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An efficient fault-tolerant routing algorithm in bijective connection networks with restricted faulty edges

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ABSTRACT

In this paper, we study fault-tolerant routing in bijective connection networks with restricted faulty edges. First, we prove that the probability that all the incident edges of an arbitrary node become faulty in an n-dimensional bijective connection network, denoted by X_n, is extremely low when n becomes sufficient large. Then, we give an $O(n)$ algorithm to find a fault-free path of length at most $n + 3\lceil \log_2 |F| \rceil + 1$ between any two different nodes in X_n if each node of X_n has at least one fault-free incident edge and the number of faulty edges is not more than $2n - 3$. In fact, we also for the first time provide an upper bound of the fault diameter of all the bijective connection networks with the restricted faulty edges. Since the family of BC networks contains hypercubes, crossed cubes, Möbius cubes, etc., all the results are appropriate for these cubes.

1. Introduction

1.1. Fault-tolerant routing and conditional connectivity

An interconnection network plays an important role in a large-scale parallel computer system. There are a lot of processors and links in interconnection networks of modern parallel computer systems such that it is unavoidable that the processors and links become faulty in such a system. Therefore, fault-tolerant communication has been being an important issue in interconnection networks with faulty processors or links.

An interconnection network can be represented by a simple graph $G = (V, E)$, where V is the node set and E is the edge set of graph G. In this paper, we use graphs and interconnection networks (networks for short), nodes and processors, and edges and links interchangeably. Fault-tolerant routing is a basic communication mode in interconnection networks with faulty processors or links. Let F denote a set of faulty nodes/edges in G. Given two different nodes u and v in $G - F$, fault-tolerant routing is finding a fault-free path between u and v in $G - F$. Such a fault-free path can be used to transmit data packets between u and v in $G - F$. Clearly, a shorter path between u and v in $G - F$ is desirable because a delay will occur whenever a packet passes through a node. On the other hand, fault-tolerant routing should be completed as fast as possible.

Whether there is a fault-free path between u and v in $G - F$ depends on the node/edge connectivity of G. That is, if the node/edge connectivity $(\kappa(G)/\lambda(G))$ of G is n and $|F| \leq n - 1$, then there always exists a fault-free path between u and v in $G - F$. However, the node/edge connectivity of G is bounded by the minimum node degree $(\delta(G))$ of G. That is, $\kappa(G) \leq \lambda(G) \leq \delta(G)$. In order to break through this bound, Harary proposed the concept of *conditional node/edge connectivity*

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[13]. Given a restricted condition R, the edge connectivity $\lambda'(G : R)$ of G is defined as $\min\{|E'||E' \subset E(G)$ satisfies the condition R and $G - E'$ is disconnected or trivial$\}$ and the node connectivity $\kappa'(G : R)$ of G is defined as $\min\{|V'||V' \subset V(G)$ satisfies the condition R and $G - V'$ is disconnected or trivial$\}$.

1.2. Related work

Esfahanian introduced the concept of conditional node/edge connectivity in hypercubes [3], where he defined the restricted condition as "each node of the n-dimensional hypercube Q_n has at least one fault-free neighbor/incident edge". He proved that under this condition the conditional node/edge connectivity of Q_n becomes $2n-2$, which is almost as twice the node/edge connectivity n of Q_n. That is, if each node of Q_n has at least one fault-free neighbor/incident edge and the number of faulty nodes/edges does not exceed $2n - 3$, there always a fault-free path between any two different nodes in Q_n. We should point out that it does not hold true that the conditional node/edge connectivity of any graph always is greater than its node/edge connectivity if each node of it has at least one fault-free neighbor/incident edge, which can be easily verified when taking a path of length at least 4 as evidence to the contrary. Based on Esfahanian's result, Gu and Peng gave an $O(n)$ algorithm to find a fault-free path of length at most $d(s, t) + 4$ between any two different nodes s and t in Q_n if each node of Q_n has at least one fault-free neighbor and the number of faulty nodes does not exceed $2n - 3$ [24], where $d(s, t)$ is the distance between s and t in Q_n. This algorithm possesses advantageous performance because it can find a fault-free path of length approaching the distance between s and t in Q_n in less time.

So far, the variants of hypercubes, crossed cubes, Möbius cubes, and locally twisted cubes have been proposed [1,2,25,31,32]. They have the same characters as hypercubes. For example, they have the same node number, edge number and node/edge connectivity as hypercubes with the same dimensions. On the other hand, they also have different characteristics from hypercubes. For example, their diameters are about half those of hypercubes with the same dimensions; hypercubes are symmetric and bipartite graphs, while these variants are generally not, etc. These characteristics have made researchers be very interested in them [6,7,10–12,14–23,26–29]. Most of the research on the properties of these variants were respectively carried out based on their specific definitions, which provided very detailed proofs. In fact, there exist two common properties among these variants—bijective connection and recursively constructive nature. By using the two properties, a family of bijective connection networks (BC networks in brief) were defined [4], which not only include the known networks such as hypercubes, crossed cubes, Möbius cubes, locally twisted cubes, etc., but also many other unknown ones. Based on this definition, diagnosability, edge-pancyclicity, and path-embeddability of bijective connection networks were studied in [8,9].

In [30], Xu et al. studied the conditional node/edge connectivity of a family of interconnection networks. By the result in [30], it can be inferred that any n-dimensional BC network X_n has conditional node/edge connectivity $2n-2$. Gu and Peng's algorithm [24] is only appropriate for hypercubes because it is based on the symmetry of hypercubes. In order to solve this problem, [5] gave an $O(n)$ algorithm to find a fault-free path of length at most $n+3\lceil\log_2|F|\rceil+1$ between any two fault-free nodes if each node of X_n has at least one fault-free neighbor and the number of faulty nodes is not more than $2n - 3$ in X_n. However, there is still the other different problem—fault-tolerant routing in X_n with the restricted faulty edges.

1.3. Our contributions

In this paper, we will study the fault-tolerant routing of BC networks under the condition that each node has at least one fault-free incident edge. The major contributions are as follows:

(1) By proving that each minimal edge cut set (the cut of size n) must be the incident edge set of a node in X_n, we prove that the probability that all the incident edges of an arbitrary node become faulty in X_n is extremely low (n becomes sufficient big).

(2) We give an $O(n)$ algorithm to find a fault-free path of length at most $n + 3\lceil\log_2|F|\rceil + 1$ between any two fault-free nodes in an n-dimensional BC network X_n if each node of X_n has at least one fault-free incident edge and the number of faulty edges is not more than $2n - 3$.

