On Quasi Periodic Rings*

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INTRODUCTION

Let $R$ be a ring, $E$ its set of idempotents, $\langle E \rangle$ the subring generated by $E$. The ring $R$ is called:

Periodic (respectively quasi periodic) if for every element $x$ of $R$ there exists an integer $n(x)$ (respect. $q(x)$) such that $x^{\mu(x)} \in E$ (respectively $x^{\nu(x)} \in \langle E \rangle$).

Many rings are quasi periodic. This is the case of: the nil rings; the locally finite rings (in the sense generally taken that every finitely generated subring is finite); more generally, the rings in which every element generates a finite multiplicative semigroup, that is, the periodic rings; the well-known rings (necessarily periodic) in which every element is a proper power of itself [for a formal definition, see Section 4, Theorem 5, condition (F)]; the ring $Z$ of the relative integers although it is not a periodic ring; every subring of endomorphisms $\sigma$ of an indecomposable module with both chain condition such that every endomorphism $\sigma$ is a multiple of the identity endomorphism [see N. Jacobson, The Theory of Rings, Theorem 10 p. 11]; every radical extension of a quasi periodic ring; every weak product of a family of homomorphic images of quasi periodic rings.

In Section 1, we study the reduced quasi periodic rings. Incidentally such rings are commutative [see Theorem 1]. In Section 2, we show that a periodic ring is nothing else than a quasi periodic ring in which every idempotent has an additive order [see Theorem 4] and deduce a minimal set of axioms for the condition (F) [see Theorem 5]. In Section 3, we study the quasi periodic

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1 In [3], D. C. Rees has showed that every finite semigroup possesses an idempotent.
rings with a unique nonzero idempotent. We show in Section 4 that this class of rings can describe every quasi periodic ring in which every idempotent is central [see Theorem 9]. Some properties are derived from this characterization which extend a work we have done on "certains anneaux periodiques" to be published in the "Bulletin de la Société Mathématique de Belgique". [see Theorem 10, 11 and 12].

Conventions. Throughout this paper, integers used as exponents are assumed to be \( \neq 0 \); rings without nonzero nilpotent elements are assumed to be \( \neq \{0\} \); subrings generated by a singleton \( \{x\} \) are denoted \( \langle x \rangle \).

1. Reduced\(^2\) Quasi Periodic Rings

It is well known that every idempotent of a reduced ring \( R \) is in the center, therefore, the subring \( \langle E \rangle \) itself is contained in the center. In [I], Herstein has showed that every ring \( R \) with no nil-ideal \( \neq (0) \), which is a radical extension\(^3\) of its center is commutative. From these results one can deduce easily the following theorem.

**Theorem 1.** A reduced quasi periodic ring is commutative.

*Proof.* \( R \) does not possess obviously nil ideals \( \neq (0) \) and is a radical extension of its center. In fact, it is a radical extension of some subset contained in the center namely, the subset \( \langle E \rangle \).

**Theorem 2.** In order for a quasi periodic ring to be reduced, it is necessary and sufficient to be a subdirect product of quasi periodic rings without divisors of zero.

*Proof.* By Theorem 1, the ring is commutative. Hence the prime radical coincides with the set of the nilpotent elements [2] which set is \( \{0\} \) by the hypothesis, therefore, the prime radical is the zero ideal \( (0) \), this shows the theorem.

Theorem 2 suggests the study of those rings which are quasi periodic and without divisors of zero. The following theorem brings an answer.

**Theorem 3.** A quasi periodic ring without divisors of zero is nothing less than a periodic field or (i) a unitary reduced commutative ring radical extension of the ring of the integers (up to an isomorphism) with characteristic 0.

\(^2\) Following P. Samuel we call "reduced", a ring with nonzero nilpotent elements.

