin V_\rho, I_\rho(x) = I_\rho(y)\}$$

if $\alpha = \beta + 1$ is successor. Now if we let $E' = E_\kappa$, we get clearly that $G' = G_\kappa$ is a linear ordering and therefore that $G'$ is a path. Furthermore, for $\alpha < \beta < \kappa$, let $\gamma < \kappa$ be the least ordinal such that either $\gamma = \alpha$ or $I_{\gamma+1}(x_\alpha) \neq I_{\gamma+1}(x_\beta)$, (note also that the least ordinal $\delta < \kappa$ such that $I_\delta(x_\alpha) \neq I_\delta(x_\beta)$ is a successor ordinal).

Assuming for example that $x_\alpha < x_\beta$, we get in the first case that $(x_\alpha, x_\beta) \in E_{\gamma+1} \subseteq E'$ and in the second case that $(x_\alpha, x_\gamma), (x_\gamma, x_\beta) \in E_{\gamma+1} \subseteq E'$.

Problem 1 has a positive answer for transitive graphs.

**Fact 2.** Let $k \geq 1$ be an integer and $G = (V, E)$ be a transitive directed graph. Then $G$ has no independent set of size $k + 1$ if and only if $V$ is the union of $\leq k$ (pairwise disjoint) paths of $G$. 

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Proof. The main point is that in a transitive graph the vertex set of a path is a clique and conversely, as we already observed in Section 2.4. Thus a subset of the vertex set of a path is the vertex set of a path and $P \subseteq V$ is the vertex set of a path if each finite subset of $P$ is the vertex set of a path. Therefore a standard compactness argument using the finite classical Gallai-Milgram theorem will prove the nontrivial implication. The reverse implication follows immediately from the transitivity of the graph $G$. Note also that Fact 2 easily reduces to Dilworth's theorem. □

A natural generalization of Fact 2 to arbitrary graph is the following:

Fact 3. Let $k \geq 1$ be an integer and $G = (V, E)$ be a directed graph. Then $V$ is the union of $\leq k$ chains if and only if each finite subset of $V$ is the union of $\leq k$ chains.

Proof. We have only to note that $C \subseteq V$ is a chain if only if each finite subset of $C$ is a chain. One implication is obvious, the other one follows from a standard compactness argument: For each finite subset $F$ of $V$, let $\mathcal{U}_F = \{F \subseteq V \times V : R \cap (F \times F) \text{ is an equivalence relation on } F \text{ with at most } k \text{ classes which are chains of } G\}$. The $\mathcal{U}_F$'s, $(F \text{ finite subset of } V)$, are closed non-empty subsets of the compact space $2^{V \times V}$, and they finitely intersect. Hence there is $R \in \bigcap \{\mathcal{U}_F : F \text{ finite subset of } V\}$, and $R$ is an equivalence relation on $V$ with at most $k$ classes which are chains of $G$. □

Remark. In Fact 3, we can replace the notion of 'chain', by the notion of 'clique in the transitive closure'; but the corresponding statement is then Fact 2, (as shown by the next sentence).

Now we can note that, using the classical Gallai-Milgram theorem, the hypothesis of Problem 1 that there is no independent set of size $k + 1$ in the graph $G = (V, E)$ is equivalent to the fact that each finite subset of $V$ is the union of $\leq k$ (pairwise disjoint) paths. But this hypothesis is not a necessary condition for the conclusion of Problem 1, (except if the graph $G$ is transitive). A necessary condition for the conclusion of Problem 1 is that each finite subset is contained in $\leq k$ pairwise disjoint finite paths, (which is the same as the original condition if $G$ is transitive). Note also that the hypothesis of Fact 3 that each finite subset of $V$ is the union of $\leq k$ chains is equivalent to the fact that each finite subset of $V$ is contained in $\leq k$ (finite) paths. From these considerations we can be tempted to strengthen both Problem 1 and Fact 3 in the following.

Problem 2. If $1 \leq k < \omega$ and $G = (V, E)$ is a graph such that each finite subset of $V$ is contained in $\leq k$, (respectively $\leq k$ pairwise disjoint), paths, $(1 \leq k < \omega)$, then $V$ is the union of $\leq k$, (respectively $\leq k$ pairwise disjoint), paths.
We can weaken the conclusion of Problems 1 and 2 asking only that $V$ be the union of $f(k)$ paths for some function $f : \omega \rightarrow \omega$, or even of finitely many paths, and also by removing the condition that these paths be pairwise disjoint. The answer to Problem 2 is trivially positive for $k = 1$ if we consider acyclic graphs. In Section 4.2, we shall obtain a positive result in this direction, for the special case of an undirected graph which has no infinite independent set. But the next examples, (cf. [1]), which are given here without proof, show that the answer to Problem 2 is negative in general.

**Example 1.** An undirected (bipartite) graph $\mathcal{G} = (V, E)$ with no maximal vertex set of a path and such that any finite subset of $V$ is contained in a finite path but $V$ is not the union of fewer than $|V|$ paths.

Let $\mathcal{G} = K_{\lambda, \mu} = (V, E)$ with $V = X \cup Y$ and $X = \{x_\alpha : \alpha < \lambda\}$, $(x_\alpha = (\alpha, 0))$, $Y = \{y_\beta : \beta < \mu\}$, $(y_\beta = (\beta, 1))$, $E = (X \times Y) \cup (Y \times X)$, where $\lambda$ and $\mu$ are two infinite cardinals. The graph $\mathcal{G}$ is a (bipartite) undirected graph (which is $(\min(\lambda, \mu))$-connected, as defined in Section 4.1) such that any finite subset of $V$ is contained in a finite path. More precisely if $V' = X' \cup Y'$ with $X' \subseteq X$ and $Y' \subseteq Y$, then, for $E' \subseteq E$, $\mathcal{G}' = (V', E')$ is a path of $\mathcal{G}$ if and only if $Y'$ is dense in $X'$ and $X'$ is dense in $Y'$ in $\mathcal{G}'$, (i.e. in $\mathcal{G}'$ there is an element of $Y'$ between any two distinct elements of $X'$ and there is an element of $X'$ between any two distinct elements of $Y'$). Therefore if $V'$ is the vertex set of a path, then $X'$ and $Y'$ are either both finite and $|X'| \leq |Y'| + 1$, $|Y'| \leq |X'| + 1$, or both infinite and $|X'| \leq 2^{|Y'|}$, $|Y'| \leq 2^{|X'|}$; conversely if $X'$ is infinite and $|Y'| = 2^{|X'|}$, then $V'$ is the vertex set of a path. Thus $V$ is not the union of fewer than $\mu$ paths provided that $2^\lambda < \mu$. We can also note that there is no maximal vertex set of a path under this assumption that $2^\lambda < \mu$.

**Example 2** (Fig. 7). A countable undirected (bipartite) graph $\mathcal{G} = (V, E)$ such that any finite subset of $V$ is contained in a finite path but $V$ is not the union of finitely many pairwise disjoint paths.

Independently, we can remark that $\{x_{2n+1} : n < \omega\} \cup \{y_n : n < \omega\}$ (Fig. 7) is the vertex set of a path but is not contained in a maximal one; however each vertex set of a finite path is contained in a maximal vertex set of a path. Note also that $V$ is the union of the two paths $(x_n : n < \omega)$ and $(x_1, y_0, x_3, y_1, \ldots, x_{n+1}, y_n, \ldots : n < \omega)$. We can in fact build a countable undirected (bipartite) graph $\mathcal{G} = (V, E)$ such that each finite subset of $V$ is contained in a (finite) path but $V$ is not the union of finitely many (even nondisjoint) paths, (see [1] or [3]).

