Determination of Certain Parameters in Heat Conduction Problems

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I. INTRODUCTION

If one of the physical properties of a conductor is unknown, then it will be shown for the case of unknown diffusivity of a homogeneous finite conductor, the case of unknown length of a finite homogeneous conductor, and the case of an unknown distance from the boundary to the interface between a finite homogeneous conductor and a semi-infinite homogeneous conductor that a measurement of the flow rate on the boundary at one instant in time is sufficient to determine uniquely the unknown physical property and the temperature distribution.

II. UNKNOWN DIFFUSIVITIES OF HOMOGENEOUS CONDUCTORS

Consider the problem of determining a positive constant $K$ and a function $u(x, t)$ which is $C^1$ in $0 < x < 1$, $0 < t < T$, such that the pair $(K, u)$ satisfies

\begin{align*}
\text{(a)} & \quad u_t = K u_{xx}, \quad 0 < x < 1, \quad 0 < t < T, \\
\text{(b)} & \quad u(x, 0) = 0, \quad 0 < x < 1, \\
\text{(c)} & \quad u(0, t) = f(t), \quad 0 \leq t \leq T, \quad f(0) = 0, \\
\text{(d)} & \quad u(1, t) = g(t), \quad 0 \leq t \leq T, \quad g(0) = 0, \\
\text{(e)} & \quad -\rho cKu_x(0, t_0) = h, \quad 0 < t_0 < T,
\end{align*}

where $\kappa$ is the unknown diffusivity, $\rho$ is the known density, $c$ is the known specific heat, $h$ is a known constant, and $f(t)$ and $g(t)$ are known functions of $t$.

REMARK. Note that it is necessary to measure the heat flow rate for just a single time. Also, as $\kappa$ is regarded here as a constant, (2.1) is really a special
case of the problem of variable unknown diffusivity \( a(t) \) which has been treated by Jones and Douglas [1, 2]. The method given in [1, 2] seems unduly complicated in the opinion of the author to be applied to this simple case. Hence, the author is proposing an alternate approach for (2.1).

Assume that a solution of (2.1) exists. Then [3, 4], for continuous \( f(t) \) and \( g(t) \),

\[
\begin{align*}
\frac{\partial M(x, \kappa(t - \tau))}{\partial x} f(t) \kappa d\tau + \frac{\partial M(x - 1, \kappa(t - \tau))}{\partial x} g(t) \kappa d\tau,
\end{align*}
\]

where

\[
M(\xi, \sigma) = \pi^{-1/2} \sigma^{-1/2} \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{(\xi + 2n)^2}{4\sigma} \right], \quad \sigma > 0.
\]  

If \( f(t) \) and \( g(t) \) are continuously differentiable, then Leibnitz's rule can be applied for \( 0 < x < 1, 0 < t < T \) to obtain

\[
\begin{align*}
\frac{\partial}{\partial x} M(x, \kappa(t - \tau)) = -\frac{\partial}{\partial x} M(x - \xi, \kappa(t - \tau)), \quad x \neq \xi, \quad t \neq \tau, \tag{2.5}
\end{align*}
\]

it follows that

\[
\begin{align*}
u_x(x, t) = \int_0^t \frac{\partial M(x, \kappa(t - \tau))}{\partial \tau} f(t) \kappa d\tau + \int_0^t \frac{\partial M(x - 1, \kappa(t - \tau))}{\partial \tau} g(t) \kappa d\tau. \tag{2.6}
\end{align*}
\]

As \( f(0) = g(0) = 0 \) and

\[
\lim_{t \to \xi} M(x - \xi, \kappa(t - \tau)) = 0, \quad x \neq \xi, \tag{2.7}
\]

then

\[
\begin{align*}
u_x(x, t) & = -\int_0^t M(x, \kappa(t - \tau)) f'(\tau) d\tau + \int_0^t M(x - 1, \kappa(t - \tau)) g'(\tau) d\tau, \quad 0 < x < 1. \tag{2.8}
\end{align*}
\]
By Lebesgue's dominated convergence theorem,

\[
\lim_{t \to 0} u_2(x, t) = -\int_0^t M(0, \kappa(t - \tau)) f'(\tau) \, d\tau + \int_0^t M(-1, \kappa(t - \tau)) g'(\tau) \, d\tau.
\]

