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Dixmier traces on noncompact isospectral deformations

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Abstract

We extend the isospectral deformations of Connes, Landi and Dubois-Violette to the case of Riemannian spin manifolds carrying a proper action of the noncompact abelian group $\mathbb{R}^l$. Under deformation by a torus action, a standard formula relates Dixmier traces of measurable operators to integrals of functions on the manifold. We show that this relation persists for actions of $\mathbb{R}^l$, under mild restrictions on the geometry of the manifold which guarantee the Dixmier traceability of those operators.

Keywords: Dixmier trace; Dirac operator; Moyal product; Isospectral deformation

1. Introduction

The primary example of a noncommutative differential geometry is the noncommutative torus [9,42]; its coordinate algebra may be reconstructed from the algebra of smooth functions on an ordinary torus $\mathbb{T}^l$ by deforming the product compatibly with the rotation action of the torus, regarded as a compact abelian group, on itself. The group $\mathbb{T}^l$ acts ergodically on the resulting deformed algebra. Given a spin structure on $\mathbb{T}^l$, there is a Dirac operator $D\!\!\!/$ on the Hilbert space $\mathcal{H}$

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of square-integrable spinors, which is invariant under a lifted action of $\mathbb{T}^l$; the deformed algebra is also represented on this Hilbert space, giving rise to a spectral triple [11] with the same Dirac operator: one speaks of an *isospectral deformation* of the triple $(C^\infty(\mathbb{T}^l), \mathcal{H}, D\!\!\!/)$.

This example was generalized by Connes and Landi [13] to the case of a $\mathbb{T}^l$-action, for $l \geqslant 2$, on a compact Riemannian spin manifold. It was further refined by Connes and Dubois-Violette [12] to encompass the case where the spin manifold need not be compact but still carries a smooth torus action. In all such cases, the Dirac operator interacts with the deformed algebra to provide a isospectral deformation of the standard commutative spectral triple.

Isospectral deformations arising from noncompact group actions provide a more challenging analytic framework. It was established by Rieffel [43] that Moyal deformations under actions of $\mathbb{R}^l$ have good analytic properties, both at the level of C^*-algebras and in terms of the smooth subalgebras for the action. This deformation construction goes through when the symmetry group is abelian, so that $\mathbb{T}^l$ and $\mathbb{R}^l$ are the cases of primary interest. However, the compatibility of Moyal deformations with (invariant) Dirac operators on noncompact spin manifolds poses additional issues for the construction of deformed spectral triples. These issues have been addressed and resolved in the "flat" case of the affine space $\mathbb{R}^{2m}$ with translation action, whose deformations are Moyal "planes," in our [29] and in [28].

In this paper, we consider proper actions of a connected abelian group $\mathbb{T}^k \times \mathbb{R}^{l-k}$ on a (not necessarily compact) n-dimensional Riemannian spin manifold M. This can be thought of as a proper, hence free, action of $\mathbb{R}^{l-k}$ on a $\mathbb{T}^k$-twisted Connes–Landi spectral triple; we therefore deal mainly with the subcase where $k = 0$. The detailed geometry of the manifold (isoperimetry, curvature bounds) plays a role in establishing the heat-kernel properties of M and in determining the interplay of the isospectrally deformed algebra with the Laplacian and the Dirac operator. From Connes' trace theorem [10] for the case of compact manifolds, one expects that operators such as $L_f |D\!\!\!/|^{-n}$ or $L_f \Delta^{-n/2}$, where L_f denotes the deformed product by a function $f \in C_c^\infty(M)$, should lie in the Dixmier trace-class, and their Dixmier traces should be proportional to the integral of f with respect to the Riemannian volume form. We show that this hope is fulfilled in the noncompact case, under suitable conditions on the geometry of M. This general result was foreshadowed in the flat case in [29] and is extended here to a more general setting.

In Section 2, we review the Moyal products on manifolds with an $\mathbb{R}^l$-action, to fix the notation. In Section 3, we show that the Hilbert–Schmidt norm of operators of the form $L_f h(D\!\!\!/)$ is independent of the deformation. In Section 4, after discussing how the required geometric properties yield bounds on the heat kernel, we identify the Schatten classes $\mathcal{L}^p$ to which several such operators belong, and show that certain important cases they lie in the weak Schatten class $\mathcal{L}^{n,\infty}$, so that $L_f(1 + D\!\!\!/^2)^{-n/2}$ belongs to the Dixmier trace class $\mathcal{L}^{1,\infty}$. The proof extends and simplifies the argument of [29], based on Cwikel's inequality [46].

In the final Sections 5 and 6, we compute the desired Dixmier traces, for both periodic and aperiodic actions of $\mathbb{R}^l$. In the aperiodic case, the geometry is straightforward but the analysis is not, since Dixmier traces, unlike ordinary integrals, do not admit monotone or dominated convergence theorems: the heuristic extension of the compact case put forward in [23, Theorem 4.2] is therefore unsupported in general. We show, nevertheless, how to overcome this objection for algebras arising from Moyal deformations.

2. Moyal products on manifolds

Definition 2.1. Let (M, g) be an n-dimensional (not necessarily compact) Riemannian spin manifold which is geodesically complete and without boundary. Let G be a connected abelian Lie group of rank l (so that $G \simeq \mathbb{T}^k \times \mathbb{R}^{l-k}$ for some $k = 0, 1, \dots, l$), with $l \geqslant 2$. Assume that M is endowed with an isometric effective action of G, denoted $\alpha : G \to \mathrm{Isom}(M, g)$, which is *smooth* (i.e., the map $G \times M \to M : (z, p) \mapsto \alpha_z(p)$ is smooth) and *proper*. Thus M is a proper G-manifold in the sense of [39].

For brevity, we often write $z \cdot p := \alpha_z(p)$. We also denote by α the induced action by automorphisms on $C^\infty(M)$, i.e., $\alpha_z f(p) := f(\alpha_{-z}(p))$ for $p \in M$. Let $X_1, \dots, X_l$ be the infinitesimal vector fields associated to the action, namely $X_j(f) := \frac{\partial}{\partial z^j}(\alpha_z f)|_{z=0}$, for $f \in C^\infty(M)$.

Let $S \to M$ be the spinor bundle and $\mathcal{H} := L^2(M, S)$ be the separable Hilbert space of its square integrable sections. Each compactly supported smooth function $f \in C_c^\infty(M)$ defines a bounded operator M_f on $\mathcal{H}$ by pointwise multiplication, $M_f(\psi) := f\psi$.

The isometric action α lifts to S modulo ± 1, as is pointed out in [12]: for a suitable double covering $p : \widetilde{G} \to G$, where $\widetilde{G}$ is also isomorphic to $\mathbb{T}^k \times \mathbb{R}^{l-k}$, we can find a group of unitary operators $\{V_{\tilde{z}}\colon \tilde{z} \in \widetilde{G}\}$ on $\mathcal{H}$ which covers the group of isometries $\{\alpha_z\colon z \in G\}$ in the sense that

$$V_{\tilde{z}}(f\psi) = (\alpha_z f) V_{\tilde{z}}\psi, \tag{2.1}$$

whenever $\psi \in \mathcal{H}$, $f \in C_c^\infty(M)$, and $p(\tilde{z}) = z$. In general, unless $k = 0$, this spin lifting does not split: if $p(\tilde{z}) = p(\tilde{z}')$ then $V_{\tilde{z}} = \pm V_{\tilde{z}'}$ but the sign cannot be taken globally to be $+1$. In what follows, we shall ignore this nuance and shall suppose (in the notation) that the spin lifting does split, writing V_z instead of $V_{\tilde{z}}$; thus (2.1) will be rewritten here as $V_z(f\psi) = (\alpha_z f) V_z \psi$.

Definition 2.2. Let $\Theta \in M_l(\mathbb{R})$ be a fixed real skew-symmetric matrix. For $f, h \in C_c^\infty(M)$, the usual pointwise product of f and h can be deformed by the group action α, as follows [43]:

$$f \star h := (2\pi)^{-l} \int_{\mathbb{R}^l}\int_{\mathbb{R}^l} e^{-iyz} \alpha_{\frac{1}{2}\Theta y}(f) \alpha_{-z}(h)\, d^l y\, d^l z. \tag{2.2}$$

Thus, $\star$ is a bilinear product on $C_c^\infty(M)$ with values in $C^\infty(M)$; its associativity can be checked directly.

Remark 2.3. We could have written $\star_\Theta$ instead of $\star$, had we needed to emphasize the dependence of the deformation on the parameter matrix Θ. When $\Theta = 0$, the oscillatory integral (2.2) collapses to the usual pointwise product of functions.

When Θ is not invertible, the product (2.2) reduces to a twisted product associated to the action $\sigma := \alpha|_{V^\perp}$, where V is the nullspace of Θ, as in [43, Proposition 2.7]. In what follows, we shall take Θ to be a fixed *invertible* matrix. In particular, this implies that the rank l *is even*.

Remark 2.4. Herein Θ is taken to be fixed, but this restriction is not forced: it has been shown by one of us, together with Gracia-Bondía and Ruiz Ruiz [30], that Rieffel's approach is compatible with some variable noncommutativity matrices $\Theta(x)$. Giving such a Θ determines a Poisson structure Π_Θ on M; and one expects to find an associative star-product reproducing any given Π_Θ, insofar as Kontsevich's formality theorem [38] for perturbative deformations carries over to the present context. Physics would demand that this Poisson tensor should be a dynamical field, interacting with the gravity background. Among the papers that already invoke a variable Θ, we may mention [1–3,5,21,34,36], although most of the treatments so far have been kinematical.

From now on, we shall treat separately $\mathbb{T}^l$-actions and effective $\mathbb{R}^l$-actions, and we respectively talk about *periodic* and *aperiodic* deformations; we obtain similar results in both cases, albeit with different techniques. We do not deal directly with mixed cases where the $\mathbb{R}^l$-action factors through an effective action of $\mathbb{T}^k \times \mathbb{R}^{l-k}$ with $k = 1, \dots, l-1$; since the action of the toral and vectorial factors commute, one may reach the general case by composing a periodic and an aperiodic deformation.

In the aperiodic case, by the assumption of properness, the action is also free: proper actions possess compact isotropy groups, but $(\mathbb{R}^l, +)$ has no nontrivial compact subgroups. Under a proper, free action, the orbit space $M/\mathbb{R}^l$ is a (Hausdorff) smooth manifold, and the quotient map $\pi : M \to M/\mathbb{R}^l$ defines an $\mathbb{R}^l$-principal bundle projection [22, Theorem 1.11.4]. Even though this bundle is trivializable (see Section 6) and some of our results could thereby be extracted from [29,41], we adopt here an intrinsic approach more compatible with eventual generalizations. In fact, the crucial Dixmier trace computation in Theorem 6.1 requires new techniques.

In the periodic case, the action is obviously not free in general; in [27], one of us has shown that the set of singular points for the action (i.e., points with nontrivial isotropy group) may give rise to a new type of UV/IR mixing phenomenon for isospectral deformations.

Note also that on noncompact manifolds, both periodic and aperiodic deformations may occur; whereas when M is compact, to be proper, the action α must be periodic.

For torus actions, each $f \in C_c^\infty(M)$ can be isotypically decomposed via Peter–Weyl decomposition as a $\|\cdot\|_\infty$-norm convergent sequence (see [12,13] for further details):

$$f = \sum_{r \in \mathbb{Z}^l} f_r, \tag{2.3}$$

where each homogeneous component f_r satisfies $\alpha_z(f_r) = e^{-izr} f_r$, for all $z \in \mathbb{T}^l$. In this case, the twisted product reproduces the canonical commutation relations for the noncommutative l-torus, since

$$f \star h = \sum_{r,s \in \mathbb{Z}^l} e^{-\frac{1}{2} ir\cdot\Theta s} f_r h_s.$$

This computation shows that in the periodic noncompact case, $(C_c^\infty(M), \star)$ closes to an algebra: while this product is nonlocal on the orbits of the action, the twisted product of two functions $f, h \in C_c^\infty(M)$ is again smooth and compactly supported because $\operatorname{supp} f_r \subset \mathbb{T}^l \cdot (\operatorname{supp} f)$ and thus $\operatorname{supp}(f \star h) \subset \mathbb{T}^l \cdot (\operatorname{supp} f \cap \operatorname{supp} h)$.