It should be pointed out that the smallest upper bound of the fault diameter of all the BC networks has been unknown and the result (2) actually for the first time provides an upper bound of the fault diameter of all the BC networks under the condition of the above restricted faulty edges. On the other hand, in fact, we can use the BFS algorithm to obtain the shortest path between any two different nodes x and y in X_n, but the time complexity of this algorithm is as high as $O(n2^n)$. Therefore, it is important to give a fault-tolerant routing algorithm with a tradeoff between the length of the obtained path between x and y and the time complexity of the algorithm. Our algorithm actually considers this tradeoff—it has lower time complexity $O(n)$ and finds a fault-free path of length n plus a logarithm term. We may conjecture that the smallest upper bound of the fault diameter of all the BC networks with the restricted faulty edges be n plus a constant. Under this circumstance, although our algorithm possibly increases the length of fault-free path between x and y (note that the smallest upper bound of the fault diameter of all the BC networks with the restricted faulty edges has so far been unknown), its time complexity is as low as $O(n)$, which is much smaller than the time complexity $O(n2^n)$ of the BFS algorithm.

The rest of this paper is organized as follows: Section 2 provides some definitions and notations. Section 3 gives an algorithm to find a fault-free path between any two different nodes in a BC network with the restricted faulty edges and the analysis of the algorithm. In Section 4, we give the conclusions.

2. Preliminaries

Given a simple graph G, a path P between nodes u and v in G is defined as a node sequence P: $u = u^{(0)}, u^{(1)}, \ldots, u^{(k)} = v$, where any two nodes are different from each other except the beginning node u and the end node v. We use $\text{rev}(P)$ to denote the path $v = u^{(k)}, u^{(k-1)}, \ldots, u^{(0)} = u$, which is the path to reverse the path P. Let $V(P)$ and $E(P)$ denote the node set and edge set, respectively, in P.

If $V' \subseteq V(G)$, we use $G[V']$ to denote the subgraph of G induced by the node subset V'. Furthermore, we use $G - V'$ to denote $G[V(G) - V']$. For each node $v \in V(G)$, if $(u, v) \in E(G)$, we denote v to be a neighbor of u or v to be adjacent to v; we also denote u to be incident with the edge (u, v) or (u, v) is an incident edge of u. The set of all the neighbors of v is called the neighbor set of v, denoted by $\Gamma(G, v)$, that is,

$$\Gamma(G, v) = \{u \in V(G) | (v, u) \in E(G)\}.$$

Furthermore, for a set of nodes $V' \subseteq V(G)$, we define the neighbor set of V' as

$$\Gamma(G, V') = \bigcup_{x \in V'} \Gamma(G, x) - V'.$$

Moreover, for any $u \in V(G)$, and any $(x, y) \in E(G)$, let $N_e(G, u) = \{(u, v) | (u, v) \in E(G)\}$ and $CoN_e(G, x, y) = (N_e(G, x) \bigcup N_e(G, y)) - E'$.

Given two graphs G' and G'', if there exists a bijection φ from $V(G')$ to $V(G'')$ such that $(u', v') \in E(G')$ if and only if $(\varphi(u'), \varphi(v')) \in E(G'')$ for any two nodes $u', v' \in V(G')$, then we say that G' is *isomorphic* to G'' and φ is an *isomorphic mapping* from G' to G''. If the graphs G' and G'' are two isomorphic graphs, we write $G' \cong G''$. The isomorphic graphs can be regarded as identical graphs.

For an integer $n \geq 1$, a binary string u of length n is denoted by $u_{n-1}u_{n-2}\ldots u_0$, where $u_i \in \{0, 1\}$ for any integer $i = 0, 1, \ldots, n-1$. The ith bit u_i of u can also be written as $\text{bit}(u, i)$. If $x = x_{n-1}x_{n-2}\ldots x_0$ is a binary string of length n, we can use ix to denote the binary string $ix_{n-1}x_{n-2}\ldots x_0$ of length $n+1$ for any integer $i \in \{0, 1\}$. Furthermore, let $U \subseteq \{0, 1\}^n$, that is, U is a set of some binary strings of length n. Then, we use iU to denote the set $\{iu | u \in U\}$ for any integer $i \in \{0, 1\}$. Letting P: $u = u^{(0)}, u^{(1)}, \ldots, u^{(k)} = v$ denote a sequence between u and v, where $u^{(i)}$ is a binary string of length n for any $i \in \{0, 1\}$, we use iP to denote the sequence $iu = iu^{(0)}, iu^{(1)}, \ldots, iu^{(k)} = iv$ for any integer $i \in \{0, 1\}$.

Before introducing the definition of BC networks, we first give the definition of bijective connection in the following [4]:

Definition 1 (*[4]*)**.** Let G be a graph. If $V(G) = V_1 \bigcup V_2$, $V_1 \neq \phi$, $V_2 \neq \phi$, and $V_1 \bigcap V_2 = \phi$. We say that there exists a bijective connection between the subsets V_1 and V_2 in G, denoted by $V_1 \overset{G}{\longleftrightarrow} V_2$, if G satisfies the two following conditions:

(1) For every $u \in V_1$, there exists a unique $v \in V_2$ such that $\{u, v\} \in E(G)$; and
(2) For every $u \in V_2$, there exists a unique $v \in V_1$ such that $\{u, v\} \in E(G)$.

A definition of *bijective connection networks* (in brief, BC networks) without labels in their nodes was given in [4]. In this paper, for the sake of our design of algorithm in Section 3, we adopt the definition of BC networks with labels in their nodes [5]. An n-dimensional BC network, denoted by X_n, is an n-regular graph with 2^n nodes. We identify each node of X_n by a unique binary string of length n. The set of all the n-dimensional BC networks is called the *family of the n-dimensional BC networks*, denoted by $\mathcal{L}_n$. X_n and $\mathcal{L}_n$ may be recursively defined as below.

Definition 2 (*[4]*)**.** The 1-dimensional BC network X_1 is a complete graph on two nodes 0 and 1. The family of the 1-dimensional BC network is defined as $\mathcal{L}_1 = \{X_1\}$. Let G be a graph. G is an n-dimensional BC network, denoted by X_n, if there exist $V_0, V_1 \subset V(G)$ such that the following three conditions hold:

(1) $V_0 = 0V_0'$ and $V_1 = 1V_1'$, where $V_0' = V_1' = \{0, 1\}^{n-1}$;
(2) $V(G) = V_0 \bigcup V_1$, $V_0 \neq \emptyset$, $V_1 \neq \emptyset$, and $V_0 \bigcap V_1 = \emptyset$; and
(3) $V_0 \overset{G}{\longleftrightarrow} V_1$, $G[V_0] \in \mathcal{L}_{n-1}$, and $G[V_1] \in \mathcal{L}_{n-1}$.

The family of the n-dimensional BC networks is defined as $\mathcal{L}_n = \{G | G \text{ is an } n\text{-dimensional BC network}\}$.

Fig. 1 demonstrates two 3-dimensional BC networks with labels, in which (a) is isomorphic to Q_3 and (b) is isomorphic to CQ_3, MQ_3, and TQ_3, respectively. Fig. 2 demonstrates two 4-dimensional BC networks with labels, in which (a) is isomorphic to Q_4 and (b) is isomorphic to CQ_4.