\(^3\) Following Faith, a ring \( R \) is a radical extension of a subset \( B \) if for every \( x \in R \) there exists \( n(x) \) such that \( x^{n(x)} \in B \).
Proof. Assuming that $R$ is a quasi periodic ring without divisors of zero, one can immediately deduce that it is commutative. The set $E$ cannot be the singleton $\{0\}$. In fact, if this was the case, then $R$ is a radical extension of $\{0\}$, thus $R = \{0\}$. Moreover, every nonzero idempotent is a neutral element for $R$, it follows that $R$ possesses a neutral element $1$ and $\langle E \rangle = \langle 1 \rangle = \mathbb{Z} \cdot 1$. For the remainder of the proof, consider two disjoint cases:

Case 1. $\langle 1 \rangle$ is finite, that is $R$ has a nonzero characteristic. The subring $\langle 1 \rangle$ is finite, thus a periodic subring, therefore, $R$ radical extension of a periodic subring, is also periodic. But a periodic integral domain with a unity is necessarily a periodic field.

Case 2. $\langle 1 \rangle$ is infinite so that it is isomorphic to $\mathbb{Z}$. This case is immediate.

Conversely, if $R$ is a periodic field there is nothing to show, if $R$ satisfies the conditions (i) it is obviously a quasi periodic ring and does not possess divisors of zero. In fact, $R$ is commutative (Theorem 2.). The relation $ab = 0$ implies $a^n b^m = 0$ for every $n$ and $m$, but for some $n$, $m$, $k$, $t$ we have $a^n = m \cdot 1$ and $b^k = k \cdot 1$ so that $mk \cdot 1 = 0$, therefore $a^n = 0$ or $b^k = 0$, so that $a = 0$ or $b = 0$.

2. Quasi Periodic Rings with a Characteristic $\neq 0$

The following theorem determine those quasi periodic rings which are periodic under the assumption that the idempotents commute pairwise.

Theorem 4. In order for a ring, in which the idempotents commute pairwise, to be periodic, it is necessary and sufficient that it is a quasi periodic ring in which every idempotent generates a finite subring.

Necessity. Obviously, a periodic ring is quasi periodic. Let $e$ be an idempotent, the subring $\langle e \rangle$ is isomorphic to $\mathbb{Z}/\#\langle e \rangle$ so that $\mathbb{Z}/\#\langle e \rangle$ is periodic, but $\mathbb{Z}$ is not periodic, therefore $\langle e \rangle$ is finite.

Sufficiency. It suffices to prove that $\langle E \rangle$ is periodic. Let $x \in \langle E \rangle$. For some sequence $(e_i)_{i}$ of $E$ we have $x = \sum_{i=1}^{p} e_i \cdot e_{i}$. The power of $x$ of exponent $n$ shall be:

$$x^n = \sum_{e=1}^{p} c_{e,n} e_{e}^{*} \tag{1}$$

\#A, where $A$ is a set, denotes the cardinal number of $A$ if it is finite, and 0 in the contrary case.
where

\[ c_{1,n} = 1 \forall n; \quad c_{s,n} = \sum_{i_1 + \cdots + i_s = n, i_j \neq 0} \frac{n!}{i_1! \cdots i_s!} = s^n - \sum_{k=1}^{s-1} \binom{s}{k} c_{k,n}; \]

\[ e_s = \sum_{K \subseteq \{1, \ldots, p\} \mid \#K = s} \prod_{i \in K} e_i. \]

Now, every term \( e_i \) generates the finite subring \( \langle e_i \rangle \) so that for every couple \( s, i \in \{1, \ldots, p\} \), \( e_{i,s} = s \cdot e_i \) generates a finite subsemigroup. Hence, there exists \( n_{i,s} \) and \( m_{i,s} \) such that

\[ e^{n_{i,s} + m_{i,s}} = e^{m_{i,s}}; \quad (2) \]

let \( n \) and \( m \) be respectively the lowest common multiple of the \( n_{i,s}, m_{i,s} \) where \((i, s)\) run over \( \{1, \ldots, p\}^2 \). It is easy to show that

\[ e^{n+m} = e^m \quad \forall i, s \in \{1, \ldots, p\}. \quad (3) \]

