JOURNAL OF COMBINATORIAL THEORY, Series B 22, 236–238 (1977)

Note

Two Infinite Classes of Perfect Codes in Metrically Regular Graphs

J. A. THAS

Seminar of Higher Geometry, University of Ghent, Ghent, Belgium

Received January 9, 1976

1. INTRODUCTION

In a graph (assumed to be connected, undirected, and without loops or multiple edges), the distance $d(\alpha, \beta)$ between two vertices α and β is the length of the shortest path joining them. For a positive integer e, a perfect e code is a nonempty subset C of the vertex set with the property that any vertex lies at distance at most e from a unique vertex in C. Biggs [1] has shown that necessary conditions for the existence of a perfect 1 code in a regular graph of valency k on v vertices are that $k + 1$ divides v and that -1 is an eigenvalue of the adjacency matrix of the graph. These results have been extended to perfect e codes in metrically regular graphs by Delsarte [5] (see also Biggs [1]). (Metrically regular graphs are defined in [5].)

In [1] Biggs gives several examples of "nonclassical" perfect 1 codes in metrically regular graphs. There he mentions the following situation which generalizes the "repetition" codes in the classical case. A graph Γ, of diameter d (the diameter of a graph is the largest value assumed by $d(\alpha, \beta)$), is said to be antipodal if $d(\alpha, \beta) = d$ and $d(\alpha, \gamma) = d$ implies that $\gamma = \beta$ or $d(\beta, \gamma) = d$. Consequently in an antipodal graph the relation D, defined by $\alpha D \beta$ if $\alpha = \beta$ or $d(\alpha, \beta) = d$, is an equivalence relation. If $d = 2e + 1$, then any equivalence class of this relation is a perfect e code in the graph. Several examples of perfect codes in antipodal graphs are known.

Perfect 1 codes in graphs defined by generalized hexagons were discovered by Cameron, Thas, and Payne [4]. To their knowledge this was the first infinite class of perfect e codes, apart from the classical repetition and Hamming codes and their analogs. A new such infinite class of perfect 1 codes, and also a new infinite class of antipodal graphs, will be defined in this paper.

2. AN INFINITE CLASS OF ANTIPODAL GRAPHS

Let H be a nonsingular hyperquadric ([7, p. 139]) of $PG(2n, 2^h)$, $n > 1$, or an oval [6] in $PG(2, 2^h)$. The nucleus ([7, p. 139]) of H is denoted by t

ISSN 0095-8956

(t is the point for which any line tx, $x \in H$, is a tangent of H). Let $PG(2n, 2^h) = P$ be embedded in a $PG(2n+1, 2^h) = P'$. The vertices of the graph Γ are the elements of $P' - P$ (the number of vertices equals $2^{h(2n+1)}$), and two vertices x and y ($x \neq y$) are adjacent if the line xy contains a point of H. Then clearly $d(x, y) = 2$ if the line xy contains a point of $P - (H \cup \{t\})$, and $d(x, y) = 3$ if the line xy contains the nucleus t of H. The graph Γ is metrically regular with intersection array [2]

$$\begin{bmatrix} * & 1 & q^{2n-1} & q^{2n}-1 \\ 0 & q^{2n-1}-2 & q^{2n}-q^{2n-1}-2 & 0 \\ q^{2n}-1 & q^{2n}-q^{2n-1} & 1 & * \end{bmatrix},$$

where $q = 2^h$. Moreover, Γ is antipodal with diameter 3. The subset C of the vertex set is a perfect 1 code if and only if $C \cup \{t\}$ is a line of P'.

Finally we remark that the eigenvalues of the adjacency matrix of Γ are $q^{2n}-1$, -1, q^n-1, $-q^n-1$ ($q = 2^h$).