Notation 3. *For any $X_n \in \mathcal{L}_n$ and $i \in \{0, 1\}$, let $H_i = (V_i, E_i)$, where $V(X_{n-1}^i) = iV_i$ and $E_i = \{(u, v) | (iu, iv) \in E(X_{n-1}^i)\}$. By Definition 2, $iV_i \overset{X_n}{\longleftrightarrow} (1-i)V_{1-i}$. Then, X_{n-1}^i can be denoted by iH_i. Furthermore, if $F' \subset E_i$, then we use iF'' to denote F', where $F'' = \{(u', v') | (iu', iv') \in F'\}$.*

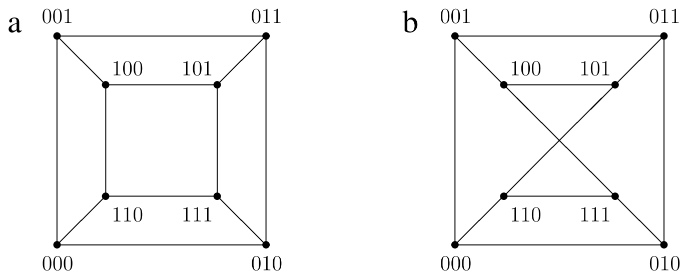

Fig. 1. Two 3-dimensional BC networks with labels.

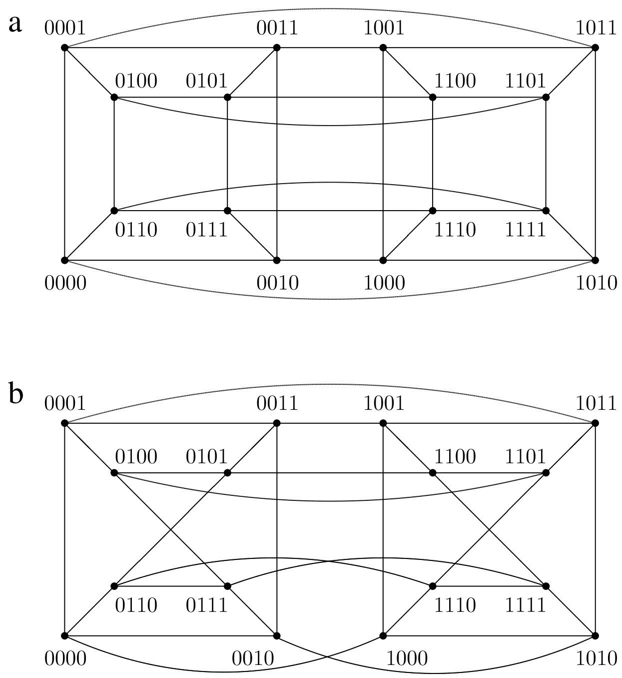

Fig. 2. Two 4-dimensional BC networks with labels.

3. Fault-tolerant routing algorithm

In this section, we will first prove that it is reasonable to introduce the concept of restricted faulty edge set into BC networks. We will then give an algorithm to find a fault-free path between any two different nodes in any n-dimensional BC network ($n \geq 2$). Finally, we will analyze the time complexity and the length of the fault-free path between the two different nodes found by the algorithm.

Lemma 4 (*[4]*). *For any integer $n \geq 1$ and $X_n \in \mathcal{L}_n$, $\lambda(X_n) = n$.*

Lemma 5. *For any $X_n \in \mathcal{L}_n$ and $F \subset E(X_3)$ with $|F| = 3$, if $X_3 - F$ is disconnected, then there is a $u \in V(X_3)$ such that $F = N_e(X_3, u)$ and $X_3 - F$ has exactly two connected components, one is $X_3[\{u\}]$ and the other is $X_3 - \{u\}$.*

Proof. Let $F = F_0 \bigcup F_1 \bigcup F_2$, where $F_0 \subset E(X_2^0)$, $F_1 \subset E(X_2^1)$, and $F_2 = F \bigcap \{(u, v) \in E(X_3) | \, u \in V(X_2^0)$ and $v \in V(X_2^1)\}$. Without loss of generality, we assume that $|F_0| \leq |F_1|$. Then, $|F_0| \leq \lfloor \frac{|F|}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor \leq 1$. By Lemma 4, $X_2^0 - F_0$ is connected. If $F_2 = \emptyset$, by Definition 2, each node in $X_2^1 - F_1$ is adjacent to one node in $X_2^0 - F_0$ in $X_3 - F$ and thus $X_3 - F$ is connected, a contradiction. Therefore, $F_2 \neq \emptyset$. Without loss of generality, we deal with the following cases.

Case 1. $|F_1| \leq 1$. Then, $X_2^1 - F_1$ is connected and $|V(X_2^1)| > 3 > |F_2|$. Hence, there exists one node in $X_2^1 - F_1$ that is adjacent to one node in $X_2^0 - F_0$ in $X_3 - F$ and thus $X_3 - F$ is connected, a contradiction.

Case 2. $|F_1| = 2$. Then, $|F_0| = 0$ and $|F_2| = 1$. If $X_2^1 - F_1$ is connected, similar to Case 1, we can claim that $X_2 - F$ is connected, a contradiction. Hence, $X_2^1 - F_1$ is disconnected. Obviously, F_1 contains two edges in X_2^1. And, the two edges in F_1 have no common end node or have exact one end node.

If the former case holds, then $X_2^1 - F_1$ has exact two connected components, each of which is a complete graph on two nodes. Clearly, at least one node in each connected component of $X_2^1 - F_1$ is adjacent to one node in $X_2^0 - F_0 = X_2^0$ in $X_3 - F$ and thus $X_3 - F$ is connected, a contradiction.

Otherwise, that is, there exists a node u in $X_2^1 - F_1$, such that $N_e(X_2^1, u) = F_1$. Then, $X_2^1 - F_1$ has exactly two connected components, one is $X_2^1[\{u\}]$ and the other is $X_2^1 - \{u\}$. Clearly, at least one node in the connected component $X_2^1 - \{u\}$ is adjacent to one node in $X_2^0 - F_0 = X_2^0$ in $X_3 - F$. Thus, u must be incident with the unique edge in F_2. That is, $N_e(X_3, u) = F$ and $X_3 - F$ has exactly two connected components, one is $X_2[\{u\}]$ and the other is $X_3 - \{u\}$.

In summary, the lemma holds. □

Then, we have the following theorem, which demonstrates the reasonability that we introduce the concept of restricted faulty edge set into bijective connection networks.