(2.9)

Set

\[
G(\kappa) = -\int_0^t M(0, \kappa(t - \tau)) f'(\tau) \, d\tau + \int_0^t M(-1, \kappa(t - \tau)) g'(\tau) \, d\tau
\]

(2.10)

and

\[
F(\kappa) = -\rho \kappa G(t_0, \kappa).
\]

(2.11)

Then, from (e) of (2.1), it follows that

\[
h = F(\kappa).
\]

(2.12)

Thus, if (2.1) possesses a solution, then the diffusivity \( \kappa \) must satisfy (2.12).

Now, (2.1) and (2.12) are equivalent in the following sense.

**Theorem 2.1.** If \( f(t) \) and \( g(t) \) are continuously differentiable for \( 0 \leq t \leq T \), then problem (2.1) possesses a unique solution if and only if (2.12) possesses exactly one positive solution \( \kappa \).

**Proof.** Assume that (2.1) possesses a unique solution. By previous analysis, it follows that (2.12) possesses a positive solution \( \kappa \). If \( \kappa \) is not the only positive solution of (2.12), then there exists \( \kappa_1 > 0 \) and \( \kappa_2 > 0 \), \( \kappa_1 \neq \kappa_2 \), such that each is a solution of (2.12). Set

\[
u_i(x, t) = -\int_0^t M(x, \kappa_i(t - \tau)) f(\tau) \kappa_i \, d\tau + \int_0^t M(x - 1, \kappa_i(t - \tau)) g(\tau) \kappa_i \, d\tau, \quad i = 1, 2.
\]

(2.13)

Clearly, the pairs \((\kappa_i, \nu_i), i = 1, 2\), form two distinct solutions of (2.1). This contradicts the hypothesis that (2.1) possesses a unique solution. Thus, \( \kappa \) is the only positive solution of (2.12).

Assume now that (2.12) possesses exactly one positive solution \( \kappa \). Then, \( \kappa \) and \( u(x, t) \), defined by (2.2), form a solution of (2.1). The uniqueness follows as above.

Consider the equation (2.12).

**Theorem 2.2.** If \( f(t) \) and \( g(t) \) are \( C^1 \) for \( 0 \leq t \leq T \) with \( f(t) \neq 0 \), \( f'(t) \geq 0 \), and \( g'(t) \leq 0 \) for \( 0 \leq t \leq t_0 \) and if \( h > 0 \), then (2.12) possesses exactly one positive solution \( \kappa \).
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PROOF. Since $\kappa M(0, \kappa(t_0 - \tau))$ and $\kappa M(-1, \kappa(t_0 - \tau))$ possess continuous integrable derivatives with respect to $\kappa$ for $\kappa > 0$, it follows from [5] that $F(\kappa)$ is continuously differentiable for $\kappa > 0$ and that $F'(\kappa) > 0, \kappa > 0$. Thus, $F(\kappa)$ is a strictly increasing function. Hence, in order to show that (2.12) has exactly one positive solution $\kappa$, it suffices to show that

$$\lim_{\kappa \downarrow 0} F(\kappa) = 0$$  \hspace{1cm} (2.14)

and

$$\lim_{\kappa \uparrow \infty} F(\kappa) = \infty.$$  \hspace{1cm} (2.15)

As

$$\exp \{-x\} < x^{-3}, \quad x > 0,$$

it follows from (2.3) and (2.11) that

$$F(\kappa) \leq C\kappa^{1/2},$$  \hspace{1cm} (2.16)

where $C$ is some positive constant. Thus, (2.14) is valid. As

$$F(\kappa) \geq \left\{ \rho c \int_0^t \frac{f'(\tau)}{\sqrt{\pi(t_0 - \tau)}} d\tau \right\}^{1/2},$$  \hspace{1cm} (2.17)

relation to (2.15) holds.

REMARK. Note that constructing a graph of $F(\kappa)$ suffices to provide a numerical solution for $\kappa$. Moreover, the graph of $F(\kappa)$ can also be obtained approximately by solving (2.1, a-d) numerically [6] for various $\kappa$ and evaluating $-h[\rho cG(t_0, \kappa)]^{-1}$ from the numerical approximation.