**Example 3.** A countable acyclic graph $\mathcal{G} = (V, E)$ with no infinite independent set, such that each finite subset of $V$ is contained in the union of two disjoint finite paths, but $V$ is not the union of finitely many paths.
Let $\mathcal{P} = (P, <)$ be the ordering such that $P = X \cup Y$ with $X = \{x_n : n \in \omega^* + \omega\}$, $(x_n = (n, 0))$, and $Y = \{y_n : n \in \omega^* + \omega\}$, $(y_n = (n, 1))$, $x_n < x_m$, $y_n < y_m$, $y_n < x_m$ if $n < m$, $y_n$ and $x_m$ are incomparable if $m < n$. For $k \geq 1$, we define a directed graph $\mathcal{G}_k = (V_k, E_k)$ (Fig. 8) by

$$V_k = P \times \{l : 0 \leq l < k\}$$

and

$$E_k = \{(z, l), (z', l) : 0 \leq l < k, z, z' \in P, z < z'\}$$
$$\cup \{(x, l), (y, l + 1) : 0 \leq l < k - 1, x \in X, y \in Y\}.$$
A directed graph \( G = (V, E) \) by

\[
V = \bigcup \{ V_k \times \{ k \} : 1 \leq k < \omega \}
\]

\[
E = \{ ((z, k), (z', k)) : 1 \leq k < \omega, \ z, z' \in V_k, \ (z, z') \in E_k \}
\]

\[
\cup \{ ((z, k), (z', k')) : 1 \leq k < k' < \omega, \ z \in V_k, \ z' \in V_{k'} \}.
\]
4. Positive results for undirected graphs

In Section 4.2, we show that Problem 2 of Section 3 holds for any undirected graph with no infinite independent set, (cf. Theorem 2). Such a graph has the following property: For any cardinal $\kappa \geq \omega$, the connected components of the induced graph obtained by removing fewer than $\kappa$ vertices properly chosen from the original graph are $\kappa$-connected, i.e. they always remain connected when we remove again fewer than $\kappa$ vertices, (Lemma 2). This leads us to study first the $\kappa$-connectivity in Section 4.1. The main tool, with Lemma 2, which we use to establish Theorem 2, is the following property: If the undirected graph $\mathcal{G} = (V, E)$ has size $\kappa \geq \omega$ and is $\kappa$-connected, then $V$ is covered by a path with arbitrarily prescribed extremities (Lemma 1). In Section 4.3 we survey generalizations of some results of Section 4.2 to the case where the size of the independent sets is bounded by a given infinite cardinal which may be large. The case where the size of the independent sets is bounded by an integer is also investigated.

4.1. $\kappa$-Connected undirected graphs. We say that an undirected graph $\mathcal{G} = (V, E)$ is connected if $\{x, y\}$ is contained in a finite path of $\mathcal{G}$ for any $x, y \in V$, i.e. if the transitive closure of $\mathcal{G}$ is complete. A connected set of an undirected graph $\mathcal{G} = (V, E)$ is a subset $C$ of $V$ such that $\mathcal{G} \upharpoonright C$ is connected. For an undirected graph $\mathcal{G} = (V, E)$, we define an equivalence relation $R$ on $V$ by $(x, y) \in R$ if $\{x, y\}$ is contained in a finite path of $\mathcal{G}$, $(x, y \in V)$. The $R$-equivalence classes are the connected components of the undirected graph $\mathcal{G}$. They are the maximal connected subsets of $V$.

If $\kappa$ is a cardinal, the undirected graph $\mathcal{G} = (V, E)$ is $\kappa$-connected if $\mathcal{G} \upharpoonright (V \setminus V')$ is connected for any subset $V'$ of $V$ of size $|V'| < \kappa$. A $\kappa$-connected set of an undirected graph $\mathcal{G} = (V, E)$ is a subset $C$ of $V$ such that $\mathcal{G} \upharpoonright C$ is $\kappa$-connected. A complete graph is $\kappa$-connected for any cardinal $\kappa$. But the size of a $\kappa$-connected graph which is not complete is at least $\kappa + 2$.

For our purpose, the main property of $\kappa$-connected undirected graphs is Proposition 1 below.

**Proposition 1.** Let $\kappa$ be an infinite cardinal and $\mathcal{G} = (V, E)$ be a $\kappa$-connected undirected graph. Let $(W, \prec)$ be a linear ordering where $W \subseteq V$ and $|W| < \kappa$. Also, let $W' \subseteq V$, $|W'| \leq \kappa$. Then there is a path $\mathcal{G}' = (V', E')$ of $\mathcal{G}$ such that:

(i) $W \cup W' \subseteq V'$;
(ii) the linear ordering $(W, \prec)$ is induced by the linear ordering defined by $\mathcal{G}'$;
(iii) $W$ is coinitial in $\mathcal{G}'$ if $W \neq \emptyset$ and cofinal if $|W| > 1$;
(iv) $V'$ is finite or $|V'| = |W \cup W'|$.

We shall need in Section 4.2 the following obvious corollary.
Lemma 1. If $G = (V, E)$ is a $\kappa$-connected undirected graph of size $\kappa$, where $\kappa$ is an infinite cardinal, if $x, y$ are two distinct elements of $V$, then there is $E' \subseteq E$ such that $G' = (V, E')$ is a path with first vertex $x$ and last vertex $y$.

Remarks. (1) Proposition 1 and Lemma 1 can be improved. For example, we can ensure in Lemma 1 that $G'$ is scattered (see [1] or [3]), and even well ordered of order type $\kappa + 1$ if $\kappa = \omega$ or if $G$ has no infinite independent set (see [3]) (density conditions can also be ensured). However we cannot ensure that $G'$ is well ordered in general (see [3]).

(2) Some additional basic properties on $\kappa$-connected graphs have been obtained, (in [1]). For example the following.

If $G = (V, E)$ is a $\kappa$-connected graph of size $\kappa \geq \omega$, then $V$ is the union of $\kappa$ pairwise disjoint $\kappa$-connected sets of size $\kappa$.

Note also that any $\kappa$-connected set of an undirected graph is contained in a maximal one.