This need not be the case for aperiodic deformations, whose orbits are noncompact. At this level of generality, one can only prove, using Lemma 3.4, that $f \star h \in C^\infty(M) \cap L^\infty(M, \mu_g)$.

3. Hilbertian analysis of deformed products

The Moyal product (2.2) is defined on functions, but the operator of left-twisted multiplication $L_f : h \mapsto f \star h$ may be lifted to spinors by replacing $\alpha_{-z} h$ by $V_{-z}\psi$ in the defining formula. We find it convenient to work at both levels, on the "reduced" Hilbert space of functions $\mathcal{H}_r := L^2(M, \mu_g)$ and the "full" Hilbert space of square-integrable spinors $\mathcal{H} = L^2(M, S)$. (Here μ_g is the Riemannian volume form.) Somewhat abusively, we denote the left multiplication operators on both spaces by the same symbol L_f, trusting that the context will make clear which is which.

3.1. Kernel properties of the $\star$ product

We begin by showing that, for $f \in C_c^\infty(M)$, the operator of left-twisted multiplication L_f is a bounded kernel operator on $\mathcal{H}$ (or on $\mathcal{H}_r$). The same properties hold for the right-twisted multiplication operator R_f. We adopt the notation M_f for the (left or right) ordinary multiplication operator by f, corresponding to the case $\Theta = 0$.

Definition 3.1. For $f \in C_c^\infty(M)$, the operator of left-twisted multiplication L_f acting on $\mathcal{H} = L^2(M, S)$ is defined for $p \in M$ by

$$L_f\psi(p) := (2\pi)^{-l} \int_{\mathbb{R}^l}\int_{\mathbb{R}^l} e^{-iyz}(\alpha_{\frac{1}{2}\Theta y} f)(p) V_{-z}\psi(p)\, d^l y\, d^l z. \tag{3.1}$$

(When the spin lifting of the action α does not split, the right-hand side must be replaced by

$$(2\pi)^{-l} \int_{\mathbb{R}^l}\int_{\widetilde{\mathbb{R}}^l} e^{-iyp(\tilde{z})}(\alpha_{\frac{1}{2}\Theta y} f)(p) V_{-\tilde{z}}\psi(p)\, d^l y\, d^l \tilde{z},$$

but we shall keep the version (3.1) to simplify the notation.)

Definition 3.2. For any $p \in M$, let $\delta_p^g \in \mathcal{D}'(M)$ be the distribution defined for $\phi \in C_c^\infty(M)$ by

$$\langle \delta_p^g, \phi \rangle = \int_M \delta_p^g(p')\phi(p')\mu_g(p') := \phi(p).$$

The distribution δ_p^g is represented by $(\det g(x))^{-1/2}\delta(x - x')$ in a local coordinate system, and the product $\delta_p^g \mu_g$ can also be thought of as a de Rham n-current [44].

Proposition 3.3. *Let α be a smooth proper and isometric action of $\mathbb{R}^l$. When $f \in C_c^\infty(M)$, L_f is a bounded kernel operator on $\mathcal{H}$ (or on $\mathcal{H}_r$), with Schwartz kernel*

$$K_{L_f}(p, p') = (2\pi)^{-l} \int_{\mathbb{R}^{2l}} e^{-iyz} f\big(\big(-\tfrac{1}{2}\Theta y\big) \cdot p\big)\delta_{z\cdot p}^g(p')\, d^l y\, d^l z. \tag{3.2}$$

Before giving a proof, we need the following lemma.

Lemma 3.4. *If $f \in C_c^\infty(M)$ and the action α of $\mathbb{R}^l$ is free, then for all $k \in \mathbb{N}$,*

$$\sup_{p \in M} \int_{\mathbb{R}^l} \big|\Delta_\alpha^k(\alpha_y f)(p)\big|\, d^l y < \infty,$$

where $\Delta_\alpha := -\sum_{j=1}^l X_j^2$ is the Casimir operator.

Proof. For any fixed k, the map $\tilde{f}(p) := \int_{\mathbb{R}^l} |\Delta_\alpha^k(\alpha_y f)(p)|\, d^l y$ is well defined since $\{y \in \mathbb{R}^l\colon \alpha_y(p) \in \operatorname{supp} f\}$ is compact for each $p \in M$ [39, p. 41] because f has compact support. This gives rises to a finite y-integration and $\tilde{f} \in C^\infty(M)^G$. Let $\pi : M \to M/\mathbb{R}^l$ be the projection on the orbit space. Then $\tilde{f}$ factors through π to give a map $\bar{f}$ defined by $\bar{f}(\pi(p)) := \tilde{f}(p)$. This yields the result since $\bar{f} \in C_c^\infty(M/\mathbb{R}^l)$, because if $p \notin \alpha_{\mathbb{R}^l}(\operatorname{supp} f)$, so that $\pi(p)$ is not in the compact set $\pi(\operatorname{supp} f)$, then $\bar{f}(\pi(p)) = 0$. □

Proof of Proposition 3.3. For $\psi \in \mathcal{H}$, we can write, according to (3.1)

$$L_f \psi(p) = (2\pi)^{-l} \int_{\mathbb{R}^l}\int_{\mathbb{R}^l} e^{-iyz} \alpha_{\frac{1}{2}\Theta y}(f)(p) \int_M \delta^g_{z\cdot p}(p')\psi(p')\mu_g(p')\, d^l y\, d^l z.$$

The form of the kernel (3.2) is then obtained by interchange of integrals. In the aperiodic case, that α is proper is equivalent (see [39, Definition 5.1]) to the compactness of $\{y \in \mathbb{R}^l\colon y \cdot K \cap L \neq \emptyset\}$ for any compact subsets K and L of M. So for $K = L = \{p\}$, for any $p \in M$, this implies that its isotropy subgroup $H_p \subset \mathbb{R}^l$ is compact. Hence $H_p = \{0\}$ for all $p \in M$ since α is free.

Boundedness of L_f follows by a standard oscillatory-integral trick [26,37,43]:

$$\begin{aligned}
L_f \psi(p) &= (2\pi)^{-l} \int_{\mathbb{R}^l}\int_{\mathbb{R}^l} e^{-iyz} \alpha_{\frac{1}{2}\Theta y}(f) V_{-z}\psi(p)\, d^l y\, d^l z \\
&= (2\pi)^{-l} \int_{\mathbb{R}^l} \big(1 + |z|^2\big)^{-r} \int_{\mathbb{R}^l} \big(1 + |z|^2\big)^r e^{-iyz} \alpha_{\frac{1}{2}\Theta y}(f)\, d^l y V_{-z}\psi(p)\, d^l z \\
&= (2\pi)^{-l} \int_{\mathbb{R}^l} \big(1 + |z|^2\big)^{-r} \int_{\mathbb{R}^l} \big((1 + \Delta_y)^r e^{-iyz}\big) \alpha_{\frac{1}{2}\Theta y}(f)\, d^l y V_{-z}\psi(p)\, d^l z \\
&= (2\pi)^{-l} \int_{\mathbb{R}^l} \big(1 + |z|^2\big)^{-r} \int_{\mathbb{R}^l} e^{-iyz} \big((1 + \Delta_\alpha)^r \alpha_{\frac{1}{2}\Theta y}(f)\big)\, d^l y V_{-z}\psi(p)\, d^l z,
\end{aligned}$$

where boundary terms vanish due to the compactness of supp f. Hence,

$$\|L_f\psi\| \leqslant (2\pi)^{-l}\|\psi\| \bigg(\int_{\mathbb{R}^l} \big(1 + |z|^2\big)^{-r} d^l z \bigg) \sup_{p \in M} \int_{\mathbb{R}^l} \big|(1 + \Delta_\alpha)^r \alpha_{\frac{1}{2}\Theta y}(f)(p)\big|\, d^l y \tag{3.3}$$

is finite for $r > l/2$, thanks to Lemma 3.4. □

Remark 3.5. In the periodic (compact or not) case, that L_f is bounded for $f \in C_c^\infty(M)$ is a direct consequence of the Peter–Weyl decomposition (2.3). Indeed, the relation

$$L_{f_r} = M_{f_r} V_{-\frac{1}{2}\Theta r} \tag{3.4}$$

implies

$$\|L_f\| \leqslant \sum_{r \in \mathbb{Z}^l} \|M_{f_r} V_{-\frac{1}{2}\Theta r}\| \leqslant \sum_{r \in \mathbb{Z}^l} \|f_r\|_\infty,$$

which is finite since decomposition (2.3) is convergent in the sup norm.

Furthermore, the Schwartz kernel of L_f is

$$K_{L_f}(p,p') = \sum_{r\in\mathbb{Z}^l} f_r(p)\delta^g_{\frac{1}{2}\Theta r\cdot p}(p').$$

Remark 3.6. From the estimates used in the proof of Proposition 3.3, it is clear that this result, as well as all the statements of this section, holds for a wider class of smooth functions decreasing fast enough at infinity. It is in particular the case for smooth functions which satisfy (with an obvious abuse of notation) $\int_{\mathbb{R}^l} \alpha_y(f)\,d^l y \in C_c^\infty(M/\mathbb{R}^l)$ and $\int_{\mathbb{R}^l} |y^\beta X^\gamma \alpha_y(f)|\,d^l y < \infty$, for all $\beta,\gamma \in \mathbb{N}^l$, i.e., which are compactly supported once projected on the orbit space, and which are in the Schwartz space of the orbits. However, for the sake of simplicity, we only consider in the sequel functions in $C_c^\infty(M)$, which of course have those properties.

3.2. Hilbert–Schmidt norm invariance

We are now concerned with invariance properties for kernels of operators of type $h(\not{D})$, where h is any bounded positive smooth function, and $\not{D}$ is the Dirac operator on S. Since we want $\not{D}$ to be essentially selfadjoint with domain $C_c^\infty(M)$ (and we still denote by $\not{D}$ its selfadjoint closure), it is sufficient, by a result of Wolf [49], to assume from now on that M *is geodesically complete*.

Lemma 3.7. *Let h be a bounded positive smooth function on $\mathbb{R}$. Then the kernel $K_{h(\not{D})}$ is α-invariant: for all $z \in \mathbb{R}^l$, $p, p' \in M$,*

$$K_{h(\not{D})}(z\cdot p, z\cdot p') = K_{h(\not{D})}(p,p'),$$

except possibly on a nullset of $M\times M$.

Proof. This is a direct consequence of the isometry property of α; indeed, the invariance of the Levi-Civita connection for g entails invariance of the spin connection under the lifted action on spinors, so that $V_z \not{D} V_{-z} = \not{D}$ for all z.

This implies that $[V_z, h(\not{D})] = 0$ for all $z \in \mathbb{R}^l$. Thus, for $\psi \in \mathcal{H}$, the invariance of the Riemannian volume form under the diffeomorphism α_{-z} yields

$$\int_M K_{h(\not{D})}(z\cdot p, z\cdot p')\psi(p')\mu_g(p') = \int_M K_{h(\not{D})}(z\cdot p, p')\psi\big((-z)\cdot p'\big)\mu_g(p').$$

The right-hand side equals $(h(\not{D})V_z\psi)(z\cdot p) = (V_{-z}h(\not{D})V_z\psi)(p) = (h(\not{D})\psi)(p)$.