3. An Infinite Class of Perfect 1 Codes in Nonantipodal Graphs

Let π be a symplectic polarity [6] of the projective space $PG(5, q)$. The polarity π has exactly $(q^3+1)(q^2+1)(q+1)$ absolute planes [6]. The vertices of the graph Γ are the absolute planes of π, and two vertices P_1 and P_2 ($P_1 \neq P_2$) are adjacent if $P_1 \cap P_2$ is a line. Clearly $d(P_1, P_2) = 2$ if the planes P_1, P_2 have exactly one point in common, and $d(P_1, P_2) = 3$ if $P_1 \cap P_2 = \varnothing$. The graph Γ is metrically regular with intersection array

$$\begin{bmatrix} * & 1 & q+1 & q^2+q+1 \\ 0 & q-1 & q^2-1 & q^3-1 \\ q^3+q^2+q & q^2(q+1) & q^3 & * \end{bmatrix}.$$

The eigenvalues of the adjacency matrix of Γ are q^3+q^2+q, -1, q^2+q-1, $-q^2-q-1$. Since $k+1 = (q^2+1)(q+1)$ divides v and since -1 is an eigenvalue of the adjacency matrix, the necessary conditions for the existence of a perfect 1 code are satisfied.

The subset C of the vertex set of Γ is a perfect 1 code if and only if C consists of q^3+1 mutually disjoint absolute planes. Consequently a perfect 1 code in Γ is the same thing as a spread [6] of $PG(5, q)$, consisting entirely of absolute planes. Now we show that the set of absolute planes of any symplectic polarity of $PG(5, q)$ always contains a spread of planes.

Let $C = \{P_1, P_2, \ldots, P_{q^3+1}\}$ be a regular spread [3, 6] of planes of $PG(5, q)$. Then there are exactly three lines l_1, l_2, l_3 of the extension $PG(5, q^3)$ of $PG(5, q)$ which meet every plane of C [3, 8]. Moreover, these lines are conjugated with respect to the cubic extension $GF(q^3)$ of $GF(q)$ (there is a collineation τ of order 3 of $PG(5, q^3)$ which fixes every point of $PG(5, q)$, for which $l_1^\tau = l_2$, $l_2^\tau = l_3$, $l_3^\tau = l_1$ [3]). Let π be a symplectic polarity of $PG(5, q)$ for which P_1, P_2, P_3, P_4 are absolute planes. The extension to $PG(5, q^3)$ of this polarity is also denoted by π. The polarity π maps l_i onto a threespace which has a line in common with each of the planes P_1, P_2, P_3, P_4. Consequently this three space contains two of the lines l_1, l_2, l_3. If π maps l_i onto the threespace $l_j l_k$, then π maps l_i^τ onto the threespace $l_j^\tau l_k^\tau$. From this remark it follows easily that π maps l_i onto the threespace $l_j l_k$, where $\{i, j, k\} = \{1, 2, 3\}$. Consequently the $q^3 + 1$ planes P_1, P_2,..., P_{q^3+1} of $PG(5, q)$ which meet l_1, l_2, l_3 are absolute with respect to π. Since any two symplectic polarities of $PG(5, q)$ are projectively equivalent, we conclude that the set of absolute planes of any symplectic polarity of $PG(5, q)$ always contains a regular spread of planes.

From the preceding follows immediately that the metrically regular graph Γ always contains a perfect 1 code.

References

1. N. L. Biggs, Perfect codes and distance-transitive graphs, *in* "Combinatorics" (Proceedings of the British Combinatorics Conference, 1973) (T. P. McDonough and V. C. Mavron, Eds.), pp. 1–8, Cambridge Univ. Press, London/New York, 1974.
2. N. L. Biggs, Perfect codes in graphs, *J. Combinatorial Theory* (*B*) **15** (1973), 289–296.
3. R. H. Bruck, Circle geometry in higher dimensions, II, *Geometriæ Dedicata* **2** (1973), 133–188.
4. P. J. Cameron, J. A. Thas, and S. E. Payne, Polarities of generalized hexagons and perfect codes, *Geometriæ Dedicata*, to appear.
5. P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Res. Rep. Suppl.* **10** (1973).
6. P. Dembowski, "Finite Geometries," Springer-Verlag, Berlin, 1968.
7. B. Segre, Introduction to Galois geometries, *Acad. Naz. Lincei, Mem. Cl. Sc. Fis. Mat. Nat.* **8**, Sez.I[a] (1967), 133–236.
8. B. Segre, Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane, *Ann. Mat. Pura Appl.* **64** (1964), 1–76.