Decomposition in Ordered Semigroups

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A certain type of partial orderings of positive measures has recently attracted
attention in connection with Choquet's theorem. It had been known in special
cases for a long time that these orderings have the property: If a measure \( \nu \) is
"more diffuse" than a sum of measures \( \sum \mu_i \), then \( \nu \) can be written as a sum
\( \nu = \sum \nu_i \) such that each summand \( \nu_i \) is more diffuse than the corresponding
\( \mu_i \). A similar problem of decomposition can be formulated for instance for
convex functions on some convex set: Let \( l \) be convex with \( l \geq \sum k_i \) where
the \( k_i \) are convex. Do there exist convex \( l_i \) with \( \sum l_i = l \) and \( k_i \leq l_i \)?

The present paper gives a systematic account on problems of this type.
It develops a fairly general method of finding inequalities for the given quan-
tities \( (\nu, \mu_1, \ldots, \mu_n, \text{respectively}, l, k_1, \ldots, k_n) \) which are necessary and sufficient
for decomposability.

A powerful generalization of the Hahn–Banach theorem plays the crucial part
in the proof of existence. It seems to be of importance in other contexts too and
is discussed in some detail.

Measure theoretic extensions of the existence arguments suggested in
particular by V. Strassen's work are not treated in the present paper. However,
some results along these lines obtained before by ad hoc methods are listed
in the introduction, to give an idea of the relevance of decomposition problems.

The paper is completely self-contained.

I. INTRODUCTION

In [5] V. Strassen has proved and demonstrated the wide applicability of

THEOREM 1.1. Let \( X \) be a separable Banach space and \((\Omega, \mathcal{B}, \mu)\) a
probability space. Let \( \omega \sim h_\omega \) be a map from \( \Omega \) into the set of continuous

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support functions (i.e., sublinear real functions on X) such that for every $x \in X$ $\omega \rightsquigarrow h_\omega(x)$ is $\mathcal{B}$-measurable and

$$\int \| h_\omega \| \, d\mu(\omega) < \infty \quad \text{with} \quad \| h_\omega \| = \sup\{ |h(x)| : \|x\| \leq 1 \}.$$  

The integral

$$h(x) = \int h_\omega(x) \, d\mu$$

is then a continuous support function.

For every element $x^*$ in the dual $X^*$ of $X$

$$x^*(x) \leq h(x) \quad \text{for all} \ x \in X$$

is equivalent with the condition: There exists a map $\omega \rightsquigarrow x_\omega^*$ such that for every $x \in X$

1. $x_\omega^*(x)$ is measurable
2. $x_\omega^*(x) \leq h_\omega(x)$
3. $x^*(x) = \int x_\omega^*(x) \, d\mu(\omega)$.

As an application, Strassen got the celebrated

**Theorem 1.2** (Hardy–Littlewood–Polya–Blackwell–Sherman–Stein–Cartier–Fell–Meyer). Let $H$ be the set of all finite Borel measures on a metrizable compact and convex subset $\Omega$ of a locally convex space. For a pair $k, l \in H$ the following conditions are equivalent

1. $\int y \, dk \leq \int y \, dl$ for all continuous convex $y$ on $\Omega$.
2. There exists a mapping $\omega \rightsquigarrow l_\omega$ from $\Omega$ into $H$ such that
3. $y(\omega) \leq \int y \, dl_\omega$ for every continuous convex $y$ and $\omega \in \Omega$
4. $\int x \, dl_\omega = \int (Sx) \, dk$ for every continuous $x$ if $(Sx)(\omega) = \int x \, dl_\omega$.

In [2] H. Rost results of the following type proved

**Theorem 1.3.** Let $H$ be the set of those Borel measures on $\mathbb{R}^n$ for which the linear functions are summable and let $(\Omega, \mathcal{B}, \mu)$ be a probability space. Denote by $\mathcal{F}$ the set of all those nonnegative functions on $\mathbb{R}^n$ which are the maximum of finitely many affine functions.

If $\omega \rightsquigarrow k_\omega$ is a mapping of $\Omega$ into $H$ such that for every $f \in \mathcal{F}$

$$\int f \, dk_\omega$$

is $\mu$-summable;
then for an \( l \in H \) the following conditions are equivalent

(a) \( \int f \, dl \geq \int (\int f \, dk_\omega) \, d\mu(\omega) \) for all \( f \in \mathcal{F} \).

(b) There exists a mapping \( \omega \rightarrow l_\omega \) of \( \Omega \) into \( H \) and an \( l_0 \in H \) such that for \( f \in \mathcal{F} \cup \{ \text{affine function on } \mathbb{R}^n \} \)

1. \( \int f \, dl_\omega \) is measurable
2. \( \int f \, dl_\omega \geq \int f \, dk_\omega \) for all \( \omega \in \Omega \)
3. \( \int f \, dl = \int f \, dl_0 + \int (\int f \, dk_\omega) \, d\mu(\omega) \).

A similar result, which comes up in connection with positive contractions of an \( L^1(\Omega, \mathcal{B}, \mu) \) is

**Theorem 1.4.** Let \( (\Omega, \mathcal{B}, m) \) and \( (\Omega', \mathcal{B}', m') \) be measure spaces and let be \( x_1, x_2, \ldots \) \( m \)-summable, \( y_1, y_2, \ldots \) \( m' \)-summable real valued with the property that for every nonnegative \( f \) of the form

\[
f(x_1, x_2, \ldots) = \max \left( \sum_j x_j \xi_j \right), \quad \xi_j \text{ real}
\]

\[
\int f(x_1, x_2, \ldots) \, dm \geq \int f(y_1, y_2, \ldots) \, dm'.
\]

(a) There exists a \( \mathcal{B} \)-measurable function \( \psi \) with \( 0 \leq \psi \leq 1 \) and

\[
\int f(x_1, x_2, \ldots) \psi \, dm \geq \int f(y_1, y_2, \ldots) \, dm'.
\]

for all positively homogeneous convex \( f \), in particular

\[
\int x^4 \psi \, dm = \int y^4 \, dm'.
\]

(b) If \( \chi(t) \) is a family of \( \mathcal{B}' \)-measurable functions with

\[
0 = \chi(0) \leq \chi(s) \leq \chi(t) \leq \chi(1) = 1 \quad \text{for } 0 \leq s \leq t \leq 1,
\]

then there exists a family \( \psi(t) \) of \( \mathcal{B} \)-measurable functions with

\[
0 = \psi(0) \leq \psi(s) \leq \psi(t) \leq \psi(1) \leq 1 \quad \text{for } 0 \leq s \leq t \leq 1,
\]

and

\[
\int f(x_1, x_2, \ldots) (\psi(t) - \psi(s)) \, dm \geq \int f(y_1, y_2, \ldots) (\chi(t) - \chi(s)) \, dm'.
\]

for \( f \) positively homogeneous convex and \( s \leq t \).

We shall study the following nonmeasure theoretic versions of similar problems in a systematic approach:
A semigroup \((H, +)\) is a commutative semigroup with neutral element 0. The cancellation rule is said to be valid in \((H, +)\) if, for every \(h \in H\), \(h + l = k + l\) implies \(h = k\). 0 is said to be extremal in \((H, +)\) if \(h + k = 0\) implies \(h = 0 = k\). If \(h, k \in H\), then \(h\) is called a summand of \(k\) iff there exists an \(l \in H\) with \(h + l = k\). A subsemigroup of \((H, +)\) is always assumed to contain the neutral element of \(H\). If \(K_1\) and \(K_2\) are subsets of \((H, +)\) then \(K_1 + K_2 = \{k_1 + k_2 : k_1 \in K_1, k_2 \in K_2\}\).

An ordered set \((H, <)\) is a set \(H\) with a reflexive and transitive ordering \(<\), i.e., \(h < h\) for \(h \in H\), \(h < k\), \(k < l\) implies \(h < l\). The ordering is called asymmetric if \(h < k < h\) implies \(h = k\). If \(h < k\) holds, then we say that \(k\) is to the right of \(h\) and \(h\) is to the left of \(k\) for \(<\). If \(\leq\) and \(<\) are two orderings on \(H\) such that \(h \leq k\) implies \(h < k\), then \(\leq\) is called stricter than \(<\) and \(<\) is called weaker than \(\leq\).

\((H, +, <)\) is called an ordered semigroup if \((H, +)\) is a semigroup and < is an ordering on \(H\) such that for every \(l \in H\) \(h < k\) implies \(h + l < k + l\). In this case < is called an ordering on \((H, +)\). An ordering < on \((H, +)\) is called regular if for every \(l \in H\) \(h + l < k + l\) implies \(h < k\).

**Definition 2.1.** A subsemigroup \(K\) of \((H, +)\) is called full if for \(k \in H, k \in K\) holds whenever \(k + k + \cdots + k \in K\) for a sufficiently large number of summands. (We write \(m \cdot k\) for \(k + \cdots + k\) with \(m\) summands for \(m = 1, 2, \ldots\).)
DEFINITION 2.2. If $K$ is a full semigroup of $(H, +)$, then a subsemigroup $L$ with $L \subseteq K$ is called a face of $K$ if

$$k_1, k_2 \in K \quad \text{and} \quad k_1 + k_2 \in L \quad \text{implies} \quad k_1, k_2 \in L.$$ 

The proofs of the following lemmas are obvious.

**Lemma 2.1.** Let $(H, +)$ be a semigroup

(a) The intersection of an arbitrary collection of full subsemigroups is a full subsemigroup.

Denote for an arbitrary subset $K$ of $(H, +)$ by $F(K)$ the smallest full subsemigroup which contains $K$ and call $F(K)$ the semigroup generated by $K$; then,

(b) If $K$ is a subsemigroup of $(H, +)$, then

$$F(K) = \{k : k \in H \text{ and } mk \in K \text{ for some natural } m\}.$$ 

(c) For arbitrary $K$ holds $K \subseteq F(K) = F(F(K)).$

(d) If $K_1, K_2,$ and $K$ are subsemigroups with $K \supseteq K_1 + K_2$, then $F(K) \supseteq F(K_1) + F(K_2).$

**Lemma 2.2.** Let $K$ be a semigroup of $(H, +)$.

(a) Every face of $F(K)$ is a full subsemigroup of $(H, +)$.

(b) If $L$ is a face of a face of $F(K)$, then $L$ is also a face of $F(K)$.

(c) The intersection of an arbitrary collection of faces of $F(K)$ is a face of $F(K)$.

We denote, for an arbitrary subset $L$ of $F(K)$, by $F_K(L)$, the smallest face of $F(K)$ which contains $L$.

(d) For arbitrary $L \subseteq F(K)$ holds, $L \subseteq F_K(L) = F_K(F_K(L));$

(e) If $L_1, L_2,$ and $L$ are subsemigroups with $F(K) \supseteq L \supseteq L_1 + L_2$, then $F_K(L) \supseteq F_K(L_1) + F_K(L_2).$

**Lemma 2.3.** Let $H$ be a set.

(a) The system of all orderings on $H$ is a complete lattice with respect to the stricter-relation. Its strictest element is the equality on $H$.

If $\{<_i : i \in I\}$ is a collection of orderings on $H$, then

(b) the weakest ordering among those which are stricter than every $<_i$ is given by

$$h \ll k \quad \text{iff} \quad h \prec_i k \quad \text{for all } i \in I.$$
(c) The strictest ordering among those which are weaker than every
\(<_{1}\), is given by
\[ h < k \text{ iff there exist } i_1, i_2, \ldots, i_n \in I \text{ and } h_1, \ldots, h_{n-1} \in H \]
\[ \text{with } h < h_1 < h_2 < \cdots < h_{n-1} < k. \]

Let \((H, +)\) be a semigroup. Then

(d) The system of all orderings on \((H, +)\) is a complete sublattice of
the system of all orderings on \(H\).

(e) If the cancellation rule is valid in \((H, +)\), then the system of all
regular orderings is again a complete lattice.

**Lemma 2.4.** Let \(H\) be a set with an asymmetric ordering \(<\). An
element \(l \in H\) is called infimum of a subset \(K \subseteq H\) if

1. \(l < k\) for all \(k \in K\).
2. If \(l' \in H\) is such that \(l' < k\) for all \(k \in K\), then \(l' < l\).

(a) If an infimum of \(K\) exists it is uniquely determined. It will be
denoted by \(\inf\{k : k \in K\}\) or \(\inf K\).

(b) If \(K\) and \(L\) are such that \(\inf K\) and \(\inf(\inf K \cup L)\) exist, then
\(\inf(\inf K \cup L) = \inf(K \cup L)\).

**Lemma 2.5.**

(a) A semigroup \((H, +)\) can be imbedded in a group \((E, +)\) iff the
cancellation rule is valid in \((H, +)\).

(b) Let \((H, +)\) be imbedded in the group \((E, +)\). An ordering \(<\) on
\((H, +)\) can be extended to an ordering on \((E, +)\) iff \(<\) is regular.

**Lemma 2.6.** Let \((H, +)\) be a semigroup.

(a) The “intrinsic ordering on \((H, +)\)” defined by \(h < k\) iff there
exists an \(l \in H\) with \(h + l = k\), is the strictest ordering on \((H, +)\) with
\(0 < h\) for all \(h \in H\).

(b) The intrinsic ordering is regular if the cancellation rule is valid in
\((H, +)\). It is also asymmetric if \(0\) is extremal in \((H, +)\).

(c) If the intrinsic ordering is asymmetric, then \(0\) is extremal in \((H, +)\).
If it is also regular, then the cancellation rule is valid.

**Definition 2.3.** Let \((H, +)\) be a semigroup, for which \(0\) is
extremal and in which the cancellation rule is valid. The decomposition
lemma is said to be valid in \((H, +)\) if for every system \(h_1, \ldots, h_n, k_1, \ldots, k_n\) with \(\sum h_i = \sum k_j\) there exist \(l_{ij} \in H\) such that
\[
\sum_j l_{ij} = h_i \quad \text{for every } i, \quad \sum_i l_{ij} = k_j \quad \text{for every } j.
\]

**Proposition 2.1.** The decomposition lemma is valid in the semigroup \((H, +)\) iff the intrinsic ordering \(<\) has the properties

1. \(<\) is regular and asymmetric.
2. Whenever \(k < l_1 + l_2\) with \(k, l_1, l_2 \in H\), then there exist \(k_1, k_2 \in H\) with \(k_1 < l_i\) and \(k = k_1 + k_2\).

**Proof.** (a) By Lemma 2.6, the property (1) of the intrinsic ordering is equivalent with the fact that the cancellation rule is valid in \((H, +)\) and 0 is extremal.

(b) Assume that the decomposition lemma is valid and \(k < l_1 + l_2\). Then \(l_1 + l_2 = h + k\) for some \(h \in H\) and there exist \(h_1, h_2, k_1, k_2\) with \(k_1 + k_2 = k, h_1 + h_2 = h, k_1 + h_1 = l_1, k_2 + h_2 = l_2\). Therefore there exist \(k_i < l_i\) with \(k_1 + k_2 = k\).