Proof of Proposition 1. We prove the result by induction on the cardinal $\lambda = |W \cup W'| \leq \kappa$. Let $\{x_\alpha : \alpha < \lambda\}$ be a well ordering of $W \cup W'$, such that $x_0 \in W$ if $W \neq \emptyset$ and $x_1 \in W$ if $|W| \geq 2$. We inductively define some paths $P_\alpha = (P_\alpha, F_\alpha)$ of $G$ ($1 \leq \alpha \leq \lambda$), such that $P_\alpha$ is finite or $|P_\alpha| = |\alpha|$, the linear ordering defined by $P_\alpha$ coincides with $<$ on $W \cap P_\alpha$, and $\{x_\beta : \beta < \alpha\} \subseteq W \cap P_\alpha$, in the following way. $P_1 = ((x_0), \emptyset)$. For $\alpha$ limit, $P_\alpha = \bigcup\{P_\beta : \beta < \alpha\}$ and $F_\alpha = \bigcup\{F_\beta : \beta < \alpha\}$. For $\alpha = \beta + 1$ successor ($\beta \geq 1$), let $<_\beta$ be a linear ordering of $P_\beta \cup \{x_\beta\}$ which extends the linear ordering defined by $P_\beta$, and also which coincides with $<$ on $W \cap (P_\beta \cup \{x_\beta\})$, and such that $x_\beta$ is not the first element in this order if $x_\beta \notin W$, and is not the last element if $x_\beta \notin W$ and $\beta \geq 2$. If $\lambda$ is infinite, then, by the induction hypothesis, there is a path $P' = (P, F)$ of the $\kappa$-connected graph $G \downarrow (V \setminus (W \cup \{x_\beta\}))$ such that $P_\beta \cup \{x_\beta\} \subseteq P$, the linear ordering defined by $P$ coincides with $<_\beta$ on $P_\beta \cup \{x_\beta\}$, $P$ is finite or $|P| = |P_\beta|$. Moreover, we can assume that $P_\beta \cup \{x_\beta\}$ is coinitial and cofinal in $P$. We let $P_\alpha = P'$. If $\lambda$ is finite, then $P_\beta$ is finite. Let $x'$ be the last element lower than $x_\beta$ for $<_\beta$ and $x''$ be the first element higher than $x_\beta$ (if $x_\beta$ is an extremal element of $<_\beta$, the construction is similar but simpler). There is a finite path $P'' = (P'', F'')$ from $x'$ to $x_\beta$ in the connected graph $G \downarrow (V \setminus (W \cup \{x_\beta\}))$ and there is a finite path $P'' = (P'', F'')$ from $x_\beta$ to $x''$ in the connected graph $G \downarrow (V \setminus (W \cup \{x_\beta\})) \setminus \{x_\beta, x''\})$. We let $P_\alpha = P_\beta \cup P'' \cup P''$ and $F_\alpha = F_\beta \cup F' \cup F''$. Clearly $G' = P_\lambda$ satisfies the conditions of Proposition 1. □

4.2. Undirected graphs with no infinite independent set. The main results of this section are the following ones.

Theorem 1. Let $G = (V, E)$ be an undirected graph with no infinite independent set. Then $V$ is the union of some pairwise disjoint paths such that there is an independent set with a vertex in each of these paths.
Theorem 2. Let $\mathcal{G} = (V, E)$ be an undirected graph with no infinite independent set, $1 \leq k < \omega$ and $f : [k]^2 \to [\omega]$ be a map which assigns a set of nonnegative integers to each (unordered) pair of nonnegative integers smaller than $k$. If each finite subset of $V$ is contained in $\leq k$ finite paths $P_i$ $(i < 1 \leq k)$ such that $|P_i \cap P_j| \in f([i, j])$ for $i \neq j$, then $V$ is the union of $\leq k$ paths $P_i$ $(i < p \leq k)$ such that $|P_i \cap P_j| \in f([i, j])$ for $i \neq j$.

Theorem 1 shows in particular that $V$ is the union of finitely many pairwise disjoint paths of $\mathcal{G}$ if the undirected graph $\mathcal{G}$ has no infinite independent set; and that $V$ is the union of $\leq k$ pairwise disjoint paths if $\mathcal{G}$ has no independent set of size $k + 1$ ($1 < k < \omega$). Hence it shows that Problem 1 of Section 3 has a positive answer for the undirected graphs.

In Theorem 2, we can ensure in particular that the paths $P_i$ $(i < p)$ are pairwise disjoint, provided that each finite subset of $V$ is contained in $\leq k$ pairwise disjoint paths. This shows that Problem 2 of Section 3 has a positive answer for the undirected graphs with no infinite independent set.

Theorem 2, for $k = 1$, shows immediately that the union of an up-directed family of vertex sets of paths of an undirected graph $\mathcal{G}$ with no infinite independent set is the vertex set of a path of $\mathcal{G}$. Hence, by Zorn’s lemma, any vertex set of a path of $\mathcal{G}$ is contained in a maximal one. In fact, this follows from the fact that any chain of an undirected graph with no infinite independent set is contained in the vertex set of a path. This last property is easy to check directly by applying Lemma 3’ (and Lemma 2) in the same way as it will be used in the proof of Theorem 2 below. More precisely it is proved in [1] or [3] the following.

Theorem 3. There are only finitely many maximal vertex sets of paths in an undirected graph with no infinite independent set, and hence the notions of maximal chain and of maximal vertex set of a path are identical in such a graph.

Note finally that Theorem 1 is false for directed graphs in general. Any ordering (or transitive graph) with no infinite antichain but with arbitrarily large finite antichains will provide a counter-example. We recall a well known example, (see [8]).

Example. For an infinite cardinal $\kappa$, let the ordering $\mathcal{P} = (P, \leq)$ where $P = \{((\alpha, \beta) : \alpha, \beta < \kappa\}$ and $\leq = \{((\alpha, \beta), (\alpha', \beta')) \in P \times P : \alpha \leq \alpha'$ and $\beta \leq \beta'\}$. $\mathcal{P}$ has no infinite antichain but $P$ is not the union of fewer than $\kappa$ chains.

Besides Lemma 1 of Section 4.1, the principal tool that we use to prove Theorems 1 and 2 is the following result.

Lemma 2. If $\mathcal{G} = (V, E)$ is an undirected graph with no infinite independent set, then for any infinite cardinal $\kappa$, there is a subset $V'$ of $V$ of size $|V'| < \kappa$ such that...
any connected component of the graph \( \mathcal{G} \upharpoonright (V \setminus V') \) is \( \kappa \)-connected, (note that there are only finitely many such connected components).

**Proof of Lemma 2.** We shall use the following self-evident fact: If \( \{ V_i : i \in I \} \) is the set of the connected components of the undirected graph \( \mathcal{G} = (V, E) \) then, for any set \( \{ W_i : i \in I \} \) such that \( W_i \subseteq V_i \) for \( i \in I \), the set of the connected components of the graph \( \mathcal{G} \upharpoonright (V \setminus \bigcup \{ W_i : i \in I \}) \) is \( \{ W_{ij} : i \in I, j \in J_i \} \), where \( \{ W_{ij} : j \in J_i \} \) is the set of the connected components of the graph \( \mathcal{G} \upharpoonright (V_i \setminus W_i) \), \( i \in I \). Assume for contradiction that, for a given infinite cardinal \( \kappa \), \( \mathcal{G} \) fails to satisfy the conclusion of Lemma 2. Then this is also the case for \( \mathcal{G} \upharpoonright C \), for at least one connected component \( C \) of \( \mathcal{G} \upharpoonright (V \setminus V') \) whenever \( V' \subseteq V \) and \( |V'| < \kappa \). Hence we can inductively define \( V_n, V'_n \subseteq V \) \((n < \omega)\) such that \( V_0 = V \), \( \mathcal{G} \upharpoonright V_n \) does not satisfy the conclusion of Lemma 2, \( |V'_n| < \kappa \) and \( V_N \setminus V'_n \) is not connected in \( \mathcal{G} \), \( V^{n+1}_n \) is a connected component of \( \mathcal{G} \upharpoonright (V_n \setminus V'_n) \), (such that \( \mathcal{G} \upharpoonright V^{n+1}_n \) does not satisfy the conclusion of Lemma 2). Choose \( x_n \in V \setminus (V'_n \cup V^{n+1}_n) \) \((n < \omega)\). The set \( \{ x_n : n < \omega \} \) so defined is an infinite independent set of \( \mathcal{G} \) and this is a contradiction. \( \square \)

**Proof of Theorem 1.** We first show by induction on the size of the graph \( \mathcal{G} = (V, E) \) that \( V \) is the union of finitely many pairwise disjoint paths. If \( V \) is finite, the result is obvious. Otherwise, let \( |V| = \kappa > \omega \). There is \( V' \subseteq V \) such that \( |V'| < \kappa \) and the connected components of \( \mathcal{G} \upharpoonright (V \setminus V') \) are \( \kappa \)-connected. They are only finitely many and they are vertex sets of paths of \( \mathcal{G} \) by Lemma 1. To conclude, it is enough to apply the induction hypothesis to \( \mathcal{G} \upharpoonright V' \).