Theorem 6. *For any integer $n \geq 3$, $X_n \in \mathscr{L}_n$, and $F \subset E(X_n)$ with $|F| = n$, if $X_n - F$ is disconnected, then there is a $u \in V(X_n)$ such that $F = N_e(X_n, u)$ and $X_n - F$ has exactly two connected components, one is $X_n[\{u\}]$ and the other is $X_n - \{u\}$.*

Proof. We prove the lemma by induction on n. By Lemma 5, the lemma holds for $n = 3$. Supposing that the lemma holds for $n = \tau - 1$ ($\tau \geq 4$). For $n = \tau$, let $F = F_0 \bigcup F_1 \bigcup F_2$, where $F_0 \subset E(X_{\tau-1}^0)$, $F_1 \subset E(X_{\tau-1}^1)$, and $F_2 = F \bigcap \{(u, v) \in E(X_\tau) | u \in V(X_{\tau-1}^0)$ and $v \in V(X_{\tau-1}^1)\}$. Without loss of generality, we assume that $|F_0| \leq |F_1|$. Then, $|F_0| \leq \lfloor \frac{|F|}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor \leq n - 2$. By Lemma 4, $X_{\tau-1}^0 - F_0$ is connected. If $F_2 = \emptyset$, by Definition 2, each node in $X_{\tau-1}^1 - F_1$ is adjacent to one node in $X_{\tau-1}^0 - F_0$ in $X_\tau - F$ and thus $X_\tau - F$ is connected, a contradiction. Therefore, $F_2 \neq \emptyset$. Without loss of generality, we deal with the following cases.

Case 1. $|F_1| \leq \tau - 2$. Then, $X_{\tau-1}^1 - F_1$ is connected and $|V(X_{\tau-1}^1)| = 2^{\tau-1} > \tau > |F_2|$. Hence, at least one node in $X_{\tau-1}^1 - F_1$ is adjacent to one node in $X_{\tau-1}^0 - F_0$ in $X_\tau - F$ and thus $X_\tau - F$ is connected, a contradiction.

Case 2. $|F_1| = \tau - 1$. Then $|F_0| = 0$ and $|F_2| = 1$. If $X_{\tau-1}^1 - F_1$ is connected, similar to Case 1, we can claim that $X_\tau - F$ is connected, a contradiction. Hence, $X_{\tau-1}^1 - F_1$ is disconnected. According to the induction hypothesis, there exists a node u in $X_{\tau-1}^1$, such that $N_e(X_{\tau-1}^1, u) = F_1$ and $X_{\tau-1}^1 - F_1$ has exactly two connected components, one is $X_{\tau-1}^1[\{u\}]$ and the other is $X_{\tau-1}^1 - \{u\}$. Clearly, at least one node in the connected component of $X_{\tau-1}^1 - \{u\}$ is adjacent to one node in $X_{\tau-1}^0 - F_0 = X_{\tau-1}^0$ in $X_\tau - F$. Thus, u must be incident with the unique edge in F_2. That is, $N_e(X_\tau, u) = F$ and $X_\tau - F$ has exactly two connected components, one is $X_\tau[\{u\}]$ and the other is $X_\tau - \{u\}$.

In summary, the lemma holds. □

Remark. Theorem 6 implies that each minimal edge cut set (the cut of size n) must be the incident edge set of a node in X_n. Since X_n has n nodes and $n2^{n-1}$ edges, Theorem 6 actually proves that there exist exactly 2^n minimal edge cut sets among $\binom{n2^{n-1}}{n}$ edge sub-sets of size n. This fact shows that the probability that all the incident edges of an arbitrary node becomes faulty in X_n is extremely low when n becomes sufficient large. For example, even selecting $n = 8$, the probability that all the incident edges of an arbitrary node becomes faulty in X_n is

$$\frac{2^8}{\binom{8\times 2^7}{8}} \leq 10^{-16}.$$

As a result, it is reasonable that we introduce the concept of restricted faulty edge set into bijective connection networks.

Then, we will give an fault-tolerant routing algorithm in BC networks with restricted faulty edges, which based the following two theorems.

Theorem 7. *For any integers $n \geq 3$ and $k \in \{0, 1\}$, $X_n \in \mathscr{L}_n$, faulty edge set $F \subset E(X_n)$ with $|F| \leq n - 1$, and $x \in V(X_{n-1}^k)$, there exists a path P of length 1 or 2 in $X_n - F$ from x into some node in X_{n-1}^{1-k} such that $|V(P) \bigcap V(X_{n-1}^{1-k})| = 1$.*

Proof. Without loss of generality, we only need consider the case for $k = 0$. Let z be the neighbor of x in X_{n-1}^1 and $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be the $n-1$ neighbors of x in X_{n-1}^0. Furthermore, let $\beta_1, \beta_2, \ldots, \beta_{n-1}$ be the $n-1$ neighbors of $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ in X_{n-1}^1, respectively. Then,

$$\begin{array}{rlll} P_1: & x, & \alpha_1, & \beta_1 \\ P_2: & x, & \alpha_2, & \beta_2 \\ \cdots & \cdots & \cdots & \\ P_{n-1}: & x, & \alpha_{n-1}, & \beta_{n-1} \\ P_n: & x, & & z \end{array}$$

are n paths of length 1 or 2 from x into X_{n-1}^1.

Since $x, \alpha_1, \alpha_2, \ldots,$ and α_{n-1} are different from each other, by Definition 2, $z, \beta_1, \beta_2, \ldots, \beta_{n-1}$ are also different from each other. Hence, $P_1, P_2, \ldots, P_n$ are n node-disjoint (except x) and edge-disjoint paths from x into X_{n-1}^1. Thus, each edge in F lies in at most one of the n paths $P_1, P_2, \ldots, P_n$, which implies that there exists a path P_j of length 1 or 2 in $X_n - F$ from x into some node in $X_{n-1}^1 - F$ such that $|V(P_j) \bigcap V(X_{n-1}^1)| = 1$, where $1 \leq j \leq n$. □

Lemma 8 ([4]). *There is no cycle of length 3 in any BC network.*

The following two theorems will help us design a fault-tolerant routing algorithm in bijective connection networks with the restricted faulty edges.

Theorem 9. *For any integers $n \geq 3$ and $k \in \{0, 1\}$, $X_n \in \mathcal{L}_n$, faulty edge set $F \subset E(X_n)$ with $|F| \leq 2n-3$, and $u \in V(X_{n-1}^k)$, if each node has at least one fault-free incident edge in X_n, then there exists at least one path $P : \alpha_0 = u, \alpha_1, \ldots, \alpha_l$, of length l with $1 \leq l \leq 3$ in $X_n - F$ from u into X_{n-1}^{1-k}, such that $\{\alpha_0, \alpha_1, \ldots, \alpha_{l-1}\} \subset V(X_{n-1}^k)$ and $\alpha_l \in V(X_{n-1}^{1-k})$.*

Proof. Without loss of generality, we only need consider the case for $k = 0$. Let $P_1, P_2, \ldots, P_n$ be the n node-disjoint and edge-disjoint paths from x into X_{n-1}^1 listed as in the proof of Theorem 7. If there exists a fault-free path P_i among them, $1 \leq i \leq n$, then P_i is such a path that the conditions hold in the theorem.