(c) The decomposition lemma stated as above for \(h_1, \ldots, h_n, k_1, \ldots, k_n\) is easily derived by induction from the special case \(m = 2 = n\). Assume that the intrinsic ordering has the properties (1) and (2) and \(h_1 + h_2 = k_1 + k_2\). Since \(h_1 < k_1 + k_2\) there exist \(l_{ij}\) with \(l_{ij} < k_j\) and \(l_{11} + l_{12} = h_1\). \(l_{21} = k_1 - l_{11}, l_{22} = k_2 - l_{12}\) are uniquely determined elements of \(H\) since the cancellation rule is valid. We have \(l_{ij} < k_j\) and \(l_{21} + l_{22} + h_1 = \sum l_{ij} = k_1 + k_2 = h_1 + h_2\); hence \(l_{21} + l_{22} = h_2\) and the \(l_{ij}\) fulfill all requirements.

**Proposition 2.2.** (F. Riesz). Let \((H, +)\) be a semigroup, such that the intrinsic ordering is asymmetric and regular. The decomposition lemma is valid if \(\inf\{h, k\}\) exists for every pair \(h, k \in H\).

**Proof.** (a) If \(<\) is an asymmetric ordering on the semigroup \((H, +)\) and if \(\inf\{k : k \in K\}\) exists, then for every \(l \in H\)
\[
\inf\{k + l : k \in K\} = l + \inf\{k : k \in K\}.
\]
If \(<\) is also regular, then \(\inf\{k + l : k \in K\}\) exists iff \(\inf\{k : k \in K\}\) exists.

(b) If \(K\) and \(L\) are subsets of \(H\) such that \(\inf K\) and \(\inf L\) exist and for every \(l \in L\) there exists a \(k \in K\) with \(k < l\), then
\[
\inf K \leq \inf L.
\]
(c) If $h, k, l \in H$ are such that $\inf\{h, k, l\}$, $\inf\{h, l\}$, $\inf\{k, l\}$, and $\inf\{h + k, l\}$ exist and $\inf\{h, k, l\} > 0$, then
\[ \inf\{h + k, l\} \leq \inf\{h, l\} + \inf\{k, l\}. \]

In fact
\[
\begin{align*}
\inf\{h, l\} + \inf\{k, l\} &= \inf\{h + \inf\{k, l\}, l + \inf\{k, l\}\} \\
&= \inf\{h + k, h + l, l + k, l + l\} \\
&= \inf\{h + k, l + \inf\{h, k, l\}\}.
\end{align*}
\]

(d) According to Proposition 2.1 we have to show that, for $h, k, l$ with $l < h + k$, there exist $l_1 < h$, $l_2 < h$ with $l_1 + l_2 = l$. $l_1 = \inf\{h, l\} < l$, $l_2 = l - l_1$ satisfies these requirements since by (c)
\[ \inf\{h, l\} + l_2 = l = \inf\{h + k, l\} < \inf\{h, l\} + \inf\{k, l\} \]
and by the regularity of $<$
\[ l_2 < \inf\{k, l\}. \]

Remark. In the semigroup of all nonnegative differentiable functions on a differentiable manifold the decomposition lemma holds while infima fail in general to exist. If $h_1 + h_2 = k_1 + k_2 = l$, then the functions $l_{ij} = l^{-1} \cdot h_i \cdot k_j$ have the desired properties if the quotient is defined to be zero in the zeros of $l$.

III. Monotone and subadditive functions

A function on a set $H$ is here assumed to have values in the extended real line $[-\infty, +\infty]$, a positive function takes values in $[0, +\infty]$. If $\lambda$ and $\mu$ are functions on $H$ with $\lambda(h) \leq \mu(h)$ for all $h \in H$, then we say that $\lambda$ is below $\mu$ and that $\mu$ is above $\lambda$; we write then $\lambda \leq \mu$. If $\rho$ is such that $\rho(h) \leq \lambda(h) + \mu(h)$ for all $h \in H$ for which the sum $\lambda(h) + \mu(h)$ is meaningful, we write $\rho \leq \lambda + \mu$ or equivalently $\lambda + \mu \geq \rho$. If $\rho \leq \lambda + \mu \leq \rho$ we write $\rho \approx \lambda + \mu$.

A function $\rho$ on an ordered set $(H, <)$ is called monotone (for $<$) if $h < k$ implies $\rho(h) \leq \rho(k)$. Clearly a function which is monotone for $<$ is monotone for every stricter ordering on $H$.

A function $\rho$ on a semigroup $(H, +)$ is called weakly subadditive if $\rho(\sum h_i) \leq \sum \rho(h_i)$ whenever the sum is meaningful. It is called subadditive if it is weakly subadditive, $\rho(0) = 0$ and $\rho(h) > -\infty$.
for all $h \in H$. \( \rho \) is called (weakly) superadditive if \(-\rho\) is (weakly) subadditive. \( \rho \) is called additive if it is both subadditive and superadditive, weakly additive if \( \sum \rho(h_i) \) is meaningful whenever \( \rho(\sum h_i) \) is finite and \( \rho(\sum h_i) = \sum \rho(h_i) \). The function \( \rho \) is called (sub)additive in \( h \), if \( \sum \rho(h_i) \) is meaningful and larger (equal to) than \( \rho(\sum h_i) \) whenever \( \sum h_i = h \). \( \rho \) is called subadditive to the right (to the left) of \( h \) if \( \rho \) is subadditive in all \( K \) with \( h < k \) (\( K < h \)). Notice that \( \rho \) is additive in every summand of \( h \) if \( \rho \) is additive in \( h \).

**Lemma 3.1.**

(a) If \((H, <)\) is an ordered set, then the pointwise supremum and the pointwise infimum of a collection of monotone functions is monotone.

(b) If \((H, +)\) is a semigroup, then the supremum of an arbitrary collection of (weakly) subadditive functions is (weakly) subadditive.

**Lemma 3.2.**

(a) If \( \rho \) is a function on an ordered set \((H, <)\) then the smallest monotone function above \( \rho \) is given by

\[
\rho^{(+)}(h) = \sup\{\rho(k) : H \ni k < h\}.
\]

The largest monotone function below \( \rho \) is given by

\[
\rho^{(-)}(h) = \inf\{\rho(k) : h < k \in H\}.
\]

(b) If \( \rho \) is a function on a semigroup \((H, +)\), then the largest weakly subadditive function below \( \rho \) is given by

\[
\bar{\rho}(h) = \inf\left\{\sum \rho(h_i) : \sum h_i = h \text{ and } \sum \rho(h_i) \text{ is meaningful}\right\}.
\]

The smallest weakly superadditive function above \( \rho \) is given by

\[
\bar{\rho}(h) = \sup\left\{\sum \rho(h_i) : \sum h_i = h \text{ and } \sum \rho(h_i) \text{ meaningful}\right\}.
\]

**Proof.** (a) \( \rho^{(+)} \) is monotone since for \( h < k \) the set of elements to the left of \( h \) is contained in the set of elements to the left of \( k \), \( \rho^{(+)} \) is monotone for a similar reason. If \( \mu \) is monotone and above \( \rho \), then for \( k < h \) we have \( \rho(k) \leq \mu(k) \leq \mu(h) \); hence

\[
\rho^{(+)}(h) = \sup\{\rho(k) : H \ni k < h\} \leq \mu(h).
\]
If \( \mu \) is monotone and below \( \rho \), then for \( h < k \), \( \mu(h) \leq \mu(k) \leq \rho(k) \); hence \( \mu(h) \leq \inf \{ \rho(k) : h < k \in H \} = \rho(\{+\})(h) \).

(b) \( \rho \) is clearly below \( \rho \). \( \rho(h) = +\infty \) holds iff \( h = \sum h_i \) implies that \( \rho(h_i) = +\infty \) for some \( i \); in these points \( h \), \( \rho \) is therefore weakly subadditive. If \( \rho(h_i) < +\infty \) for all \( i, j \),

\[
h_i = \sum h_{ij}, \quad h = \sum h_i = \sum_{i,j} h_{ij},
\]

then

\[
\rho(h) \leq \sum_{i,j} \rho(h_{ij}) = \sum_i \left( \sum_j \rho(h_{ij}) \right).
\]

Since this holds for every system \( \{h_{ij}\} \) with \( h_i = \sum_j h_{ij} \) for all \( i \), we have \( \rho(h) \leq \sum_i \rho(h_i) \). If \( \mu \) is weakly subadditive and below \( \rho \), then \( \sum_i h_i = h \) and \( \rho(h_i) < +\infty \) implies \( \mu(h_i) < \infty \). If \( \mu(h_i) < +\infty \) for all \( i \), then \( \sum \mu(h_i) \) is meaningful and

\[
\mu \left( \sum h_i \right) \leq \sum \mu(h_i) \leq \sum \rho(h_i).
\]

This shows \( \mu(h) \leq \rho(h) \) for all \( h \) for which there exists a representation \( h = \sum h_i \) with \( \rho(h_i) < \infty \) for all \( i \). For all other elements \( h \), \( \rho(h) = +\infty \).

**Example:** Let \((H, +)\) be a semigroup and let \(<\) be the intrinsic ordering. Let \( \rho \) be weakly additive and \( \rho(0) = 0 \).

(a) \( \rho \) is monotone iff \( \rho \) is positive

(b) If \( \rho \) is not positive, then

\( \rho(\{+\}) \) is infinite and \( \rho(\{+\}) \approx \rho + (-\rho)(\{+\}) \).

**Lemma 3.3.**

(I) Let \( \lambda, \mu, \) and \( \rho \) be functions on the set \( H \) such that \( \rho \leq \lambda + \mu \) and \( \lambda(h) + \mu(h) \) is meaningful whenever \( \rho(h) > -\infty \).

(a) If \(<\) is an ordering on \( H \), then

\( \rho(\{+\}) \leq \lambda(\{+\}) + \mu(\{+\}) \).

(b) If \( H \) has a semigroup structure \((H, +)\), then

\( \hat{\rho} \leq \hat{\lambda} + \hat{\mu} \).

(II) Let \( \lambda, \mu, \) and \( \rho \) be functions on the set \( H \) such that \( \rho \geq \lambda + \mu \) and \( \lambda(h) + \mu(h) \) is meaningful whenever \( \rho(h) < +\infty \).
(a) If $<$ is an ordering on $H$, then
\[ \rho(+) \geq \lambda(+) + \mu(+) . \]

(b) If $H$ has a semigroup structure $(H, +)$, then
\[ \tilde{\rho} \geq \tilde{\lambda} + \tilde{\mu} . \]

**Proof.** (Ia) We distinguish two cases: If $h$ is such that $\lambda(k) + \mu(k)$ is meaningless for some $k < h$, then $\lambda(+)h + \mu(+)h$ is meaningless or equal $+\infty$. For other elements we may assume $\lambda(+)h < \infty$ and $\mu(+)h < \infty$. Since $\lambda(k) + \mu(k)$ is meaningful for all $k < h$
\[ \rho(+)h = \sup\{\rho(k) : k < h\} \leq \sup\{\lambda(k) + \mu(k) : k < h\} \leq \lambda(+)h + \mu(+)h . \]

(Ib) If $\beta(h) = -\infty$ nothing is to be proved. Hence we can assume that $\beta(h) = \sup\{\rho(h_i) : \sum h_i = h \text{ and } \rho(h_i) > -\infty \text{ for all } i\}$. Hence
\[ \beta(h) \leq \sup \left( \sum (\lambda(h_i) + \mu(h_i)) : \sum h_i = h \right) \leq \lambda(h) + \mu(h) . \]

The statements (IIa) and (IIb) can be proved in a similar way. They can, however, also be reduced to (Ia), respectively (Ib). In fact $-\beta = (-\tilde{\beta})$ on $(H, +)$. $-\rho(+)h$ is monotone with respect to the ordering reverse to $<$ and $-\rho(+)h$ is in fact the largest such function below $-\rho$.

**Lemma 3.4.** Let $(H, +, <)$ be an ordered semigroup and $h \in H$.

(a) If $\rho$ is weakly superadditive to the left of $h$, then
\[ \rho(+)h \text{ is weakly superadditive to the left of } h . \]

(b) If $\rho$ is weakly subadditive to the right of $h$, then
\[ \rho(+)h \text{ is weakly subadditive to the right of } h . \]

**Proof of (a).** It suffices to show that $\rho(+)h$ is weakly superadditive in $h$, since every $k$ with $k < h$ meets the requirement imposed on $h$. We have to show that $\rho(+)\left(\sum h_i\right) \geq \sum \rho(+)h_i$ whenever $\sum h_i = h$ and $\sum \rho(+)h_i$ are meaningful.

If for some $i$, $\rho(+)h_i = -\infty$, then $\sum \rho(+)h_i$ is meaningless or equal $-\infty$. Hence we can assume that for every $i$ there exists a $k_i$ with $k_i < h_i$ and $\rho(k_i) > -\infty$. We have
\[ \rho(+)\left(\sum h_i\right) = \sup \left\{ \rho(k) : k < \sum h_i \right\} \geq \sup \left\{ \rho \left( \sum k_i \right) : k_i < h_i \text{ for every } i \right\} \geq \sup \left\{ \sum \rho(k_i) : h_i < h_i \right\} = \sum \rho(+)h_i . \]
DEFINITION 3.1. Let \((H, +)\) be a semigroup and \(h \in H\). \(h\) is called completely decomposable if for any \(h_1, \ldots, h_n, k_1, \ldots, k_n\) with \(\sum h_i = \sum k_j\) there exist \(l_{ij} \in H\) with
\[
\sum_j l_{ij} = h_i \quad \text{for all } i, \quad \sum_i l_{ij} = k_j \quad \text{for all } j.
\]

DEFINITION 3.2. Let \((H, +, \prec)\) be an ordered semigroup and \(h \in H\). We say that all decompositions of \(h\) are hereditary to the left (to the right) if, for every \(k, h_1, \ldots, h_n\) with \(k \prec h = \sum h_i\) (\(\sum h_i = h < k\)), there exist \(k_1, \ldots, k_n\) with \(k_i < h_i\) (\(h_i < k_i\)) and \(\sum k_i = k\).

PROPOSITION 3.1. Let \((H, +)\) be a semigroup and let \(h \in H\) be completely decomposable.

(a) If \(\rho\) is subadditive in \(h\), then \(\hat{\rho}\) is subadditive in \(h\).