Assume now that \( n \) is the minimal integer such that \( V \) is the union of \( n \) pairwise disjoint paths \( \mathcal{P}_i = (P_i, E_i) \) of \( \mathcal{G} \) \((i < n)\). Then, for \( i \neq j < n \), since \( (P_i \cup P_j, E_i \cup E_j^{-1} \cup (E \cap (P_i \times P_j))) \) is not a path of \( \mathcal{G} \), there are \( x_{ij} \in P_i, x_{ji} \in P_j \) such that \((y, z) \notin E \) when \( y \) does not precede \( x_{ij} \) on \( \mathcal{P}_i \) and \( z \) does not precede \( x_{ji} \) on \( \mathcal{P}_j \). It is enough to note that if \( x_i \in P_i \) is the last element amongst \( \{ x_{ij} : j < n, j \neq i \} \) along \( \mathcal{P}_i \) \((i < n)\), then \( \{ x_i : i < n \} \) is an independent set of \( \mathcal{G} \). \( \square \)

To prove Theorem 2, we first study the case where each finite subset of \( V \) is contained in a path of \( \mathcal{G} \). If \( \mathcal{G} = (V, E) \) is an undirected graph and \( x, y \in V \), \( W \subseteq V \), then we denote by \( C(\mathcal{G}, x, y, W) \) the following assertion: If \( n < \omega \) and \( x_i, y_l \) \((l < n)\) are pairwise distinct elements of \( V \setminus \{ x, y \} \) such that \((x_i, y_l) \in E \) for \( l < n \), then there is a path \( \mathcal{G} = (W', E') \) of \( \mathcal{G} \) such that (i) \( W' \supseteq W \), (ii) \( x \) and \( y \) are respectively the first and last vertices of \( \mathcal{G}' \), (iii) for each \( l < n \), \( x_i \) and \( y_l \) are consecutive vertices of \( \mathcal{G}' \), and (iv) \( W' \) is finite or \( |W'| = |W| \).

We need the following compactness theorem.

**Lemma 3.** Let \( \mathcal{G} = (V, E) \) be an undirected graph having no infinite independent set and \( x, y \in V \). If \( C(\mathcal{G}, x, y, W) \) holds for every finite set \( W \subseteq V \), then \( C(\mathcal{G}, x, y, V) \) holds.
Proof of Lemma 3. We prove by induction on $|W|$ that $C(\mathcal{G}; x, y, W)$ holds for $W \subseteq V$. Suppose that $|W| = \kappa \geq \omega$. By Lemma 2, there are $W' \subseteq W$, $|W'| < \kappa$ such that each connected component $V'$ $(i < p, p \leq \omega)$ of $\mathcal{G} \upharpoonright (W \setminus W')$ is $\kappa$-connected and of size $|V'| = \kappa$. Let $n < \omega$ and $x_i, y_i$ $(i < n)$ be pairwise distinct elements of $V \setminus \{x, y\}$ such that $(x_i, y_i) \in E$, $(i < n)$. For each $i < p$ choose pairwise distinct elements $x_{n+i}, y_{n+i}$ of $V \setminus \{(x, y) \cup \{x_i, y_i; l < n\}\}$ such that $(x_{n+i}, y_{n+i}) \in E$. By the induction hypothesis, $C(\mathcal{G}; x, y, W')$ holds and so there is a path $\mathcal{G}' = (W', E')$ such that $W' \supseteq W'$, $x$ and $y$ are the first and last vertices of $\mathcal{G}'$ and $x_i$ and $y_i$ are consecutive vertices of $\mathcal{G}'$ for $l < n + p$, and $|W'| < \kappa$. Since $V \setminus (W' \setminus \{x_{n+i}, y_{n+i}\})$ is $\kappa$-connected, it follows by Corollary 1 that there is $E_i \subseteq E$ such that $\mathcal{G}_i = (V \setminus (W' \setminus \{x_{n+i}, y_{n+i}\}), E_i)$ is a path having $x_{n+i}$ and $y_{n+i}$ as first and last vertices if $x_{n+i}$ precedes $y_{n+i}$ in $V'$, as last and first vertices otherwise, $(i < p)$. Then $\mathcal{G}' = (V', E')$, with $V' = W' \cup W$ and $E' = E'' \cup \bigcup \{E_i; i < p\}$ is a witness that $C(\mathcal{G}; x, y, W)$ holds. \qed

Note that, by Proposition 1 of Section 4.1, if $\mathcal{G} = (V, E)$ is an $\omega$-connected undirected graph, then $C(\mathcal{G}; x, y, W)$ holds for every $x, y \in V$ and every finite set $W \subseteq V$. Hence we have the following corollary of Lemma 3.

Lemma 3'. If $\mathcal{G} = (V, E)$ is an $\omega$-connected undirected graph with no infinite independent set, then $C(\mathcal{G}; x, y, V)$ holds for every $x, y \in V$.

Proof of Theorem 2. For $|V| < \omega$ the result is obvious, and so we assume that $|V| \geq \omega$. By Lemma 2, there are a finite set $V' \subseteq V$ and a finite number of infinite sets $V_i$ $(i < \kappa)$ which are the distinct connected components of $\mathcal{G} \upharpoonright (V \setminus V')$ and are moreover $\omega$-connected. For each $i < \kappa$, let $X_i$ be a finite subset of $V_i$ of size $|X_i| = k \cdot |V'| + k + 1$.

By the hypothesis of Theorem 2, there are a finite set $X \subseteq V$ and $p \leq k$ such that $V' \cup \bigcup \{X_i; i < \kappa\} \subseteq X$, and $X$ is the union of $p$ paths $\mathcal{H}_j = (H_j, E_j)$ of $\mathcal{G}$ $(j < p)$ such that $|H_j \cap H_j'| \leq f((j, j'))$ if $j < j' < p$.

Let $i < \kappa$. Since $X_i \subseteq X \subseteq \bigcup \{H_j; j < p\}$, $|X_i| = k \cdot |V'| + k + 1$ and $p \leq k$, it follows that there is some index $j < p$ such that $|X_i \cap H_j| > |V'| + 2$. Hence there is some element $x \in X_i \cap H_j$ which is not the last element of $\mathcal{H}_j$ and is such that its successor $x'$ on $\mathcal{H}_j$ is not an element of $V'$. Since $(x, x') \in E$, $x \in X_i \subseteq V$ and $x' \notin V'$, and since $V_i$ is a connected component of $\mathcal{G} \upharpoonright (V \setminus V')$, it follows that $x'$ also belongs to the set $V_i$. Put $V_i' = V_i \setminus \{x, x'\}$.