Assuming that for some \( s > 1 \) we have

\[ s^ne_i = s^{n+m}e_i \quad \forall i \in \{1, \ldots, p\} \]
\[ c_{t,n}e_i = c_{t,n+m}e_i \quad \forall t = 1, 2, \ldots, s - 1 \quad (4) \]

once can deduce:

\[ c_{n,n}e_i = (s^n - \sum_{k=1}^{s-1} \binom{s}{k} c_{k,n}) e_i \]
\[ = s^ne_i - \sum_{k} \binom{s}{k} c_{k,n}e_i \]
\[ = s^{n+m}e_i - \sum_{k} \binom{s}{k} c_{k,n+m}e_i \]
\[ = c_{n,n+m}e_i \quad \forall i \in \{1, \ldots, p\}. \]

By (3), \( c_{2,n}e_i = c_{2,n+m}e_i \forall i = 1, \ldots, p \). By induction on \( s \) we have \( c_{s,n}e_i = c_{s,n+m}e_i \forall i, s \in \{1, \ldots, p\} \) so that \( x^n \) and \( x^{n+m} \) coincide for every \( x \in \langle E \rangle \).

As a corollary of Theorem 4, we shall now determine those rings in which (F) every element \( x \) is equal to a certain power \( x^n (n > 1) \).
Theorem 5. In order for a ring $R$ to satisfy the condition\(^5\)
\[ \forall x \exists n > 1x = x^n, \]
\[ (F) \]
it is necessary and sufficient to be a reduced quasi periodic ring in which every idempotent generates a finite subring.

Proof. Clearly, the condition $(F)$ implies the conditions in the statement. Conversely, by Theorem 4, $R$ is periodic. By Theorem 2, $R$ is a subdirect of integral domains. By Theorem 3, the factors are periodic fields. Therefore, the relation $x^n = x^{n+m}$ carried over every component will bring $x = x^{m+1}$.

Remark. One can show easily that the underlined conditions above are independent.

3. Quasi Periodic Rings With a Unique Idempotent

More generally, we begin by examining the condition:

(C) \[ (\forall e, f \in E)(efe = fe). \]

Note that $(C)$ is equivalent to

(C') \[ u \cdot \langle E \rangle \supseteq \langle E \rangle \cdot u \quad \forall u \in \langle E \rangle. \]

In [4], G. Thierrin has shown that every primitive ring $R$ which is a radical extension of a certain subring $B \neq R$, is a (commutative) field provided that:

(i) \[ uR \supseteq Bu \quad \forall u \in R \]
or that

(ii) \[ uR \supseteq Ru \quad \forall u \in B. \]

Note that in case of the equality $B = R$, $R$ is a division ring. From this result one can show immediately:

Theorem 6. A primitive quasi periodic field which satisfies $(C)$ is a periodic field.

Proof. The ring is a division ring. By Theorem 1, it is a field.

\(^5\) Condition $(F)$ has been studied by N. Jacobson, he showed that it implies the commutativity [see, Structure of Rings, p. 217]. Many other authors among which, I. Herstein, L. Lesieur, has given a variant of his proof.
As any homomorphic image of a quasi periodic ring with the condition (C) is also a quasi periodic ring with condition (C) we shall have:

**Theorem 7.** In order for a quasi periodic semisimple ring $R$ to satisfy (C), it is necessary (and sufficient) that it is a reduced commutative ring.

**Proof.** By Theorem 6, the ring is a subdirect product of (periodic) fields. From Theorem 7 we shall now deduce:

**Theorem 8.** In order for a quasi periodic ring $R$ to possess a unique nonzero idempotent, it is necessary and sufficient that it is an integral domain modulo the radical and that the idempotents commute pairwise.

**Proof of the necessity.** Obviously, $R$ satisfies (C). If $\mathcal{R}$ is the radical, $R/\mathcal{R}$ satisfies (C) also. By Theorem 7, $R/\mathcal{R}$ is reduced and commutative, it follows that every intersection $I$ of onesided modular maximal ideals of $R$ is completely semi-prime (i.e. $x^n \in I \Rightarrow x \in I$), in particular $\mathcal{R}$ is completely semi-prime, therefore, every nilpotent element is in $\mathcal{R}$.