(b) If \(\lambda\) and \(\mu\) are subadditive in \(h\) and \(\lambda + \mu = \rho\), then
\[
\hat{\rho}(h) = \hat{\lambda}(h) + \hat{\mu}(h).
\]

Proof. (a) Let \(h = \sum h_i\)
\[
\hat{\rho}\left(\sum h_i\right) = \sup \left\{\sum_j \rho(k_j) : \sum k_j = \sum h_i\right\}
\]
\[
= \sup \left\{\sum_j \rho\left(\sum_i l_{ij}\right) : \sum_j l_{ij} = h_i \text{ for every } i\right\}
\]
\[
= \sup \left\{\sum_{ij} \rho(l_{ij}) : \sum_i l_{ij} = h_i \text{ for every } i\right\} = \sum \hat{\rho}(h_i).
\]

(b) \(\hat{\lambda}(h) + \hat{\mu}(h) = \sup \left\{\sum_i \lambda(h_i) : \sum h_i = h\right\} + \sup \left(\sum \mu(k_j) : \sum k_j = h\right)
\]
\[
- \sup \left\{\sum_i \lambda\left(\sum_j l_{ij}\right) + \sum_j \mu\left(\sum_i l_{ij}\right) : \sum_j l_{ij} = h_i \text{ for every } i, \sum_i l_{ij} = k_j \text{ for every } j\right\}
\]
\[
\leq \sup \left\{\sum_{ij} \left[\lambda(l_{ij}) + \mu(l_{ij})\right] : \sum_i l_{ij} = h\right\} = \hat{\rho}(h).
\]

Remark. \(\rho(k) = \hat{\lambda}(k) + \hat{\mu}(k)\) holds for every summand \(k\) of \(h\), if it holds for \(h\).
Proposition 3.2. Let \((H, <)\) be an ordered set and let \(h \in H\) be such that the set of elements to the left of \(h\), \(\{k : k < h\}\), is filtered to the left.

(a) If \(\rho\) restricted to \(\{k : k < h\}\) is antitone, then \(\rho^{(+)}\) is constant on \(\{k : k < h\}\).

(b) If \(\lambda\) and \(\mu\) are antitone to the left of \(h\), then \((\lambda + \mu)^{(+)}(k) = \lambda^{(+)}(k) + \mu^{(+)}(k)\) for every \(k < h\).

The proof is obvious.

Proposition 3.3. Let \((H, +, <)\) be an ordered semigroup and \(k \in H\).

(a) If all decompositions of \(k\) are hereditary to the left and \(\rho\) is subadditive to the left of \(k\), then \(\rho^{(+)}\) is subadditive in \(k\).

(b) If all decompositions of \(k\) are hereditary to the right and \(\rho\) is superadditive to the right of \(k\), then \(\rho^{(+)}\) is superadditive in \(k\).

Proof. (a) Let \(k = \sum k_i\). If \(l_i < k_i\) for all \(i\), then \(\sum l_i < k\) and by the subadditivity of \(\rho\) we have \(\rho(\sum l_i) \leq \sum \rho(l_i)\). Hence

\[
\rho^{(+)}(k) = \rho^{(+)}(\sum k_i) = \sup \left\{ \rho(l) : l < \sum k_i \right\} \\
= \sup \left\{ \rho \left( \sum l_i \right) : l_i < k_i \text{ for every } i \right\} \\
\leq \sup \left\{ \sum \rho(l_i) : l_i < k_i \text{ for every } i \right\} = \sum \rho^{(+)}(k_i).
\]

The proof of (b) is dual.

Definition 3.3. Let \((H, +, <)\) be an ordered semigroup. The decomposition problem, \("l < \sum k_i"\), is the problem to find \(l_1, \ldots, l_n\) such that \(l_i < k_i\) and \(\sum l_i = l\). \("l < \sum k_i"\) is called the decomposition problem to the left. \(\sum k_i < l\) denotes the decomposition problem to the right: To find \(l_i\) with \(k_i < l_i\) and \(\sum l_i = l\).

For later reference we formulate a necessary condition for the solvability of a more general decomposition problem:

Proposition 3.4. Let \((H, +)\) be a semigroup and let \(L_1, L_2, \ldots, L_n\) be subsets of \(H\).

If for an \(l \in H\) there exist \(l_1, \ldots, l_n\) with \(l_i \in L_i\) such that \(\sum l_i = l\), then for every \(\rho\) on \((H, +)\) which is weakly subadditive in \(l\)

\[
\rho(l) \leq \sum \sup \{ \rho(l_i) : l_i \in L_i \}.
\]
Remark. Let \( k_1, \ldots, k_n \in (H, +, \prec) \) be given. If \( L_i = \{ l : l \prec k_i \} \), then the condition reads

\[
\rho(l) \leq \sum \rho^{(+)}(k_i)
\]

for every \( \rho \), which is weakly subadditive in \( l \).

If \( L_i = \{ l : k_i \prec l \} \), then the condition reads

\[
\sum \rho_{(+)}(k_i) \leq \rho(l)
\]

for every \( \rho \), which is weakly superadditive in \( l \).

Proof. If \( l_i \in L_i \), then \( \rho(l_i) \leq \sup \{ \rho(l) : l \in L_i \} \). If \( l = \sum l_i \) and \( \rho \) is subadditive in \( l \), then

\[
\rho(l) \leq \sum \rho(l_i) \leq \sum \sup \{ \rho(l) : l \in L_i \}.
\]

If \( L_i = \{ l : l \prec k_i \} \), then

\[
\sup \{ \rho(l_i) : l_i \in L_i \} = \sup \{ \rho(l) : l \prec k_i \} = \rho^{(+)}(k_i).
\]

If \( L_i = \{ l : k_i \prec l \} \), then

\[
\sup \{ (-\rho)(l) : l \in L_i \} = -\inf \{ \rho(l) : k_i \prec l \} = -\rho_{(+)}(k_i).
\]

\( -\rho \) is subadditive in \( l \) iff \( \rho \) is superadditive in \( l \), therefore

\[
(-\rho)(l) \leq \sum_i \sup \{ (-\rho)(l_i) : l_i \in L_i \} \quad \text{iff} \quad \sum \rho_{(+)}(k_i) \leq \rho(l).
\]

Proposition 3.5. Let \( (H, +, \prec) \) be an ordered semigroup and \( k_1, k_2, \ldots, k_n \in H \). If \( \rho \) is subadditive and such that \( \rho^{(+)} \) is also subadditive, then \( l \prec \sum k_i \) implies

\[
\rho(l) \leq \sum \rho^{(+)}(k_i).
\]

Proof. \( l \prec \sum k_i \) implies \( \rho(l) \leq \rho^{(+)}(\sum k_i) \). If, furthermore, \( \rho^{(+)}(\sum k_i) \leq \sum \rho^{(+)}(k_i) \), then

\[
\rho(l) \leq \sum \rho^{(+)}(k_i).
\]

Corollary. If \( \rho \) is subadditive and monotone, then \( l \prec \sum k_i \) implies \( \rho(l) \leq \sum \rho^{(+)}(k_i) \).
IV. EXAMPLES: CONIC DISTRIBUTIONS

Notations. A semigroup \((H, +)\) is called a cone, if for \(h \in H\) and \(\alpha \geq 0\) real \(\alpha \cdot h \in H\) is defined with the properties

1. \(0 \cdot h = 0 = \alpha \cdot 0, 1 \cdot h = h\)
2. \(\alpha(h + k) = \alpha h + \alpha k\)
3. \((\alpha + \beta)h = \alpha h + \beta h\), for \(\alpha, \beta \geq 0\) real, \(h, k \in H\).

An ordering on a cone is always assumed to be compatible with the scalar multiplication, i.e., \(h < k\) iff \(\alpha h < \alpha k\) for some \(\alpha > 0\). Functions on a cone will, unless the contrary is explicitly stated, be positively homogeneous, i.e.,

\[
\rho(\alpha h) = \alpha \cdot \rho(h) \quad \text{for} \quad \alpha \geq 0 \text{ real}, \quad h \in H.
\]

An additive positively homogeneous functions on a cone is shortly called a linear function. Correspondingly (weakly) sublinear and superlinear functions are defined.

Time does not seem ripe to give a good definition of a conic distribution. We therefore treat three typical examples in their geometrical aspects.

E.1. If \(b\) is a point in \(B = \mathbb{R}^n\), then we denote by \([b]\) the probability measure concentrated in \(b\). A discrete measure on \(B\) is a positive linear combination \(\sum \alpha_i[b_i]\) of denumerably many point measures \([b_i]\). If the summands with \(\alpha_i = 0\) are suppressed and the \(b_i\) are chosen to be distinct, \(\alpha[b] + \beta[b] = (\alpha + \beta) \cdot [b]\), then the representation \(\sum \alpha_i[b_i]\) of a discrete measure is unique. If \(\sum \alpha_i = 1\), then \(\sum \alpha_i[b_i]\) is a probability measure; if \(\int u d(\sum \alpha_i[b_i]) = \sum \alpha_i u(b_i)\) is meaningful for every affine function \(u\) on \(B\), then the probability measure \(\sum \alpha_i[b_i]\) is said to have the barycenter \(\sum \alpha_i b_i\); it is determined by the equalities \(u(b) = \sum \alpha_i u(b_i)\). If \(k = \alpha \cdot h\) with \(\alpha > 0\) and a probability measure \(h\), then \(\alpha\) is called the total mass of \(k\) and the barycenter of \(h\) is also called the barycenter of \(k\). The nullmeasure has no barycenter.

Let \(H\) be the cone of all nonnegative multiples of the discrete probability measures with barycenter in \(B\). A real function \(f\) on \(\mathbb{R}^n\) is said to grow not faster than linear at infinity, if

\[
|f(b)| \leq C_1 + C_2 \cdot |b| \quad \text{for all } b \in B = \mathbb{R}^n,
\]

with constants \(C_1, C_2\) and some norm \(|b|\) on \(\mathbb{R}^n\). For such functions \(f\) and \(h = \sum \alpha_i[b_i] \in H\), \(\int f \, dh = \sum \alpha_i \cdot f(b_i)\) is well defined.
Every such $f$ defines a linear functional on $(H, +)$. If $g$ is a function on $B$ which is bounded below by an affine function $u$, then $\int g \, dh$ is well defined for every $h \in H$; It may, however, be $+ \infty$ for some $h$ and therefore not define an additive function on $H$. In particular $\int f \, dh$ is well defined for every convex function $f$ and is finite for convex functions which are the (pointwise on $B$) maximum of finitely many affine functions.

For $k, l \in H$ we call $k$ more concentrated than $l$ if $\int f \, dk \leq \int f \, dl$ for all nonnegative convex functions $f$ on $B$, and we write $k < l$.

The decomposition problems in the ordered cone $(H, +, <)$ will, in Chapter VI, be solved in the following sense. All decomposition problems to the right "$\sum k_i < l$" have a solution; in other words, every decomposition $k = \sum k_i$ is hereditary to the right. The necessary conditions for the solvability of a decomposition problem to the left "$l < \sum k_i$" are not a consequence of the order relation $l < \sum k_i$. It will, however, be shown that "$l < k_i$" is solvable, if for a sufficiently large class of functions $\rho$ on $(H, +)$ which are additive in $l$ the necessary inequalities hold: $\rho(l) \leq \sum \rho^{(+)}(k_i)$.

**Proposition E.1.1.** If $k$ and $l$ in $H$ are probability measures with $k < l$, then their barycenters coincide. If $b$ is the barycenter of $l$, then $[b] < l$.

**Proof.** By Jensen's inequality we have for $l$ with barycenter $b$

\[
\int f d[b] = f(b) \leq \int f \, dl \quad \text{for every convex } f.
\]

If $k$ and $l$ are probability measures with $k < l$ and $u$ is affine, then for every constant $\delta$ the functions $(u + \delta)^+$ and $(-u + \delta)^+$ are nonnegative and convex. Since $\max(u, -\delta) = -\delta + (u + \delta)^+$ and $\int (-\delta) \, dk = -\delta = \int (-\delta) \, dl$ we have

\[
\int \max(u, -\delta) \, dk \leq \int \max(u, -\delta) \, dl
\]

\[
\int \max(-u, -\delta) \, dk \leq \int \max(-u, -\delta) \, dl.
\]

Since $k$ and $l$ have a barycenter, these integrals tend to the integrals of $u$, respectively $-u$, for $\delta \to +\infty$. Hence

\[
\int u \, dk = \int u \, dl \quad \text{for all affine } u.
\]

**Proposition E.1.2.** If $[b] < l$, then there exists a summand $l'$ of $l$ which is a probability measure with barycenter $b$. 
Remark. This result can be generalized with the help of the theorem announced above which asserts that every decomposition problem to the right is solvable in \((H, +, \prec)\). One gets, with Proposition E.1.2 and a solution of \(\sum \alpha_k [b_k] \prec l'\) for \(k = \sum \alpha_k [b_k] \prec l\), a summand \(l'\) of \(l\) such that \(k \prec l'\) and the total masses of \(k\) and \(l'\) are equal.

Construction. Let \(u_0\) be a linear function not identically 0 on some \(\mathbb{R}^{n+1}\), and let \(A\) be the cone

\[
A = \{u_0 > 0\} \cup \{0\}.
\]

Let the space \(B\) on which the measures \(h \in H\) are given be imbedded in \(A\) in some affine way as the hypersurface \(\{u_0 = 1\} = B\). For every function \(f\) on \(B\), there exists a unique positively homogeneous extension to \(A\) which will again be denoted by \(f\), if no confusion seems to be possible. By this identification every \(h \in H\) defines a positive linear functional on the nonnegative positively homogeneous functions \(f\) on the cone \(A\). Such functionals will be called conic distributions on \(A\).

Proof of E.1.2. (a) For every \(k \in H\) we define the resultant \(r(k)\) to be the point of \(A\) with

\[
u(r(k)) = \int u \, dk \quad \text{for every linear } u \text{ on } A.
\]

Clearly \(r(k) = \alpha \cdot b\) if \(\alpha\) is the total mass of \(k\) and \(b \in B\) is the barycenter of \(k\). \(\alpha = \int u_0 \, dk\). The null measure is the only element of \(H\) with resultant \(\emptyset\).

(b) If \(h\) is a summand of \(l \in H\), i.e., \(l = h + h'\) for some \(h' \in H\), then we shortly write \(h \ll l\). Clearly \(h \ll l\) implies \(h < l\), \(\ll\) is a stricter ordering than \(<\). The statement to be proved can be reformulated: If \([b] < l\), then there exists \(l' \in H\) with \([b] < l' \ll l\) and \(r(l') = b\).

Consider the subset of \(A\)

\[
S(l) = \{r(h) : h \ll l\}.
\]

It is obviously convex and compact. For \(a \in A\) outside \(S(l)\) there exists a linear form \(u\) on \(\mathbb{R}^{n+1}\) such that

\[
u(a) > 1 \geq \sup\{u(a') : a' \in S(l)\} = \sup \left\{ \int u \, dh : h \ll l \right\}.
\]
If \( I_u \) is the indicator function of \( \{ u \geq 0 \} \subseteq A \), then for \( h = l \cdot I_u \ll l \) we have
\[
    u(a) > \int u \, dh = \int (u(a))^+ \, dl(a).
\]

If \( [b] < l \), however, implies
\[
    (u(b))^+ = \int (u(a))^+ \, d([b]) \leq \int (u(a))^+ \, dl(a)
\]
for every linear \( u \), since \( u^+ \) is positive and sublinear. Therefore \( b \in S(l) \). By a trivial compactness argument one gets an \( l' \ll l \) with \( r(l') = b \), which according to E.1.1 satisfies \( [b] < l' \).