Before the dependence of $\kappa$ upon the data can be considered, an a priori bound for $\kappa$ must be derived. Since

$$\kappa = -h[\rho cG(t_0, \kappa)]^{-1}$$  \hspace{1cm} (2.18)

it follows from the hypothesis of Theorem 2.2 that

$$\kappa \leq h \left\{ \rho c \int_0^t \frac{f'(\tau)}{\sqrt{\pi\kappa(t_0 - \tau)}} d\tau \right\}^{-1}.$$  \hspace{1cm} (2.19)

As $\kappa > 0$,

$$\kappa \leq h^2 \left\{ \rho c \int_0^t \frac{f'(\tau)}{\sqrt{\pi(t_0 - \tau)}} d\tau \right\}^{-2} = \gamma.$$  \hspace{1cm} (2.20)
Let
\[ ||f||_0 = \sup_{0 \leq t \leq t_0} |f(t)| \] (2.21)

for any function \( f(t) \) defined on \( 0 \leq t \leq t_0 \). Let \( f_i(t), g_i(t), \) and \( h_i, \ i = 1, 2 \) satisfy the hypothesis of Theorem 2.2. Consider
\[ h_i = F_i(\kappa), \quad i = 1, 2. \] (2.22)

Now
\[ h_1 - h_2 = [F_1(\kappa_1) - F_1(\kappa_2)] + [F_2(\kappa_1) - F_2(\kappa_2)]. \] (2.23)

As
\[
|F_1(\kappa_2) - F_2(\kappa_2)| \leq 2\pi^{-1/2} \rho c \gamma_2^{1/2} t_0^{1/2} ||f'_1 - f'_2||_{t_0} \\
+ \left( \frac{8}{3} \right) \rho c \pi^{-1/2} \gamma_2^{3/2} t_0^{3/2} ||f'_1 - f'_2||_{t_0} \\
+ \left( \frac{32}{3} \right) \rho c \pi^{-1/2} \gamma_2^{3/2} t_0^{3/2} ||g'_1 - g'_2||_{t_0} \\
= C_1 ||f'_1 - f'_2||_{t_0} + C_2 ||g'_1 - g'_2||_{t_0},
\] (2.24)

it follows from the differentiability of \( F_1(\kappa) \) that
\[
|\kappa_1 - \kappa_2| \leq C_3 \{ |h_1 - h_2| + C_1 ||f'_1 - f'_2||_{t_0} + C_2 ||g'_1 - g'_2||_{t_0} \}, \] (2.25)

where
\[
C_3 = \left\{ \frac{\rho c}{2} \int_0^{t_0} \min \left\{ f'_1(\tau), f'_2(\tau) \right\} \frac{d\tau}{\sqrt{\pi(t_0 - \tau)} \max (\gamma_1, \gamma_2)} \right\}^{-1}.
\] (2.26)

This implies that the positive solution
\[
\kappa = \kappa(h, f, g)
\] (2.27)

is locally Lipschitz continuous in its arguments. Therefore, the positive solution of (2.12) depends continuously upon the data.

In concluding this section, it should be noted that similar analysis for determining the thermal diffusivity of a part of a composite infinite medium and a part of a composite semi-infinite medium has been done [7]. In both cases, the determination of the diffusivity is reduced to plotting the graph of a monotone function.
III. Unknown Length of a Finite Homogeneous Conductor

Consider the problem of determining a positive constant \( v \) and a function \( u(x, t) \) which is \( C^1 \) in \( 0 \leq x \leq v, \ 0 \leq t \leq T \) such that the pair \( (v, u) \) satisfies

\[
\begin{align*}
(a) & \quad u_t = u_{xx}, \quad 0 < x < v, \quad 0 < t < T, \\
(b) & \quad u(x, 0) = 0, \quad 0 < x < v, \\
(c) & \quad u(0, t) = f(t), \quad 0 \leq t \leq T, \quad f(0) = 0, \\
(d) & \quad u(v, t) = g(t), \quad 0 \leq t \leq T, \quad g(0) = 0, \\
(e) & \quad -u_x(0, t) = h, \quad 0 < t \leq T,
\end{align*}
\]

where \( v \) is the unknown length of the conductor. It is assumed that the boundary at \( x = 0 \) is accessible for temperature and heat flow measurements, while the behavior of these quantities at \( x = v \) must be assumed to be known a priori or to be measurable in some fashion which does not require the location of this boundary to be known.