Since $V_i'$ $(i < \kappa)$ is $\omega$-connected, it follows from Lemma 3' that $C(\mathcal{G} \upharpoonright V_i', x_i, x_i', V_i')$ holds. Hence there is a path $\mathcal{P}_i = (V_i, E_i)$ having first vertex $x_i$ and last vertex $x_i'$ $(i < \kappa)$. Put $P_j = H_j \cup \bigcup \{V_i'; j = i\}$ and $E_j = E_j \cup \bigcup \{E_i'; j = i\}$ $(j < p)$. Then $\mathcal{P}_j = (P_j, E_j)$ is a path of $\mathcal{G}$ for $j < p$ and $V = X \cup \bigcup \{V_i'; i < n\} = \bigcup \{P_j; j < p\}$. Moreover, since the sets $V_i'$ $(i < \kappa)$ are pairwise disjoint, it follows that $|P_i \cap P_j| = |H_j \cap H_j'| \leq f((j, j'))$ for $j < j' < p$. \qed
Remark. (1) If the undirected graph $G = (V, E)$ has no infinite independent set and if $W \subseteq V$ is the vertex set of a path of $G$, then $W$ is the vertex set of a well ordered path of $G$ with order type smaller than $|W| \cdot \omega$, (see [3]). Hence in Theorems 1 and 2 we can ensure that the paths are well ordered. In Theorem 1 we can in fact even ensure that the order types of the paths are cardinals.

(2) It is easy to prove directly that Problem 1 of Section 3 has a positive answer for undirected graphs. Namely let $1 \leq k < \omega$ and $G = (V, E)$ be an undirected graph with no independent set of size $k + 1$. We inductively define $X_\alpha \subseteq V$, some 1-1 maps $\sigma_{\beta, \alpha}: X_\alpha \to X_\beta$ for $\beta < \alpha$ and a cofinal subset $I_\alpha$ of $\alpha$, ($\alpha < \lambda$), in the following way: If $\bigcup \{X_\beta: \beta < \alpha\} = V$, we let $\lambda = \alpha$; otherwise $X_\alpha$ is defined to be an independent set of $G \upharpoonright (V \setminus \bigcup \{X_\beta: \beta < \alpha\})$ of maximal size. For $\beta < \alpha < \lambda$, by the classical Gallai–Milgram theorem applied to the finite directed graph $\mathcal{G}_{\beta, \alpha} = (X_\beta \cup X_\alpha, E \cap (X_\beta \times X_\alpha))$, there is a 1-1 map $\sigma_{\beta, \alpha}: X_\alpha \to X_\beta$ such that $(\sigma_{\beta, \alpha}(x), x) \in E$ for any $x \in X_\alpha$. Since all the sets $X_\beta$ ($\beta < \alpha$) have their size bounded by $k < \omega$, there is a cofinal subset $I_\alpha$ of $\alpha$ such that $\sigma_{\gamma, \alpha} = \sigma_{\gamma, \beta} \circ \sigma_{\beta, \alpha}$ for $\beta, \gamma \in I_\alpha$ and $\gamma \in I_\beta$. Let us define $R = \{(x, y) \in V \times V: x \in X_\beta, y \in X_\alpha$ for some $\alpha, \beta < \lambda$ such that $\beta \in I_\alpha$ and $\sigma_{\alpha, \beta}(y) = x$, or $\alpha \in I_\beta$ and $\sigma_{\alpha, \beta}(x) = y\}$. The transitive closure of $R$ is an equivalence relation on $V$ which has $|X_\alpha| = k$ classes and whose classes are obviously vertex sets of well-ordered paths of $G$.

A slight modification of the above proof would show that $V$ is the union of $\leq k$ pairwise disjoint well ordered paths whose order types are cardinals.

4.3. Undirected graphs with no independent set of size $v$. The ideas used in Section 4.2 have been applied in [2] and [3] to study the class of the undirected graphs having no independent set of size $v$, where $v$ is an arbitrary, finite in [2], or infinite in [3], cardinal.

For example we show in [2] the following finite version of Theorem 2.

Theorem 2'. There is a map $f: \omega \to \omega$ such that for $1 \leq n, k < \omega$ and any undirected graph $G = (V, E)$ with no independent set of size $n + 1$, $V$ is the union of $\leq k$ pairwise disjoint paths if and only if each subset of $V$ of size $\leq f(n)$ is contained in $\leq k$ pairwise disjoint paths.

For $k = 1$ this gives a criterion for the existence of a hamiltonian path (or similarly a hamiltonian cycle), and enables to show that this existence problem is solvable in polynomial time for the class of the infinite undirected graphs with no independent set of size $n + 1$, ($1 \leq n < \omega$), (whereas this problem is NP-complete in general).

To present the general infinite version of Theorem 2, we first need some definitions. A tree is a well founded ordering $\mathcal{T} = (T, <)$ which has a minimum element and is such that for any $x \in T$, the set $(\leftarrow x[ = \{y \in T: y < x\}$ is a chain of $\mathcal{T}$. The height of $x$ is the order type $ht(x)$ of $(\leftarrow x[$, (which is an ordinal).
For an ordinal $\alpha$, the $\alpha$th level of $\mathcal{T}$ is the set $T_\alpha = \{ x \in T : \text{ht}(x) = \alpha \}$ and the height of $\mathcal{T}$ is the ordinal $\text{ht}(\mathcal{T}) = \sup \{ \text{ht}(x) + 1 : x \in T \}$, or the least $\alpha$ such that $T_\alpha = \emptyset$. We remind the reader that an ordinal is equal to the set of the ordinals smaller than it. A typical example of a tree is the following one, where, for two sets $A$, $B$, $^B A$ denotes the set of the maps $f : A \to B$.

**Example.** Let $\lambda$, $\mu$ be ordinals and let the ordering $\mathcal{T} = (T, \prec)$ where $T = \bigcup \{ ^{\alpha \mu} : \alpha < \lambda \}$ and, for $f \in ^{\alpha \mu}$, $g \in ^{\beta \mu}$, $f \prec g$ if $\alpha < \beta$ and $f = g \upharpoonright a$. $\mathcal{T}$ is a tree of height $\lambda$ and for $\alpha < \lambda$, $T_\alpha = ^{\alpha \mu}$.

The cofinality $\text{cf}(\kappa)$ of an infinite cardinal $\kappa$ is the smallest cardinal of a cofinal subset of $\kappa$. $\kappa$ is regular if $\text{cf}(\kappa) = \kappa$. For an infinite regular cardinal $\kappa$, let $\text{SH}(\kappa)$ denote the statement: *There is no Suslin tree of size $\kappa$, i.e. any tree of height $\kappa$ has a chain or an antichain of size $\kappa$.* We show in [3] the following extension, (this is an extension since, by Ramsey's theorem [9], $\text{SH}(\omega)$ is true), of Theorem 2.

**Theorem 2.** Let $\nu > \omega$ and $\lambda > 2$ be cardinals and let $\mathcal{G} = (V, E)$ be an undirected graph with no independent set of size $\nu$. If each subset of $V$ of size $\leq \nu$ is contained in $< \lambda$ pairwise disjoint paths, (respectively if $\nu$ is regular, $\text{SH}(\nu)$ holds and each subset of $V$ of size $< \nu$ is contained in $< \lambda$ pairwise disjoint paths), then $V$ is the union of $< \lambda$ pairwise disjoint paths.

As already observed, Theorem 1 shows that $V$ is the union of finitely many pairwise disjoint paths of $\mathcal{G}$ whenever the undirected graph $\mathcal{G} = (V, E)$ has no infinite independent set. Note that the following obvious Corollary of Theorem 2 generalizes this weak form of Theorem 1.