**Proposition E.1.3.** If \( f \) is a nonnegative positively homogeneous function on \( A \) and \( f \) is the largest subadditive function below \( f \) on \( A \), then the functional \( \rho \) on \((H, +, <)\) given by
\[
    \rho(h) = \int f \, dh \quad \text{holds} \quad \rho_+(h) = \int f \, dh \quad \text{for all } h \in H.
\]

**Remark.** If \( f \) assumes strictly negative values, then \( \rho \) assumes negative values and \( 0 < h \) for all \( h \in H \) implies \( \rho_+ \) is infinite. (Compare the example following Lemma 3.2.)

**Proof.** (a) Since \( f \) is sublinear and positive
\[
    h \rightsquigarrow \int f \, dh
\]
is monotone on \((H, <)\). Since it is below \( \rho \) we have
\[
    \rho_+(h) \geq \int f \, dh \quad \text{for all } h.
\]

(b) Since \( \rho \) is positive and additive \( \rho(l' + l'') \geq \rho(l') \) for \( l', l'' \in H \). According to E.1.2, therefore, for \( b \in B \)
\[
    \rho_+([b]) = \inf \{ \rho(l) : [b] < l \} = \inf \{ \int f \, dl : r(l) = b \}.
\]
As is well known there exists a support function of \( f \) in a given \( b \), i.e., a linear function \( u \) with \( u(b) = f(b) \) and \( u \leq f \leq u \) on \( A \). For every \( \varepsilon > 0 \) the cone
\[
    \{ (u + \varepsilon \cdot u_0) \geq f \}
\]
contains points \( b_1, \ldots, b_n \in B \) with \( \sum \alpha_i \cdot b_i - b \) for certain \( \alpha_i \geq 0. \sum \alpha_i \cdot [b_i] = l \) has barycenter \( b \) and
\[
    \rho_+([b]) \leq \int f \, dl \leq \int (u + \varepsilon u_0) \, dl - \varepsilon + \int u \, dl - \varepsilon + u(b) = \varepsilon + f(b).
\]
(c) According to Lemma 3.4, \( \rho_+ \) is sublinear on \((H, +)\) and hence for \( h = \sum \alpha_i [b_i] \in H \) we have

\[
\rho_+(h) = \rho_+ \left( \sum \alpha_i [b_i] \right) \leq \sum \alpha_i \cdot \rho_+([b_i]) = \sum \alpha_i \cdot f(b_i) = \int f \, dh.
\]

**Proposition E.1.4.** Let \( f \) be a positively homogeneous function on \( A \) which does not grow faster than linearly at infinity and let

\[
\rho(h) = \int f \, dh.
\]

Denote by \( f^+ \) the maximum of \( f \) and 0 on \( A \). Then

(a) \( \rho_+([b]) = f^+(b) \) for \( b \in A \)

(b) If \( f \) is sublinear, then \( \rho_+(h) = \int f^+ \, dh \) for all \( h \in H \).

(c) If \( h \) has support \( A(h) \subset A \) and \( f^h \) is the smallest positive weakly superadditive function which is above \( f \) on \( A(h) \), then

\[
\int f^+ \, dh \leq \rho_+(h) \leq f^h(\rho(h)).
\]

**Proof.** (a) \( \rho_+([b]) = \sup \{ \rho(k) : k < [b] \} \). According to Propositions E.1.2 and E.1.1, \( k < [b] \) holds iff \( k = \alpha \cdot [b] \) with \( 0 \leq \alpha \leq 1 \).

(b) If \( f \) is sublinear, then \( f^+ \) is sublinear and positive and \( \int f^+ \, dh \) is monotone and above \( \rho \), therefore above \( \rho_+ \). By Lemma 3.4 \( \rho_+(\cdot) \) is weakly superadditive. Hence \( \rho_+(h) \geq \int f^+ \, dh \) for all \( h \in H \) is implied by \( \rho_+([b]) \geq \int f^+ \, d([b]) = f^+(b) \) in (a).

(c) If \( k < h \) and \( h \) is concentrated on the convex set \( A(h) \), then \( k \) is clearly also concentrated on \( A(h) \). If \( f^h \) is superadditive on \( A(h) \), then the function on \( H \) \( k \sim \int f^h \, dk \) is antitone to the left of \( h \). Hence for \( l < k < h \)

\[
\rho(k) = \int f \, dk \leq \int f^h \, dk \leq \int f^h \, dl
\]

and if \( b \) is the resultant of \( h \)

\[
\rho_+(h) = \sup \{ \rho(k) : k < h \} \leq \int f^h d([b]) = f^h(b).
\]

**Proposition E.1.5.** Let \( l, k_1, \ldots, k_n \) be probability measures on \( B \) with barycenters \( b, b_1, \ldots, b_n \), respectively. If \( l < \sum \alpha_i k_i \) with \( \alpha_i > 0 \) and \( \sum \alpha_i = 1 \), then necessary for the solvability of \( "l < \sum \alpha_i \cdot k_i" \) is the condition

\[
\sum \alpha_i f(b_i) \leq \int f \, dl \quad \text{for every convex } f \text{ on } B.
\]
Proof. (a) We replace the ordering $<$ on $(H, +)$ considered above by the stricter ordering $\leq$:

$$h \leq k \iff \int f \, dh \leq \int f \, dk \quad \text{for all convex } f.$$ 

Clearly $h \leq k$ is equivalent with the pair of conditions, $h < k$ and the total masses are equal (Proposition E.1.2). If $b$ is the barycenter of the probability measure $k$, then for every $h \leq k$ we have $[b] \leq h \leq k$.

(b) If $f$ is convex function of $B$, then $\rho(h) = \int (-f) \, dh$ is weakly superadditive and antitone on $(H, +, \leq)$. By the remark above therefore on $(H, \leq)$

$$\rho^{(+)}(k) = \sup(\rho(h) : h \leq k) = \rho([b]) = -f(b)$$

if $b$ is the barycenter of the probability measure.

(c) Since, by assumption, $l$ and $\sum \alpha_i \cdot k_i$ are probability measures, the problems

"$l \leq \sum \alpha_i \cdot k_i$" and "$l \leq \sum \alpha_i k_i$" are equivalent.

A necessary condition for the solvability of "$l \leq \sum \alpha_i k_i$" is according to Proposition 3.4, that for every weakly sublinear $\rho$

$$\rho(l) \leq \sum \alpha_i \rho^{(+)}(k_i).$$

In particular, for $\rho(h) = \int (-f) \, dh$ we get

$$\int (-f) \, dl \leq \sum \alpha_i (-f(b_i)).$$

In the ordered cone of measures just considered, measure theoretical or topological complications did not arise so far. In the description of closely related situations topology or measure theory tends to obscure the purely geometrical features. It seems that this is so only by tradition and that the facts which do not refer to the geometry of ordered semigroups can very well be separated. Here are two such similar situations:

E.2. Let $A$ be a cone in a locally convex vectorspace which has a compact base $B$. Let $F$ be the vector space of all positively homogeneous functions on $A$ which are continuous on $B$, let $V$ be the cone of all (finite valued) sublinear functions on $A$ and $W$ the cone of all those continuous linear functions on $A$ which are nonnegative on $A$. 


Let $F$, $V$, and $W$ be ordered by the pointwise ordering. An additive and monotone functional on $F$ is called a conic measure. The set of all conic measures on $A$ is a cone, $(H, +)$, by the pointwise addition on $F$. Choquet was the first to define the ordering on $(H, +)$

$$h < k \iff h(v) \leq k(v) \quad \text{for all } v \in V.$$ 

Remark. The theorem of Blackwell–Sherman–Stein–Cartier cited above is a measure theoretical version of the assertion that every decomposition problem to the right in $(H, +, <)$ is solvable.

The basic topological facts in this situation are:

E.2.1. The elements of $W$ separate the points in $A$. Moreover, if $\tilde{a}$ is a positive linear functional on $W$, then there exists exactly one point $a \in A$ with

$$\tilde{a}(w) = w(a) \quad \text{for all } w \in W.$$ 

E.2.2. A conic measure on $A$ is uniquely determined by its values on $V$. Moreover, if $h$ is an additive and monotone functional on the ordered cone $V$, then $h$ has a unique extension to a conic measure $H$.

E.3. Let $\Theta$ be an index set and let $A$ be the set of all positive functions on $\Theta$, $\{\xi_\theta : \theta \in \Theta\}$. If $\theta_1, ..., \theta_n \in \Theta$ and $f$ is a function of $n$ real variables then we denote by $f_{\theta_1, ..., \theta_n}$ the function on $A$

$$f_{\theta_1, ..., \theta_n}(\{\xi_\theta\}) = f(\xi_{\theta_1}, ..., \xi_{\theta_n}).$$

The set of all those functions $f_{\theta_1, ..., \theta_n}$ with $f$ nonnegative and sublinear is denoted by $V$. $V$ is then an ordered cone with respect to pointwise operations.

If $\mathcal{B}$ is a Boolean algebra with maximal element $\Omega$, and $\{\mu_\theta : \theta \in \Theta\}$ is a family of finitely additive “set” functions on $\mathcal{B}$, then the joint (conic) distribution $\mathcal{D}\{\mu_\theta\}$ is defined as the function on $V$

$$\mathcal{D}\{\mu_\theta\}(f_{\theta_1, ..., \theta_n}) = \sup \left\{ \sum_i f(\mu_{\theta_i}(B_i), ..., \mu_{\theta_i}(B_i)) : \Omega = \cup_i B_i \text{ disjoint} \right\}.$$ 

Every $\mathcal{D}\{\mu_\theta\}$ is clearly a monotone additive function on $(V, +, <)$. The cone of all such functions equipped with pointwise (on $V$) addition and ordering is called the cone of all joint conic distributions. Here are some facts which explain why joint conic distributions are of interest to statisticians:
E.3.1. If \( \Omega = \bigcup_1^n B_i \) is disjoint and \( \mu_\vartheta(B) = \mu_\vartheta(B \cap B_i) \) for \( B \in \mathcal{B} \), then

\[
\mathcal{D}\{\mu_\vartheta\} = \sum_{i=1}^n \mathcal{D}\{\mu_\vartheta^{(i)}\},
\]

E.3.2. If \( \mathcal{B}' \) is a subalgebra of \( \mathcal{B} \) and \( \mu_\vartheta' \) is the restriction of \( \mu_\vartheta \) to \( \mathcal{B}' \) for all \( \vartheta \in \Theta \), then

\[
\mathcal{D}\{\mu_\vartheta'\} \leq \mathcal{D}\{\mu_\vartheta\}
\]

and \( \mathcal{B}' \) is a sufficient subalgebra of \( \mathcal{B} \) iff \( \mathcal{D}\{\mu_\vartheta'\} = \mathcal{D}\{\mu_\vartheta\} \).

E.3.3. If \( (\Omega, \mathcal{B}) \) and \( (\Omega', \mathcal{B}') \) are pointsets with an algebra of subsets and \( \varphi \) is a measurable from \( \Omega \) into \( \Omega' \), then for the \( \varphi \)-images \( \mu_\vartheta' \) of the measures \( \mu_\vartheta \) holds

\[
\mathcal{D}\{\mu_\vartheta\} \geq \mathcal{D}\{\mu_\vartheta'\}.
\]

E.3.4. If \( \mathcal{B} \) is a \( \sigma \)-algebra of subsets of \( \Omega \) and all \( \mu_\vartheta \) are \( \sigma \)-additive, then \( \mathcal{D}\{\mu_\vartheta\} \) \((f_{\vartheta_1}, \ldots, f_{\vartheta_n})\) can be evaluated in a way which suggested the name "joint distribution" for \( \mathcal{D}\{\mu_\vartheta\} \): Choose a measure \( \mu \) on \( \mathcal{B} \) with respect to which \( \mu_{\vartheta_{\vartheta_1}} \ldots, \mu_{\vartheta_n} \) are absolutely continuous and have Radon–Nikodym derivatives \( x_{\vartheta_1}, \ldots, x_{\vartheta_n} \), then

\[
\mathcal{D}\{\mu_\vartheta\}(f_{\vartheta_1}, \ldots, f_{\vartheta_n}) = \int_{\Omega} f(x_{\vartheta_{\vartheta_1}}(\omega), \ldots, x_{\vartheta_n}(\omega)) \, d\mu.
\]

E.3.5. In the ordered cone of joint conic distributions (\( \Theta \) is fixed) decomposition problems to the right are always solvable.\(^1\)

If \( (\Omega, \mathcal{B}, m) \) and \( (\Omega', \mathcal{B}', m') \) are measure spaces with \( \mathcal{B} \) and \( \mathcal{B}' \) countably generated, \( x_1, x_2, \ldots \) are \( m \)-summable, \( x_1', x_2', \ldots \) \( m' \)-summable with

\[
\mathcal{D}\{x_i \, dm\} \geq \mathcal{D}\{x_i' \, dm'\}
\]

then there exists a positive contraction \( T \):

\[
L^1(\Omega, \mathcal{B}, m) \to L^1(\Omega', \mathcal{B}', m') \quad \text{with} \quad T x_i = x_i' \quad \text{for} \quad i = 1, 2, \ldots.
\]

\(^1\) A "measurable" generalization of this statement yields a converse of E.3.3 due to H. Rost [2].
V. EXTENSION OF ADDITIVE MONOTONE FUNCTIONS TO G. AUMANN'S HULL

In this chapter we shall extend additive monotone functions from a subsemigroup $Y$ of $(X, +, \ll)$ to larger subsemigroups $U \subseteq X$. As an additional property we want the functions to be below a given subadditive function $p$.

**Remark 5.1.** If $f$ is additive, monotone, and below $p$ on $U \subset (X, +, \ll)$ and $f$ is its restriction to $Y \subseteq U$, then

$$f(y) \leq f(y') + p(u)$$

holds whenever $y, y' \in Y$, $u \in U$ are such that $y \ll y' + u$. In fact

$$f(y) = f(y') + p(u) = f(y') + f(u) \leq f(y') + p(u).$$

These inequalities are the natural generalizations of the assumption known from the Hahn–Banach theorem that $f$ has to be below $p$ on $Y$ in order to admit an extension below $p$ on a larger subspace $U$.

The weaker the ordering $\ll$ on $(X, +)$ is, the more inequalities $f(y) \leq f(y') + p(u)$ have to be verified.

**Remark 5.2.** The obstacle for extension, which is presented by the fact that $p$ may assume the value $+\infty$, is of a different nature.

G. Aumann gave an example: $p(x) = +\infty$ for $x \neq 0$, $f$ is additive and monotone on a subgroup $Y$ of $X$, so that certainly all inequalities are satisfied. $f$ admits, however, no monotone additive extension.

H. Rost gave an extremely simple example for the case where the ordering is trivial, $p$, however, assumes finite values as well as the value $+\infty$:

$$X = \{(x, y), x, y \text{ real}\}$$

$$p(x, y) = \begin{cases} +\infty & \text{if } x > 0 \text{ or } y > 0 \\ 0 & \text{if } x = 0 \text{ and } y \leq 0 \\ x + y & \text{if } x < 0 \text{ and } y \leq 0 \end{cases}$$

$p$ is sublinear on $X$ and nonnegative on $Y = \{(0, y) : y \text{ real}\}$. The function $f$, which is zero on $Y$, has no additive extension beyond $Y$.