Consider the transformation \( x = v\zeta, \ 0 \leq \zeta \leq 1 \). Set

\[
U(\zeta, t) = u(v\zeta, t).
\]

Then, it is clear from

\[
U_\zeta = vu
\]

that (3.1) is equivalent with

\[
\begin{align*}
(a) & \quad U_t = (1/v^2) U_{\zeta\zeta}, \quad 0 < \zeta < 1, \quad 0 < t \leq T, \\
(b) & \quad U(\zeta, 0) = 0, \quad 0 \leq \zeta \leq 1, \\
(c) & \quad U(0, t) = f(t), \quad 0 \leq t \leq T, \quad f(0) = 0, \\
(d) & \quad U(1, t) = g(t), \quad 0 \leq t \leq T, \quad g(0) = 0, \\
(e) & \quad -U_\zeta(0, t_0) = h, \quad 0 < t_0 \leq T.
\end{align*}
\]

By the analysis of Section II, it follows that (3.4) is equivalent to the equation

\[
h = - \left( \frac{1}{v} \right) G \left( t_0, \frac{1}{v^2} \right) = H(v).
\]

Now, if \( f(t) \) and \( g(t) \) satisfy the hypothesis of Theorem 2.2, then it can be shown by elementary techniques that \( H(v) \) is a strictly decreasing function for \( v > 0 \) such that \( a < H(v) < \infty \), where

\[
a = \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi(t_0 - \tau)}} d\tau.
\]
Therefore, if \( f(t) \) and \( g(t) \) satisfy the hypothesis of Theorem 2.2 and if \( a < h < \infty \), then (3.5) possesses exactly one positive solution \( v \). Moreover, by an argument similar to that of Section II [7], the positive solution

\[
\nu = v(h, f, g)
\]  

of (3.5) is locally Lipschitz continuous.

For an alternate approach to the determination of the length of a finite conductor, consider

\[
\begin{align*}
(a) & \quad u_t = u_{xx}, \quad 0 < x < v, \quad 0 < t < T, \\
(b) & \quad u(x, 0) = 0, \quad 0 < x < v, \\
(c) & \quad -u_x(0, t) - h(t), \quad 0 < t < T, \\
(d) & \quad u_x(v, t) = i(t), \quad 0 < t < T \\
(e) & \quad u(0, t_0) = f, \quad 0 < t_0 \leq T,
\end{align*}
\]

where \( v \) is the unknown length, \( u(x, t) \) is unknown, and all of the rest is known data. For continuous \( h(t) \) and \( i(t) \), the solution of (3.8, a-d) is given by

\[
u(x, t) = \int_0^t \{M(x, t - \tau, v) h(\tau) + M(x - v, t - \tau, v) i(\tau)\} d\tau,
\]

where

\[
M(\xi, \sigma, \nu) = \pi^{-1/2} \sigma^{-1/2} \sum_{n=0}^{\infty} \exp \left\{ -\frac{(\xi + 2n\nu)^2}{4\sigma} \right\}, \quad \sigma > 0.
\]

Hence, it follows immediately that \( \nu \) must satisfy

\[
f = \int_0^{t_0} \{M(0, t_0 - \tau, \nu) h(\tau) + M(- \nu, t_0 - \tau, \nu) i(\tau)\} d\tau.
\]

Moreover, if

\[
\begin{align*}
(a) & \quad h(t) \geq 0, \quad h(t) \neq 0 \quad \text{for} \quad 0 \leq t \leq t_0; \\
(b) & \quad i(t) \geq 0, \quad \text{for} \quad 0 \leq t \leq t_0; \\
(c) & \quad b < f < \infty,
\end{align*}
\]

where

\[
b = \int_0^{t_0} \frac{h(\tau)}{\sqrt{\pi(t_0 - \tau)}} d\tau.
\]
then

\[ F_\ast(v) = \int_0^{t_0} \{ M(0, t_0 - \tau, v) h(\tau) + M(- v, t_0 - \tau, v) i(\tau) \} \, d\tau \]  

(3.14)

is a strictly decreasing function of \( v \). Hence, (3.11) possesses exactly one positive solution \( v \).