Thus, $K_{h(\not{D})}(\alpha_z(\cdot), \alpha_z(\cdot))$ and $K_{h(\not{D})}$ represent the same operator on $\mathcal{H}$. $\square$

The main result of this section is the following equality, which shows that the Hilbert–Schmidt norm of $L_f h(\not{D})$ is independent of the deformation parameters in Θ.

Theorem 3.8. *Let $f \in C_c^\infty(M)$ and h be a bounded positive function on $\mathbb{R}$ such that $M_f h(\not{D})$ is a Hilbert–Schmidt operator. Then the operator $L_f h(\not{D})$ is also Hilbert–Schmidt, with*

$$\big\|L_f h(\not{D})\big\|_2 = \big\|M_f h(\not{D})\big\|_2.$$

Proof. First, by Proposition 3.3, one can compute the kernel of $L_f h(\not{p})$ in terms of $K_{h(\not{p})}$:

$$\begin{aligned}
K_{L_f h(\not{p})}(p,p') &= \int_M K_{L_f}(p,q) K_{h(\not{p})}(q,p')\mu_g(q) \\
&= (2\pi)^{-l} \int_M \int_{\mathbb{R}^{2l}} e^{-iyz} f\big(\big(-\tfrac{1}{2}\Theta y\big)\cdot p\big)\delta^g_{z\cdot p}(q) K_{h(\not{p})}(q,p')\, d^l y\, d^l z\, \mu_g(q) \\
&= (2\pi)^{-l} \int_{\mathbb{R}^{2l}} e^{-iyz} f\big(\big(-\tfrac{1}{2}\Theta y\big)\cdot p\big) K_{h(\not{p})}(z\cdot p,p')\, d^l y\, d^l z. \qquad (3.5)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big\|L_f h(\not{p})\big\|_2^2 &= \int_{M\times M} \big|K_{L_f h(\not{p})}(p,p')\big|^2 \mu_g(p)\mu_g(p') \\
&= (2\pi)^{-2l} \int_{M\times M} \int_{\mathbb{R}^{4l}} e^{i(y_1 z_1 - y_2 z_2)} \bar{f}\big(\big(-\tfrac{1}{2}\Theta y_1\big)\cdot p\big) f\big(\big(-\tfrac{1}{2}\Theta y_2\big)\cdot p\big) \\
&\quad \times \overline{K_{h(\not{p})}}(z_1\cdot p,p') K_{h(\not{p})}(z_2\cdot p,p')\, d^l y_1\, d^l z_1\, d^l y_2\, d^l z_2\, \mu_g(p)\mu_g(p') \\
&= (2\pi)^{-2l} \int_{M\times M} \int_{\mathbb{R}^{4l}} e^{i(y_1 z_1 - y_2 z_2)} \bar{f}\big(\big(-\tfrac{1}{2}\Theta y_1 - z_2\big)\cdot p\big) f\big(\big(-\tfrac{1}{2}\Theta y_2 - z_2\big)\cdot p\big) \\
&\quad \times \overline{K_{h(\not{p})}}\big((z_1 - z_2)\cdot p, (z_1 - z_2)\cdot p'\big) \\
&\quad \times K_{h(\not{p})}\big(p, (z_1 - z_2)\cdot p'\big)\, d^l y_1\, d^l z_1\, d^l y_2\, d^l z_2\, \mu_g(p)\mu_g(p'),
\end{aligned}$$

where we used the invariance of μ_g under the isometries $p \mapsto (-z_2)\cdot p$ and $p' \mapsto (z_1 - z_2)\cdot p'$. Now by Lemma 3.7, using the translation $z_1 \mapsto z_1 + z_2$, the last expression becomes

$$\begin{aligned}
&(2\pi)^{-2l} \int_{M\times M} \int_{\mathbb{R}^{4l}} e^{i(y_1(z_1+z_2) - y_2 z_2)} \bar{f}\big(\big(-\tfrac{1}{2}\Theta y_1 - z_2\big)\cdot p\big) f\big(\big(-\tfrac{1}{2}\Theta y_2 - z_2\big)\cdot p\big) \\
&\quad \times \overline{K_{h(\not{p})}}(p,p') K_{h(\not{p})}(p, z_1\cdot p')\, d^l y_1\, d^l z_1\, d^l y_2\, d^l z_2\, \mu_g(p)\mu_g(p') \\
&= (2\pi)^{-2l} \int_{M\times M} \int_{\mathbb{R}^{4l}} e^{i((y_1 - 2\Theta^{-1} z_2)(z_1+z_2) - y_2 z_2)} \bar{f}\big(\big(-\tfrac{1}{2}\Theta y_1\big)\cdot p\big) f\big(\big(-\tfrac{1}{2}\Theta y_2\big)\cdot p\big) \\
&\quad \times \overline{K_{h(\not{p})}}(p,p') K_{h(\not{p})}(p, z_1\cdot p')\, d^l y_1\, d^l z_1\, d^l y_2\, d^l z_2\, \mu_g(p)\mu_g(p'),
\end{aligned}$$

on making the translations $y_1 \mapsto y_1 - 2\Theta^{-1} z_2$ and $y_2 \mapsto y_2 - 2\Theta^{-1} z_2$. This yields

$$\begin{aligned}
&(2\pi)^{-l} \int_{M\times M} \int_{\mathbb{R}^{2l}} e^{iyz} \bar{f}\big(\big(-\tfrac{1}{2}\Theta y\big)\cdot p\big) f\big(\big(-\tfrac{1}{2}\Theta y - z\big)\cdot p\big) \\
&\quad \times \overline{K_{h(\not{p})}}(p,p') K_{h(\not{p})}(p, z\cdot p')\, d^l y\, d^l z\, \mu_g(p)\mu_g(p')
\end{aligned}$$

$$= (2\pi)^{-l} \int\limits_{M\times M} \int\limits_{\mathbb{R}^{2l}} e^{iyz} \bar{f}(p) f\big((-z)\cdot p\big) \overline{K_{h(\not{D})}}(p,p') K_{h(\not{D})}(p, z\cdot p')\, d^l y\, d^l z\, \mu_g(p)\mu_g(p')$$

$$= \int\limits_{M\times M} \big|f(p)\big|^2 \big|K_{h(\not{D})}(p,p')\big|^2 \mu_g(p)\mu_g(p') = \big\|M_f h(\not{D})\big\|_2^2.$$

The second equality uses the isometries $p \mapsto (\frac{1}{2}\Theta y)\cdot p$ and $p' \mapsto (\frac{1}{2}\Theta y)\cdot p'$. □

Remark 3.9. Naturally, this result is still true in the restricted case of a scalar Laplacian, i.e., for $L_f h(\Delta_r)$, with Δ_r the scalar Laplacian acting on the reduced Hilbert space $\mathcal{H}_r = L^2(M,\mu_g)$.

We shall see in the next section sufficient conditions on h implying that $M_f h(\not{D})$ lies in the Hilbert–Schmidt ideal.

Corollary 3.10. *If $L_f h(\not{D})$ and $M_f h(\not{D})$ are trace-class operators, then their traces coincide*:

$$\operatorname{Tr}\big(L_f h(\not{D})\big) = \operatorname{Tr}\big(M_f h(\not{D})\big).$$

Proof. The translation-invariance property of Lemma 3.7 and expression (3.5) for the kernel of $L_f h(\not{D})$ yield the equalities

$$\begin{aligned}
\operatorname{Tr}\big(L_f h(\not{D})\big) &= \int\limits_M K_{L_f h(\not{D})}(p,p)\mu_g(p) \\
&= (2\pi)^{-l} \int\limits_M \int\limits_{\mathbb{R}^{2l}} e^{-iyz} f\big(\big(-\tfrac{1}{2}\Theta y\big)\cdot p\big) K_{h(\not{D})}(z\cdot p, p)\, d^l y\, d^l z\, \mu_g(p) \\
&= (2\pi)^{-l} \int\limits_M \int\limits_{\mathbb{R}^{2l}} e^{-iyz} f(p') K_{h(\not{D})}(z\cdot p', p')\, d^l y\, d^l z\, \mu_g(p') \\
&= \int\limits_M f(p') K_{h(\not{D})}(p',p')\mu_g(p') = \operatorname{Tr}\big(M_f h(\not{D})\big). \qquad \square
\end{aligned}$$

The Riemannian volume form gives a natural trace for the twisted product.

Lemma 3.11. *For $f, h \in C_c^\infty(M)$,*

$$\int\limits_M (f\star h)\mu_g = \int\limits_M f h \mu_g.$$

Proof. It is enough to notice that, with $p\in M$,

$$\int\limits_M f\star h(p)\mu_g(p) = (2\pi)^{-l} \int\limits_M \int\limits_{\mathbb{R}^{2l}} e^{-iyz} f\big(\big(-\tfrac{1}{2}\Theta y\big)\cdot p\big) h(z\cdot p)\, d^l y\, d^l z\, \mu_g(p)$$

$$= (2\pi)^{-l} \int_M \int_{\mathbb{R}^{2l}} e^{-iyz} f\big(\big(-\tfrac{1}{2}\Theta y - z\big) \cdot p\big) h(p)\, d^l y\, d^l z\, \mu_g(p)$$

$$= (2\pi)^{-l} \int_M \int_{\mathbb{R}^{2l}} e^{-iyz} f\big((-z) \cdot p\big) h(p)\, d^l y\, d^l z\, \mu_g(p)$$

$$= \int_M f(p) h(p) \mu_g(p),$$

using the isometry $p \mapsto (-z) \cdot p$ and the translation $z \mapsto z - \frac{1}{2}\Theta y$. $\square$

Remark 3.12. For formal deformations, Felder and Shoikhet [25] have shown that a divergenceless Poisson bivector field yields a star-product which is tracial. The divergence of Π_Θ is a vector field, given in local coordinates by

$$\operatorname{div} \Pi_\Theta = \big(\partial_j \Theta^{ij} + \Gamma^l_{lk} \Theta^{ik}\big) \partial_i,$$

where Γ^i_{jk} are the Christoffel symbols for the metric g. Thus, Π_Θ will be divergenceless if and only if [33, Chapter 7]:

$$\Theta^{ij} \partial_i \big(\log \sqrt{\det g}\,\big) + \partial_l \Theta^{lj} = 0.$$

This implies that Θ must be of constant rank [24]. A result parallel to that of [25], in our context, would suggest that variable noncommutativity matrices should prevail in non-flat backgrounds, although one may admit a nonconstant, divergenceless Θ in a flat background (the case considered in [30]) or a constant, divergenceless Θ in a non-flat background (as we do here).

In what follows, we shall take advantage of the possibility of viewing L_f, for $f \in C_c^\infty(M)$, as an integral of bounded operators:

$$L_f = (2\pi)^{-l} \int_{\mathbb{R}^{2l}} e^{-iyz} V_{\frac{1}{2}\Theta y} M_f V_{-\frac{1}{2}\Theta y - z}\, d^l y\, d^l z. \tag{3.6}$$

This is not a Bochner integral (the integral of the norm of the integrand is not absolutely convergent), but rather a $\mathcal{B}(\mathcal{H})$-valued oscillatory integral, as shown in the proof of Proposition 3.3.

The invariance property of the Hilbert–Schmidt norm can be generalized as follows. One can construct left and right twists for a wider class of bounded operators. For $A \in \mathcal{B}(\mathcal{H})$ we formally define its left and right twists by

$$L_A := (2\pi)^{-l} \int_{\mathbb{R}^{2l}} e^{-iyz} V_{\frac{1}{2}\Theta y} A V_{-\frac{1}{2}\Theta y - z}\, d^l y\, d^l z,$$

$$R_A := (2\pi)^{-l} \int_{\mathbb{R}^{2l}} e^{-iyz} V_{-z} A V_{z + \frac{1}{2}\Theta y}\, d^l y\, d^l z.$$

These expressions are well defined, at least, for Hilbert–Schmidt operators thanks to the following generalization of Theorem 3.8.