It will be shown that the obstacle for extension is the fact that $Y$ does not penetrate the subsemigroup $\{p < \infty\}$. The principal ideas of this chapter are due to G. Aumann [1].

We shall use the notations $F(Z)$ and $F_Z(Y)$ defined in II: $F(Z)$ is the
smallest full subsemigroup of \((X, +)\) which contains \(Z\). \(F_Z(Y)\) is the smallest face of \(F(Z)\) which contains \(Y \cap Z\).

**Definition 5.1.** Let \(\leq\) be an ordering in the semigroup \((X, +)\) and let \(Z\) be a subsemigroup of \(X\). Then we denote

\[
Z^\rightarrow = \{x: \text{there exists } z \in Z \text{ and } m \text{ natural with } z \geq mx\}.
\]

If \(Y\) and \(Z\) are subsemigroups then we denote

\[
Y^\leftarrow = \{x: \text{there exist } y \in Y, z \in Z \text{ and } m \text{ natural with } y \leq mx + z\}.
\]

If \(\leq\) is the equality, we write \(Y_Z\) instead of \(Y^\leftarrow_Z\).

**Remark 5.3.**

1. \(Z^\rightarrow = Z^\leftarrow_{\{0\}}\)
2. \(Z = F(Z) = Z^\leftarrow_{\{0\}}\).

**Proposition 5.1.** Let \((X, +)\) be a semigroup

(a) For every ordering \(\leq\) and every pair \(Y, Z\) of subsemigroups, \(Y^\leftarrow_Z\) is a full subsemigroup with \(Y \subseteq Y_Z \subseteq Y^\leftarrow_Z = (Y^\leftarrow_Z)_Z\).

(b) If \(Y, Y', Z, Z'\) are subsemigroups with \(Y \subseteq Y'\) and \(Z \subseteq Z'\), then

\[
Y^\leftarrow_{Z'} \subseteq Y^\leftarrow_Z.
\]

(c) If \(\leq\) and \(<\) are orderings on \((X, +)\) such that \(x \leq x'\) implies \(x < x'\), then

\[
Y^\leftarrow_Z \subseteq Y^\leftarrow_Z.
\]

**Proof.** (a) \(Y^\leftarrow\) and the special case \(Y^\leftarrow_{\{0\}} = Y^\leftarrow\) are full subsemigroups:

\[
y \leq x + z \quad \text{and} \quad y' \leq x' + z' \implies (y + y') \leq (x + x') + (z + z')
\]

and by the definition \(x \in Y^\leftarrow Z\) iff some natural multiple \(x\) "can overtake \(Y\) with the help of \(Z\)", i.e.,

\[
y \leq x + z.
\]
Assume $\xi \in (Y_Z)^2$. Then for a certain natural multiple $\lambda$ we have

\[ y' \ll x + z' \text{ with } y' \in Y_Z^\lambda \text{ and } z' \in Z, \]

and

\[ y \ll y' + z \text{ with } y \in Y, z \in Z. \]

This gives $y \ll x + (z + z')$ and therefore $x \in Y_Z^\lambda$.

(b) Assume $\tilde{y} \ll m\tilde{x} + \tilde{z}$ with $\tilde{y} \in \tilde{Y}$, $\tilde{z} \in \tilde{Z}$.

There exists $m$ natural, $y \in Y$, $z \in Z$ with

\[ y \ll m\tilde{y} \text{ and } z \gg m\tilde{z}. \]

Hence

\[ y \ll m\tilde{y} \ll m\tilde{x} + \tilde{z} \ll m\tilde{x} + z. \]

This shows $\tilde{Y}_Z^\lambda \subset Y_Z^\lambda$.

(c) is obvious

Remark 5.4. The proposition shows in particular

1. $Y_Z^\lambda = (Y^\lambda)^2$
2. $Y_Z^\lambda$ is monotone in both arguments $Y$ and $Z$.
3. If $x \in Y_Z^\lambda$ and $x \ll x'$, then $x' \in Y_Z^\lambda$; $Y_Z^\lambda$ contains $Y$ and all elements to the right of its elements.

For later reference we prove the

Lemma 5.1. Let $<_1, <_2, \ldots, <_n$ be orderings in $(X, +)$; let $Z_1, Z_2, \ldots, Z_n$ be subsemigroups and let $Y$ be a further semigroup with $Y \subset \bigcap_i Z_i$. For an $n$-tuple of elements in $X$ the following two conditions are equivalent:

(a) $x_i \in Y_{Z_i}^\preceq$ for all $i$ ($x_i \in (Z_i)^\preceq$ for all $i$).
(b) There exist $y \in Y$, $Z_1, \ldots, Z_n$ with $z_i \in Z_i$ and $m$ natural with

\[ y \preceq mx_i + z_i \text{ for all } i. \quad (z_i \preceq mx_i + y \text{ for all } i). \]

Proof. (b) clearly implies (a). If $y_i <_i m_i x_i + z_i'$ holds with $y_i \in Y$, $z_i' \in Z_i$, then put

\[ m = m_1 \cdot m_2 \cdot \cdots \cdot m_n; \quad y = \sum_{i} \frac{m}{m_i} y_i \in Y; \quad z_i = \frac{m}{m_i} z_i' + \sum_{j \neq i} \frac{m}{m_j} y_j \in Z_i. \]

By multiplying the $i$-th equation with $m/m_i$ and by adding
\( \sum_{j \neq i} m_j/m_j \in Y \subset Z_i \) to both sides we get the result. The second result (in brackets) is proved by the same method.

**Proposition 5.2.** Let \((X, +, \preccurlyeq)\) be an ordered semigroup. Assume \(Z = Z^*\) and \(Y = Y^*\). \(Y^* \subset Z^*\) intersects \(Z\) in the smallest face of \(Z\) which contains \(Y \cap Z\),

i.e., \( Y^* \cap Z = F_Z(Y \cap Z) \).

**Proof.** (a) Assume \( z_1, z_2 \in Z \) and \( z_1 + z_2 \in Z \cap Y^* \). There exist \( m \) natural, \( y \in Y \) and \( z \in Z \) with

\[
y \preccurlyeq m(z_1 + z_2) + z = mz_1 + (mz_2 + z) = mz_2 + (mz_1 + z).
\]

Since \( mz_1 + z \in Z \) we have \( z_i \in Y^* \). \( Z \cap Y^* \) is therefore a face of \( Z \). It clearly contains \( Y \cap Z \).

(b) Assume \( x \in Z \) and \( y \preccurlyeq x + z \) for certain \( y \in Y \) and \( z \in Z \). This shows \( x + z \in Y^* = Y \) with \( x, z \in Z \). Hence \( x \) is in every face of \( Z \) which contains \( Y \cap Z \).

Of particular importance for the applications is the case when the cancellation rule is valid in \((X, +, \preccurlyeq)\) (compare Lemma 2.5).

**Proposition 5.3.** Let \((X, +, \preccurlyeq)\) be imbedded in the ordered group \(E\) and let \(P\) be the semigroup of all positive elements in \(E\). Then

\[
Y^* \preccurlyeq X \cap (Y + P - Z).
\]

**Proof.** \( x \preccurlyeq x' \) holds iff \( x' - x \in P \). For \( x \in X \) the following conditions are equivalent

(1) \( y \preccurlyeq mx + z \) with \( y \in Y, z \in Z \),

(2) \( mx + z - y = w \in P \) with \( y \in Y, z \in Z \),

(3) \( mx = y + w - z \) with \( y \in Y, z \in Z, w \in P \).

**Proposition 5.4.** For a pair of subsemigroups \( Y, Z \) of \((X, +, \preccurlyeq)\) let

\[
U(Y, \preccurlyeq, Z) = Y^* \cap Z^* = U(Z, \succ, Y).
\]

1. \( U \) depends monotonically on \( Y \) and \( Z \), it increases if the ordering \( \preccurlyeq \) is weakened. \( U(Y, \preccurlyeq, Z) \succ Y \cap Z \).
2. \( U(Y, \preccurlyeq, Z) = U(Y^*, \preccurlyeq, Z^*) \)
3. If \( Y^* = Y \) and \( Z^* = Z \), then

\[
U(Y, \preccurlyeq, Z) = U(Y \cap Z^*, \preccurlyeq, Z \cap Y^*) = U(F_Y(Y \cap Z), \preccurlyeq, F_Z(Y \cap Z)).
\]
(4) If \( W = U(Y, \leq, Z) \), then
\[
U(Y^\leq, \leq, Z^\geq) = W = U(W, \leq, W) = W^\leq \cap W^\geq.
\]

Proof. (1) is obvious. To show (2) we prove first
\[
Y^\leq = Y_{Z^\geq}^\leq.
\]
By passage to a natural multiple one can assume that for \( x \in Y_{Z^\geq}^\leq \) there exist \( y \in Y, \ w \in Z^\geq \) with \( y \leq x + w \) and \( z \in Z, \ y' \in Y \) with \( z \geq w + y' \). This implies
\[
y + y' \leq x + w + y' \leq x + z \quad \text{with} \quad y + y' \in Y, \ z \in Z, \ i.e., \ x \in Y^\leq.
\]
The other inclusion relation is clear from the monotonicity of \( Y^\leq \) in
the argument \( Z \) (Proposition 5.1).

(2) Is now a straightforward calculation.

(3) By passage to natural multiples one can assume that for \( x \in U(Y, \leq, Z) = Y^\leq \cap Z^\geq \) there exist \( y, y' \in Y, \ z, z' \in Z \) with
\( y \leq x + z, \ z' \geq x + y' \). This shows
\[
y + y' \leq x + z + y' \leq z + z' \quad \text{with} \quad y + y' \in Z^\leq \cap Y.
\]
\( z, z' \in Z \) and \( z + z' \in F_z(Y^\leq \cap Z) \) implies \( z, z' \in F_z(Z^\leq \cap Z) \).
\( y, y' \in Y \) and \( y + y' \in F_Y(Z^\geq \cap Y) \) implies \( y, y' \in F_Y(Z^\geq \cap Y) \).
\( y \leq x + z \) then shows \( x \in Y^\leq_z, \ z' \geq x + y' \) shows \( x \in Z^\geq_z \).
\( Y, Z \) denote semigroups with \( Y \supseteq F_Y(Z^\geq \cap Y) \) and \( Z \supseteq F_Z(Y^\leq \cap Z) \).
Hence
\[
U(Y, \leq, Z) \subseteq U(Y^\leq, \leq, Z^\geq).
\]
Proposition 5.2 is applicable and gives
\[
U(Y, \leq, Z) \subseteq U(Y \cap Z^\geq, \leq, Z \cap Y^\leq).
\]
The inclusion, \( \supseteq \), is trivial.

(4) From (1) and (2) we have
\[
U(Y^\leq \cap Z^\geq, \leq, Y^\leq \cap Z^\geq) \subseteq U(Y^\leq, \leq, Z^\geq) = W.
\]
Since \( Y = Y^\leq \) and \( Z = Z^\geq \) can be assumed (3) is applicable
\[
U(Y^\leq \cap Z^\geq, \leq, Y^\leq \cap Z^\geq) \supseteq U(Y \cap Z^\geq, \leq, Z \cap Y^\leq)
\]
\[
= U(Y, \leq, Z) = U(Y^\leq, \leq, Z^\geq).
\]
EXAMPLE. Let \( Y \) be a linear subspace of the ordered vector space \((X, +, \langle \rangle)\). Denote \( P = \{ x : x \geq 0 \} \). Then for every cone \( Z \), \( U(Y, \langle \rangle, Z) \) is the largest vector space contained in \((Y + P - Z)\). In fact \( Y_Z^* = Y + P - Z \), \( Z_Y^* = Z - P - Y = -(Y + P - Z) \).

**Lemma 5.2.** Let \( Y \) be a subsemigroup of the ordered semigroup \((X, +, \langle \rangle)\). If \( f \) is additive on \( Y \) then it has a unique additive extension to the semigroup \( Y_Y \). This extension is monotone iff \( f \) is monotone on \( Y \).

**Proof.** \( x \in Y_Y \) iff there exist \( m \) natural and \( y, y' \in Y \) with \( y = mx + y' \). For every additive extension \( \hat{f} \) we have

\[
\hat{f}(x) = \frac{1}{m} \cdot (f(y) - f(y')).
\]

\( \hat{f} \) is well-defined on \( Y_Y \) by this formula. In fact, if \( z = mx + z' \) with \( z, z' \in Y \) and we assume for convenience \( m = n \) (we can always pass on to \( m \cdot n \cdot x \)), then

\[
y + z' = mx + y' + z' = z + y' \in Y \quad \text{and} \quad f(y) - f(y') = f(z) - f(z').
\]

\( \hat{f} \) is clearly additive.

Assume that \( f \) is monotone on \( Y \) and \( x, x' \in Y_Y \) are in the relation \( x \preceq x' \). We can assume

\[
y = mx + z \quad y' = mx' + z' \quad \text{with} \quad y, y', z, z' \in Y.
\]

\[
y + z' = mx + z + z' \preceq mx' + z + z' = y' + z
\]

implies

\[
f(y + z') \preceq f(y' + z)
\]

and

\[
mf(x) - f(y) - f(z) \preceq f(y') - f(z') = mf(x').
\]

**Remark 5.5.** If \((X, +, \langle \rangle)\) is an ordered cone and \( Y, Z \) are subcones, then \( Y_{\preceq} \) is a subcone. If the function \( f \) on \( Y \) in Lemma 5.3 is linear, then the unique additive extension to \( Y_Y \) is linear.

**Theorem 5.5.** Let \((X, +, \langle \rangle)\) be a regularly ordered semigroup. Let \( Y, Z, \) and \( U \) be subsemigroups of \( X \) with

\[
\{0\} \subseteq Y \subseteq U \subseteq (Y + Z)_Y \cap Y_{(Y + Z)}.
\]
If $f$ is an additive function on $Y$ and $p$ a subadditive function on $X$ with $p(0) = 0$ and $p(x) < \infty$ for $x \in Z$, then the following conditions on the pair $f$, $p$ are equivalent:

1. For $y, y', u \in U$ with $y \leq y' + u$ holds
   \[ f(y) \leq f(y') + p(u). \]

2. There exists an additive monotone function $f$ on $U$ with
   \[ f(Y) = \int_Y f(y) \text{ for } y \in Y \]
   \[ f(u) \leq p(u) \text{ for } u \in U. \]

Proof. (2) implies (1):

\[ f(y) = f(Y) \leq f(y') + \int_Y f(u) \leq f(y') + p(u). \]

To show that (1) implies (2), consider the collection $\mathcal{F}$ of all pairs $(W, g)$ where $W$ is a semigroup with $Y \subseteq W \subseteq U$ and $g$ is an additive monotone function on $W$ which extends $f$ in such a way that $g(w) \leq g(w') + p(u)$ whenever $w, w' \in W$, $u \in U$ satisfy $w \leq w' + u$. The system $\mathcal{F}$ is inductive for the ordering defined by $(W_1, g_1) \leq (W_2, g_2)$ iff $W_1 \subseteq W_2$ and $g_2$ extends $g_1$. Let $(W^*, g^*)$ be a maximal element in $\mathcal{F}$ (from Zorn's lemma). The theorem will be established, if we show $W^* = U$, which follows from the next lemma.