IV. UNKNOWN DISTANCE TO AN INTERFACE BETWEEN A FINITE CONDUCTOR AND A SEMI-INFINITE CONDUCTOR

Consider the problem of determining an unknown positive constant \( v \) and a bounded function \( u(x, t) \) which is \( C^1 \) in \( 0 < x < v, \ v < x < \infty, \ 0 < t < T \), such that the pair \( (v, u) \) satisfies

(a) \( u_t = \kappa_1 u_{xx}, \quad 0 < x < v, \quad 0 < t < T, \)

(b) \( u_t = \kappa_2 u_{xx}, \quad \nu < x < \infty, \quad 0 < t < T, \)

(c) \( \lim_{x \to v} u(x, t) = \lim_{x \to v} u(x, t), \quad 0 < t < T, \)

(d) \( \lim_{x \to v} \rho_1 \kappa_1 u(x, t) = \lim_{x \to v} \rho_2 \kappa_2 u(x, t), \quad 0 < t < T, \)

(e) \( u(x, 0) = 0, \quad 0 < x < \infty, \)

(f) \( u(0, t) = f(t), \quad 0 < t < T, \)

(g) \( \rho_1 \kappa_1 u_x(0, t_0) = h, \quad 0 < t_0 < T, \)

Remark. A problem similar to (4.1) arises from the acid treatment of an oil well.

Assume that \( f(t) \) is continuously differentiable and Lebesgue integrable for \( 0 \leq t \leq T \) with \( f(0) = 0 \). If problem (4.1) possessed a solution, then using the representation of the solution of (4.1, a-f) which is given by (1.7) of the appendix and an argument which is step by step similar to that of Section II, it follows that \( \nu \) must satisfy the nonlinear equation

\[ h = -\rho_1 \kappa_1 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi} \kappa_1 (t_0 - \tau)} \left( 1 + 2 \sum_{n=1}^{\infty} \alpha^n \exp \left[ -\frac{\alpha^2 \nu^3}{\kappa_1 (t_0 - \tau)} \right] \right) \, d\tau. \]  

(4.2)

By an argument similar to that of Theorem 2.1, (4.1) and (4.2) are equivalent in the sense that (4.1) possesses a unique solution if and only if (4.2) possesses exactly one positive solution \( v \). The existence of exactly one positive solution of (4.2) will be broken into the cases: \( \alpha > 0, \ \alpha < 0, \ \alpha = 0 \) (\( \alpha \) defined by (1.5) in the Appendix).
Solution of the Nonlinear Equation for the Case $\alpha > 0$

In addition to the previous hypothesis on $f(t)$, assume that

(a) $f(t) \neq 0, \quad f'(t) \geq 0 \quad \text{for} \quad 0 < t < t_0,$ \hspace{1cm} \hspace{1cm} (4.3)

(b) $a < h < b,$

where

$$a = - \rho_1 e_1 \kappa_1 \left(1 + \frac{\alpha}{1 - \alpha}\right) \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi \kappa_1 (t_0 - \tau)}} \, d\tau$$ \hspace{1cm} (4.4)

and

$$b = - \rho_1 e_1 \kappa_1 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi \kappa_1 (t_0 - \tau)}} \, d\tau.$$ \hspace{1cm} (4.5)

Define the function $F(v)$ by the relation

$$F(v) = - \rho_1 e_1 \kappa_1 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi \kappa_1 (t_0 - \tau)}} \left[1 + 2 \sum_{n=1}^{\infty} \alpha^n \exp \left[ \frac{- n^2 \nu^2}{\kappa_1(t_0 - \tau)} \right] \right] \, d\tau,$$

$$v > 0.$$ \hspace{1cm} (4.6)

Since $0 < \alpha < 1$, it follows that $F(v)$ is continuously differentiable for $0 < v < \infty$ with derivative

$$F'(v) = 4 \rho_1 e_1 \kappa_1 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi} \left(\kappa_1(t_0 - \tau)\right)^{3/2}} \left[\sum_{n=1}^{\infty} vn^2 \alpha^n \exp \left[ \frac{- n^2 \nu^2}{\kappa_1(t_0 - \tau)} \right] \right] \, d\tau > 0.$$ \hspace{1cm} (4.7)

Thus, $F(v)$ is strictly increasing. As

$$\lim_{v \downarrow 0} F(v) = \alpha$$ \hspace{1cm} (4.8)

and

$$\lim_{v \uparrow \infty} F(v) = b,$$ \hspace{1cm} (4.9)

it follows that (4.2) possesses exactly one positive solution.