Theorem 3.13. *Let A be a Hilbert–Schmidt operator. Then L_A and R_A are also Hilbert–Schmidt operators and*

$$\|L_A\|_2 = \|R_A\|_2 = \|A\|_2.$$

Proof. We treat L_A only. The kernel K_A of A lies in $L^2(M \times M, \mu_g \times \mu_g)$, and we can express K_{L_A} in terms of K_A:

$$K_{L_A}(p, p') = (2\pi)^{-l} \int\limits_{\mathbb{R}^{2l}} e^{-iyz} K_A\big(\tfrac{1}{2}\Theta y \cdot p, z \cdot p'\big)\, d^l y\, d^l z.$$

Thus, routine computations yield that the map $K_A \mapsto K_{L_A}$ is an isometry on $L^2(M \times M, \mu_g \times \mu_g)$:

$$\|L_A\|_2 = \int\limits_{M \times M} \big|K_{L_A}(p, p')\big|^2 \mu_g(p)\mu_g(p') = \|A\|_2. \qquad \square$$

4. Schatten-class estimates for twisted multiplication operators

In this section, we give Schatten-norm estimates for the operators $M_f(1 + \Delta_r)^{-k}$ and $L_f(1 + \Delta_r)^{-k}$ acting on the reduced Hilbert space $\mathcal{H}_r = L^2(M, \mu_g)$, where Δ_r is the Laplacian $(d + d^*)^2$ reduced to 0-forms (in our convention, the Laplacian is a positive operator). This will be done using heat kernel estimates and the Laplace transform for $(1 + \Delta_r)^{-k}$, together with Proposition 3.3 and Theorem 3.8. For convenience and when no ambiguity can occur, we shall omit the subscript r for the reduced Laplacian.

We use the notations $\mathcal{L}^p(\mathcal{H})$, $p \geqslant 1$, for the p-Schatten class of operators on the Hilbert space $\mathcal{H}$ and $\mathcal{L}^{n,\infty}(\mathcal{H})$ for the n^+-summable operators on $\mathcal{H}$.

4.1. Some heat-kernel estimates

We now need to make some more precise assumptions on the geometry of M, which give some (mild) controls on the asymptotics of the heat kernel.

Let $K_t(p, p')$ denote the heat kernel, associated to the operator $e^{-t\Delta_r}$, defined on $M \times M$ for $0 < t < \infty$. Recall that, in full generality, each $K_t(p, p')$ is a smooth strictly positive symmetric function on $M \times M$ [16, Theorem 5.2.1].

For the remainder of the article, we shall suppose that the manifold M satisfies the following hypothesis.

Condition 4.1. M is a complete connected Riemannian spin manifold of dimension $n \geqslant 2$ without boundary such that

$$\sup_{p \in M} \int\limits_0^\infty t^k e^{-t} K_t(p, p)\, dt < \infty \quad \text{for all } k > \frac{n}{2} - 1, \tag{4.1}$$

and for some $c > 0$,

$$\sup_{p\in M}\int_m^\infty t^{-1/2}e^{-t}K_t(p,p)\,dt < cm^{-(n-1)/2} \quad \text{for all } m \in (0,1]. \tag{4.2}$$

These constraints imply a control of the heat kernel near 0 and ∞ which is sufficient for the Dixmier trace computations. They are not too severe, as the next lemma shows. Some such controls are necessary because for any complete Riemannian manifold of finite volume $V(M)$, in particular for a compact manifold, $\int_1^\infty K_t(p,p)\,dt = \infty$ holds since $\lim_{t\to\infty} K_t(p,p') = V(M)^{-1}$.

Let $B(p,r) := \{p' \in M\colon d_g(p,p') < r\}$ denote the geodesic ball centered at p with radius r. The isoperimetric constant $\mathcal{I}(M)$ is given [7, p. 96] by

$$\mathcal{I}(M) := \inf_{\Omega} \frac{A(\partial\Omega)^n}{V(\Omega)^{n-1}},$$

where Ω ranges over all open submanifolds with compact closure in M and with smooth boundary, $V(\Omega)$ and $A(\partial\Omega)$ are the Riemannian volume and area of Ω and $\partial\Omega$, respectively.

Lemma 4.2. *Let M be a complete Riemannian manifold satisfying one of the following*:

(1) *M has Ricci curvature bounded from below, that is,* $\mathrm{Ric}(p) \geqslant (n-1)\beta$, *for all $p \in M$ and some constant β. Moreover,* $\sup_{p\in M} V(B(p,a))^{-1} < \infty$ *for some $a > 0$.*
(2) *M is noncompact with a positive injectivity radius, and there exists $a > 0$ such that* $\sup_{p\in M} \mathcal{I}(B(p,a))^{-1} < \infty$. (*This last property holds if M has a positive isoperimetric constant*: $\mathcal{I}(M) > 0$.)

Then inequalities (4.1) *and* (4.2) *hold for M.*

Proof. Assume the first condition. In [17, Lemma 15]—see also [16]—we get the following estimates. Given $\varepsilon > 0$, there exists a constant c_ε such that, for all $t > 0$ and $p \in M$,

$$0 \leqslant K_t(p,p) \leqslant c_\varepsilon(n) V\big(B\big(p,t^{1/2}\big)\big)^{-1} e^{(\varepsilon-E)t},$$

where $E := \inf \mathrm{sp}(\Delta) \geqslant 0$. Since, by [8, Proposition 4.1],

$$V\big(B(p,r)\big) \geqslant cr^n V\big(B(p,1)\big) \quad \text{for } 0 < r < 1,$$

we get

$$K_t(p,p) \leqslant \begin{cases} C_2(\varepsilon)t^{-n/2}V(B(p,1))^{-1}e^{(\varepsilon-E)t}, & t \leqslant 1, \\ C_3(\varepsilon)V(B(p,1))^{-1}e^{(\varepsilon-E)t}, & t > 1. \end{cases} \tag{4.3}$$

Now suppose instead that the second condition holds. In [7, Theorem 8, p. 198], it is proved that the heat kernel has an upper bound: for all $p \in M$ and $r > 0$ for which $\overline{B(p,r)}$ lies in the image of the exponential map $\exp_p$, the following estimate holds:

$$K_t(p,p) \leqslant C_1(n)\big(t^{-n/2} + r^{-(n+2)}t\big)\mathcal{I}\big(B(p,r)\big)^{-1}. \tag{4.4}$$

In case (1), we assumed that $\sup_{p\in M} V(B(p,a))^{-1} < \infty$ for some $a > 0$. Similarly, the constraint on the injectivity radius in case (2) implies that for some r_0, $\overline{B(p,r_0)}$ lies in the image of the exponential maps $\exp_p$ for all $p \in M$.

Thus estimates (4.4) and (4.3) yield

$$K_t(p,p) \leqslant c_1\big(t^{-n/2} + c_2 t\big)\max\big(e^{(\varepsilon_0 - E)t}, 1\big)$$

for some positive constants c_1, c_2, independent of p, for a fixed $\varepsilon = \varepsilon_0 < 1$.

Let $b = \max(\varepsilon_0 - E - 1, -1)$. Then $b < 0$ and

$$\begin{aligned}\sup_{p\in M}\int_0^\infty t^k e^{-t} K_t(p,p)\,dt &\leqslant c_1\int_0^\infty t^{k-n/2}e^{bt}\,dt + c_2\int_0^\infty t^{k+1}e^{bt}\,dt\\ &= c_1\Gamma(k-n/2+1)b^{-(k-n/2+1)} + c_2\Gamma(k+2)b^{-(k+2)}\end{aligned}$$

is finite and (4.1) holds.

Similarly, we get

$$\begin{aligned}\sup_{p\in M}\int_m^\infty t^{-1/2}e^{-t}K_t(p,p)\,dt &\leqslant c_1\sup_{p\in M}\int_m^\infty t^{-(n+1)/2}\,dt + c_2\sup_{p\in M}\int_m^\infty te^{bt}\,dt\\ &= c_1\frac{2}{n-1}m^{-(n-1)/2} + c_2\frac{1-mb}{b^2}e^{mb}.\end{aligned}$$

Since $(1 - mb)e^{mb} < 1 - mb < (1 - mb)m^{-(n-1)/2}$ for $0 < m < 1$, inequality (4.2) also holds. $\square$

Remark 4.3. Since $\sup_{p\in M} K_t(p,p)$ is decreasing in t, condition (4.1) is satisfied if, for some $c' > 0$,

$$\sup_{p\in M} K_t(p,p) < c'e^t t^{-n/2} \quad \text{for all } 0 < t < 1.$$

It is known (see [15], for instance) that

$$\big\|e^{-t\Delta}\big\|_{1\to\infty} = \sup_{p\in M} K_t(p,p).$$

Thus, changing Δ to $1 + \Delta$, condition (4.1) is guaranteed by

$$\big\|e^{-t(1+\Delta)}\big\|_{1\to\infty} < c't^{-n/2} \quad \text{for all } 0 < t < 1. \tag{4.5}$$

This can be reformulated in many different ways, according to [14]. For $n > 2$, (4.5) is equivalent to the boundedness of the operator $(1 + \Delta)^{-1/2} : L^2(M,\mu) \to L^{2n/(n-2)}(M,\mu)$, or of the operator $(1 + \Delta)^{-\alpha/2} : L^p(M,\mu) \to L^q(M,\mu)$, for $1 < p < \infty$, $\alpha p < n$ and $1/q = 1/p - \alpha/n$. This can be used in the next subsection.

According to [15, Proposition 1.2], this implies that $\sup_{p\in M} V(B(p,1))^{-1} < \infty$, for $n > 2$.

Note that a strictly positive isoperimetric constant is a stronger condition than (4.1): see [15]. For instance, when $M = \mathbb{R}^n$ with its Euclidean metric, $K_t(p,p) = (4\pi t)^{-n/2}$ for all $p \in M$ and $t > 0$.

Remark 4.4. Recall that a *bounded geometry* on a connected manifold M is a Riemannian metric on M whose injectivity radius is positive and satisfies $|\nabla^k R| \leqslant C_k$, $k \in \mathbb{N}$, i.e., every covariant derivative of the Riemann curvature tensor is bounded: see [7,45,47]. Such a Riemannian manifold is automatically complete and satisfies Condition 4.1. In fact, any n-dimensional manifold with positive injectivity radius and Ricci curvature uniformly bounded below obeys an upper bound: $\sup_{p\in M} K_t(p,p) \leqslant C \max(t^{-n/2}, t^{-1/2})$ for all $t > 0$: see [35, Theorem 7.9]. Thus (4.1) and (4.2) are valid.

Examples of manifolds with bounded geometry are given by Lie groups, homogeneous manifolds with invariant metrics, covering manifolds of compact manifolds with the lifted Riemannian metric, leaves of a foliation on a compact manifold with a metric induced by the Riemannian metric on the compact manifold. In particular, all manifolds with a transitive group of isometries have C^∞-bounded geometry.

4.2. Schatten-class estimates

We start with a straightforward consequence of (4.1).