**Lemma 5.3.** Let be $(W, g) \in \mathcal{F}$ and $v \in U$ such that

(a) There exist $y, y' \in Y$, $z \in Z$ with
   \[ y \leq y' + v + z. \]

(b) There exist $y, y' \in Y$, $z \in Z$ with
   \[ y + z \geq v + y'. \]

Then there exists a number $\bar{g}(v)$ such that $\bar{g}$ defined by

\[ \bar{g}(w + mv) = g(w) + mg(v) \text{ for } w \in W, \text{ m natural} \]

satisfies the conditions:

If for $w, w' \in W$, $m, m'$ (nonnegative integers),

\[ u \in U \quad w + mv \leq w' + m'v + u, \]

\[ \bar{g}(w + mv) = \bar{g}(w') + p(u). \]
then
\[ g(w + mv) \leq g(w' + m'v) + p(u). \]

**Proof.** (a) An inequality for \( \tilde{g}(v) \): Let \( w, w' \in W, u \in U \) such that
\[ w \ll w' + mv + u. \]
The desired inequality for \( \tilde{g} \) yields
\[ g(w) = \tilde{g}(w) \leq \tilde{g}(w' + m'v) + p(u) = g(w') + mg(v) + p(u). \]
\[ \tilde{g}(v) = \sup \left\{ \frac{1}{m} \left[ g(w) - g(w') - p(u) \right] : w \ll w' + mv + u \right\} = \delta. \]

(b) The supremum \( \delta \) is finite.

By condition (b) \( y' + v \ll y + z \) for certain \( y, y' \in Y, z \in Z \). From \((W, g) \in \mathcal{F}\) we get for
\[ w < w' + mv + u \]
\[ w + my' < w' + m(y' + v) + u < w' + m(y + z) + u \]
\[ g(w + my') \leq g(w' + my) + p(mx + u) \]
\[ \leq g(w') + mg(y) + mp(z) + p(u) \]
\[ \frac{1}{m} (g(w) - g(w') - p(u)) \leq g(y) - g(y') + p(z) \]
which is a finite constant depending only on \( v \).

To make sure that \( \delta \) is not \(-\infty\) we have to find a \( z \in Z \) such that for certain \( w, w' \in W \) and some natural \( m \)
\[ w \ll w' + mv + z. \]
Condition (a) on \( v \) yields such elements.

(c) \( \tilde{g}(v) = \delta \) satisfies the inequalities for \( \tilde{g} \).

For \( \epsilon > 0 \) there exist \( w, w' \in W, \tilde{u} \in U, \) and \( m \) natural with
\[ \tilde{w} \ll \tilde{w}' + \tilde{m}v + \tilde{u} \quad \text{and} \quad \delta - \epsilon \leq \frac{1}{m} (g(\tilde{w}) - g(\tilde{w}') - p(\tilde{u})) \]

Assume \( w, w' \in W, u \in U, m, m' \) nonnegative integers with
\[ w + mv \ll w' + m'v + u. \]
By addition of multiples of the inequalities above, we get
\[ m(w + mv) + mw \ll mw' + mm'v + mu + mw' + mmv + mu. \]
By the regularity of the ordering the term $\tilde{m}mv$ can be cancelled

$$\tilde{m}w + m\tilde{w} \subseteq (\tilde{m}w' + m\tilde{w'}) + \tilde{m}m'v + (\tilde{m}u + m\tilde{u}).$$

By definition of $\delta$ [or by $(W, g) \in \mathfrak{F}$ in the case $m' = 0$], we get

$$\tilde{m}m'b > g(\tilde{m}w + m\tilde{w}) - g(\tilde{m}w' + m\tilde{w'}) - p(\tilde{m}u + m\tilde{u})$$

$$\geq m[g(\tilde{w}) - g(\tilde{w'}) - p(\tilde{u})] + \tilde{m}[g(w) - g(w') - p(u)]$$

This proves the wanted inequality

$$g(w) + m\delta - me \leq g(w') + m'\delta + p(\tilde{u}).$$

The lemma is now applied to complete the proof of the theorem. The inequalities holding for $\tilde{g}$ show that $\tilde{g}$ is well-defined on a subsemigroup $W$ containing $W$. $g$ is monotone and additive on $W$, therefore $(W, g) \leq (W, \tilde{g})$.

If $(W^*, g^*)$ is maximal in $\mathcal{F}$ and $v \in U$, then for some natural number $m$ conditions (a) and (b) are satisfied for $mv$, because

$$U \subseteq U_{\max} = Y_{Y+Z} \cap (Y + Z)^\vee.$$

The lemma implies $mv \in W^*$, and Lemma 5.3 together with $p(v) \geq (1/m) p(mv)$ shows $v \in W^*$. Hence we have $W^* = U$.

Remarks. (1) The same proof can be used to extend a linear monotone function $f$ given on a cone to a monotone linear function $f$ below a sublinear function $p$.

(2) If the cancellation rule holds in $(X, +)$, then one can assume that $X$ is an ordered group and $Y$ is a subgroup. In fact, an additive function $f$ on $Y$ has a unique additive extension to $Y - Y$ and this extension is monotone iff $f$ is monotone on $Y$. A subadditive function $p$ on a subsemigroup $Z$ has a trivial subadditive extension, namely the function which is $+\infty$ on the complement of $Z$.

(3) If $X$ is a vector space and $p(\alpha x) = \alpha \cdot p(x)$ for all nonnegative $\alpha$, then every additive function $f$ on a cone $Y$, which is below $p$, is linear on $Y \cap \{p < \infty\}$. In fact, for every fixed $x$ with $p(x) < \infty$, the function $f(\alpha \cdot x)$ is additive in $\alpha$ and bounded for $\alpha$ in a compact interval, hence of the form

$$f(\alpha \cdot x) = \alpha \cdot f(x).$$
Specializations. In most applications of the extension theorem at least one of the following assumptions are fulfilled.

(A) $Z = X$, i.e., $p$ is finite valued on $X$.

(B) $X$ is a real vector space, $p(\alpha x) = \alpha \cdot p(x)$ for $\alpha \geq 0$ and $Y$ is a linear subspace.

(C) The ordering in $(X, +, \leq)$ is the equality (i.e., the strictest possible ordering; it makes every function monotone).

(The ordinary Hahn–Banach theorem of course requires all three of these assumptions).

E.5.1. G. Aumann was the first to prove an extension theorem which required only (A). Another result of his paper shows, moreover, that his approach yields more general results of the type proved here. In fact, as Satz 3 in [1] he treats in disguised form the other extreme case $Z = \{0\}$, i.e., $p(x) = +\infty$ for $x \neq 0$ and gets

**Proposition.** Every monotone additive function $f$ on a subsemigroup $Y$ of the regularly ordered semigroup $(X, +, \leq)$ can be extended to a monotone additive function $\tilde{f}$ on the "Vergleicherrahnbgruppe"

$$V(Y) = \{v: \text{there exist } y_1, y_2 \in Y, m \text{ natural with } v \leq y_2, y_1 \leq mv\}.$$

**Remark.** It is easily checked that our theorem yields an extension to a slightly larger semigroup, namely

$$Y_{Y}^{\leq} \cap Y_{Y}^{\geq} = \{v: \text{there exist } y, y_1, y_2 \in Y, m \text{ natural with } y_1 \leq mv + y \leq y_2\}.$$

**Remark.** If in particular $(X, +, \leq)$ is an ordered vectorspace with positive cone $P$, then every monotone linear function $f$, defined on a linear subspace $Y$, has a monotone linear extension $\tilde{f}$ on

$$(Y + P) \cap (Y - P) = (Y + P) \cap -(Y + P).$$

E.5.2. Let conditions (B) and (C) be fulfilled.

**Proposition.** Let $Y$ be a linear subspace of a vector space $(X, +)$ on which a sublinear function $p$ is given. Then every linear function $f$ defined on $Y$ which is dominated by $p$, has a linear monotone extension $\tilde{f}$ on

$$(Y - \{p < \infty\}) \cap (Y + \{p < \infty\}) = (Y + \{p < \infty\}) \cap -(Y + \{p < \infty\})$$.
Proof. Put \( Z = \{ p < \infty \} \). Then
\[
Y_{Y+Z} \cap (Y + Z)^Y = (Y - (Y + Z)) \cap (Y + Z - Y).
\]
Since \( \preceq \) is the equality in this case
\[
y \preceq y' + u \quad \text{is equivalent with } u = y - y'.
\]
For \( y, y' \in Y \) and \( u = y - y' \in U \), clearly
\[
f(y) \leq f(y') + p(u) \quad \text{since } p(u) = p(y - y') \geq f(y - y').
\]

E.5.3. Assume (B) alone.

**Proposition.** \((X, +, \ll)\) is an ordered vector space. \( Y \) is a linear subspace. \( p \) is sublinear on \( X \), \( f \) is linear on \( Y \). Then there exists a linear monotone extension \( f \) on
\[
U \subseteq (Y + K) \cap -(Y + K)
\]
with \( K = \{ p(x) < \infty \} = \{ x : \text{for some } z \gg x \text{ holds } p(z) < \infty \} \) iff \( y \ll u \in U \) implies \( f(y) \leq p(u) \) for every \( y \in Y \).

The following special case has been considered by H. Bauer:

**Proposition.** Let \( Y \) be linear subspace of the ordered vector space \((X, +, \ll)\) with positive cone \( P = \{ x : x \gg 0 \} \). Let \( p \) be sublinear and finite valued on \( X \) and \( f \) additive on \( Y \).

A necessary and sufficient condition for a monotone linear extension \( f \) of \( f \) on \( X \) to exist is that
\[
f(y) > -1 \quad \text{whenever } y \in P - \{ p < 1 \}.
\]

Remark. This last condition is indeed equivalent with the condition
\[
Y \ni y \ll x \in X \quad \text{implies } f(y) \leq p(x).
\]

Namely for \( y \in Y \), \( y \in P - \{ p < 1 \} \) iff \( -y = w - x \) with \( p(w) < 1 \) and \( x \gg 0 \), or \( -y \ll w \) with \( p(w) < 1 \). Since \( f \) and \( p \) are positively homogeneous Bauer's condition says
\[
f(-y) \leq p(w) \quad \text{whenever } -y \ll w, \ y \in Y, \ w \in X.
\]

E.5.4.

**Proposition.** Let \((X, +, \ll)\) be an ordered vector space and \( p \)
sublinear such that \( x \ll 0 \) implies \( p(x) \ll 0 \). Every linear function \( f \) on a linear subspace \( Y \) which is below \( p \) can be extended to a monotone linear function \( \tilde{f} \) below \( p \) on the subspace

\[ U - \{ u: \text{there exist } y_1, y_2 \in Y \text{ with } y_1 \ll u \ll y_2 \}. \]

**Proof.** Clearly \( U \subseteq (Y + K) \cap -(Y + K) \) with \( K \) as in E.5.3. If \( y \ll u \) then

\[
y = u + w \quad \text{with} \quad w = y - u \ll 0.
\]

\[
f(y) = p(u + w) \leq p(u) + p(w) \ll p(u)
\]

proves the condition for extendability.

**E.5.5.**

**Proposition.** Let \((X, +)\) be a semigroup in which the cancellation rule is valid and let \( p \) be subadditive on \( X \). Let, further, \( Y \) and \( U \) be subsemigroups with \( Y \subseteq U \subseteq F_X(\{ p_+ < \infty \}) \), where

\[
p_+(x) = \inf\{ p(x') : x' \in x + X \}.
\]

An additive function \( f \) on \( Y \) admits an additive and positive extension \( \tilde{f} \) on \( U \) iff

\[
y + u = y' + u' \quad \text{with} \quad y, y' \in Y, \ u, u' \in U
\]

implies

\[
f(y) \ll f(y') + p(u').
\]

**Proof.** (a) Let \( \tilde{f} \) be an additive positive extension of \( f \) which is below \( p \) on \( U \). Then \( y + u = y' + u' \) implies

\[
f(y) \leq \tilde{f}(y) + f(u) = \tilde{f}(y' + u') = \tilde{f}(y') + f(u') \leq f(y') + p(u').
\]

(b) Let \( \ll \) be the intrinsic ordering on \((X, +)\). It is regular since the cancellation rule is valid in \((X, +)\). Every monotone additive function on \( U \) is positive, since \( 0 \ll x \) for all \( x \in X \). It suffices therefore to construct a monotone additive extension \( \tilde{f} \) on \( U \), given the inequalities (*) which can be written

for \( y, y' \in Y, u \in U \) with \( y \ll y' + u \) holds \( f(y) \leq f(y') + p(u) \). (*)

(c) The extension theorem applies if we show that

\[ U \subseteq F_X(\{ p_+ < \infty \}) \subseteq U(Y, \ll, \{ p < \infty \}). \]
Clearly \( Y^\preceq = X, \{ p < \infty \}^\preceq = \{ p(+) < \infty \} \). Hence

\[ U(Y, \preceq, \{ p < \infty \}) = U(X, \preceq, \{ p(+) < \infty \}). \]

By Proposition 5.4

\[ U(X, \preceq, \{ p(+) < \infty \}) = U(F_X(\{ p(+) < \infty \}), \preceq, \{ p(+) < \infty \}) \]

VI. EXISTENCE OF DECOMPOSITIONS

For many important ordered semigroups \((H, +, <)\) (\(<\) not necessarily regular), one can associate monotone additive functions \(l\) on a regularly ordered semigroup \((X, +, \preceq)\) with certain (sometimes with all) elements \(l \in H\). The extension theorem 5.5 provides, then, a tool to solve decomposition problems in \((H, +, <)\).

It is convenient to study more general decomposition problems than those of the form \("l < C A_i"\), namely: Let \(L_1, \ldots, L_n\) be subsets of \(H\) with \((H, +)\) a semigroup. Characterize those \(l \in H\) which can be written in the form

\[ l = l_1 + \cdots + l_n \quad \text{with} \quad l_i \in L_i \quad \text{for all} \ i. \]

DEFINITION 6.1. If \((H, +)\) is a semigroup with distinguished subsets \(L_1, \ldots, L_n\), and \((X, +)\) is a semigroup with distinguished regular orderings \(<_1, \ldots, <_n\), then \((X, +, \{<_i\})\) is called admissible for \((H, +, \{L_i\})\) if

(1) With every \(x \in X\) there is associated a real-valued function on \(H\) which is weakly additive on \(H\); its value in \(h\) is denoted by \(x(h) = \hat{h}(x)\).