**Remark.** Using the techniques of Section II, it can be shown [7] that the solution $v$ depends continuously upon the data.
Solution of the Nonlinear Equation for the Case $\alpha < 0$

In order to treat this case, assume the existence of an a priori lower bound for $\nu$; i.e., to (4.1) add the condition that $0 < \beta < \nu$. This means that in order for (4.1) to have a unique solution, exactly one solution $\nu \geq \beta$ of (4.2) must be found. Assume in addition to the hypothesis on $f(t)$ that

(a) \quad 0 < t_0 < \beta^2/\kappa_1,

(b) \quad b < h \leq d,

where

\[ d = -\rho c \kappa_1 \int_0^{t_0} \frac{f'(t)}{\sqrt{\pi \kappa_1(t_0 - \tau)}} \left\{ 1 + 2 \sum_{n=1}^{\infty} \alpha^n \exp \left[ \frac{-n^2 \beta^2}{\kappa_1(t_0 - \tau)} \right] \right\} d\tau. \]  

(4.11)

As above, $F(\nu)$ defined by (4.6) for $\nu \geq \beta > 0$ is continuously differentiable with its derivative given by (4.7) for $\nu \geq \beta > 0$. Note here that the derivative certainly exists since $-1 < \alpha < 0$. Consider the series

\[ \sum_{n=1}^{\infty} n^2 \alpha^n \exp \left[ \frac{-n^2 \beta^2}{\kappa_1(t_0 - \tau)} \right] \]  

(4.12)

which is an alternating series since $\alpha < 0$. Now, the function

\[ S(x) = x^2 \exp \{-\mu x^2\}, \quad \mu > 0, \quad x > 0, \]  

(4.13)

has its maximum at

\[ x = \mu^{-1/2}. \]  

(4.14)

Setting

\[ \mu = \frac{\nu^2}{\kappa_1(t_0 - \tau)}, \]  

(4.15)

the function

\[ x^2 \exp \left[ \frac{-x^2 \nu^2}{\kappa_1(t_0 - \tau)} \right], \quad x > 0, \]  

(4.16)

has its maximum at

\[ x = \frac{\sqrt{\kappa_1(t_0 - \tau)}}{\nu} < \frac{\sqrt{\kappa_1 \beta^2}}{\nu} < \frac{\sqrt{\kappa_1 \beta^2}}{\nu} = \frac{\beta}{\nu} \leq 1. \]  

(4.17)
Hence, for \( x \geq 1 \), the function \( S(x) \) is strictly decreasing. Thus, the series in expression (4.7) converges and is negative. Therefore, \( F'(\nu) < 0 \). Hence, for this case \( F(\nu) \) is a strictly decreasing function. As

\[
\lim_{\nu \to \beta} F(\nu) = d \tag{4.18}
\]

and

\[
\lim_{\nu \to \infty} F(\nu) = b, \tag{4.19}
\]

it follows that (4.2) possesses exactly one positive solution \( \nu \geq \beta > 0 \).

Remark. The solution \( \nu \geq \beta > 0 \) depends continuously upon the data [7].

Discussion of the Case \( \alpha = 0 \)

In this case (note \( \sigma = 1 \leftrightarrow \alpha = 0 \)) it can be shown that the solution of (4.1, a-f) is

\[
u(x, t) = \frac{1}{2 \sqrt{\pi k_1}} \int_0^t \frac{x}{(t - \tau)^{3/2}} \exp \left\{ - \frac{x^2}{4k_1(t - \tau)} \right\} f(\tau) d\tau,
\]

\[0 < x < \nu, \quad 0 < t < T,
\]

\[
u(x, t) = \frac{1}{2 \sqrt{\pi k_1}} \int_0^t \frac{\nu + kx - kv}{(t - \tau)^{3/2}} \exp \left[ - \frac{(\nu + kx - kv)^2}{4k_1(t - \tau)} \right] f(\tau) d\tau,
\]

\[v < x < \infty, \quad 0 < t < T,
\]

where \( k \) is defined by (1.3) of the Appendix. It can be seen immediately that the position of the interface \( \nu \) has no effect upon the behavior of the solution of (4.1, a-f) in the neighborhood of the boundary \( x = 0 \). Hence, no heat flow test can be devised which will locate the interface from data taken at the boundary for the case \( \alpha = 0 \) (\( \sigma = 1 \)).