**Corollary.** Let $\nu$ be an infinite cardinal and let $\mathcal{G} = (V, E)$ be an undirected graph with no independent set of size $\nu$. Then (i) $V$ is the union of $< \nu$ pairwise disjoint paths of $\mathcal{G}$; (ii) $V$ is the union of $< \nu$ pairwise disjoint paths of $\mathcal{G}$ if $\nu$ is regular and $\text{SH}(\nu)$ holds.

The following extension of Lemma 2, whose proof involves a tree argument, is needed to establish Theorem 2. It explains how the statement $\text{SH}(\nu)$ occurs in Theorem 2. Note that the above Corollary follows obviously (modulo Lemma 1) from Lemma 2 by induction on the size of $\mathcal{G}$. We conclude this section by presenting the proof of Lemma 2.

**Lemma 2.** Let $\nu$, $\kappa$ be two infinite cardinals and let $\mathcal{G} = (V, E)$ be an undirected graph with no independent set of size $\nu$. If $\nu < \text{cf}(\kappa)$ or $\nu = \text{cf}(\kappa)$ and $\text{SH}(\text{cf}(\kappa))$ holds, then there is $V' \subseteq V$ such that $|V'| < \kappa$ and the connected components of $\mathcal{G} \upharpoonright (V \setminus V')$ are $\kappa$-connected.
Proof of Lemma 2". For an ordinal α and \( f \in {}^\alpha \nu \), we define a subset \( V_f \) of \( V \) in the following way. \( V_f = V \) if \( \alpha = 0 \). \( V_f = \cap \{ V_{f \uparrow \beta}: \beta < \alpha \} \) if \( \alpha \) is limit. Assume now that \( \alpha = \beta + 1 \) is a successor and let \( f' = f \uparrow \beta \). We define \( V_f = \phi \) if \( |V_f| < \kappa \) or if \( V_f \) is \( \kappa \)-connected. Otherwise there is \( W_f \subseteq V_f \) such that \( |W_f| < \kappa \) and \( V_f \setminus W_f \) is not connected. Let \( \{ C_\alpha: \alpha < \lambda \} \) be the set of the connected components of \( \mathcal{G} \upharpoonright (V_f \setminus W_f) \); then \( 2 \leq \lambda < \nu \) since there is no independent set of size \( \nu \). We put \( V_f = C_{f(\beta)} \) if \( f(\beta) < \lambda \), and \( V_f = \phi \) if \( \lambda \leq f(\beta) < \nu \). This completes the definition of the sets \( V_f \). Assume that \( f \in {}^\alpha \nu \) for some limit ordinal \( \alpha \) and \( V_{f \uparrow \beta} \neq \phi \) for all \( \beta < \alpha \). For each \( \beta < \alpha \), choose

\[
x_\beta \in V_f \setminus (V_{f \uparrow (\beta + 1)} \cup W_f \setminus \beta).
\]

\( \{ x_\beta: \beta < \alpha \} \) is clearly an independent set of \( \mathcal{G} \) of size \( |\alpha| \). In particular \( V_f = \phi \) if \( f \in {}^\alpha \nu \) and \( \alpha \geq \nu \). Let

\[
T = \{ f \in \cup \{ {}^\alpha \nu: \alpha < \nu \}: V_f \neq \phi \}
\]

which is an initial segment of the tree \( (\cup \{ {}^\alpha \nu: \alpha < \nu \}, \langle \rangle) \) defined in the previous example, and let \( \mathcal{T} = (T, \langle \rangle) \) be the induced tree. We just showed that \( \mathcal{T} \) has no chain of size \( \nu \). Assume that \( f, g \in T \) are incomparable in \( \mathcal{T} \). Then \( V_f, V_g \) are non-empty and \( \{ x, y \} \) is an independent set of \( \mathcal{G} \) if \( x \in V_f, y \in V_g \). Indeed, if \( y < \min(\alpha, \beta) \) is the minimal ordinal such that \( f(y) \neq g(y) \), where \( f \in {}^\alpha \nu, g \in {}^\beta \nu \), then \( V_f \upharpoonright (\gamma + 1) \) and \( V_g \upharpoonright (\gamma + 1) \) are two distinct connected components of \( \mathcal{G} \upharpoonright (V_f \setminus W_f) \) which contain respectively \( x \) and \( y \). Hence to any antichain of \( \mathcal{G} \) there corresponds an independent set of \( \mathcal{G} \) of the same size. Therefore \( \mathcal{T} \) has no antichain of size \( \nu \). Let \( X = \{ f \in T: f \) is a maximal element of \( \mathcal{T} \) and \( |V_f| \geq \kappa \} \), which is an antichain of \( \mathcal{T} \). Hence if \( f, g \) are two distinct elements of \( X \), and if \( x \in V_f, y \in V_g \), we know that \( (x, y) \notin E \). Moreover, from the definition of \( \mathcal{T} \), it follows also that \( V_f \) is \( \kappa \)-connected if \( f \in X \), and so \( \{ V_f: f \in X \} \) is the set of the connected components of \( \mathcal{G} \upharpoonright (\bigcup \{ V_f: f \in X \}) \). It is enough to show that \( |V \setminus \bigcup \{ V_f: f \in X \}| < \kappa \) to end the proof. Note that if \( f' \in T \cap \{ \beta \nu \} \) is not a maximal element of \( \mathcal{T} \), (i.e. if \( f' \in {}^\beta \nu \) and \( W_f \) is defined), then \( V_f = W_f \cup \{ V_f: f \in T \cap \{ \beta \nu \}, f \uparrow \beta = f' \} \). Thus \( V \setminus \bigcup \{ V_f: f \in X \} = \bigcup \{ W_f: f \in T \) and \( f' \) is not maximal in \( \mathcal{T} \} \cup \bigcup \{ V_f: f \in T \) and \( |V_f| < \kappa \} \). Since \( |W_f| < \kappa \) when \( W_f \) is defined, it is enough to check that \( |T| < \text{cf}(\kappa) \). Since \( \mathcal{T} \) is a tree with no chain or antichain of size \( \nu \), we get that \( |T| < \nu \) and that \( |T| < \nu \) if \( \nu = \text{cf}(\kappa) \) since \( \text{SH}(\text{cf}(\kappa)) \) holds. In any case, we get that \( |T| < \text{cf}(\kappa) \).

5. A Gallai–Milgram property for ‘regular’ acyclic countable graphs

The Gallai–Milgram theorem for infinite acyclic directed (well-founded) graphs would generalize the Gallai–Milgram theorem that we proved for infinite undirected graphs (Corollary 2). Indeed, if \( \mathcal{G} = (V, E) \) is an undirected graph and if \( V = \{ x_\alpha: \alpha < \kappa \} \) is a well ordering of \( V \), then \( \mathcal{G}' = (V, E') \), where \( E' = \)
$\{(x_\alpha, x_\beta) \in E: \alpha < \beta\}$ is an acyclic graph such that $\mathcal{G}$ and $\mathcal{G}'$ have the same independent sets and such that any path of $\mathcal{G}'$ is a path of $\mathcal{G}$, (but, of course, not conversely).