(2) The ordering \(<_i\) is so strict that for every \(l_i \in L_i\) the function

\[ x \rightsquigarrow l_i(x) \quad \text{is monotone on} \quad (X, <_i). \]

(3) The addition in \(X\) is such that for every \(h \in \bigcup_i L_i\) the function

\[ x \rightsquigarrow \hat{h}(x) \quad \text{is subadditive on} \quad (X, +). \]

The following proposition shows how the existence of a decomposition of \(l\) (for a certain type of \(l\)), is reflected by properties of functions \(\hat{h}\) associated by an admissible \((X, +, \{<_i\})\).
PROPOSITION 6.1. Let \((X, +, \{<_i\})\) be admissible for \((H, +, \{L_i\})\). If for some \(l \in L\) the function \(l\) is additive on \((X, +)\) and

\[ l = l_1 + \cdots + l_n \quad \text{with} \quad l_i \in L_i, \]

then

1. \(l = l_1 + \cdots + l_n\) on \(X\)
2. \(l_i\) is additive and monotone on \((X, +, <_i)\)
3. \(l_i(x) \leq \sup\{h(x) : h \in L_i\}\) for \(x \in X\).

Proof. (1) Uses the fact that for every \(x\) the function \(x \mapsto l(x) < \infty\) is additive in \(l\).

(2) If a sum of subadditive functions is additive, then each summand is additive. Hence conditions (2) and (3) of Definition 6.1 imply (2).

(3) is obvious. It should be remarked that the right side \(\sup\{h(x) : h \in L_i\}\) is subadditive on \((X, +)\) and monotone with respect to \(<_i\). If \(L_i\) is small, then the set where it is equal + \(\infty\) may also be small.

THEOREM 6.2. Let \((X, +, \{<_i\})\) be admissible for \((H, +, \{L_i\})\). Denote \(p_i = \sup\{l_i : l_i \in L_i\}\), \(Z_i = \{p_i < \infty\}\), \(Z = \bigcap_i Z_i\),

\[ X_i = U(Z_i, <_i, Z_i) = Z_{<i}^i \cap Z_{>i}^i \]

and \(Y = \bigcap_i X_i\). For an element \(l \in L\), for which \(l\) is additive on \((X, +)\), the following properties are then equivalent:

(a) Whenever \(y, y' \in Y, x_1, \ldots, x_n\), with \(x_i \in X_i\), are such that

\[ y < y' + x_i, \quad \text{then} \quad l(y) \leq l(y') + \sum_i p_i(x_i). \]

(b) There exist functions \(f_1, \ldots, f_n\) such that

1. \(f_i\) is additive and monotone on \((X_i, +, <_i)\)
2. \(f_i \leq p_i\) on \(X_i\).
3. \(l(y) = \sum f_i(y)\) for all \(y \in Y = \bigcap_i X_i\).

Proof. The existence of the \(f_i\) implies the inequalities in (a): If \(y < y' + x_i\), then

\[ f_i(y) \leq f_i(y' + x_i) \leq f_i(y') + p_i(x_i). \]

Addition gives

\[ l(y) = \sum f_i(y) \leq \sum f_i(y') + \sum p_i(x_i) = l(y') + \sum p_i(x_i). \]
The converse is now proved by the extension theorem proved above.

1. Consider

\[ E = X_1 \times X_2 \times \cdots \times X_n \]

\(E\) is a regularly ordered semigroup if one defines

\[(x_1, x_2, \ldots, x_n) + (x_1', x_2', \ldots, x_n') = (x_1 + x_1', \ldots, x_n + x_n')\]

\[(x_1, x_2, \ldots, x_n) \leq (x_1', x_2', \ldots, x_n') \iff x_1 \leq x_1', x_2 \leq x_2', \ldots, x_n \leq x_n'.\]

On \(E\) we get a subadditive function \(p\) by the definition

\[ p(x_1, x_2, \ldots, x_n) = \sum_i p_1(x_i) .\]

\(p\) is finite valued on

\[ G = (Z_1 \times \cdots \times Z_n) .\]

On the subsemigroup \(F \subseteq E\) of all constant \(n\)-tuples

\[ F = \{(y, \ldots, y) \text{ with } y \in Y\} \]

we define an additive function \(f\) by

\[ f(y, \ldots, y) = \tilde{l}(y) \quad \text{for } y \in Y.\]

2. The theorem is proved if we can show that \(f\) is extendable to an additive monotone function \(\tilde{f}\) below \(p\) on \(X_1 \times X_2 \times \cdots \times X_n\). In fact such an extension \(\tilde{f}\) is of the form

\[ \tilde{f}(x_1, x_2, \ldots, x_n) = \sum_i f_i(x_i) \]

where \(f_i\) is defined on \(X_i\) and is additive and monotone with respect to \(<_i\). \(\tilde{f} \leq p\) is equivalent with \(f_i \leq p_i\) on \(X_i\) for every \(i\),

\[ f(y, \ldots, y) = \tilde{l}(y) = \sum f_i(y) \quad \text{for } y \in \bigcap_i X_i \]

proves property (b) (3).

3. According to Theorem 1 the desired extension exists on the subsemigroup

\[(F + G)^F \cap F^{F+G} .\]
iff the inequalities hold:

\[ f(y_1, ..., y_n) \leq f(y'_1, ..., y'_n) + p(x_1, ..., x_n) \]

whenever \((y_1, ..., y_n), (y'_1, ..., y'_n) \in F\) and \((x_1, ..., x_n) \in U\) satisfy

\[ (y_1, ..., y_n) \ll (y'_1, ..., y'_n) + (x_1, ..., x_n). \]

This is just condition (a) of Theorem 6.2.

(4) What remains to be proved is

\[ E \subseteq (F + G)_F \quad \text{and} \quad E \subseteq F^G_F. \]

Let \((x_1, ..., x_n) \in E. Z_1, Z_2, ..., Z_n \text{ and } \prec_1, \prec_2, ..., \prec_n \text{ satisfy the assumption of Lemma 5.1. Since } x_i \in Z_{2i}^* \text{ for all } i, \text{ there exist } z \in Z, z_1, z_2, ..., z_n \text{ with } z_i \in Z_i \text{ and } m \text{ natural such that}

\[ z \preceq m(x_1, ..., x_n) \text{ for all } i, \]

or equivalently

\[ (z, z, ..., z) \ll m(x_1, ..., x_n) + (z_1, ..., z_n) \]

with

\[ (z, z, ..., z) \in F \quad \text{since} \quad Z \subseteq Y \quad \text{and} \quad (z_1, ..., z_n) \in G. \]

This shows \( E \subseteq F^G_G \).

Since \( x_i \in (Z_i^*)^{2i} \) for all \( i \) we get, again by Lemma 5.1, that there exist \( z \in Z, z_1, z_2, ..., z_n \) with \( z_i \in Z_i \) and \( m \) natural such that

\[ z_i \succ m x_i + z \quad \text{for all } i, \]

or equivalently

\[ (z_1, z_2, ..., z_n) \gg m(x_1, x_2, ..., x_n) + (z, z, ..., z). \]

This shows \( E \subseteq G^G_p \).

Here are two examples where Theorem 6.2 applies a straightforward way:

**E.6.1.**

**Proposition.** Let \( X \) be a vector space and let \( k_1, ..., k_n \) be sublinear
on $X$ such that the vector space $W$ spanned by $\{\sum k_i < \infty\}$ has the property

$$W + \{k_i < \infty\} = X \quad \text{for} \quad i = 1, 2, \ldots, n.$$ 

If $l$ is linear on $X$ with $l \leq \sum k_i$, then there exist $l_1, \ldots, l_n$ with $l_i \leq k_i$ on $X$ and $\sum l_i = l$.

Proof. (1) Consider the ordered semigroup $(H, +, \leq)$ of all sublinear functions on $X$ with pointwise addition and ordering. (Since sublinear functions may assume the value $+\infty$, the cancellation rule is not valid in $(H, +, \leq)$; neither is the pointwise ordering, $\leq$, regular in $(H, +)$). In the terminology of Chapter III "$l \leq \sum k_i$" is the decomposition problem to the left, to find $l_1, \ldots, l_n$ on $X$ with $l_i \leq k_i$ and $\sum l_i = l$ on $X$. The Proposition E.6.1 asserts that "$l \leq \sum k_i$" is solvable if the cones $\{k_i < \infty\}$ and $\{\sum k_i < \infty\}$ are as big as assumed above.

(2) Consider the subsets of $H$

$$L_i = \{h : h \leq k_i\}.$$ 

We show that $(X, +, \{<_i\})$ is admissible for $(H, +, \{L_i\})$ if $<_i$ is the equality in $X$ for all $i$, and $+$ is the vector space addition in $X$:

In fact, for every $x \in X$ the evaluation function

$$h \mapsto \tilde{h}(x) = h(x)$$

is a real valued function on $(H, +)$, which is additive in every $h \in H$ with $h(x) < \infty$. $(h + k \mapsto (h + k)(x) = h(x) + k(x))$; for every $l_i \in L_i$ the function $l_i$ ($l_i(x) = l_i(x)$ for $x \in X$) is (monotone and) subadditive, since all elements of $H$ are subadditive functions on the vector space $X$.

Theorem 6.2 is applicable and gives:

(3) $\sup\{l_i : l_i \in L_i\} = k_i$ since $L_i$ contains $k_i$ and all other elements of $L_i$ are smaller. We have

$$\bigcap_i \{k_i < \infty\} = \{\sum k_i < \infty\}$$

therefore and

$$X_i = U \left(\bigcap_i \{k_i < \infty\} = \{k_i < \infty\}\right)$$

$$= \left(\sum k_i < \infty\right) - Z_i \cap Z_i - \left(\sum k_i < \infty\right)$$

$$= (W - Z_i) \cap (Z_i - W) = X \quad \text{for all} \quad i.$$
(4) If $l$ is linear on $X$ with $l \leq \sum k_i$, then for $y, y', x_i \in X$ such that $y - y' + x_i$, we have

$$k_i(x_i) = k_i(y - y') \quad \text{and} \quad \sum k_i(x_i) = \sum k_i(y - y') \geq l(y - y').$$

The condition (a) is therefore satisfied.

Hence there exist additive functions on $\bigcap X_i = X$, $l_i \leq k_i$ with $l = \sum l_i$. These $l_i$ are the wanted elements of $H$.

E.6.2.

Proposition. Let $X$ be a cone in a vector space and let $k_1, \ldots, k_n$ be positive sublinear functions on $X$. Assume that for every $i$ the cone $Z_i = \{x : k_i(y) < \infty \text{ for some } y \in x + X\}$ penetrates $X$ (i.e., $Z_i$ is not contained in a proper face of $X$).

For an additive function $l$ on $X$ the following condition (*) is necessary and sufficient for the existence of positive linear functions $l_1, l_2, \ldots, l_n$ with $0 \leq l_i \leq k_i$ and $\sum l_i = l$ on $X$.

(*) For every pair $x, x' \in X$ and for every $n$-tuple of pairs $y_i, x_i \in X$ with $x + y_i = x' + x_i$ for $i = 1, 2, \ldots, n$ holds $l(x) \leq l(x') + \sum k_i(x_i)$.

Proof. (1) Consider again the ordered semigroup $(H, +, \leq)$ of all sublinear functions on $X$. Define the subsets $L_i$ of $H$

$$L_i = \{h : h \leq k_i \text{ and } h(x) \leq h(x + y) \text{ for all } x, y \in X\},$$

and the orderings $\leq_i$, $i = 1, 2, \ldots, n$ all equal to the intrinsic ordering $\leq$ in $(X, +)$. [$\leq$ is regular since the cancellation rule is valid in $(X, +)$]. As in E.6.1 one shows that

$$(X, +, \{\leq_i\}) \text{ is admissible for } (H, +, \{L_i\}).$$

Theorem 6.2 is therefore applicable and yields

(2) sup$\{l_i : l_i \in L_i\} = k_{i(\cdot)} \in L_i$ with $k_{i(\cdot)}(x) = \inf\{k_i(x + y) : y \in X\}$ since $k_{i(\cdot)}$ is monotone, nonnegative, and therefore by Lemma 3.2, subadditive. We calculate for $i = 1, 2, \ldots, n$

$$U\left(\sum k_{i(\cdot)} < \infty, \leq_i, \{k_i < \infty\}\right) = U(\bigcap Z_i, \leq, Z_i) \supseteq U(\{0\}^*, \leq, Z_i) = U(X, \leq, Z_i) = Z_i^*.$$ 

The monotonicity of $k_{i(\cdot)}$ implies $Z_i = Z_i^*$ and Proposition 5.2 yields, together with the finiteness assumption above,

$$X_i = Z_i^{\#} = X \cap Z_i^{\#} = F_x(Z_i) = X.$$

By Theorem 6.2 the linear function $l$ on $X$ can be written as a sum

$$l = l_1 + \cdots + l_n$$

with $0 \leq l_i \leq k_i$, $l_i$ linear for $i = 1, 2, \ldots, n$,

iff for every $(n + 2)$-tuple $x, x', x_1, \ldots, x_n$ with $x \leq x' + x_i$ for every $i$ we have

$$l(x) \leq l(x') + \sum k_i(x_i).$$

To finish the proof, we have to show that this condition on $l$ is equivalent with the condition (*). Clearly it is stronger than (*): $x \leq x' + x_i$ holds iff there exists $y_i \in X$ with $x + y_i = x' + x_i$ and $k_{i(+)i} \leq k_i$.

On the other hand (*) implies our condition above: If for $x, x'$ each of the elements $x_1, \ldots, x_n$ has the property $x \leq x' + x_i$, then $x \leq x' + (x_i + x_i)$ for every $x_i \in X$. By (*) we have for every choice of $x_1, \ldots, x_n$

$$l(x) \leq l(x') + \sum k_i(x_i + x_i)$$

and therefore,

$$l(x) \leq l(x') + \sum k_i(x_i + x_i).$$

For the next examples, E.6.3 and E.6.4, it is not quite so obvious how to find a suitable semigroup $(X, +)$ with regular orderings $<_{1, \ldots, n}$. We follow the

Procedure. $l, k_1, \ldots, k_n \in (H, +, <)$ are given with $l < \sum k_i$.

1. Choose subsets $L_i \subset H$ with the property that $L_i$ contains the candidates for $l_i$ in the desired decomposition

$$l = l_1 + \cdots + l_n$$

with $l_i \leq k_i$.

It may be useful (for the computation of $p_i$) to include in $L_i$ also elements which are not summands of $l$.

2. Choose a set $X$ of real valued functions on $(H, +)$ which are additive in the given $l$. One may pass from $H$ to a subsemigroup which contains $\bigcup_i L_i$.

3. Find a commutative addition in $X$ which makes $\bar{h}$ subadditive on $(X, +)$ for all $\bar{h} \in \bigcup_i L_i$ and makes moreover $l$ additive. Hereby $\bar{h}$ is the function on $X$ defined by $\bar{h}(x) = x(h)$. 
Find furthermore regular orderings \(<_1,\ldots,<_n\) on \((X,+)\) such that \(l_i\) is monotone on \((X,<_i)\) for every \(l_i \in L_i\).