No onesided modular maximal ideal $I$ can contain the nonzero idempotent $e$. In fact, if this was not the case, then $R$ is radical extension of $\langle e \rangle \subseteq I$ and $I$ is completely semi-prime, so that $R = I$.

We are now ready to prove that every element $x$ in the radical is a nilpotent element, so that we will prove that the radical coincides with the set of nilpotent elements.

**Case 1.** $\langle e \rangle$ is infinite. Let $x \in \mathcal{R}$. For some $n, m$ we have

$$y = x^n = m \cdot e \in \mathcal{R}.$$  

The set $(1 - y) R = \{t - yt; t \in R\}$ is a right modular ideal with the element $e - ye = e - y (ye = mee = me = y)$. If a right modular maximal ideal $I$ contains $(1 - y) R$, then it contains $\mathcal{R} + (1 - y) R$ and $e \in I$, which is impossible, therefore, $(1 - y) R = R$. Hence, for some $u \in R$ we have $u - yu = e$, but

$$e = e^2 = eu - eyu = eu - e(m \cdot e)u = eu - m(eu) = (1 - m) eu$$

and for some $r, s$ we have $(eu)^r = s \cdot e$, so that

$$e = ((1 - m) eu)^r = (1 - m)^r (eu)^r = (1 - m)^r s \cdot e$$

it follows that $m = 0$ and $x^n = m \cdot e = 0$.

**Case 2.** $\langle e \rangle$ is finite, thus periodic. $R$, which is a radical extension of $\langle e \rangle$ is periodic. But $\mathcal{R} \cap E = (0)$, therefore $\mathcal{R}$ is a nil-ideal.
Now, we shall prove that $\mathcal{R}$ is completely prime (that is, $x \cdot y \in \mathcal{R} \Rightarrow x \in \mathcal{R}$ or $y \in \mathcal{R}$)

Case 1. Let $xy \in \mathcal{R}$. For some $k, k', t, t'$ we have $x^k = t \cdot e, y^{k'} = t' \cdot e$, so that

$$tt' \cdot e = te \cdot t' \cdot e = y^k y^{k'} \in \mathcal{R}$$

therefore, $tt' \cdot e$ is nilpotent, this shows that $tt' = 0$ in other words $x$ or $y$ are in $\mathcal{R}$.

Case 2. Let $xy \in \mathcal{R}$. Again if $x_1 = x^k = t \cdot e, y_1 = y^{k'} = t' \cdot e$ we shall have

$$x_1 y_1 \in \mathcal{R}$$

Let $c$ be the order of $\langle e \rangle$. For some $r, r', x_1^r, y_1^{r'} \in E = \{e, 0\}$. If $x_1^r$ and $y_1^{r'}$ were both $\neq 0$, then $x_1^r = y_1^{r'} = e$ so that

$$t^r \cdot e = (te)^r = x_1^r = e$$
$$t'^{r'} \cdot e = (t' \cdot e)^{r'} = y_1^{r'} = e;$$

iff

$$t^r \equiv 1 \pmod{1}$$
$$t'^{r'} \equiv 1 \pmod{c}$$

consequently $c$ and $t, c$ and $t'$ are relatively prime, so that $c$ is not a divisor of $tt'$. But from $x_1 y_1 \in \mathcal{R}$ it follows that for some $k$ $$(x_1 y_1)^k = 0$$

this is contradictory to the fact that $e$ is not, again, a divisor of $tt'$. Thus $x_1^r = 0$ or $y_1^{r'} = 0$, hence $x$ or $y$ are in $\mathcal{R}$.

It remains to show that $\mathcal{R}/\mathcal{R}$ possesses a unit. In fact, the idempotent $e$ is not in $\mathcal{R}$ and $\mathcal{R}/\mathcal{R}$ does not possess divisors of zero.