Otherwise, arbitrarily choose a fault-free incident edge (u, v) of u. Then, $v \in V(X_{n-1}^0)$ (the reason is that there does not exist a fault-free path of length 1 from u into X_{n-1}^1). Let $\gamma_1, \gamma_2, \ldots, \gamma_{n-2}$ be all the neighbors, except u, of v in X_{n-1}^0. Furthermore, let $\delta_1, \delta_2, \ldots$, and δ_{n-2} be the neighbors of $\gamma_1, \gamma_2, \ldots$, and γ_{n-2} in X_{n-1}^1, respectively. Since $\gamma_1, \gamma_2, \ldots$, and γ_{n-2} are different from each other, by Definition 2, $\delta_1, \delta_2, \ldots, \delta_{n-2}$ are different from each other. Hence,

$$\begin{array}{rlll} P_1' : & v, \gamma_1, & \delta_1 \\ P_2' : & v, \gamma_2, & \delta_2 \\ \ldots & \ldots & \ldots \\ P_{n-2}' : & v, \gamma_{n-2}, & \delta_{n-2} \end{array}$$

are $n-2$ node-disjoint and edge-disjoint paths from v into X_{n-1}^1, each of which is of length 2.

Since $P_1, P_2, \ldots, P_n$ are n faulty paths, which means that $|F \bigcap E(P_j)| \geq 1$ for any j with $1 \leq j \leq n$. Also considering that $P_1, P_2, \ldots, P_n$ are n node-disjoint and edge-disjoint paths,

$$\left|F \bigcap \bigcup_{j=1}^{n} E(P_j)\right| = \left|\bigcup_{j=1}^{n} \left(F \bigcap E(P_j)\right)\right| = \sum_{j=1}^{n} \left|F \bigcap E(P_j)\right| \geq n.$$

Therefore,

$$\left|F - \left(F \bigcap \bigcup_{j=1}^{n} E(P_j)\right)\right| \leq n-3.$$

Considering that each edge in $F - (F \bigcap \bigcup_{j=1}^{n} E(P_j))$ lies in at most one of the $n-2$ paths $P_1', P_2', \ldots, P_{n-2}'$, there exists a fault-free path, say P_t', among these $n-2$ paths, where $1 \leq t \leq n-2$. Then, u, v, γ_t, δ_t is a path of length 3 in $X_n - F$ from u into X_{n-1}^1, such that $\{u, v, \gamma_t\} \subset V(X_{n-1}^0)$ and $\delta_t \in V(X_{n-1}^1)$. In summary, the theorem holds. □

In the following, according to the above results, we will give an algorithm to find a fault-free path between any two different nodes in any n-dimensional BC network X_n with $n \geq 2$.

Algorithm: Fault-Free-Path(X_n, F, x, y)
Input: An n-dimensional BC network X_n with $n \geq 2$, a faulty edge set $F \subset E(X_n)$ satisfying that any node has at least one fault-free incident edge and $|F| \leq 2n-3$, and two different nodes x and y in X_n
Output: A path between x and y in $X_n - F$.

```
if (x, y) ∈ E(X_n) − F
   then return (x, y)
if n = 2
   then return (A fault-free path between x and y in X_2)
if F = ∅
   then return (Path-3(X_n, x, y))
let i ← bit(x, n − 1) and j ← bit(y, n − 1)
let F_k ← F ⋂ E(X^k_{n−1}) for any k ∈ {0, 1} and F_2 ← (F − F_0) − F_1
switch
   case i = 1 − j :
        if |F_i| ≤ |F_{1−i}|
           then return (Path-1(X_n, i, F_i, F_{1−i}, F_2, x, y))
           else let P ← Path-1(X_n, 1 − i, F_{1−i}, F_i, F_2, y, x)
                return (rev(P))
   case i = j :
        if |F_i| ≤ |F_{1−i}|
           then let x ← ix′, y ← iy′, F_i ← iF′, and X^i_{n−1} ← iX_{n−1}
                let P ← Fault-Free-Path(X_{n−1}, F′, x′, y′)
                return (iP)
```

else **let** $P \leftarrow$ Path-2$(X_n, i, F_i, F_{1-i}, F_2, x)$
let $P' \leftarrow$ Path-2$(X_n, i, F_i, F_{1-i}, F_2, y)$
let P be $x = \alpha_0, \alpha_1, \ldots, \alpha_l$ and P' be $y = \beta_0, \beta_1, \ldots, \beta_m$, where $1 \leq l, m \leq 3$
if $V(P) \bigcap V(P') \neq \emptyset$
then let $\alpha_k = \beta_{k'}$ be the first common node of P and P', where $0 \leq k, k' \leq 2$
return $(x, \alpha_1, \alpha_2, \ldots, \alpha_k, \beta_{k'-1}, \beta_{k'-2}, \ldots, \beta_1, y)$
else **let** $\alpha_l \leftarrow (1-i)x'$, $\beta_m \leftarrow (1-i)y'$, $F_{1-i} \leftarrow (1-i)F'$,
and $X_{n-1}^{1-i} \leftarrow (1-i)X_{n-1}$
let $P \leftarrow$ Fault-Free-Path(X_{n-1}, F', x', y')
return $(x, \alpha_1, \alpha_2, \ldots, \alpha_{l-1}, (1-i)P, \beta_{m-1}, \beta_{m-2}, \ldots, \beta_1, y)$

Path-1$(X_n, i, F, F_{1-i}, F_2, x, y)$
let z be the neighbor of y in X_{n-1}^i
if $(y, z) \notin F_2$
then let $z \leftarrow iz'$, $x \leftarrow ix'$, $F \leftarrow iF'$, and $X_{n-1}^i \leftarrow iX_{n-1}$
let $P \leftarrow$ Fault-Free-Path(X_{n-1}, F', x', z')
return (iP, y)
else **if** There exists a neighbor v of y in X_{n-1}^{1-i} and the neighbor u of v in X_{n-1}^i such that
$(y, v) \notin F_{1-i}$ and $(v, u) \notin F_2$
then let $u \leftarrow iu'$, $x \leftarrow ix'$, $F \leftarrow iF'$, and $X_{n-1}^i \leftarrow iX_{n-1}$
let $P \leftarrow$ Fault-Free-Path(X_{n-1}, F', x', u')
return (iP, v, y)
else Select a neighbor v of y in X_{n-1}^{1-i} such that $(y, v) \notin F_{1-i}$
Select a neighbor u of v in $X_{n-1}^{1-i} - \{y\}$ and the neighbor w of u in X_{n-1}^i such
that $(u, v) \notin F_{1-i}$ and $(u, w) \notin F_2$
let $w \leftarrow iw'$, $x \leftarrow ix'$, $F \leftarrow iF'$, and $X_{n-1}^i \leftarrow iX_{n-1}$
let $P \leftarrow$ Fault-Free-Path(X_{n-1}, F', x', w')
return (iP, u, v, y)

Path-2$(X_n, i, F_i, F_{1-i}, F_2, x)$
let z be the neighbor of x in X_{n-1}^{1-i}
if $(x, z) \notin F_2$
then return (x, z)
else **if** There exists a neighbor v of x in X_{n-1}^i and the neighbor u of v in $V(X_{n-1}^{1-i})$ such
that $(x, v) \notin F_i$ and $(v, w) \notin F_2$
then return (x, v, u)
else Select a neighbor v of x in X_{n-1}^i such that $(x, v) \notin F_i$
Select a neighbor u of v in $X_{n-1}^i - \{x\}$ and the neighbor w of u in X_{n-1}^{1-i} such
that $(u, v) \notin F_i$ and $(u, w) \notin F_2$
return (x, v, u, w)