It is possible that an a priori upper bound \( \gamma \) for \( \nu \) might be known. Also, it is possible that at \( x = x_* \geq \gamma \) heat flow measurements could be taken. Since for \( \alpha = 0 \) the solution of (4.1, a-f) in the neighborhood of \( x = x_* \) depends upon the location of the interface \( \nu \), it is reasonable to consider the following problem:

(a) \( u_t = k_1u_{xx}, \quad 0 < x < \nu, \quad 0 < t < T; \quad \nu \leq \gamma \),

(b) \( u_t = k_2u_{xx}, \quad \nu < x < \infty, \quad 0 < t < T, \)
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(c) \( \lim_{x \to \nu} u(x, t) = \lim_{x \to \nu} u(x, t), \quad 0 < t < T, \)

(d) \( \lim_{x \to \nu} \rho_1 c_1 \kappa_1 u_x(x, t) = \lim_{x \to \nu} \rho_2 c_2 \kappa_2 u_x(x, t) \quad 0 < t < T, \) (4.21)

(e) \( u(x, 0) = 0, \quad 0 < x < \infty, \)

(f) \( u(0, t) = f(t), \quad 0 < t < T, \)

(g) \( \rho_2 c_2 \kappa_2 u_x(x_*, t_0) = h, \quad 0 < t_0 < T, \quad x_* \geq \gamma, \)

where \( u(x, t) \) and \( v \) are unknowns.

The equivalent nonlinear equation for \( v \) is

\[
h = -\rho_2 c_2 \kappa_2 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\kappa_2(t_0 - \tau)}} \exp \left\{ -\frac{[x_* - v + (v/h)]^2}{4\kappa_2(t_0 - \tau)} \right\} d\tau. \] (4.22)

This equation possesses exactly one positive solution \( v \leq \gamma \) if in addition to the hypothesis on \( f(t) \) above, it is assumed that

(a) \( \text{if } \kappa_1 < \kappa_2, \quad p < h < q, \quad \text{or} \)

(b) \( \text{if } \kappa_2 < \kappa_1, \quad q < h < p, \) (4.23)

where

\[
p = -\rho_2 c_2 \kappa_2 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\kappa_2(t_0 - \tau)}} \exp \left\{ -\frac{x_*^2}{4\kappa_2(t_0 - \tau)} \right\} d\tau \] (4.24)

and

\[
q = -\rho_2 c_2 \kappa_2 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\kappa_2(t_0 - \tau)}} \exp \left\{ -\frac{[x_* - \gamma + (\gamma/h)]^2}{4\kappa_2(t_0 - \tau)} \right\} d\tau. \] (4.25)

Finally, it can be shown [7] that the solution \( v \) of (4.22) depends continuously upon the data.

REMARK. As in Section II, results similar to those in this section have been obtained for a composite infinite medium [7].

APPENDIX

1. A Representation

The object of this appendix is to find an explicit representation of a solution of the problem

(a) \( u_t = \kappa_1 u_{xx}, \quad 0 < x < \nu, \quad 0 < t < T, \)

(b) \( u_t = \kappa_2 u_{xx}, \quad \nu < x < \infty, \quad 0 < t < T, \)
\[(c) \quad \lim_{x \to x_+} u(x, t) = \lim_{x \to x_-} u(x, t), \quad 0 < t < T,\]

\[(d) \quad \lim_{x \to x_+} \rho_1 c_1 \kappa_1 u_\nu(x, t) = \lim_{x \to x_-} \rho_2 c_2 \kappa_2 u_\nu(x, t), \quad 0 < t < T, \quad (1.1)\]

\[(e) \quad u(x, 0) = 0, \quad 0 < x < \infty,\]

\[(f) \quad u(0, t) = f(t), \quad 0 < t < T,\]

where \(\rho_i, c_i,\) and \(\kappa_i, i = 1, 2,\) are the densities, the specific heats, and the thermal diffusivities respectively, and to specify conditions on solutions of (1.1) which will imply the uniqueness of the explicitly represented solution.