Proof of the sufficiency. The quotient ring $\mathcal{R}/\mathcal{R}$ possesses by definition a unit $e^* + \mathcal{R} \neq \mathcal{R}$. By the well-known lifting idempotent property, there exists an idempotent $e$ in $\mathcal{R}$ congruent to $e^*$ modulo $\mathcal{R}$. Since $e^* \notin \mathcal{R}$ we have $e \notin \mathcal{R}$ and $e \neq 0$. Let $f$ be a non zero idempotent, since $\mathcal{R}/\mathcal{R}$ has a
unique non zero idempotent, we have \( f = e(\mathcal{R}) \), that is \( f - e \in \mathcal{R} \), so that \( fe - e \in \mathcal{R} \) but

\[
(fe - e)^2 = (fe - e)(fe - e) = fe - fe - fe + e = e - fe
\]

so that \( e = fe \), similarly \( f = ef \), therefore \( e = f \). This shows the theorem.

*Remark.* Instead of the condition that the idempotent commute pairwise, we can take the condition \((c')\) together with the condition that the binary relation on \( E \) defined by:

\[
e \leq f \iff e \cdot fe
\]

is antisymmetric.

**Corollary 1.** In order for a quasi periodic ring to be local, it is necessary and sufficient that it is a periodic ring with a unique nonzero idempotent.

*Proof of the necessity.* The quotient ring \( R/\mathcal{R} \) is quasi periodic and a division ring so that it is a periodic field, it follows that for some integer \( n \geq 0 \), if \( e \) is the nonzero idempotent, we have \( n \cdot e \in \mathcal{R} \), but \( \mathcal{R} \) is a nil-ideal, therefore \( n \cdot e \) is nilpotent, this shows that \( \langle e \rangle \) is finite and \( R \) is periodic.

**Corollary 2.** In order for a quasi periodic ring to be an integral domain, it is necessary and sufficient that it is a reduced ring with a unique nonzero idempotent.

4. **Quasi Periodic Rings in Which Every Idempotent is Central**

As every ring is a subdirect product of subdirectly irreducible rings and that every homomorphic image of a quasi periodic ring is again a quasi periodic ring we shall have:

**Theorem 8.** Every quasi periodic ring is a subdirect product of subdirectly irreducible quasi periodic rings.

If \( R \) is a ring in which every idempotent is central, then every homomorphic image \( \overline{R} \) of \( R \) satisfies the same condition. For if \( \overline{e} \) is an idempotent of \( \overline{R} \), there exists \( x \in R \) such that \( \overline{e} = f(x) = f(x^n) = \cdots = f(x^n) = \cdots \), and for some \( n \), \( x^n \) is a sum of idempotents. From this result one can show:

**Theorem 9.** In order for a quasi periodic ring to be such that every idempotent is central, it is necessary and sufficient that is a subdirect product of nil
rings or quasi periodic unitary rings without proper idempotents (that is, idempotents different of the trivial idempotents 0 and 1).

Necessity. Let: $R \rightarrow \prod R_s$ be a subdirectly irreducible representation of $R$. Each $R_s$ is homomorphic to $R$, hence, its idempotents are central. On the other hand, if $R_s$ is a subdirectly irreducible ring, every idempotent $e_s$ such that

$$\forall y \in R_s, y = xe_s, y = xe_s$$

in particular a central idempotent, is necessarily trivial. (In fact, the mapping $x \in R_s \mapsto (xe_s, x - xe_s)$ is, then, an isomorphism of $R_s$ onto $R_s \cdot e_s \times R_s (1 - e_s)$ ($R_s \cdot e_s = \{x - xe_s, x \in R_s\}$) by the left Pierce decomposition, so that if $e_s \neq 0$, the natural epimorphism $x \mapsto x - xe_s$ is not a monomorphism

$$(0 = e_s - e_s e_s = 0 - e_s, 0),$$

hence the natural epimorphism $x \mapsto xe_s$ is a monomorphism, that is, $e_s$ is a right unity and, similarly, a left unity, thus the neutral element of $R$. Therefore, either $R_s$ is a quasi periodic ring with the unique idempotent 0, that is, $R_s$ is a nilring, or, $R_s$ is a unitary quasi periodic ring without proper idempotents.