Path-3(X_n, x, y)
if $(x, y) \in E(X_n)$
then return (x, y)
if $n = 2$
then return (A path between x and y in X_2)
let $i \leftarrow$ bit $(x, n-1)$ and $j \leftarrow$ bit $(y, n-1)$
switch
case $i = j$:
let $x \leftarrow ix'$, $y \leftarrow iy'$, and $X_{n-1}^i \leftarrow iX_{n-1}$
let $P \leftarrow$ Path-3(X_{n-1}, x', y')
return (iP)
case $i = 1 - j$:
let z be the neighbor of x in X_{n-1}^j
let $z \leftarrow jz'$, $y \leftarrow jy'$, and $X_{n-1}^j \leftarrow jX_{n-1}$
let $P \leftarrow$ Path-3(X_{n-1}, z', y')
return (x, jP)

In order to analyze the performance of Algorithm Fault-Free-Path, we need the following results.

Lemma 10 ([9]). *For any integers $k \geq 1$ and $n \geq \lceil \frac{k+1}{2} \rceil$, $X_n \in \mathcal{L}_n$, and $V' \subset V(X_n)$ with $|V'| = k$, $|\Gamma(X_n, V')| \geq kn - \frac{k(k+1)}{2} + 1$.*

Theorem 11. *For any integers $n \geq 3$ and $k \in \{0, 1\}$, $X_n \in \mathcal{L}_n$, faulty edge set $F \subset E(X_n)$ with $|F| \leq 2n-3$, and $u, v \in V(X^k_{n-1})$ with $u \neq v$, if each node has at least one fault-free incident edge in X_n and there does not exist a fault-free path of length 1 or 2 from u into X^{1-k}_{n-1} among the n disjoint paths listed as in the proof of Theorem 7, then one of the following two results holds:*

(1) *There exists a fault-free path of length l with $1 \leq l \leq 3$ between u and v in X^k_{n-1}.*
(2) *There exists a fault-free path P_1 of length 3 from u into X^{1-k}_{n-1} and a fault-free path P_2 of length 1 or 2 from v into X^{1-k}_{n-1}, such that $V(P_1) \bigcap V(P_2) = \emptyset$.*

Proof. Without loss of generality, we only need consider the case for $k = 0$. Let $F_0 = F \bigcap E(X^0_{n-1})$. If $(u, v) \in E(X^0_{n-1}) - F_0$ or there exists a node w in X^0_{n-1} such that $(u, w), (v, w) \in E(X^0_{n-1}) - F_0$, then there exists a fault-free path of length 1 or 2 between u and v in X^0_{n-1} and thus the theorem holds.

Otherwise, if there does not exist a fault-free path of length 1 or 2 from u into X^1_{n-1} among the n node-disjoint and edge-disjoint paths $P'_1, P'_2, \ldots, P'_n$ listed as in the proof of Theorem 7, each of the n paths contains at least one faulty edge. Thus, at least n faulty edges in F lie in these n paths. By the proof of Theorem 7, let

$$\begin{array}{rlll} P_1: & v, \gamma_1, & \delta_1 \\ P_2: & v, \gamma_2, & \delta_2 \\ \cdots & \cdots & \cdots \\ P_{n-1}: & v, \gamma_{n-1}, & \delta_{n-1} \\ P_n: & v, & z \end{array}$$

be the n node-disjoint and edge-disjoint paths from v into X^1_{n-1}, where $\gamma_i \in E(X^0_{n-1})$ and $z, \delta_i \in E(X^1_{n-1})$, $1 \leq i \leq n-1$. By Lemma 10, $|\Gamma(X^0_{n-1}, \{u, v\})| \geq 2n-4$. Then,

$$\begin{aligned} |\Gamma(X^0_{n-1}, u) \bigcap \Gamma(X^0_{n-1}, v)| &= |\Gamma(X^0_{n-1}, u)| + |\Gamma(X^0_{n-1}, v)| - |\Gamma(X^0_{n-1}, \{u, v\})| \\ &\leq 2(n-1) - (2n-4) = 2. \end{aligned}$$

That is, u and v have at most two common neighbors (Notice that the path passing through a common neighbor of u and v between u and v is faulty and of length 2, if any) in X^0_{n-1}. Hence, by Definition 2, there are at least $n-2$ paths among $P_1, P_2, \ldots, P_n$ that are node-disjoint and edge-disjoint with the n paths $P'_1, P'_2, \ldots, P'_n$ from u into X^1_{n-1}. Since at least n faulty edges lie in the n paths $P'_1, P'_2, \ldots, P'_n$ from u into X^1_{n-1} and each edge in F lies in at most one of the n paths $P_1, P_2, \ldots, P_n$, and $|F| - n \leq (2n-3) - n = n-3$, there exists at least one fault-free path, say $P_j : v, \gamma_j, \delta_j$ or $P_j : v, z$, among the paths $P_1, P_2, \ldots, P_n$ such that P_j is node-disjoint and edge-disjoint with each of the n paths $P'_1, P'_2, \ldots, P'_n$ from u into X^1_{n-1}. Furthermore, by Theorem 9, there exists a fault-free path $P' : x^{(0)} = u, x^{(1)}, x^{(2)}, x^{(3)}$ of length 3 from u into X^1_{n-1} such that $\{x^{(0)}, x^{(1)}, x^{(2)}\} \subset V(X^0_{n-1})$ and $x^{(3)} \in V(X^1_{n-1})$. If P_j is the path v, z, then P_j is a fault-free path of length 1 from v into X^{1-k}_{n-1} such that $V(P') \bigcap V(P_j) = \emptyset$ and thus the theorem holds; otherwise, P_j must be the path v, γ_j, δ_j, which is a fault-free path of length 2 from v into X^{1-k}_{n-1}. Noticing that $\gamma_j \notin \{u, x^{(1)}\}$, if $\gamma_j = x^{(2)}$, then $u, x^{(1)}, x^{(2)}, v$ is a fault-free path of length 3 between u and v in X^0_{n-1} and the theorem also holds; otherwise, by Definition 2, $V(P') \bigcap V(P_j) = \emptyset$ and the theorem still holds. □

Lemma 12. *For any integers $n \geq 3$ and $k \in \{0, 1\}$, $X_n \in \mathcal{L}_n$, and faulty edge set $F \subset E(X_n)$ with $|F| \leq 2n-3$, if each node has at least one fault-free incident edge in X_n, then there exists a node $z \in \{x, y\}$, such that there is a fault-free path of length 1 or 2 from z into X^{1-k}_{n-1} for any two different nodes $x, y \in V(X^k_{n-1})$ and $(x, y) \notin E(X_n)$.*

Proof. Without loss of generality, we only need consider the case for $k = 0$. Let $P_1, P_2, \ldots, P_n$ be the n node-disjoint and edge-disjoint paths from x into X^1_{n-1} listed as in the proof of Theorem 7. If one of the n paths $P_1, P_2, \ldots,$ and P_n are fault-free, then the lemma holds; otherwise, that is, each of the n paths contains at least one faulty edge. Then, $\sum_{i=1}^{n} |F \bigcap E(P_i)| \geq n$. Furthermore, let $P'_1, P'_2, \ldots, P'_n$ be the n node-disjoint and edge-disjoint paths from y into X^1_{n-1} listed as in the proof of Theorem 7. By Lemma 10, $|\Gamma(X^0_{n-1}, \{x, y\})| \geq 2n-4$. Similar to the proof in Theorem 11, x and y have at most two common neighbors.