(4) Calculate the subadditive monotone functions on \((X,+,<_i)\)

\[ p_i = \sup\{l_i : l_i \in L_i\} \]

and the semigroups

\[ X_i = U\left(\{\sum p_i < \infty\}, \leq_i, \{p_i < \infty\}\right), \quad \text{for } i = 1, 2,\ldots, n. \]

(5) Adapt the inequalities of Theorem 6.2 (a) for \(l, p_1,\ldots, p_n\) to the special situation:

\[ \text{If these inequalities are satisfied, then Theorem 6.2 proves the existence of } f_1,\ldots,f_n \text{ such that} \]

\[ l = f_1 + \cdots + f_n \text{ on } \bigcap_i X_i \]

(1) \(f_i\) is additive and monotone on \((X_i,+,<_i)\)

(2) \(f_i \leq p_i = \sup\{l_i : l_i \in L_i\} \) or \(X_i\)

(6) Check if, for these \(f_i\), there exist \(l_i \in L_i\) such that \(l_i = f_i\) on \(X_i\).

If such \(l_i\) exist check

\[ l_i < k_i \quad \text{and} \quad l_1 + \cdots + l_n = l. \]

**Remarks on the choices.**

(1) \(X\) should be at least so large that \(l \in H\) can be identified from the associated function \(l\) on \(X\).

Moreover, the subsemigroups \(X_1,\ldots,X_n\) should turn out so large that the restrictions of \(l_i\) to \(X_i\) determine \(l_i \in L_i\) uniquely.

(2) To get large subsemigroups \(X_i\) the set \(L_i\) should be chosen rather small and the orderings \(<_i\) rather weak. Namely, if \(L_i\) is replaced by \(L_i'\) with \(L_i' \subseteq L_i\), then

\[ p_i' = \sup\{h : h \in L_i'\} \leq \sup\{h : h \in L_i\} = p_i, \]

therefore \(\{p_i' < \infty\} \supseteq \{p_i < \infty\}\) and \(X_i' \supseteq X_i\). Moreover, a weaker ordering \(<_i'\) can be introduced in \((X,+)\) if it is only required that \(h\) is monotone for \(h \in L_i'\). In fact every set \(L\) defines a weakest ordering with the property that for all \(h \in L\) the functions \(h\) are monotone.

(3) If \(<_i\) is weakened and \(L_i\) decreased, then the set of necessary inequalities (a) for the existence of the \(f_i\) is enlarged.
In many important applications $H$ is already given as a semigroup of finite valued functions on some set $B$ (with pointwise ordering and addition). The following construction of an admissible $(X, +, \{<_i\})$ may then be useful for an attack on decomposition problems.

Construction. Let $X$ be the set of all finite discrete measures on $B$

$$X = \left\{ x = \sum \alpha_i [b_i] \text{ with } \alpha_i \text{ real, } b_i \in B \right\}$$

$([b_i]$ denotes the probability measure in $b \in B$ in accordance with the notation in Chapter III).

With an arbitrary finite valued real function $f$ on $B$, we then associate an additive function $f$ on $X$ by

$$f(x) = \sum \alpha_i f(b_i) \quad \text{for } x = \sum \alpha_i \cdot [b_i].$$

Clearly every additive function on $X$ is derived from a unique function on $B$ in that way.

**Definition 6.2.** Let $L$ be a set of finite valued functions on the set $B$. For a function $f$ on $B$ we write $f \in L$ and call $f$ $L$-monotone iff

$$f(x) \leq f(x') \text{ holds for every pair } x, x' \in X \text{ with } l(x) \leq l(x') \text{ for all } l \in L.$$  

**Remark.** Another way to define $L$ is the following: There exists a weakest ordering among all orderings $<$ on $X$ with the property that $l$ is monotone on $(X, <)$ for all $l \in L$. This ordering, denoted by $<_L$ is regular on $(X, +)$ and

$$x \leq_L y \quad \text{iff } l(y - x) \geq 0 \quad \text{for all } l \in L.$$  

For a real-valued function $f$ on $B$

$$f \in \bar{L} \quad \text{iff } f \text{ is monotone on } (X, +, \leq_L).$$

We have the trivial

**Lemma 6.2.** Let $B$ be a set and $L, M$ a set of finite valued real functions on $B$. Then

(a) $L \subseteq \bar{L}$

(b) $L \subseteq M$ implies $L \subseteq \bar{M}$.  

Examples. Let \( B \) denote a set and \( L \) a collection of finite valued real functions on \( B \).

1. If all \( l \in L \) are nonnegative then \( \hat{L} \) contains only nonnegative functions.

2. If in \( B \) the multiplication with positive reals is defined and all \( l \in L \) are positively homogeneous, then every \( f \in \hat{L} \) is positively homogeneous.

3. If \( B \) is a convex set and \( l \in L \) are convex on \( B \), then every \( L \)-monotone \( f \) is convex.

4. If \( B \) is ordered and all \( l \in L \) are monotone, then all \( L \)-monotone \( f \) are monotone on \( B \).

5. If \( B \) is a family of subsets of \( \Omega \) which is closed under finite unions and intersections and if, for all \( l \) in a class \( L \) of set functions,

\[
l(b \cup c) + l(b \cap c) \leq l(b) + l(c)
\]

for \( b, c \in B \),

then these inequalities hold for all \( f \in \hat{L} \).

6. If \( B \) is the real line and every \( l \in L \) is a polynomial of at most order \( n \), then every \( f \in \hat{L} \) is such a polynomial.

The proofs are obvious except possibly in the case (6) where one has to realize that \( f \) is a polynomial of degree not higher than \( (n - 1) \) if all the \( n \)-th differences vanish:

\[
f(b) - f(b + b_1) - \cdots - f(b + b_n)
+ f(b + b_1 + b_2) + \cdots + (-1)^n f(b + b_1 + \cdots + b_n) = 0
\]

for an arbitrary choice \( b, b_1, \ldots, b_n \in B \).

E.6.3.

Proposition. Let \( l \) and \( k_1, k_2, \ldots, k_n \) be finite-valued positive convex functions on a convex set \( B \).

There exist positive convex functions \( l_1, \ldots, l_n \) on \( B \) with \( l = l_1 + \cdots + l_n \) and \( l_i \geq k_i \) pointwise on \( B \) iff the inequalities are satisfied

\[
p(l) \geq \sum_i p(+) (k_i)
\]
for every finite signed measure $\rho = \sum \alpha_j [b_j]$ ($\alpha_j$ real, $b_j \in B$), where the notation is used $\rho(h) = \sum \alpha_j \cdot h(b_j)$ for $h$ convex on $B$

$$\rho_{(+)}(k) = \inf \left\{ \sum \alpha_j h(b_j) : k \leq h, h \text{ convex} \right\}.$$ 

Proof. We have to solve the decomposition problems $-l \leq \sum (-k_i)$ in the ordered semigroup $-H$ of all negative finite valued concave functions on $B$ (with pointwise addition and ordering).

1. Define

$$L_i = \{-h : h \text{ positive convex and } -h \leq -k_i \} \subseteq -H.$$ 

2. Let $X$ be the vector space of all signed measures $x = \sum \alpha_j [b_j]$ with $\alpha_j$ real, $b_j \in B$. The $x$ act as linear functions on $-H$ by integration over $B$

$$x(-h) = -\sum \alpha_j h(b_j) = -\rho(h) \quad \text{for } h \text{ positive and convex.}$$ 

($\rho$ denotes, as in the assertion E.6.3, the integral with respect to $x$ restricted to the cone $H$ of positive convex functions ordered by the pointwise ordering.)

3. The addition in $X$ is the obvious one, the ordering $<$ was introduced in E.3.1.

$$x < y \iff \int h \, dx \leq \int h \, dy \quad \text{for every positive convex } h.$$ 

$<$ and also the reverse ordering $>$ are clearly regular on $(X, +)$. For every $h \in H$ the integral of $h$ is a monotone function on $(X, +, <)$ and every additive monotone function on $(X, +, <)$ is the integral of a certain convex positive function $h$ [by the examples (1) and (3) for the hull operation $L \to L$].

If we put $<_i$ equal $>$ on $X$ for all $i$, then $(X, +, \{<_i\})$ is admissible for $(-H, +, \{L_i\})$. In fact every element in $L_i$ defines a monotone linear function on $(X, +, >)$.

4. (a) $p_i(x) = \sup \{ l_i(x) : l_i \in L_i \} = \sup \{ \int l_i \, dx : l_i \leq -k_i \text{ concave} \}$

$$= -\inf \{ \rho(h) : h \text{ convex } k_i \leq h \} = -\rho_{(+)}(k_i),$$

where $\rho$ is again the integral with respect to $x$ restricted to the ordered semigroup $(H, +, \leq)$ of positive convex functions.

(b) $\rho_{(+)}(k_i) = -p_i(x)$ is finite for those $x$ for which $\int x \, dh \geq 0$ for
all \( h \in H \), i.e., \( x > 0 \). (Not all of these \( x \) are positive measures in the ordinary sense.) Hence

\[
X_i = U \left( \left\{ \sum p_i < \infty \right\}, >, \{ p_i < \infty \} \right) \geq \{ x > 0 \} - \{ x > 0 \} = X.
\]

(5) The inequalities (a) in Theorem 6.2 have the form: If \( x, x', x_1, \ldots, x_n \) satisfy \( x > x' + x_i \) for all \( i \), then

\[
\int (-l) \, dx \leq \int (-l) \, dx' + \sum p_i(x_i).
\]

This system of inequalities is equivalent with the system of inequalities

\[
\rho(l) \geq \sum \rho_{(+)}(k_i) \quad \text{for all integrals } \rho \text{ on } (H, +, \leq).
\]

In fact, since the cancellation rule is valid, \( x - x' \) is well defined, and \( x_i < (x - x') \) implies \( \rho_{(+)}(k) \leq \rho_{(+)}(k) \) for all \( k \in H \), if \( \rho_{(+)} \) is the function on \((H, +, \leq)\) derived from \( x_i \) while \( \rho \) is derived from \((x - x') \geq x_i \). In particular

\[
\sum \rho_{(+)}(k_i) > \sum \rho_{(+)}(k_i).
\]

\[
-\rho(l) = \int (-l) \, d(x - x') \leq \sum p_i(x_i) = -\sum \rho_{(+)}(k_i),
\]

for all \( x - x', x_1, \ldots, x_n \) with \( x - x' > x_i \) for all \( i \), is therefore equivalent with the special case

\[
-\rho(l) \leq -\sum \rho_{(+)}(k_i) \quad \text{for all } \rho : \rho(h) = \int h \, dx, \quad x \in X.
\]

(6) Theorem 6.2 yields monotone linear functions on \((X, +, >)\) \( f_1, \ldots, f_n \); these are uniquely determined by the values in the point-measures \([b]\) and \(-l_i(b) = f_i([b])\) defines, for every \( i \), a positive convex function \( l_i \) on \( B \). The decomposition problem is thus solved.

The inequalities \( \rho(l) \geq \sum \rho_{(+)}(k_i) \) are necessary for decomposability in \((H, +, \leq)\) of \( \sum k_i \leq l' \) (compare Proposition 3.4).

**Remark.** The following example, due to H. Rost, shows that \( \sum k_i \leq l' \) is not always solvable:
Let \( B = \mathbb{R}^n \) with the coordinate functions \( x, y \). Then

\[
"x^+ + y^+ \leq l = (\max(x^+ + y^+, \frac{1}{2}(x + y + 1))"
\]
is not solvable since

\[
\rho(+dx^+) + \rho(+dy^+) > \rho(l)
\]
for \( \rho \) defined by

\[
\rho(h) = h(1, 0) + h(-1, 0) - 2 \cdot h(0, 0) + h(0, 1) + h(0, -1).
\]

**E.6.4.**

PROPOSITION. Let \( l, k_1, \ldots, k_n \) be positive discrete measures on a convex set \( B \) with \( \sum k_i < l \) in the familiar sense. \( h < k \) iff \( \int y \, dh \leq \int y \, dk \) for all positive convex functions \( y \) on \( B \). Then there exist positive discrete measures \( \lambda_1, \ldots, \lambda_n \) on \( B \) with

\[
k_i < l_i \quad \text{for all } i \quad \text{and} \quad l_1 + \cdots + l_n = l.
\]

**Proof.** The proposition asserts the solvability of every decomposition problem \("l > \sum k_i"\) in \((H, +, >)\), where \( H \) is the cone of all positive discrete measures with the familiar ordering >.

1. Choose \( L_i \) to be set \( \{l_i : l_i \in H \text{ and } l_i > k_i\} \)
2. Choose \( X \) to be the cone of all finite valued functions on \( B \).
3. \( x \in X \) operates as a linear function on \((H, +)\) by
   \[
x(h) = \int x \, dh = \hat{h}(x).
   \]

4. \( X \) is an ordered semigroup by pointwise (on \( B \)) addition and ordering. Since the elements of \( H \) are positive measures, \( \hat{h} \) is a monotone function on \((X, \leq)\) for every \( h \in H \). All \( \hat{h} \) are moreover additive on \((X, +)\). Hence
   \[
   (X, +, \leq) \text{ is admissible for } (H, +, \{L_i\}).
   \]
5. \( p_t(x) = \sup\{\int x \, dl_i : l_i \in L_i\} = \sup\{\int x \, dl_i : k_i < l_i\} \). If \( x \) assumes a strictly positive value on \( B \), then \( p_t(x) = +\infty \). If \( x \) is nonpositive on \( B \), then the argument of Proposition E.3.1.3 yields
   \[
p_t(x) = \int \hat{x} \, dk_i \leq 0
   \]
where \( \hat{x} \) is the smallest concave function on \( B \) which is above \( X \).

\[
x_i \ni U(\{x < 0\}, =, \{x \leq 0\}) = X \quad \text{for all } i.
\]
(5) The inequalities (a) in Theorem 6.2 take the form \( x \leq x_i \) pointwise on \( B \) for \( i = 1, 2, \ldots, n \) implies
\[
\int x \, dl \leq \sum p_i(x_i).
\]

This is so, since \( X \) is a group. Now \( p_i \) is monotone and therefore it suffices to require
\[
\int x \, dl \leq \sum \int x \, dk_i = \sum p_i(x)
\]
for every nonpositive function \( x \) on \( B \). This is, however, a consequence of \( \int > \sum k_i \), since \( \int x \) is negative and concave and
\[
\int x \, dl \leq \int \int \, dl \leq \int \int \, d \left( \sum k_i \right).
\]
Therefore Theorem 6.2 yields monotone additive functions \( f_1, f_2, \ldots, f_n \) on \( (X, +, \leq) \).

(6) These \( f_i \) are the integrals with respect to certain positive discrete measures \( l_i \) on \( B \) since \( \sum f_i = I \) is derived from such a positive discrete measure \( l \). Finally, \( f_i(y) \leq p_i(y) = \int y \, dk_i \) for all negative concave \( y \) on \( B \) implies \( l_i > k_i \).

The applications described in 1 base on the geometrical facts just seen in Proposition E.6.4.

LITERATURE

2. H. Rostr, Charakterisierung einer Ordnung von konischen Maßen durch positive \( L^1 \)-Kontraktionen (to be published in Journal of Mathematical Analysis and its Applications).