Sufficiency. Let: $R \rightarrow \prod R_s$ be a representation of $R$ as in the statement above. Clearly, in every $R_s$ the idempotents are central, so that every subring of the product satisfies the same condition, hence $R$ satisfies the condition. We assume henceforth that $R$ is a unitary quasi periodic ring in which every idempotent is central. By the preceding result, $R$ possesses a faithful representation: $R \rightarrow \prod R_s$ such that each $R_s$ is a unitary quasi periodic ring without proper idempotents. We fix the representation throughout the remainder of this article and we shall be mainly concerned here with the determination of the group of the invertible elements and the radical of $R$.

**Theorem 10.** Every invertible element of $R$ is a root of the unity.

**Proof.** Let $x$ be an invertible element of $R$. For some $n$ and some sequence of idempotents $(e_i)_{i \leq n}$, we have

$$x^n = \sum_{i=1}^{n} e_i.$$

Let $y_s$ denote the projection of the element $y$ on the component $R_s$. Since
every idempotent in $R_\alpha$ is trivial and that the projection of an idempotent is an idempotent, for every subscript $\alpha$ there exists $m'(\alpha) \leq m$ such that

$$x_\alpha^n = m'(\alpha) \cdot 1_\alpha.$$ 

Now, $x^{-1}$ is also of the form:

$$(x^{-1})^r = \sum_{j=1}^n f_j; \quad f_j \in E,$$

so that

$$x^{-r} = s'(\alpha) \cdot 1_\alpha; \quad s'(\alpha) \leq s.$$ 

Now, put $l = (m' \cdot s^n)_n$, we shall prove that $x^{nl} = 1_\alpha$ for every subscript $\alpha$, so that $x$ has a finite order divisor of $n\mathbb{Z}(\neq 0)$. In fact, if $\langle 1_\alpha \rangle$ is infinite, then

$$(1) \quad 1_\alpha = x^{nr} \cdot x^{-nr} = m' \cdot s^n \cdot 1_\alpha$$

imply $m' = 1$, so that $x^{nl} = 1_\alpha$, hence $x^{nl} = 1_\alpha$. If $\langle 1_\alpha \rangle$ is finite. Let $c_\alpha$ be the cardinality of $\langle 1_\alpha \rangle$. From (1), it follows that $m' \cdot s^n \equiv 1$ (mod $c_\alpha$). If $m' = 1$, there is nothing to show, if $m' \neq 1$, than $m' > 1$, so that

$$c_\alpha \cdot m' \cdot s^n - 1 \neq 0$$

hence,

$$c_\alpha \leq m' \cdot s^n.$$ 

Now, $y_\alpha = x^{nl} = m' \cdot 1_\alpha \in \langle 1_\alpha \rangle$ generates a finite multiplicative semi group of order less or equal to $c_\alpha$, hence, its order is a divisor of $c_\alpha$! which is a divisor of $(m' \cdot s^n)! = l$, therefore, $x^{nl} = y_\alpha = 1_\alpha$.

**Theorem 11.** The radical of $R$ coincides with the set of nilpotent elements.

*Proof.* $R$ satisfies obviously to the condition (C) in Section 5.

By Theorem 7, $R/\mathcal{R}$ is reduced, that is, $x^n \in \mathcal{R} \Rightarrow x \in \mathcal{R}$, hence every nilpotent element is in $\mathcal{R}$. Conversely, let $x \in \mathcal{R}$ and let $\mathcal{R}_\alpha$ be the radical of $R_\alpha$. For some $n$, $m$ we have

$$y = x^n \in \langle E \rangle \cap \mathcal{R},$$

since $y$ is quasi-regular, there exists $y'$ such that

$$yy' + y + y' = 0,$$

therefore,

$$(y + 1)y' = -y$$
Now, \( y \in \mathcal{R} \), thus \( 1 + y \) is invertible. By Theorem 10, \( 1 + y \) has a finite order \( k \); it follows that

\[
y' = -(y + 1)^{-1} y
= -(y + 1)^{-1} (y + 1 - 1)
= (y + 1)^{-1} - 1
= (1 + y)^{k-1} - 1
= \sum_{j \neq 0} \binom{k}{j} y^j.
\]