In this section we consider a Gallai–Milgram type property for acyclic graphs of a very special kind. We say that an ordering $\mathcal{G} = (P, \prec)$ is $n$-regular, ($n < \omega$), if there is a linear ordering $\mathcal{J} = (I, \prec)$ such that $P = \bigcup \{C_i: i \in I\}$, where the sets $C_i$ are pairwise disjoint finite antichains of $\mathcal{G}$ which have same size $n$ and such that if $i, j \in I$, $i < j$, there is some bijection $\sigma_{i,j}$ from $C_i$ to $C_j$ such that $x < \sigma(x)$ for every $x \in C_i$. The width of $\mathcal{G}$ is clearly $n$, (i.e. $n$ is the maximal size of an antichain of $\mathcal{G}$), and if $x \in C_i$, $y \in C_j$ for $i, j \in I$, $i < j$, then either $x < y$ or $x$ and $y$ are incomparable. Hence by Dilworth’s theorem, $P$ is the union of $n$ pairwise disjoint maximal chains which intersect each $C_i$ in a single point and which are canonically isomorphic to $\mathcal{J}$.

An acyclic graph $\mathcal{G} = (V, E)$ is $n$-regular if its transitive closure is $n$-regular. Although $n$ is not necessarily the maximal size of an independent set of $\mathcal{G}$, we are interested in the question whether, by analogy with the above result for ordered sets, it is possible to cover $V$ by $n$ ‘coherent’ paths, (which will be necessarily pairwise disjoint). More precisely, for a given linear ordering $\mathcal{J} = (I, \prec)$ and a family $\{C_i: i \in I\}$ which witness the fact that $\mathcal{G}$ is $n$-regular, are there $n$ paths $\mathcal{G}_i = (V, E_i)$ of $\mathcal{G}$, ($l < n$), such that $V = \bigcup \{V_i: l < n\}$, (and hence $V_l \cap V_k = \emptyset$ if $l \neq k$), and such that $E_i = \{(i, j) \in I \times I: i < j \text{ and } (x, y) \in E_i \text{ for some } x \in C_i, y \in C_j\}$ does not depend on $I$? A necessary condition for this is that the graph $(I, F)$ is a path, where $F = \{(i, j) \in I \times I: i < j \text{ and there is a bijection } \sigma \text{ from } C_i \text{ to } C_j \text{ such that } (x, \sigma(x)) \in E \text{ for } x \in C_i\}$. We show now that, in the countable case, this condition is also sufficient. For any $i \in I$, we write $C_i = \{x_{i,l}: l < n\}$ and for $(i, j) \in F$, we choose a bijection $\sigma_{i,j}$ from $n$ onto $n$ such that $(x_{i,b}, x_{j,\sigma_{i,j}(b)}) \in E$ for $l < n$. We have to find $F' \subseteq F$ such that the graph $(I, F')$ is a path and such that the composed permutation of $(\sigma_{i_0,i_1}, \sigma_{i_1,i_2}, \ldots, \sigma_{i_{k-1},i_k})$ along a finite path $p = (l_0, l_1, \ldots, l_k)$ of $\mathcal{G}$ depends only upon the extremities $l_0, l_k$ of $p$. This follows from Theorem 4 where we take $\mathcal{H} = \mathcal{G}_n$, the $n$th permutation group.

**Theorem 4.** Let $\mathcal{G} = (V, E)$ be a countable path (such that $(x, x) \notin E \text{ for } x \in V$), let $\mathcal{H} = (H, \ast)$ be a finite group, and let $\theta: E \to H$. For any finite path $p = (x_0, x_1, \ldots, x_k)$ of $\mathcal{G}$, we define $\hat{\theta}(p) = \theta(x_0, x_1) \ast \theta(x_1, x_2) \ast \cdots \ast \theta(x_{k-1}, x_k) \in H$. Then there is $E' \subseteq E$ such that $\mathcal{G}' = (V, E')$ is a path and for any finite path $p$ of $\mathcal{G}'$, $\hat{\theta}(p)$ depends only upon the extremities $l_0, l_k$ of $p$.

**Remarks.** (1) Since by Cayley’s theorem any finite group is a subgroup of a permutation group, Theorem 4 is in fact equivalent to the result on regular acyclic graphs.

(2) We can define $\hat{\theta}(p)$ for any finite sequence $p = (x_0, x_1, \ldots, x_k)$ such that $(x_i, x_{i+1}) \in E \cup E^{-1}$, $(i < k)$, if we first let $\hat{\theta}(y, x) = (\theta(x, y))^{-1}$ when $(x, y) \in E$. Then Theorem 4 says that $\hat{\theta}(p) = 1$ if $x_0 = x_k$. 


(3) By Theorem 4, we can define an equivalence relation \( R \) on \( V \) by \((x, y) \in R\) if \( x = y \) or if the value \( \theta(x, y) \), \( (\tilde{\theta}(y, x) \) if \( y \) precedes \( x \) in \( \mathcal{G} \), calculated in \( \mathcal{G} \)'s, is equal to the identity of \((H, \ast)\). If we fix an origin \( x_0 \in V \), we can index the \( R\)-equivalence classes by \( H \). Conversely Theorem 4 says that there is a partition \( \{ V_\alpha : \alpha \in H \} \) of \( V \) such that \( \mathcal{G}' = (V, E') \) is a path where

\[
E' = \{(x, y) \in E : \theta(x, y) = \alpha^{-1} \ast \beta \quad \text{if} \ x \in V_\alpha, \ y \in V_\beta\}.
\]

In the case where \((H, \ast) = (\mathbb{Z}/n\mathbb{Z}, +)\) is the cyclic group of order \( n \), \((2 \leq n < \omega)\), and where \( \theta(x, y) = 1 \) for every \((x, y) \in E\), we deduce from Theorem 4 the weaker statement that for any countable path \( \mathcal{G} = (V, E) \), there is a partition \( \{ V_l : l \in \mathbb{Z}/n\mathbb{Z} \} \) of \( V \) such that, if \( l \in \mathbb{Z}/n\mathbb{Z} \) and \( x \in V_l \), then \( \{ y \in V_l+1 : (x, y) \in E \} \) is coinitial in \( \mathcal{G} \) \( \{ y \in V : x \) precedes \( y \) in \( \mathcal{G} \} \) and \( \{ y \in V_{l+(n-1)} : (y, x) \in E \} \) is cofinal in \( \mathcal{G} \) \( \{ y \in V : y \) precedes \( x \) in \( \mathcal{G} \} \). If \( \mathcal{G} = \mathcal{K} = (I, <) \) is a countable linear ordering and \( n = 2 \), then Theorem 2 says that \( I \) is the union of two disjoint sets \( I_0, I_1 \) which are dense in \( I \), (in the sense that for every \( i < 2 \) and \( x, y \in I \) such that \( x < y \), there is \( z_i \in I \) such that \( x < z_i < y \)). This very particular case is quite easy to check directly, without assuming that \( I \) is countable.

(4) It is not very difficult to check that the conclusion of Theorem 4 is also true if the path \( \mathcal{G} \) is scattered, without the assumption that \( \mathcal{G} \) is countable. (see [4]). However we prove in [4] that the conclusion of Theorem 4 is true if \( \mathcal{G} \) has cardinality \( \kappa \) and MA(\( \kappa) \) holds, (see [7] for a definition of Martin’s axiom MA), but false in general if \( \mathcal{G} \) has same order type as the real line.