Lemma 4.5. *Assume that M satisfies Condition* 4.1. *Then $(1+\Delta)^{-k}$ is a bounded operator from $L^2(M,\mu_g)$ to $L^\infty(M,\mu_g)$, for all $k > n/4$.*

Proof. Let $\phi \in L^2(M,\mu_g)$. Using the Cauchy–Schwarz inequality, positivity and symmetry of $K_{(1+\Delta)^{-k}}$, positivity of μ_g, the product rule for kernel operators and the Laplace transform $(1+\Delta)^{-2k} = \Gamma(2k)^{-1} \int_0^\infty t^{2k-1} e^{-t(1+\Delta)}\,dt$, we get

$$
\begin{aligned}
\big\|(1+\Delta)^{-k}\phi\big\|_\infty^2 &= \sup_{p\in M} \left| \int_M K_{(1+\Delta)^{-k}}(p,p')\phi(p')\mu_g(p') \right|^2 \\
&\leqslant \|\phi\|_2^2 \sup_{p\in M} \int_M \big|K_{(1+\Delta)^{-k}}(p,p')\big|^2 \mu_g(p') \\
&= \|\phi\|_2^2 \sup_{p\in M} \int_M K_{(1+\Delta)^{-k}}(p,p') K_{(1+\Delta)^{-k}}(p',p)\mu_g(p') \\
&= \|\phi\|_2^2 \sup_{p\in M} K_{(1+\Delta)^{-2k}}(p,p) \\
&= \frac{\|\phi\|_2^2}{\Gamma(2k)} \sup_{p\in M} \int_0^\infty t^{2k-1} e^{-t} K_t(p,p)\,dt.
\end{aligned}
$$

By (4.1), the t-integral is finite when $k > n/4$, so $\|(1+\Delta)^{-k}\phi\|_\infty \leqslant c(k)\|\phi\|_2$. $\square$

We now give the principal result of this subsection. Condition 4.1 is assumed throughout.

Proposition 4.6. *For $f \in L^2(M, \mu_g)$, the operator $M_f(1+\Delta)^{-k}$ is Hilbert–Schmidt for $k > n/4$ and satisfies*

$$\left\| M_f(1+\Delta)^{-k} \right\|_2 \leqslant C_k(n) \| f \|_2.$$

Proof. That the operator $M_f(1+\Delta)^{-k}$ is Hilbert–Schmidt is a consequence of the factorization principle of Grothendieck—see [18, Example 11.18], for instance—which is this context says that when two operators $B : L^2(X, \mu) \to L^\infty(X, \mu)$ and $A : L^\infty(X, \mu) \to L^2(X, \mu)$ are both bounded, their product AB is a Hilbert–Schmidt operator on $L^2(X, \mu)$.

Since for $f \in L^2(M, \mu_g)$, M_f is bounded from $L^\infty(M, \mu_g)$ into $L^2(M, \mu_g)$, Lemma 4.5 shows that $M_f(1+\Delta)^{-k}$ is Hilbert–Schmidt for $k > n/4$. For the Hilbert–Schmidt-norm estimate, we again use (4.4), (4.3) and Laplace transform techniques:

$$\begin{aligned}
\left\| M_f(1+\Delta)^{-k} \right\|_2^2 &= \int_{M \times M} \left| f(p) \right|^2 \left| K_{(1+\Delta)^{-k}}(p, p') \right|^2 \mu_g(p) \mu_g(p') \\
&= \int_M \left| f(p) \right|^2 K_{(1+\Delta)^{-2k}}(p, p) \mu_g(p) \\
&= \frac{1}{\Gamma(2k)} \int_M \left| f(p) \right|^2 \mu_g(p) \int_0^\infty t^{2k-1} e^{-t} K_t(p, p) \, dt \\
&\leqslant C_k(n)^2 \| f \|_2^2,
\end{aligned}$$

where we used again (4.1), the symmetry of $K_{(1+\Delta)^{-k}}$ and the product rule for kernels. □

Remark 4.7. The result of the previous proposition can be generalized at least for operators $M_f h(\sqrt{\Delta})$ where h is a Laplace transform of some function which behaves as t^{k-1} when $t \downarrow 0$, for $k > n/4$, and has fast enough decrease at infinity.

Theorem 4.8. *If $f \in L^p(M, \mu_g)$ with $2 \leqslant p < \infty$, then $M_f(1+\Delta)^{-k} \in \mathcal{L}^p(\mathcal{H}_r)$ for $k > n/4$.*

Proof. The case $p = 2$ is Proposition 4.6. For $p = \infty$, we use

$$\left\| M_f(1+\Delta)^{-k} \right\| \leqslant \| M_f \| \left\| (1+\Delta)^{-k} \right\| \leqslant \| f \|_\infty.$$

We use complex interpolation for $2 < p < \infty$. Firstly, note that we can always assume f to be nonnegative, since

$$\| M_f \| = \| M_{|f|} \|, \qquad \left\| M_f(1+\Delta)^{-k} \right\|_2 = \left\| M_{|f|}(1+\Delta)^{-k} \right\|_2.$$

Then, for $f \geqslant 0$ in $L^p(M, \mu_g)$, we define the map

$$F_p : z \mapsto M_f^{pz}(1+\Delta)^{-kpz},$$

for all z in the strip $S := \{z \in \mathbb{C}\colon\ 0 \leqslant \Re z \leqslant 1/2\}$. For all $y \in \mathbb{R}$, $F_p(iy) = M_f^{ipy}(1+\Delta)^{ikpy}$ is bounded with $\|F_p(iy)\| \leqslant 1$; and for $z = 1/2 + iy$, Proposition 4.6 shows that

$$\big\|F_p(1/2+iy)\big\|_2 = \big\|M_{f^{p/2}}(1+\Delta)^{-kp/2}\big\|_2 \leqslant C_{kp/2}(n)\big\|f^{p/2}\big\|_2 = C_{kp/2}(n)\|f\|_p^{p/2},$$

which is finite because $k > n/2p$. Then, interpolation [46] yields $F_p(z) \in \mathcal{L}^{1/\Re z}(\mathcal{H}_r)$ for all $z \in S$, and

$$\begin{aligned}\big\|F_p(z)\big\|_{1/\Re z} &\leqslant \big\|F_p(0)\big\|_\infty^{1-2\Re z}\big\|F_p(1/2)\big\|_2^{2\Re z} \leqslant \big\|M_f^{p/2}(1+\Delta)^{-kp/2}\big\|_2^{2\Re z} \\ &\leqslant C_{kp/2}(n)^{2\Re z}\big\|f^{p/2}\big\|_2^{2\Re z} = C_{kp/2}(n)^{2\Re z}\|f\|_p^{p\Re z}.\end{aligned}$$

So, for $z = 1/p$, we get

$$\big\|F_p(1/p)\big\|_p = \big\|M_f(1+\Delta)^{-k}\big\|_p \leqslant C_{kp/2}(n)^{2/p}\|f\|_p,$$

and the result follows. □

Proposition 4.9. *Let $2 \leqslant p < \infty$ and $f \in C_c^\infty(M)$. Then, if α is an isometric proper action of $\mathbb{R}^l$ on M, $L_f(1+\Delta)^{-k} \in \mathcal{L}^p(\mathcal{H}_r)$ for all $k > n/2p$.*

Proof. The proof is essentially the same as the previous one, so we only sketch it. Theorem 3.8 and Proposition 4.6 imply that, for $k > n/4$,

$$\big\|L_f(1+\Delta)^{-k}\big\|_2 = \big\|M_f(1+\Delta)^{-k}\big\|_2 \leqslant C_k(n)\|f\|_2.$$

Moreover, by (3.3),

$$\big\|L_f(1+\Delta)^{-k}\big\| \leqslant \|L_f\| \leqslant \widetilde{C}_r(l)\sup_{p\in M}\int_{\mathbb{R}^l}\big|(1+\Delta_y)^r\alpha_{\frac{1}{2}\Theta y}f(p)\big|\,d^ly =: \omega(f;r,l,n),$$

which is finite whenever $r > l/2$. Defining $G_p(z) := L_f(1+\Delta)^{-kpz}$ for $z \in S$ and $k > n/2p$, we conclude that, for all $y \in \mathbb{R}$,

$$\big\|G_p(iy)\big\| = \big\|L_f(1+\Delta)^{-ikpy}\big\| \leqslant \omega(f;r,l,n),$$

and

$$\big\|G_p(1/2+iy)\big\|_2 = \big\|L_f(1+\Delta)^{-kp/2}\big\|_2 \leqslant C_{kp/2}(n)\|f\|_2.$$

Again, complex interpolation gives the result:

$$\begin{aligned}\big\|L_f(1+\Delta)^{-k}\big\|_p &= \big\|G_p\big(p^{-1}\big)\big\|_p \leqslant \big\|G_p(0)\big\|_\infty^{1-2/p}\big\|G_p\big(2^{-1}\big)\big\|_2^{2/p} \\ &\leqslant \omega(f;r,l,n)^{1-2/p}C_{kp/2}(n)^{2/p}\|f\|_2^{2/p}.\end{aligned}$$ □

Remark 4.10. Using again the Peter–Weyl decomposition (2.3), one can show that this interpolation result holds also for periodic noncompact deformations.

We now show that the previous proposition extends directly to the spinor bundle.

Condition 4.11. Assume from now on that M satisfies Condition 4.1 and, moreover, has bounded scalar curvature.

This condition is satisfied for bounded geometries as noticed in Remark 4.4.

Corollary 4.12. *Let $2 \leqslant p < \infty$ and $f \in C_c^\infty(M)$. If α is an isometric proper action of $\mathbb{R}^l$ on M, then $L_f(1+\not{D}^2)^{-k}$ and $L_f(1+|\not{D}|)^{-2k}$ are in $\mathcal{L}^p(\mathcal{H})$ for all $k > n/2p$.*

Proof. Since $(1+\not{D}^2)^k(1+|\not{D}|)^{-2k}$ is bounded, it suffices to consider only $L_f(1+\not{D}^2)^{-k}$. For this operator, the result follows from a simple comparison argument using the Lichnerowicz formula

$$\not{D}^2 = \Delta + \tfrac{1}{4}R, \tag{4.6}$$

where R is the scalar curvature, bounded by hypothesis. Thus, the result follows from

$$\bigl(1+\not{D}^2\bigr)^{-1} = (1+\Delta)^{-1}\bigl(1 - \tfrac{1}{4}R\bigl(1+\not{D}^2\bigr)^{-1}\bigr). \qquad \square$$

Before finishing this subsection, we show for later use that the following commutators have the same summability properties as $L_f(1+\not{D}^2)^{-k}$.

Lemma 4.13. *If $f \in C_c^\infty(M)$ and $2 \leqslant p < \infty$, then the operators*

$$\begin{gathered}
[\not{D}, L_f]\bigl(1+\not{D}^2\bigr)^{-k}, \qquad \bigl[|\not{D}|, L_f\bigr]\bigl(1+\not{D}^2\bigr)^{-k}, \qquad \bigl[\bigl(1+\not{D}^2\bigr)^{1/2}, L_f\bigr]\bigl(1+\not{D}^2\bigr)^{-k}, \\
[\not{D}, L_f]\bigl(1+|\not{D}|\bigr)^{-2k}, \qquad \bigl[|\not{D}|, L_f\bigr]\bigl(1+|\not{D}|\bigr)^{-2k}, \qquad \bigl[\bigl(1+\not{D}^2\bigr)^{1/2}, L_f\bigr]\bigl(1+|\not{D}|\bigr)^{-2k}
\end{gathered}$$

all lie in $\mathcal{L}^p(\mathcal{H})$ whenever $k > n/2p$.

Proof. It is enough to prove this lemma in the $(1+\not{D}^2)^{-k}$ case.

For $[\not{D}, L_f](1+\not{D}^2)^{-k}$, this is a direct consequence of the isometry property of the action: since $\not{D}$ commutes with (the lift to the spinor bundle of) the action, we obtain

$$[\not{D}, L_f] = L_{[\not{D}, M_f]} = L_{\not{D}f}.$$

Hence the proof of Proposition 4.9 applies with $\not{D}f$ instead of f because $\not{D}f \in C_c^\infty(M)$.