Put \( r = (\sum_{i \neq 0} \binom{k-1}{i-1}) m^i \), where \( m \) is such that

\[
y = x^n = \sum_{i=1}^{m} e_i \quad (e_i \in E),
\]

and \( l = (rm + m + r) \); we shall prove that \( x_n^\alpha = 0 \) \( \forall \alpha \), so that \( x \) is a nilpotent element. In fact, for a fixed subscript \( \alpha \) we have

\[
y_{\alpha} = m'(\alpha) \cdot 1_{\alpha} ; \quad m'(\alpha) \leq m
\]

\[
y_{\alpha}' = \left( \sum_{i \neq 0} \binom{k}{j} m^i \right) 1_{\alpha} = r'(\alpha) \cdot 1_{\alpha} ; \quad r'(\alpha) \leq r
\]

so that

\[
(m'r' + m' + r') 1_{\alpha} = y_{\alpha} y_{\alpha}' + y_{\alpha} + y_{\alpha}' = 0.
\]

In characteristic 0, this implies \( m' = 0 \), thus \( x_{\alpha} n = y_{\alpha} = 0 \) and \( x_{\alpha} n = (x_{\alpha} n)^i = 0 \). In characteristic \( c_{\alpha} \neq 0 \), we obtain

\[
1 \equiv m'r' + m' + r' (\text{mod } c_{\alpha}).
\]

If \( m' \neq 0 \), then \( c_{\alpha} \cdot m'r' + m' + r' - 1 \neq 0 \), hence \( c_{\alpha} \leq m'r' + m' + r' \).

By Theorem 8, \( y_{\alpha} \in \mathcal{R}_{\alpha} \) is a nilpotent element; let \( h_{\alpha} \) be the first integer such that

\[
y_{\alpha}^{h_{\alpha}} = m'^{h_{\alpha}} \cdot 1_{\alpha} = 0,
\]

we have \( c_{\alpha} \mid m'^{h_{\alpha}} \) and

\[
c_{\alpha} \leq m'r' + m' + r' \leq mr + m + r = l,
\]

consequently every primary factor of \( c_{\alpha} \) has an exponent no greater than
$l$, so that $c_{\alpha} \mid m^{l}$, since $h$ is the first integer such that $c_{\alpha} \mid m^{h_{\alpha}}$, we have $h_{\alpha} \leq l$ so that

$$m^{l} \cdot l_{\alpha} = m^{h_{\alpha}} \cdot m^{l-h_{\alpha}} \cdot l_{\alpha} = 0 \cdot m^{l-h_{\alpha}} = 0.$$ 

**Remark.** A simple computation proves that the characteristic in each $R_{\alpha}$ is either 0 or a primary number.

**Theorem 12.** If $R$ is two-sided artinian, then it is a product of local rings; it follows that $R$ is periodic.

**Proof.** By the so-called chinese lemma (adapted to the noncommutative case) there exists a subdirectly irreducible exact representation: $R \leftrightarrow \prod_{i=1}^{n} R_{i}$ with a finite number of components. Each $R_{i}$ is homomorphic to $R$, hence a two-sided artinian ring. Since $R_{i}$ is a duo\(^6\) ring modulo the radical and that $R_{i}$ is artinian modulo the radical, it is an integral domain artinian modulo the radical, thus a local ring, hence periodic (Theorem 8 Corollary 1), but a finite product of periodic rings is periodic, therefore, $R$ is periodic.

**References**


\(^6\) Following G. Thierrin, a duo ring is a ring in which every one-sided ideal is a two-sided ideal.