The Cayley-Hilbert Metric and Positive Operators

P. J. Bushell*
University of Sussex
Falmer, Brighton, Sussex, England

Submitted by H. Amann

ABSTRACT

The Cayley-Hilbert metric is defined for a real Banach space containing a closed cone. By restricting the domain of a particular type of positive nonlinear operator, the Banach contraction-mapping theorem is used to prove the existence of a unique fixed point of the operator with explicit upper and lower bounds. Applications to quasi-linear elliptic partial differential equations and to matrix theory are considered.

1. INTRODUCTION

The Cayley-Hilbert metric is particularly useful in proving the existence of a unique fixed point for a positive homogeneous operator defined in a Banach space. Elementary accounts of the general theory may be found in Krasnosel'skii, Vainikko, Zabreiko, Rutitskii, and Stetsenko [7] and in Bushell [2]. In [3] the author gave some applications of the theory to the solution of a class of Fredholm and Volterra integral equations. Closely related general theory and applications are given by Potter [12].

In Sections 2 and 3 of the present paper we consider further extensions of the theory. In particular, we do not assume that the cone has a nonempty interior, and we consider the case of a self-mapping of a subset of the

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boundary of the cone. This allows us to study Volterra integral equations without using the rather unwieldy weighted norms used in [3]. Moreover, the method provides explicit upper and lower bounds for a fixed point of a class of nonlinear mappings and asserts that only one fixed point can satisfy such bounds.

The main result is given in Theorem 3.1. In Section 4 we illustrate the use of the projective metric by considering two simple quasilinear elliptic partial differential equations, and in Section 5 we give two results from matrix theory.

2. PROJECTIVE METRIC

Let X be a real Banach space, and let K be a closed cone in X. Let $\leqslant$ denote the usual induced partial ordering in X defined by $x \leqslant y$ if and only if $y - x \in K$. If $x, y \in K^+ = K \setminus \{0\}$, let

$$M(x, y) = \inf\{\lambda \in \mathbb{R} : x \leqslant \lambda y\}, \text{ or } \infty \text{ if the set is empty,}$$

and

$$m(x, y) = \sup\{\mu \in \mathbb{R} : \mu y \leqslant x\}.$$

It is easy to see that

$$0 \leqslant m(x, y) \leqslant M(x, y) \leqslant \infty \tag{2.1}$$

and

$$m(x, y) y \leqslant x \leqslant M(x, y) y. \tag{2.2}$$

Definition. The Cayley-Hilbert projective metric is defined in K^+ by

$$d(x, y) = \log \frac{M(x, y)}{m(x, y)}.$$

If $g \in K$ and $\|g\| = 1$ we define

$$K_g = \{x \in K : d(x, g) < \infty\}$$

and

$$E_g = \{ x \in K_g : \|x\| = 1 \}.$$

THEOREM 2.1. *$\{K_g, d\}$ is a pseudometric space, and $\{E_g, d\}$ is a metric space.*

Proof. It follows from

$$\frac{m(x,g)}{M(y,g)} y \leqslant m(x,g)g \leqslant x$$

and

$$x \leqslant M(x,g)g \leqslant \frac{M(x,g)}{m(y,g)} y$$

that

$$0 \leqslant d(x,y) \leqslant d(x,g) + d(y,g) < \infty.$$

The remainder of the proof is straightforward (see Bushell [2]). ■

THEOREM 2.2. *Let the norm in X be monotonic, that is, $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant \|y\|$. Then $\{E_g, d\}$ is a complete metric space.*

Proof. We note that for $x, y \in E_g$,

$$0 < m(x,y) \leqslant 1 \leqslant M(x,y) < \infty, \tag{2.3}$$

since $0 \leqslant m(x,y)y \leqslant x$ implies $m(x,y)\|y\| \leqslant \|x\|$, that is, $m(x,y) \leqslant 1$, and similarly $1 \leqslant M(x,y)$. Then

$$x - y \leqslant \{M(x,y) - m(x,y)\} y,$$

and hence

$$\|x - y\| \leqslant |M(x,y) - m(x,y)| \leqslant \exp\{d(x,y)\} - 1. \tag{2.4}$$

Let $\{x_n\}$ be a Cauchy sequence in $\{E_g, d\}$. From (2.4), $\{x_n\}$ is a Cauchy sequence in X and hence converges in norm to $x \in K$ with $\|x\| = 1$. Now

$$d(x_n, g) \leqslant d(x_n, x_m) + d(x_m, g)$$

implies that

$$1 \leqslant \frac{M(x_n, g)}{m(x_n, g)} \leqslant c < \infty$$

for $n > m$, and hence

$$c^{-1} \leqslant m(x_n, g) \leqslant 1 \leqslant M(x_n, g) \leqslant c.$$

Therefore, for $n > m$,

$$c^{-1}g \leqslant x_n \leqslant cg,$$

and letting $n \to \infty$, we see that $d(x, g) \leqslant \log c^2 < \infty$.