For $[|\not{D}|, L_f]$, we can reduce the proof to the previous case by using the following spectral identity for a positive operator A:

$$A = \frac{1}{\pi}\int_0^\infty \frac{A^2}{A^2+\lambda}\,\frac{d\lambda}{\sqrt{\lambda}}. \tag{4.7}$$

Thus, for any positive number ρ,

$$\begin{aligned}
\big[|\not{D}|, L_f\big] &= \big[|\not{D}| + \rho, L_f\big] \\
&= \frac{1}{\pi}\int_0^\infty \frac{1}{(|\not{D}|+\rho)^2+\lambda}\big[\big(|\not{D}|+\rho\big)^2, L_f\big]\frac{1}{(|\not{D}|+\rho)^2+\lambda}\sqrt{\lambda}\,d\lambda \\
&= \frac{1}{\pi}\int_0^\infty \frac{1}{(|\not{D}|+\rho)^2+\lambda}\big(\not{D}[\not{D}, L_f] + [\not{D}, L_f]\not{D} + 2\rho|\not{D}|L_f - 2\rho L_f|\not{D}|\big) \\
&\quad \times \frac{1}{(|\not{D}|+\rho)^2+\lambda}\sqrt{\lambda}\,d\lambda.
\end{aligned}$$

Let us consider the different terms: since $[\not{D}, L_f] = L_{\not{D}f}$, they are all of the same order in $\not{D}$; we treat in detail only the first term since the proof goes along the same lines for the others.

Commuting $[\not{D}, L_f]$ with the factor $((|\not{D}|+\rho)^2+\lambda)^{-1}$ to its right, the first term of the last display equals:

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\infty \frac{|\not{D}|+\rho}{((|\not{D}|+\rho)^2+\lambda)^2}\sqrt{\lambda}\,d\lambda\ \frac{\not{D}}{|\not{D}|+\rho}[\not{D}, L_f] \\
&\quad + \frac{1}{\pi}\int_0^\infty \frac{1}{((|\not{D}|+\rho)^2+\lambda)^2}\not{D}\big[\big(|\not{D}|+\rho\big)^2, [\not{D}, L_f]\big]\frac{1}{(|\not{D}|+\rho)^2+\lambda}\sqrt{\lambda}\,d\lambda \\
&= \frac{1}{2}\frac{\not{D}}{|\not{D}|+\rho}[\not{D}, L_f] + \frac{1}{\pi}\int_0^\infty \frac{1}{((|\not{D}|+\rho)^2+\lambda)^2}\not{D}\big(\not{D}\big[\not{D}, [\not{D}, L_f]\big] \\
&\quad + \big[\not{D}, [\not{D}, L_f]\big]\not{D} + 2\rho|\not{D}|[\not{D}, L_f] - 2\rho[\not{D}, L_f]|\not{D}|\big)\frac{1}{(|\not{D}|+\rho)^2+\lambda}\sqrt{\lambda}\,d\lambda.
\end{aligned}$$

Since $\not{D}(|\not{D}|+\rho)^{-1}$ is bounded, Corollary 4.12 shows that

$$\frac{\not{D}}{|\not{D}|+\rho}[\not{D}, L_f]\big(1+\not{D}^2\big)^{-k} \in \mathcal{L}^p(\mathcal{H}) \quad \text{whenever } k > n/2p.$$

For the other four summands, for example for the first one, one gets (and similarly for the three others):

$$\begin{aligned}
&\left\| \frac{1}{\pi}\int_0^\infty \frac{\not{D}^2}{((|\not{D}|+\rho)^2+\lambda)^2}\big[\not{D}, [\not{D}, L_f]\big]\big(1+|\not{D}|\big)^{-k}\frac{1}{(|\not{D}|+\rho)^2+\lambda}\sqrt{\lambda}\,d\lambda \right\|_p \\
&\quad \leqslant \big\|\big[\not{D}, [\not{D}, L_f]\big]\big(1+|\not{D}|\big)^{-k}\big\|_p \frac{1}{\pi}\int_0^\infty \left\|\frac{\not{D}^2}{(|\not{D}|+\rho)^2+\lambda}\right\| \left\|\frac{1}{(|\not{D}|+\rho)^2+\lambda}\right\|^2 \sqrt{\lambda}\,d\lambda
\end{aligned}$$

$$\leqslant \big\|[\not{D},[\not{D},L_f]](1+|\not{D}|)^{-k}\big\|_p \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\lambda}}{(\rho^2+\lambda)^2}\,d\lambda$$
$$=\frac{1}{2\rho}\big\|L_{\not{D}^2 f}(1+|\not{D}|)^{-k}\big\|_p$$

which is again finite, using the same corollary.

For $[(1+\not{D}^2)^{1/2}, L_f]$, the proof goes along the same lines, using the spectral representation (4.7) applied to the positive operator $(1+\not{D}^2)^{1/2}$. □

4.3. Weak Schatten-class estimates

We prove now that, as expected, noncompact isospectral deformations of n-dimensional spin manifolds have spectral dimension n in the sense of [29]. The following proposition uses estimate (4.2) to get an improved version of the Cwikel inequality obtained in [29].

Proposition 4.14. *Let $f \in C_c^\infty(M)$. Then*

$$L_f(1+\Delta)^{-1/2}L_{\bar f} \in \mathcal{L}^{n,\infty}(\mathcal{H}_r).$$

Proof. Choose a number m with $0<m<1$. We define positive operators

$$A_k := L_f \int_0^{m^{2k}} t^{-1/2}e^{-t(1+\Delta)}\,dt\,L_{\bar f},$$
$$B_k := L_f \int_{m^{2k}}^1 t^{-1/2}e^{-t(1+\Delta)}\,dt\,L_{\bar f},$$
$$C := L_f \int_1^\infty t^{-1/2}e^{-t(1+\Delta)}\,dt\,L_{\bar f},$$

for each $k\in\mathbb{N}$ (the most suitable value of k will be chosen later). Their sum is $A_k+B_k+C = \Gamma(1/2)L_f(1+\Delta)^{-1/2}L_{\bar f}$ for each $k\in\mathbb{N}$.

We note first that C is in all Schatten classes $\mathcal{L}^p(\mathcal{H}_r)$ for $p \geqslant 1$. Indeed, using Theorem 3.8 and (4.2), we get

$$\|C\|_1 = \left\| L_f\left(\int_1^\infty t^{-1/2}e^{-t(1+\Delta)}\,dt\right)^{1/2}\right\|_2^2 = \left\| M_f\left(\int_1^\infty t^{-1/2}e^{-t(1+\Delta)}\,dt\right)^{1/2}\right\|_2^2$$
$$=\operatorname{Tr}\left(M_{|f|^2}\int_1^\infty t^{-1/2}e^{-t(1+\Delta)}\,dt\right)$$

$$= \int_M |f(p)|^2 \int_1^\infty t^{-1/2} e^{-t} K_t(p,p) \mu_g(p)\, dt$$

$$\leqslant c \int_M |f(p)|^2 \mu_g = c\|f\|_2^2.$$

Thus $C \in \mathcal{L}^n(\mathcal{H}_r) \subset \mathcal{L}^{n,\infty}(\mathcal{H}_r)$.

Moreover, we can bound A_k in the uniform norm:

$$\|A_k\| \leqslant \|L_f\|^2 \int_0^{m^{2k}} t^{-1/2} \|e^{-t(1+\Delta)}\|\, dt \leqslant \|L_f\|^2 \int_0^{m^{2k}} t^{-1/2}\, dt = 2\|L_f\|^2 m^k.$$

By Theorem 3.8 as above and (4.2), we can also estimate B_k in the trace norm:

$$\|B_k\|_1 = \int_{m^{2k}}^1 t^{-1/2} e^{-t} \int_M |f(p)|^2 K_t(p,p) \mu_g(p)\, dt$$

$$\leqslant c \int_M |f(p)|^2 \mu_g(p) \int_{m^{2k}}^1 t^{-(n+1)/2}\, dt$$

$$= c\|f\|_2^2 \frac{2}{n-1}\left(m^{-k(n-1)} - 1\right)$$

$$\leqslant c'\|f\|_2^2 m^{-k(n-1)},$$

since $m < 1$.

By Fan's inequality, see [4, III.6.5] or [46], we can estimate the jth singular value of $D := A_k + B_k$:

$$\mu_j(D) = \mu_j(A_k + B_k) \leqslant \mu_1(A_k) + \mu_j(B_k)$$

$$\leqslant \|A_k\| + j^{-1}\|B_k\|_1$$

$$\leqslant 2\|L_f\|^2 m^k + c'\|f\|_2^2 j^{-1} m^{k(1-n)}.$$

Now, given j and $m < 1$, one can choose $k \in \mathbb{N}$ such that $m^k \leqslant j^{-1/n} < m^{k-1}$. Thus $j^{-1} m^{-k(n-1)} < m^{(k-1)n} m^{-k(n-1)} = m^{-n} m^k$ and finally

$$\mu_j(D) \leqslant c(f,n,m) j^{-1/n},$$

which concludes the proof since $L_f(1+\Delta)^{-1/2} L_{\bar{f}} = \Gamma(1/2)^{-1}(C+D)$. $\square$

This result has an immediate corollary.

Corollary 4.15. *Let $f, h \in C_c^\infty(M)$. Then $L_f(1+\Delta)^{-1/2} L_h \in \mathcal{L}^{n,\infty}(\mathcal{H}_r)$.*

Proof. Polarization: add up $L_{(f+(-i)^k\bar{h})}(1+\Delta)^{-1/2}L_{(\bar{f}+i^k h)}$ for $k=0,1,2,3$. $\square$

Again, this result lifts to the Hilbert space of square-integrable spinors.

Corollary 4.16. *Both $L_f(1+D\!\!\!/^2)^{-1/2}L_h$ and $L_f(1+|D\!\!\!/|)^{-1}L_h$ lie in $\mathcal{L}^{n,\infty}(\mathcal{H})$ whenever $f,h\in C_c^\infty(M)$.*

Proof. Decompose the second operator $L_f(1+|D\!\!\!/|)^{-1}L_h$ as

$$\begin{aligned}
&L_f\big(1+D\!\!\!/^2\big)^{-1/2}L_h\frac{(1+D\!\!\!/^2)^{1/2}}{1+|D\!\!\!/|}+L_f\big(1+D\!\!\!/^2\big)^{-1/2}\left[\frac{(1+D\!\!\!/^2)^{1/2}}{1+|D\!\!\!/|},L_h\right]\\
&\quad=L_f\big(1+D\!\!\!/^2\big)^{-1/2}L_h\frac{(1+D\!\!\!/^2)^{1/2}}{1+|D\!\!\!/|}-L_f\big(1+|D\!\!\!/|\big)^{-1}\big[|D\!\!\!/|,L_h\big]\big(1+|D\!\!\!/|\big)^{-1}\\
&\qquad+L_f\big(1+D\!\!\!/^2\big)^{-1/2}\big[\big(1+D\!\!\!/^2\big)^{1/2},L_h\big]\big(1+|D\!\!\!/|\big)^{-1}.
\end{aligned}$$

Since $L_f(1+|D\!\!\!/|)^{-1}$, $[|D\!\!\!/|,L_h](1+|D\!\!\!/|)^{-1}$, $L_f(1+D\!\!\!/^2)^{-1/2}$ and $[(1+D\!\!\!/^2)^{1/2},L_h](1+|D\!\!\!/|)^{-1}$ all lie in $\mathcal{L}^{2n}(\mathcal{H})$ by Lemma 4.13, and since $(1+D\!\!\!/^2)^{1/2}(1+|D\!\!\!/|)^{-1}$ is bounded, it is enough to prove the case of $L_f(1+D\!\!\!/^2)^{-1/2}L_h$.