Hence, there are at least $n-2$ paths of $P'_1, P'_2, \ldots, P'_n$ that are node-disjoint and edge-disjoint with $P_1, P_2, \ldots, P_n$. Furthermore, since $|F| - \sum_{i=1}^{n} |F \bigcap E(P_i)| \leq (2n-3) - n = n-3$, there is a fault-free path among $P'_1, P'_2, \ldots, P'_n$. Noticing that all the paths $P'_1, P'_2, \ldots, P'_n$ have length 1 or 2, the lemma holds. □

With the above results, we will analyze the time complexity of Algorithm Fault-Free-Path and the length of the fault-free path between two given fault-free nodes in X_n found by Algorithm Fault-Free-Path under the worst case in the following theorem,

Theorem 13. *For any integer $n \geq 3, X_n \in \mathcal{L}_n, F \subset E(X_n)$ with $|F| \leq 2n-3$ such that each node has at least one fault-free incident edge, and $x, y \in V(X_n)$ with $x \neq y$, Algorithm Fault-Free-Path can find a fault-free path of length at most $n + 3\lceil \log_2 |F| \rceil + 1$ in $O(n)$ time under the worst case, where the worst case is refer to as the scenario that the fault-free path found by Algorithm Fault-Free-Path is as long as possible and the time taken by it is as much as possible.*

Proof. We will prove this theorem by using the method in [5]. Without loss of generality, the scenario can be presented as the following processes of $n-1$ steps:

(1) At the first step, the following conditions will hold:

$|F| = 2n-3, |F \bigcap E(X_{n-1}^1)| \leq |F \bigcap E(X_{n-1}^0)|, x, y \in V(X_{n-1}^0)$, and x and y are adjacent to the nodes x' and y', respectively, in X_{n-1}^1 such that $(x, x'), (y, y') \in F$, and there exists neither a fault-free path of length 1, 2, or 3 between x and y in X_{n-1}^0 nor a fault-free path of length 2 from x into X_{n-1}^1 among the n disjoint paths listed as in the proof of Theorem 7.

By Theorem 11 and Lemma 12, Algorithm Fault-Free-Path will find a fault-free path P_{01} of length 3 from x to a node, say α_1, in X_{n-1}^1 and a fault-free path P_{02} of length 2 from y to a node, say β_1, in X_{n-1}^1 such that $V(P_{01}) \bigcap V(P_{02}) = \emptyset$. Thus, the problem to find a fault-free path between x and y for any $x, y \in V(X_n)$ with $x \neq y$ and the restricted faulty edge set $F \subset E(X_n)$ with $|F| \leq 2n-3$ in X_n is reduced to that to find a fault-free path between α_1 and β_1 with $\alpha_1 \neq \beta_1$ for the faulty edge set $F_1 \subset E(X_{n-1}^1)$ with $|F_1| \leq \lfloor \frac{|F|}{2} \rfloor$ in $X_{n-1}^1 - F_1$ by Algorithm Fault-Free-Path. Let $F_1 = F \bigcap E(X_{n-1}^1)$, then $|F_1| \leq \lfloor \frac{|F|}{2} \rfloor \leq \lfloor \frac{2n-3}{2} \rfloor = n-2$. By Lemma 4, $X_{n-1}^1 - F_1$ is connected, which means that there exists a fault-free path between any two different nodes in X_{n-1}^1.

At this step, the process to compute $F_i = F \bigcap E(X_{n-1}^i)$ for any $i \in \{0, 1\}$ and $F_2 = F - F_0 - F_1$ can be conducted as follows: check whether the left-most bits of the two end nodes of each edge in F are 0 or 1. For any $i \in \{0, 1\}$, if the left-most bits of the two end nodes of one edge (u, v) in F are both i, then (u, v) will be added into F_i; otherwise, it will be added into F_2. During this process, $|F_k|$ for any $k \in \{0, 1\}$ can be at the same time computed. Obviously, this process will take time $O(|F|)$. On the other hand, the process to check whether an edge belongs to F_0 or F_2 will take time $O(|F_0|)$ or $O(|F_2|)$, respectively. As a result, the time taken at this step is $O(|F| + |F_0| + |F_2|) = O(|F|)$.

For convenience of presentation, we use X_{n-1} to denote X_{n-1}^1, where $V(X_{n-1}) = \{s | 1s \in V(X_n)\}$ and $E(X_{n-1}) = \{(s, t) | (1s, 1t) \in E(X_n)\}$, let $F^{(1)} = \{(s, t) | (1s, 1t) \in F_1\}$, $\alpha_1 = 1x^{(1)}$, and $\beta_1 = 1y^{(1)}$.

(2) At the second step, let $F_j^{(1)} = F^{(1)} \bigcap E(X_{n-2}^j)$ for any $j \in \{0, 1\}$ and $F_2^{(1)} = (F^{(1)} - F_0^{(1)}) - F_1^{(1)}$. The following conditions will hold:

$|F^{(1)}| \leq \lfloor \frac{|F|}{2} \rfloor, x^{(1)}, y^{(1)} \in V(X_{n-2}^0), |F_1^{(1)}| \leq \lfloor \frac{|F^{(1)}|}{2} \rfloor$, $x^{(1)}$ and $y^{(1)}$ are respectively adjacent to the nodes $x'^{(1)}$ and $y'^{(1)}$ in X_{n-2}^1 such that $(x^{(1)}, x'^{(1)}), (y^{(1)}, y'^{(1)}) \in F_2^{(1)}$, and there does not exist a fault-free path of length 1 or 2 between $x^{(1)}$ and $y^{(1)}$ in X_{n-2}^0. Since $|F^{(1)}| = |F_1| \leq n-2$, by the proof of Theorem 7, Algorithm Fault-Free-Path will find a fault-free path P_{11} of length 2 from $x^{(1)}$ to a node α_2 in X_{n-2}^1 and a fault-free path P_{12} of length 2 from $y^{(1)}$ to β_2 in X_{n-2}^1 such that $V(P_{11}) \bigcap V(P_{12}) = \emptyset$ (under the worst case). Similar to the discussion in (1), this process will take time $O(|F^{(1)}| + |F_0^{(1)}| + |F_2^{(1)}|) = O(|F^{(1)}|)$.