**Definition.** A function \(v(x, t)\) is a solution of (1.1) if and only if the following conditions are satisfied:

(a) \(v(x, t)\) is continuous in \((x, t)\) for \(0 < x < \infty, 0 < t < T,\)

(b) \(v_x, v_t,\) and \(v_{xx}\) exist and are continuous in \((x, t)\) for \(0 < x < \nu, v < x < \infty,\) and \(0 < t < T;\)

(c) (1.1) is satisfied by \(v(x, t).\)

If \(f(t) = V,\) a constant, then Carslaw and Jaeger [8] showed that

\[
U(x, t) = V - V \sum_{n=0}^{\infty} \alpha^n \left\{ \text{erf} \left( \frac{(2n + 1) \nu + x - \nu}{2 \sqrt{\kappa_1 t}} \right) - \alpha \text{erf} \left[ \frac{(2n + 1) \nu - x + \nu}{2 \sqrt{\kappa_1 t}} \right] \right\}, \quad 0 < x < \nu, \quad 0 < t < T, \quad (1.2)
\]

\[
U(x, t) = V - \frac{2V}{1 + \sigma} \sum_{n=0}^{\infty} \alpha^n \text{erf} \left[ \frac{(2n + 1) \nu + kx - k\nu}{2 \sqrt{\kappa_1 t}} \right], \quad \nu < x < \infty, \quad 0 < t < T,
\]

where

\[
k = \frac{\kappa_1^{1/2}}{\kappa_2^{1/2}}, \quad (1.3)
\]

\[
\sigma = \frac{\rho_2 c_2 \kappa_2^{1/2}}{\rho_1 c_1 \kappa_1^{1/2}}, \quad (1.4)
\]

\[
\alpha = (\sigma - 1)/(\sigma + 1), \quad (1.5)
\]

and

\[
\text{erf} [x] = \frac{2}{\sqrt{\pi}} \int_{0}^{x} \exp \left[ - \xi^2 \right] d\xi, \quad (1.6)
\]

satisfied (1.1). Now, if \(f(t)\) is a continuous Lebesgue integrable function for
0 < t < T with \( f(0) = 0 \), then it can be shown [7] by the method of Duhamel [4] that the solution of (1.1) is

\[
\begin{align*}
  u(x, t) &= \int_0^t f(\tau) \left\{ \sum_{n=0}^{\infty} \alpha^n \left[ \frac{(2n+1)\nu + x - \nu}{2\sqrt{\pi\kappa_1(t-\tau)^{3/2}}} \exp \left( -\frac{[(2n+1)\nu + x - \nu]^2}{4\kappa_1(t-\tau)} \right) \right] \right. \\
  & \quad \left. - \alpha \frac{(2n+1)\nu - x + \nu}{2\sqrt{\pi\kappa_1(t-\tau)^{3/2}}} \exp \left( -\frac{[(2n+1)\nu - x + \nu]^2}{4\kappa_1(t-\tau)} \right) \right\} \, d\tau, \\
  & \quad 0 < x < \nu, \quad 0 < t < T, \\
  u(x, t) &= \int_0^t f(\tau) \left\{ \frac{2}{1 + \sum_{n=0}^{\infty} \alpha^n} \left[ \frac{(2n+1)\nu + kx - \nu}{2\sqrt{\pi\kappa_1(t-\tau)^{3/2}}} \right. \right. \\
  & \quad \left. \times \exp \left( -\frac{[(2n+1)\nu + kx - \nu]^2}{4\kappa_1(t-\tau)} \right) \right\} \, d\tau, \\
  & \quad \nu < x < \infty, \quad 0 < t < T, \quad (1.7)
\end{align*}
\]

where \( k, \sigma, \) and \( \alpha \) are defined by (1.3), (1.4), and (1.5) respectively.

**References**