Using the spectral identity (4.7) and the Lichnerowicz formula once more, we find that

$$\begin{aligned}
&L_f\big(1+D\!\!\!/^2\big)^{-1/2}L_h\\
&\quad=L_f\frac{1}{\pi}\int_0^\infty\frac{(1+D\!\!\!/^2)^{-1}}{(1+D\!\!\!/^2)^{-1}+\lambda}\frac{d\lambda}{\sqrt{\lambda}}L_h\\
&\quad=L_f\frac{1}{\pi}\int_0^\infty\frac{(1+\Delta)^{-1}(1-\frac14 R(1+D\!\!\!/^2)^{-1})}{(1+\Delta)^{-1}(1-\frac14 R(1+D\!\!\!/^2)^{-1})+\lambda}\frac{d\lambda}{\sqrt{\lambda}}L_h\\
&\quad=L_f\frac{1}{\pi}\int_0^\infty\bigg(\frac{(1+\Delta)^{-1}}{(1+\Delta)^{-1}+\lambda}+\frac14\frac{(1+\Delta)^{-2}}{(1+\Delta)^{-1}+\lambda}R\frac{(1+D\!\!\!/^2)^{-1}}{(1+D\!\!\!/^2)^{-1}+\lambda}\\
&\qquad-\frac14(1+\Delta)^{-1}R\frac{(1+D\!\!\!/^2)^{-1}}{(1+D\!\!\!/^2)^{-1}+\lambda}\bigg)\frac{d\lambda}{\sqrt{\lambda}}L_h\\
&\quad=L_f(1+\Delta)^{-1/2}L_h+\frac{1}{4\pi}L_f\int_0^\infty\bigg(\frac{(1+\Delta)^{-2}}{(1+\Delta)^{-1}+\lambda}R\frac{(1+D\!\!\!/^2)^{-1}}{(1+D\!\!\!/^2)^{-1}+\lambda}\\
&\qquad-(1+\Delta)^{-1}R\frac{(1+D\!\!\!/^2)^{-1}}{(1+D\!\!\!/^2)^{-1}+\lambda}\bigg)\frac{d\lambda}{\sqrt{\lambda}}L_h.
\end{aligned}$$

The first term lies in $\mathcal{L}^{n,\infty}(\mathcal{H})$ by Corollary 4.15 and the two others are in $\mathcal{L}^n(\mathcal{H})$ since

$$\begin{aligned}
&\left\| L_f \int_0^\infty \frac{(1+\Delta)^{-2}}{(1+\Delta)^{-1}+\lambda} R \frac{(1+\not{D}^2)^{-1}}{(1+\not{D}^2)^{-1}+\lambda} \frac{d\lambda}{\sqrt{\lambda}} L_h \right\|_n \\
&\quad \leqslant \left\| L_f (1+\Delta)^{-2} \right\|_n \|R\| \left\| \left(1+\not{D}^2\right)^{-1} L_h \right\| \\
&\qquad \times \int_0^\infty \left\| \frac{1}{(1+\Delta)^{-1}+\lambda} \right\| \left\| \frac{1}{(1+\not{D}^2)^{-1}+\lambda} \right\| \frac{d\lambda}{\sqrt{\lambda}} \\
&\quad \leqslant \left\| L_f (1+\Delta)^{-2} \right\|_n \|R\| \|L_h\| \int_0^\infty \frac{1}{(1+\lambda)^2} \frac{d\lambda}{\sqrt{\lambda}},
\end{aligned}$$

which is finite by Proposition 4.9. Also, by Proposition 4.9 and Corollary 4.12,

$$\begin{aligned}
&\left\| L_f \int_0^\infty (1+\Delta)^{-1} R \frac{(1+\not{D}^2)^{-1}}{(1+\not{D}^2)^{-1}+\lambda} \frac{d\lambda}{\sqrt{\lambda}} L_h \right\|_n \\
&\quad \leqslant \|R\| \left\| L_f (1+\Delta)^{-1} \right\|_{2n} \left\| \left(1+\not{D}^2\right)^{-1} L_h \right\|_{2n} \int_0^\infty \frac{1}{1+\lambda} \frac{d\lambda}{\sqrt{\lambda}}
\end{aligned}$$

is finite. Since $\mathcal{L}^n(\mathcal{H}) \subset \mathcal{L}^{n,\infty}(\mathcal{H})$, the proof is complete. □

5. Dixmier trace computation: Periodic case

In this section, we shall see that the Dixmier traces Tr_ω—see [11,33] for the precise definition— give rise to an invariant for the deformation, with exactly the same role as the ordinary trace for the Hilbert–Schmidt-norm as seen in Section 3. Before giving a proof of this claim, namely that

$$\mathrm{Tr}_\omega\left(L_f\left(1+\not{D}^2\right)^{-n/2}\right) = \mathrm{Tr}_\omega\left(M_f\left(1+\not{D}^2\right)^{-n/2}\right), \quad \text{for all } f \in C_c^\infty(M) \tag{5.1}$$

(or at the scalar level, i.e., when L_f is acting on $\mathcal{H}_r$, with $(1+\Delta)^{-n/2}$ replaced by $(1+\not{D}^2)^{-n/2}$), we give an heuristic argument to see why this result is plausible. To this end, we will take advantage of the possibility of viewing L_f, for $f \in C_c^\infty(M)$, as an integral of bounded operators given by (3.6). Using this presentation for L_f, the trace property of the Dixmier trace and the commutativity of the Dirac operator (or the Laplacian) with the unitaries V_z (or $V_{\tilde{z}}$), the result would be straightforward if we could swap the Dixmier trace with the Lebesgue integral:

$$\begin{aligned}
&\mathrm{Tr}_\omega\left(L_f\left(1+\not{D}^2\right)^{-n/2}\right) \\
&\quad = (2\pi)^{-l}\,\mathrm{Tr}_\omega\left(\int_{\mathbb{R}^{2l}} e^{-iyz} V_{\frac{1}{2}\Theta y} M_f V_{-\frac{1}{2}\Theta y - z}\, d^l y\, d^l z \left(1+\not{D}^2\right)^{-n/2}\right) \\
&\quad = (2\pi)^{-l} \int_{\mathbb{R}^{2l}} e^{-iyz}\,\mathrm{Tr}_\omega\left(V_{\frac{1}{2}\Theta y} M_f \left(1+\not{D}^2\right)^{-n/2} V_{-\frac{1}{2}\Theta y - z}\right) d^l y\, d^l z
\end{aligned}$$

$$= (2\pi)^{-l} \int_{\mathbb{R}^{2l}} e^{-iyz} \operatorname{Tr}_\omega\big(M_f\big(1+\not{D}^2\big)^{-n/2} V_{-z}\big)\, d^l y\, d^l z$$

$$= \operatorname{Tr}_\omega\bigg(M_f\big(1+\not{D}^2\big)^{-n/2} \int_{\mathbb{R}^l} \delta_0(z) V_{-z}\, d^l z\bigg)$$

$$= \operatorname{Tr}_\omega\big(M_f\big(1+\not{D}^2\big)^{-n/2}\big).$$

However, this exchange of the Dixmier trace with the integral is not rigorous, since the integrals are oscillatory and Dixmier traces do not in general obey dominated convergence.

For the ordinary trace, the situation is better since such an exchange can be justified by using a family of strongly convergent regularizers. For example, one can use $\{M_{u_k}\}$, where $\{u_k\}_{k\in\mathbb{N}}$ is an approximate unit for $C_c^\infty(M)$, that is, an increasing family of nonnegative compactly supported functions such that $u_k \uparrow 1$ pointwise on M, so that $\text{s-lim}\, M_{u_k} = 1$. Then, the integrals in the product $M_{u_k} L_f (1+\not{D}^2)^{-k} L_{\bar f} M_{u_k}$ (with $k > n/2$) become Bochner integrals for the trace-norm, with uniform bound on its trace-norm. Finally, by [19, Proposition 2], we obtain that the strong limit

$$\text{s-lim}\, M_{u_k} L_f\big(1+\not{D}^2\big)^{-k} L_{\bar f} M_{u_k} = L_f\big(1+\not{D}^2\big)^{-k} L_{\bar f}$$

is trace-class as well, with the same trace-norm bound as for the family $M_{u_k} L_f (1+\not{D}^2)^{-k} L_{\bar f} M_{u_k}$. This gives another proof of Theorem 3.8.

Such an approach fails for the Dixmier trace, since these natural regularizers give rise to trace-class operators in some cases. This is for instance the case for Moyal planes, since one can prove that $L_f (1+\not{D}^2)^{-k} M_{u_k}$ is trace-class for all $k \geqslant 0$ whenever $f, u_k \in \mathcal{S}(\mathbb{R}^l)$, so they have vanishing Dixmier trace.

In the aperiodic case, we shall prove this condition indirectly, using a zeta-residue argument to evaluate the left-hand side of (5.1) as the same ordinary integral which is known to give the value of the right-hand side [41, Proposition 15]. Before that, we first establish the result in the easier periodic case, for which no hard analysis is needed. The spectral subspace decomposition of $f \in C_c^\infty(M)$ gives a direct access to the Dixmier traceability of the operators $L_f(1+\Delta)^{-n/2}$ acting on $\mathcal{H}_r = L^2(M, \mu_g)$ and $L_f(1+\not{D}^2)^{-n/2}$ acting on $\mathcal{H} = L^2(M, S)$, as well as to the value of their Dixmier traces.

Proposition 5.1. *Let α be an effective isometric smooth action of $\mathbb{T}^l$ on M, with $l \geqslant 2$, and let $f \in C_c^\infty(M)$. Then the operator $L_f(1+\Delta)^{-n/2}$ is Dixmier traceable on $\mathcal{H}_r$, and the value of its Dixmier trace is independent of ω:*

$$\operatorname{Tr}_\omega\big(L_f(1+\Delta)^{-n/2}\big) = C'(n)\delta_{0,r} \int_M f_r \mu_g = C'(n) \int_M f_0 \mu_g,$$

where $C'(n) := \Omega_n / n(2\pi)^n$, Ω_n is the volume of the unit sphere in $\mathbb{R}^n$, and $f = \sum_r f_r$ is the decomposition (2.3) *of f in homogeneous components.*

Proof. Each f_r satisfies $\alpha_z(f_r) = e^{-izr} f_r$ for all $z \in \mathbb{T}^l$. Since $[M_{f_r}, V_z] = M_{f_r}(1 - e^{-izr})V_z$, we see that $[M_{f_r}, V_{-\frac{1}{2}\Theta r}] = 0$ by skew-symmetry of the deformation matrix.

By [41, Proposition 15], $M_f(1+\Delta)^{-n/2}$ lies in $\mathcal{L}^{1,\infty}(\mathcal{H}_r)$ and, moreover,

$$\left\|M_f(1+\Delta)^{-n/2}\right\|_{1,\infty} \leqslant C_1(n)\|f\|_\infty.$$

This estimate is obtained by a (finite) partition of unity on the compact set $(\operatorname{supp} f)$ on applying Weyl's theorem. We thus obtain, using (3.4),

$$\begin{aligned}\left\|L_f(1+\Delta)^{-n/2}\right\|_{1,\infty} &\leqslant \sum_{r\in\mathbb{Z}^l}\left\|M_{f_r}V_{-\frac{1}{2}\Theta r}(1+\Delta)^{-n/2}\right\|_{1,\infty} \leqslant \sum_{r\in\mathbb{Z}^l}\left\|M_{f_r}(1+\Delta)^{-n/2}\right\|_{1,\infty}\\ &\leqslant C_1(n)\sum_{r\in\mathbb{Z}^l}\|f_r\|_\infty,\end{aligned}$$

since each f_r is compactly supported with support contained in $\mathbb{T}^l\cdot(\operatorname{supp} f)$. Those estimates give the Dixmier traceability, since the spectral-subspace decomposition is $\|\cdot\|_\infty$-convergent.