Similarly, $d(x_n, x_m) < \varepsilon$ gives $d(x, x_n) \leqslant 2\varepsilon$, and hence $\{x_n\}$ converges in $\{E_g, d\}$.

If $g \in \mathring{K}$, the interior of the cone, then K_g coincides with $\mathring{K}$, the case considered at length in Bushell [2]. The definition of the projective metric in more general settings is discussed by Kohlberg and Pratt [6] and Turinici [13].

3. NONLINEAR MAPPINGS

In this section we consider the class $\mathscr{F}$ of nonlinear mappings $T: K \to K$, which are such that

(1) $0 \leqslant x \leqslant y$ implies $Tx \leqslant Ty$, and

(2) for some p, $0 < p < 1$, if $0 \leqslant x$ and λ is a positive real number, then $T(\lambda x) = \lambda^p Tx$.

Moreover, we assume throughout the section that the norm in X is monotonic.

THEOREM 3.1. *If there exists* $g \in K$ *with* $\|g\| = 1$ *such that* $d(g, Tg) < \infty$, *then* $T \in \mathcal{F}$ *has a unique fixed point* z *in* K_g, *and*

$$\{m(Tg, g)\}^{1/(1-p)} g \leqslant z \leqslant \{M(Tg, g)\}^{1/(1-p)} g.$$

Proof. Let $x \in E_g$. Then

$$m(x, g) g \leqslant x \leqslant M(x, g) g$$

and hence

$$\{m(x, g)\}^p m(Tg, g) g \leqslant Tx \leqslant \{M(x, g)\}^p M(Tg, g) g.$$

Therefore,

$$d(Tx, g) \leqslant d(Tg, g) + p d(x, g) < \infty;$$

that is,

$$T: E_g \to K_g.$$

If $x, y \in E_g$, applying T to (2.2) gives

$$d(Tx, Ty) \leqslant p d(x, y).$$

Let $F(x) = Tx / \|Tx\|$; the $F: E_g \to E_g$ and F is the composition of a strict contraction and a normalizing isometry. By the Banach contraction-mapping theorem there is a unique x in E_g such that $F(x) = x$, and if we set $z = \|Tx\|^{1/(1-p)} x$, the existence of the unique fixed point of T in K_g follows easily.

Finally,

$$z = Tz \leqslant T\{M(z, g) g\} \leqslant \{M(z, g)\}^p M(Tg, g) g,$$

and hence

$$M(z, g) \leqslant \{M(z, g)\}^p M(Tg, g).$$

the remainder of the proof is clear. ∎

4. EXAMPLES

We illustrate the use of the results of Section 3 by considering two nonlinear elliptic partial differential equations and related boundary value problems.

EXAMPLE 1. Kawano, Kusano, and Naito [5] consider the two-dimensional elliptic equation

$$\Delta u = \phi(x)u^p, \tag{4.1}$$

where $x \in \mathbb{R}^2$ and $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. They prove the existence of positive solutions with logarithmic growth at infinity for certain types of functions $\phi: \mathbb{R}^2 \to \mathbb{R}$. Assuming solutions to be functions of $r = \sqrt{x_1^2 + x_2^2}$, they reduce the partial differential equation to the ordinary differential equation

$$y'' + \frac{1}{t}y' = G(t)y^p \tag{4.2}$$

and then to the integral equation

$$y(t) = c + \int_0^t s\log\left(\frac{t}{s}\right)G(s)[y(s)]^p\,ds, \qquad t \geqslant 0. \tag{4.3}$$

We take $c = 0$, and we assume that

(1) G is continuous in $[0, \infty)$, and
(2) $0 < \alpha \leqslant G(t) \leqslant \beta < \infty$ for $0 \leqslant t < \infty$.

Let $X = C[0, R]$, with $R > 0$, and let K be the cone of nonnegative functions in X. Let

$$Ty(t) = \int_0^t s\log\left(\frac{t}{s}\right)G(s)[y(s)]^p\,ds, \tag{4.4}$$

and suppose that $0 < p < 1$.

Let $g(t) = (t/R)^{2/(1-p)}$; then

$$R^2\alpha\left(\frac{1-p}{2}\right)^2 g \leqslant Tg \leqslant R^2\beta\left(\frac{1-p}{2}\right)^2 g.$$

From Theorem 3.1, Equation (4.1) has a unique solution such that

$$\alpha^{1/(1-p)}\left(\frac{1-p}{2}\right)^{2/(1-p)} t^{2/(1-p)} \leqslant y(t) \leqslant \beta^{1/(1-p)}\left(\frac{1-p}{2}\right)^{2/(1-p)} t^{2/(1-p)}$$

for $0 \leqslant t < \infty$.

Thus, if $0 < p < 1$ and if ϕ is a positive bounded continuous function in $\mathbb{R}^2$, then Equation (4.1) has a solution that grows like $|x|^{2/(1-p)}$ as $|x| \to \infty$.