**Proof of Theorem 4.** We say that a non-empty interval \( I \) of \( \mathcal{G} \) satisfies the condition \((*)\) if there is \( E_I \subseteq E \cap (I \times I) \) such that \( \mathcal{G}_I = (I, E_I) \) is a path and for any finite path \( p \) of \( \mathcal{G} \), \( \tilde{\theta}(p) \) depends only upon the extremities of \( p \). Define a relation \( R \) on \( V \) by \((x, y) \in R\) if and only if \( [x, y] \) satisfies \((*)\), where \([x, y]\) is the interval of \( \mathcal{G} \) of the elements of \( V \) which are between \( x \) and \( y \) in \( \mathcal{G} \), (possibly equal to \( x \) or \( y \) \), \((x, y) \in V\)). Let, for some \( \lambda \in (\omega \setminus \{0\}) \cup \{ \omega, \omega^*, \omega^* + \omega \} \), \((x : l \in \lambda) \) be a strictly increasing sequence of vertices of \( \mathcal{G} \) such that \( (x_l, x_{l+1}) \in R \) \((I, I + 1 \in \lambda) \). Let \( E_l \) be a set of edges which witnesses the fact that \([x_l, x_{l+1}] \) satisfies \((*)\), \((I, I + 1 \in \lambda) \). Then the set of edges \( E_I = \bigcup \{ E_l : l + 1 \in \lambda \} \) witnesses the fact that \( I \) satisfies \((*)\) where \( I \) is the interval \( \bigcup \{ [x_l, x_{l+1}] : l + 1 \in \lambda \} \) of \( \mathcal{G} \). Indeed it is obvious that \( \mathcal{G}_I = (I, E_I) \) is a path of \( \mathcal{G} \). Now, let \( p = (z_0, z_1, \ldots, z_k) \) be a finite path of \( \mathcal{G} \), \((k \in \omega \setminus \{0\}) \). There are some unique \( l \in \lambda \) and \( r \in \omega \), (depending only upon \( z_0 \) and \( z_k \)), such that \( l + r + 1 \in \lambda \), \( z_0 \in [x_l, x_{l+1}] \setminus \{x_{l+1}\} \) and \( z_k \in [x_{l+r}, x_{l+r+1}] \setminus \{x_{l+r+1}\} \). Clearly \( x_{l+s} \in [z_i : i \in k + 1] \) for any \( s \in (r + 1) \setminus \{0\} \). Define \( y_0 = z_0 \), \( y_{r+1} = x_{l+r}, y_s = x_{l+s} \) for \( s \in (r + 1) \setminus \{0\} \), and the finite path \( p = p \upharpoonright [y_s, y_{s+1}] \) of the graph \( \mathcal{G}_I = ([x_l, x_{l+r+1}], E_I) \) for \( s = r + 1 \). Then \( \tilde{\theta}(p) = \tilde{\theta}(p_s) \ast \cdots \ast \tilde{\theta}(p_r) \) depends only upon \( z_0 \) and \( z_k \). Hence \( R \) is an equivalence relation and any \( R\)-equivalence class is an interval \( I \) which satisfies \((*)\). Indeed if \((x : l \in \lambda) \) is a strictly increasing sequence of vertices of \( \mathcal{G} \), \((\lambda \in \{1, 2, \omega, \omega^*, \omega^* + \omega\}) \), such that \( \{ x_i : l \in \lambda \} \) is coinitial and cofinal in \( \mathcal{G} \), then
$I = \bigcup \{ [x, x_{i+1}]: \lambda \leq i + 1 \}$ satisfies (\*) since $[x, x_{i+1}]$ satisfies (\*), ($l$, $l + 1 \in \lambda$).

Now it remains to prove that there is only one $R$-equivalence class. For each $R$-equivalence class $I$, let $E_I$ be a set of edges which witnesses the fact that $I$ satisfies (\*) and let $E' = \{ (x, y) \in E: x$ and $y$ are in two different $R$-equivalence classes $\} \cup \{ E_I: I$ is an $R$-equivalence class $\}$. The graph $G' = (V, E')$ is a path. Define the set $H(x, y) = \{ \partial(p): p$ is a finite path of $G'$ with extremities $x$ and $y \} \subseteq H$, and the positive integer $n(x, y) = |H(x, y)|$, $(x, y \in V, x \neq y)$. Clearly $1 \leq n(x, y) \leq |H|$ and $n(x, y) = 1$ if and only if $x$ and $y$ are in a same $R$-equivalence class. Also, since $H = (H, \circ)$ is a group (and since $G'$ is connected), $n(x', y') = n(x, y)$ if $x'$ and $y'$ are between $x$ and $y$ in $G$, $(x' \neq y'$, possibly $\{ x, y \} \cap \{ x', y' \} = \emptyset$). Assume that there are at least two $R$-equivalence classes and choose $x$ and $y$ in two different classes such that $n(x, y) = n$ is minimum amongst the values different from 1. We shall get the contradiction that $(x, y) \in R$. We can note that $n(x', y') = n$ if $x'$ and $y'$ are in two different $R$-equivalence classes and are between $x$ and $y$ in $\mathcal{G}$. Moreover if $z$ is between $x'$ and $y'$ in $\mathcal{G}$ and if $h \in H(x', y')$, then there is a finite path $p$ of $\mathcal{G}$ from $x'$ to $y'$ (or $y'$ to $x'$) containing the vertex $z$ and such that $\partial(p) = h$. This is obvious if $x'$ and $y'$ are in the same $R$-equivalence class. Otherwise, assume for example that $x'$ precedes $y'$ in $\mathcal{G}$ and that $z$ and $y'$ are in two different $R$-equivalence classes, so that $n(z, y') = n$. $\mathcal{G}$ is a path, and hence there is a finite path $p_1$ of $\mathcal{G}$ from $x'$ to $z$. Since $\{ \partial(p_1) \circ g: g \in H(z, y') \}$ is a subset of $H(x', y')$ of size $n = |H(x', y')|$, we get that $h = \partial(p_1) \circ g$ for some $g \in H(z, y')$, which gives the desired conclusion.

Let now $\{ x_n: n \in \omega \}$ be a well-ordering of $[x, y]$, (which is infinite since two consecutive vertices of $\mathcal{G}$ are in the same $R$-equivalence class), and inductively define the finite paths $G_n = (V_n, E_n)$ of $\mathcal{G}$, $(n \in \omega)$, by

\begin{align*}
V_0 &= \{ x_0 \}, & E_0 &= \emptyset, & V_{n+1} &= V_n \cup \{ y_i: i \in m + 1 \}, \\
E_{n+1} &= E_n \cup \{ (y_i, y_{i+1}): i \in m \},
\end{align*}

$(n \in \omega)$, where $p = (y_0, \ldots, y_m)$ is a finite path of $\mathcal{G}$ defined in the following way: Let $s$ be the least integer such that $x_s \notin V_n$. If $x_s$ precedes in $\mathcal{G}$ any element of $V_n$, then $y_0 = x_s$ and $y_m$ is the first element of $G_n$; if $x_s$ follows in $\mathcal{G}$ any element of $V_n$, then $y_m = x_s$ and $y_0$ is the last element of $G_n$; if $x_s$ is between two elements $z, z'$ of $V_n$ which are consecutive in $G_n$ and such that $z$ precedes $z'$, then $y_0 = z$, $y_m = z'$, $x_s \in \{ y_i: i \in m \setminus \{ 0 \} \}$ and, moreover, $\partial(p) = \partial(z, z')$. Let $E_{[x, y]} = \bigcup \{ E_n: n \in \omega \} \subseteq E'$. The graphs $G_n$, $(n \in \omega)$, and hence the graph $G_{[x, y]} = ([x, y], E_{[x, y]})$, are clearly paths of $\mathcal{G}$ such that, for any finite subpath $p$, $\partial(p)$ depends only upon the extremities of $p$. This proves the contradiction that $(x, y) \in R$. $\square$

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