Furthermore, we use X_{n-2} to denote X_{n-2}^1, where $V(X_{n-2}) = \{s | 1s \in V(X_{n-1})\}$ and $E(X_{n-2}) = \{(s, t) | (1s, 1t) \in E(X_{n-1})\}$, let $F^{(2)} = \{(s, t) | (1s, 1t) \in F_1^{(1)}\}$, $\alpha_2 = 1x^{(2)}$, and $\beta_2 = 1y^{(2)}$.

(3) At the $(m+1)$-th step for any integer m with $2 \leq m \leq \lceil \log_2 |F| \rceil - 1$, let $F_j^{(m)} = F^{(m)} \bigcap E(X_{n-m-1}^j)$ for any $j \in \{0, 1\}$ and $F_2^{(m)} = (F^{(m)} - F_0^{(m)}) - F_1^{(m)}$. The following conditions will hold:

$|F^{(m)}| \leq \lfloor \frac{|F^{(m-1)}|}{2} \rfloor, x^{(m)}, y^{(m)} \in V(X_{n-m-1}^0), |F_1^{(m)}| \leq \lfloor \frac{|F^{(m)}|}{2} \rfloor$, $x^{(m)}$ and $y^{(m)}$ are respectively adjacent to the nodes $x'^{(m)}$ and $y'^{(m)}$ in X_{n-m-1}^1 such that $(x^{(m)}, x'^{(m)}), (y^{(m)}, y'^{(m)}) \in F_2^{(m)}$, and there does not exist a fault-free path of length 1 or 2 between $x^{(m)}$ and $y^{(m)}$ in X_{n-m-1}^0. By the proof of Theorem 7, Algorithm Fault-Free-Path will find a fault-free path P_{m1} of length 2 from $x^{(m)}$ to a node α_{m+1} in X_{n-m-1}^1 and a fault-free path P_{m2} of length 2 from $y^{(m)}$ to β_{m+1} in X_{n-m-1}^1 such that $V(P_{m1}) \bigcap V(P_{m2}) = \emptyset$ (under the worst case). Similar to the discussion in (1), this process will take time $O(|F^{(m)}| + |F_0^{(m)}| + |F_2^{(m)}|) = O(|F^{(m)}|)$.

Furthermore, we use X_{n-m-1} to denote X_{n-m-1}^1, where $V(X_{n-m-1}) = \{s | 1s \in V(X_{n-m})\}$ and $E(X_{n-m-1}) = \{(s, t) | (1s, 1t) \in E(X_{n-m})\}$, and let $F^{(m+1)} = \{s | 1s \in F_1^{(m)}\}$, $\alpha_{(m+1)} = 1x^{(m+1)}$, and $\beta_{(m+1)} = 1y^{(m+1)}$.

(4) At the t-th step for $t = \lceil \log_2 |F| \rceil$. It holds that $|F^{(t)}| = 0$. Thus, Algorithm Fault-Free-Path will call the procedure Path-3, which will find a path between $x^{(t)}$ and $y^{(t)}$ with $x^{(t)} \neq y^{(t)}$ in X_{n-t}^1 without faulty edges and take time $O((n-1)-(t-1)) = O(n)$.

According to the above discussion, we analyze the time complexity and the length of the fault free path obtained by Algorithm Fault-Free-Path under the worst case as follows:

For any integer i with $1 \leq i \leq \lceil \log_2 |F| \rceil$, we have $|F^{(i)}| \leq \lfloor \frac{|F^{(i-1)}|}{2} \rfloor$, where $F^{(0)} = F$.

As a result, in the first $\lceil \log_2 |F| \rceil$ steps, Algorithm Fault-Free-Path takes time at most

$$\sum_{i=1}^{\lceil \log_2 |F| \rceil} O\left(\frac{|F^{(i-1)}|}{2}\right) \leq \sum_{i=1}^{\lceil \log_2 (2n-3) \rceil} O\left(\frac{2n-3}{2^{i-1}}\right) = O(2n-3) = O(n).$$

To sum up, under the worst case, the time complexity of Algorithm Fault-Free-Path is

$$\sum_{i=1}^{\lceil \log_2 |F| \rceil} O\left(\frac{|F^{(i-1)}|}{2}\right) + O(n) = O(n).$$

According to the above discussion, the length of the fault free path obtained by Algorithm Fault-Free-Path under the worst case is

$$5 + 4(\lceil \log_2 |F| \rceil - 1) + (n - \lceil \log_2 |F| \rceil) = n + 3\lceil \log_2 |F| \rceil + 1. \quad \square$$

By the result in [30], it can be inferred that any n-dimensional BC network X_n has conditional edge connectivity $2n-2$, which is actually a corollary from Theorem 13 as follows:

Lemma 14 ([4]). *If the isomorphic graphs are regarded as the identical graph, there are exactly two 3-dimensional BC networks: one is Q_3 and the other is CQ_3.*

Corollary 15. *For any integer $n \geq 2$ and $X_n \in \mathcal{L}_n$, $\lambda'(X_n) = 2n - 2$.*

Proof. For $n = 2$, $2n - 2 = n$. By Lemma 4, the corollary holds.

For any integer $n \geq 3$ and $X_n \in \mathcal{L}_n$, by Theorem 13, $\lambda'(X_n) \geq 2n - 2$. In what follows, we prove that $\lambda'(X_n) \leq 2n - 2$ for any integer $n \geq 3$ and $X_n \in \mathcal{L}_n$.

By Lemmas 4 and 14, the corollary can be easily verified to hold for $n = 3$.

For $n \geq 4$, select $u \in V(X^0_{n-1})$ and $v \in V(X^1_{n-1})$ such that $(u, v) \in E(X_n)$. Then we can verify that $CoN_e(X_n, x, y) \subset E(X_n)$ and $|CoN_e(X_n, x, y)| = |N_e(X_n, x) \bigcup N_e(X_n, y) - \{(x, y)\}| = 2n-2$. Clearly, $X_n - CoN_e(X_n, x, y)$ is disconnected, which implies that $\lambda'(X_n) \leq 2n - 2$. As a consequence, $\lambda'(X_n) = 2n - 2$. $\square$

4. Conclusions

In this paper, we have studied fault-tolerant routing in BC networks under the condition that each node has at least one fault-free edge. First, we have proven that the probability that all the incident edges of an arbitrary node become faulty in X_n is extremely low (n becomes sufficient big). Then, we have given an $O(n)$ algorithm to find a fault-free path of length at most $n + 3\lceil \log_2 |F| \rceil + 1$ between any two different nodes in X_n. In fact, we also for the first time provide an upper bound of the fault diameter of all the bijective connection networks under the restricted condition. Since the family of BC networks contains hypercubes, crossed cubes, Möbius cubes, locally twisted cubes, etc., all the results are appropriate for these cubes. In fact, the performance of the algorithm in this paper is similar to that under the restricted faulty nodes in [5]. We also have a conjecture that the smallest upper bound of the fault diameter of all the BC networks under the restricted faulty edges would be n plus a constant. Due to the diversity of BC networks, this problem is significant but difficult to solve, which deserves our further investigation.

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