The existence of the solutions with logarithmic growth is proved in [5] by appeals to the Schauder-Tychonoff fixed-point theorem, but if $0 < p < 1$, these results follow more readily from Theorem 2.3 of Bushell [3].

EXAMPLE 2. Let Ω be a bounded domain in $\mathbb{R}^n$. Many authors have studied special cases of the boundary-value problem

$$-\Delta u = f(u) \quad \text{in } \Omega, \qquad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

(See the review article by Lions [8] and the more recent papers by Brézis and Nirenberg [1] and Ni and Sacks [11], as well as the many references therein.)

Let $f(u) = u^p$ with $0 < p < 1$, and let Ω be the open ball with center the origin and radius unity. Assuming solutions to be functions of r, the radial distance from the origin, the boundary-value problem reduces to

$$-u''(r) - \frac{n-1}{r}u'(r) = \{u(r)\}^p, \qquad 0 < r < 1,$$

and $u(1) = 0$. We suppose that $n \geqslant 3$.

The Green's function for this problem is

$$G(r,s) = \begin{cases} \dfrac{1}{n-2}\left[s\left(\dfrac{s}{r}\right)^{n-2} - s^{n-2}\right], & 0 < s \leqslant r, \\ \dfrac{1}{n-2}\left[s - s^{n-2}\right], & r \leqslant s < 1. \end{cases}$$

We seek a fixed point of the map

$$Tu(r) = \int_0^1 G(r,s)[u(s)]^p \, ds$$

in the cone of nonnegative functions in $C[0,1]$. Let

$$Au(r) = \int_0^1 G(r,s)u(s)\,ds,$$

and let

$$g(r) = 1 - r^2.$$

Then

$$Tg(r) = A(1-r^2)^p \leqslant A(1) = \frac{1-r^2}{2n}$$

and

$$Tg(r) = A(1-r^2)^p \geqslant A(1-r^2)$$

$$= \frac{(1-r^2)\{4+n(1-r^2)\}}{4n(n+2)}.$$

Thus

$$\frac{g(r)}{n(n+2)} \leqslant Tg(r) \leqslant \frac{g(r)}{2n}.$$

From Theorem 3.1, if $0 < p < 1$, there is a unique solution of $-\Delta u = u^p$ in $B(0,1)$, $u > 0$ in $B(0,1)$, $u = 0$ when $r = 1$, such that

$$\left(\frac{1}{n(n+2)}\right)^{1/(1-p)}(1-r^2) \leqslant u(r) \leqslant (1/2n)^{1/(1-p)}(1-r^2).$$

In such special problems involving monotonic homogeneous mappings in a cone, perhaps a projective metric technique is to be preferred to more general methods such as Leray-Schauder theory or Krasnosel'ski's concave-operator theory. The technique yields bounds for the solution and the certainty of uniqueness within a given set of possible functions.

Applications to the porous-media equation, involving more lengthy analysis, will appear elsewhere.

5. RESULTS FROM MATRIX THEORY

In this section X denotes the real Banach space of real $n \times n$ symmetric matrices, and K the cone of positive semidefinite matrices in K. The $\mathring{K}$, the interior of K, is the set of positive definite matrices in X. It was shown in Bushell [4] that for $A, B \in \mathring{K}$,

$$d(A, B) = \log \frac{\lambda_M(B^{-1}A)}{\lambda_m(B^{-1}A)},$$

where $\lambda_M(C)$ and $\lambda_m(C)$ denote the maximum and minimum eigenvalues of C, respectively.

Example 1. Let T be a real nonsingular $n \times n$ matrix, and let $\alpha \in \mathbb{R}$, $\alpha \neq \pm 1$. A straightforward extension of the argument given in [4] shows that there is a unique positive definite matrix A such that $T'AT = A^\alpha$. The cases $0 < \alpha < 1$ and $1 < \alpha < \infty$ must be considered separately, making use of Loewner's theorem [9] to observe that $A \leqslant B$ implies $A^\alpha \leqslant B^\alpha$ when $0 < \alpha < 1$. The case of negative α can be dealt with by noting that the map $A \mapsto A^{-1}$ is a projective isometry in $\mathring{K}$.

Example 2. Let T be a real nonsingular $n \times n$ matrix. For $A \in \mathring{K}$ let $\tilde{A}$ denote the $(n-1)$th compound of A as defined in Marshall and Olkin [10, Chapter 16]. Let $m \geqslant n$ and let $F(A) = (T'\tilde{A}T)^{1/m}$. Then $F : \mathring{K} \to \mathring{K}$, F is monotone increasing in $\mathring{K}$, and $F(\lambda A) = \lambda^{(n-1)/m} F(A)$ (see [10] for proofs of these assertions).

Thus, for each $m \geqslant n$, there exists a unique positive definite $n \times n$ matrix A such that $T'\tilde{A}T = A^m$.

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