To compute the Dixmier trace, it remains to remark that for all $z\in\mathbb{T}^l$,

$$\begin{aligned}\operatorname{Tr}_\omega\left(L_{f_r}(1+\Delta)^{-n/2}\right) &= \operatorname{Tr}_\omega\left(V_z M_{f_r}V_{-\frac{1}{2}\Theta r}(1+\Delta)^{-n/2}V_{-z}\right)\\ &= \operatorname{Tr}_\omega\left(M_{\alpha_z(f_r)}V_{-\frac{1}{2}\Theta r}(1+\Delta)^{-n/2}\right)\\ &= e^{-izr}\operatorname{Tr}_\omega\left(M_{f_r}V_{-\frac{1}{2}\Theta r}(1+\Delta)^{-n/2}\right),\end{aligned}$$

and therefore it must vanish unless $r=0$ because of (3.4). Thus,

$$\operatorname{Tr}_\omega\left(L_{f_r}(1+\Delta)^{-n/2}\right) = \operatorname{Tr}_\omega\left(M_{f_0}(1+\Delta)^{-n/2}\right)\delta_{0,r} = C'(n)\delta_{0,r}\int_M f_0\mu_g.$$

The last equality is obtained, as in [41, Proposition 15], by computation of the Wodzicki residue of the operator $M_f(1+\Delta)^{-n/2}$. □

Corollary 5.2. *Under the same hypothesis, the operator $L_f(1+\not{D}^2)^{-n/2}$ is Dixmier traceable on $\mathcal{H}$ for $f\in C_c^\infty(M)$; furthermore, the value of its Dixmier trace is independent of ω:*

$$\operatorname{Tr}_\omega\left(L_f\left(1+\not{D}^2\right)^{-n/2}\right) = C(n)\delta_{0,r}\int_M f_r\mu_g = C(n)\int_M f_0\mu_g,$$

where $C(n) := 2^{\lfloor n/2\rfloor}\Omega_n/n(2\pi)^n$, with $2^{\lfloor n/2\rfloor}$ being the rank of the spinor bundle.

Proof. Using the Lichnerowicz formula $\not{D}^2 = \Delta^S + \frac{1}{4}R$, the Dixmier traceability is obtained by comparison:

$$\left(1+\not{D}^2\right)^{-1} = \left(1+\Delta^S\right)^{-1}\left(1-\tfrac{1}{4}R\left(1+\not{D}^2\right)^{-1}\right). \tag{5.2}$$

For the computation of the Dixmier trace, one can apply previous arguments. We obtain the result, using that, modulo the factor $2^{\lfloor n/2\rfloor}$, the principal symbols of $(1+\not{D}^2)^{-n/2}$ and $(1+\Delta)^{-n/2}$ are the same as seen in (5.2). Thus, the operators $M_{f_r}(1+\not{D}^2)^{-n/2}$ and $M_{f_r}(1+\Delta)^{-n/2}$ have the same Wodzicki residue, up to that constant factor. □

6. Dixmier trace computation: Aperiodic case

This subsection is devoted to the proof of the following theorem.

Theorem 6.1. *Let M be a noncompact, connected, complete Riemannian spin manifold satisfying Condition* 4.1, *with bounded scalar curvature. Suppose further that M is endowed with a smooth isometric and proper action of $\mathbb{R}^l$. If $f \in C_c^\infty(M)$, then $L_f(1+|\not{D}|)^{-n}$ lies in $\mathcal{L}^{1,\infty}(\mathcal{H})$ and is a measurable operator*; *the common value of its Dixmier traces is*

$$\operatorname{Tr}_\omega\big(L_f\big(1+|\not{D}|\big)^{-n}\big) = C(n)\int_M f(p)\mu_g(p),$$

where $C(n) = 2^{\lfloor n/2\rfloor}\Omega_n/n(2\pi)^n$.

In the aperiodic case, the manifold M is necessarily of the form $V \times \mathbb{R}^l$, where the group $\mathbb{R}^l$ acts by translation on the second direct factor. Indeed, proper actions of the additive group $\mathbb{R}^l$ are automatically free, because the only compact subgroup of $\mathbb{R}^l$ is the trivial subgroup $\{0\}$. Thus, the projection on the orbit space $\pi : M \to V := M/\mathbb{R}^l$ defines a principal $\mathbb{R}^l$-bundle projection [22, Theorem 1.11.4]. We remark that properness of the action was crucially used in Proposition 3.3 to show that twisted multiplication operators are bounded. But a principal $\mathbb{R}^l$-bundle has a smooth global section and so it is automatically trivializable: see [20, 16.14.5], for instance.

Thus we write $M = V \times \mathbb{R}^l$, where V is a smooth (not necessarily compact) manifold of dimension $k = n - l$, which carries a Riemannian metric, induced from that of M, and $\pi : M \to V$ is just the projection on the first factor. If $\{\phi_j\}_{j\in J}$ is any locally finite partition of unity on V consisting of smooth compactly supported functions, then by setting $\psi_j := \phi_j \circ \pi$, we obtain an α-invariant partition of unity $\{\psi_j\}$ on M. For any $f \in C_c^\infty(M)$, the sum $f = \sum_j f\psi_j$ is finite because supp f is compact; since each ψ_j is α-invariant, we directly obtain

$$L_f = \sum_j L_{f\psi_j} = \sum_j L_f M_{\psi_j}.$$

Thus, when dealing with operators of the form $L_f h(\not{D})$, we lose no generality by restricting to a single coordinate chart of V; so we shall assume from now on that V is an open ball in $\mathbb{R}^k$.

We denote by $\hat{x} := (x^1,\dots,x^k) \in V$ and $\bar{x} := (x^{k+1},\dots,x^n) \in \mathbb{R}^l$ respectively the transverse and longitudinal local coordinates on M. It is immediate that the operator L_f is pseudodifferential, with symbol

$$\sigma[L_f](\hat{x},\bar{x};\hat{\xi},\bar{\xi}) = f\big(\hat{x},\bar{x} - \tfrac{1}{2}\Theta\bar{\xi}\big). \tag{6.1}$$

Indeed, for any vector $\psi \in \mathcal{H}$, Definition 3.1 shows that

$$\begin{aligned} L_f\psi(\hat{x},\bar{x}) = (f\star\psi)(\hat{x},\bar{x}) &= (2\pi)^{-l}\int_{\mathbb{R}^{2l}} e^{-i\bar{\xi}\bar{y}}\alpha_{\frac{1}{2}\Theta\bar{\xi}}(f)(\hat{x},\bar{x})V_{-\bar{y}}\psi(\hat{x},\bar{x})\,d^l\bar{\xi}\,d^l\bar{y} \\ &= (2\pi)^{-l}\int_{\mathbb{R}^{2l}} e^{-i\bar{\xi}\bar{y}} f\big(\hat{x},\bar{x}-\tfrac{1}{2}\Theta\bar{\xi}\big)\psi(\hat{x},\bar{x}+\bar{y})\,d^l\bar{\xi}\,d^l\bar{y} \\ &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{-i\bar{\xi}(\bar{y}-\bar{x})}e^{-i\hat{\xi}(\hat{y}-\hat{x})} f\big(\hat{x},\bar{x}-\tfrac{1}{2}\Theta\bar{\xi}\big)\psi(\hat{y},\bar{y})\,d^l\bar{\xi}\,d^l\bar{y}\,d^k\hat{\xi}\,d^k\hat{y}. \end{aligned}$$

Proposition 6.2. *Under the hypotheses of Theorem* 6.1, *if* $f \in C_c^\infty(M)$ *then* $L_f(1+|\not{D}|)^{-n}$ *lies in* $\mathcal{L}^{1,\infty}(\mathcal{H})$.

Proof. For fixed $\hat{x}$, the function $\bar{x} \mapsto f(\hat{x}, \bar{x})$ lies in $C_c^\infty(\mathbb{R}^l)$, so it can be decomposed in the Wigner eigentransition basis $\{f_{mn}\}$, indexed by $m, n \in \mathbb{N}^{l/2}$ (see [29,32,48] and recall that l is even):

$$f(\hat{x}, \bar{x}) = \sum_{m,n} c_{mn}(\hat{x}) f_{mn}(\bar{x}),$$

where the matrix coefficients c_{mn} lie in $C_c^\infty(V)$.

Given two functions $f(\hat{x}, \bar{x}) = \sum c_{mn}(\hat{x}) f_{mn}(\bar{x})$, $h(\hat{x}, \bar{x}) = \sum d_{mn}(\hat{x}) f_{mn}(\bar{x})$ of this form, their twisted product may thus be expressed as a matrix product in the $\bar{x}$ variables:

$$(f \star h)(\hat{x}, \bar{x}) = \sum_{m,n,k} c_{mk}(\hat{x}) d_{kn}(\hat{x}) f_{mn}(\bar{x}). \tag{6.2}$$

The operator L_f can then be viewed as an element of the algebra $M_\infty(C^\infty(V))$ with rapidly decreasing $C^\infty(V)$-valued matrix elements.

Thus, one can extend the strong factorization property [32] of the algebra $(\mathcal{S}(\mathbb{R}^l), \star)$ to this context: for all $f \in C_c^\infty(M)$, there exist $h, k \in C^\infty(M)$ which are Schwartz functions in the $\bar{x}$ variables, such that

$$f(\hat{x}, \bar{x}) = (h \star k)(\hat{x}, \bar{x}). \tag{6.3}$$

By iterated factorization, allowing to write f as a product of n such functions, and by taking iterated commutators, exactly as in [29, Corollary 4.12 and Lemma 4.13], we can express each $L_f(1+|\not{D}|)^{-n}$ as a product of n terms of the form $L_h(1+|\not{D}|)^{-1} L_k$, each lying in $\mathcal{L}^{n,\infty}(\mathcal{H})$ by Corollary 4.16, plus an extra term in $\mathcal{L}^1(\mathcal{H})$. Finally, by the Hölder inequality for weak Schatten classes, we conclude that $L_f(1+|\not{D}|)^{-n} \in \mathcal{L}^{1,\infty}(\mathcal{H})$. $\square$

We may also introduce a system of local units [40] for the twisted product by a straightforward extension of a construction in [29].

Definition 6.3. The manifold V may be expressed as a union of compact subsets C_i with each contained in the interior of C_{i+1}; define $\chi_i := 1$ on C_i and $\chi_i := 0$, elsewhere. For each $K \in \mathbb{N}$, define a function e_K on M by

$$e_K(\hat{x}, \bar{x}) := \sum_{|n| \leqslant K} \chi_K(\hat{x}) f_{nn}(\bar{x}),$$

where $|n| = n_1 + \cdots + n_{l/2}$. Then e_K is real-valued and $e_K \star e_K = e_K$ by using (6.2) to compute the twisted product, and L_{e_K} is defined as an orthogonal projector on $\mathcal{H}$. Next, let $f_K := e_K \star f \star e_K$, or more explicitly,

$$f_K(\hat{x}, \bar{x}) := \sum_{|m|, |n| \leqslant K} c_{mn}(\hat{x}) f_{mn}(\bar{x}). \tag{6.4}$$

By construction, $e_K \star f_K = f_K \star e_K = f_K$.

The operator $L_{e_K}(1+|\not{D}|)^{-n}L_{e_K}$ is Dixmier traceable: in Proposition 4.14 and subsequently, one can replace f by e_K even though the latter is not in $C_c^\infty(M)$, since its square-integrability is guaranteed at each step, and the factorization argument following (6.3) goes through because e_K is idempotent. The trace property of the Dixmier trace now yields

$$\mathrm{Tr}_\omega\big(L_{f_K}\big(1+|\not{D}|\big)^{-n}\big)=\mathrm{Tr}_\omega\big(L_{f_K}L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big).$$

Since L_{f_K} is bounded, Theorem 5.6 of [6] shows that if the following limit exists:

$$\lim_{s\downarrow 1}(s-1)\,\mathrm{Tr}\big(L_{f_K}\big(L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big)^s\big),$$

then it will coincide with the value of any Dixmier trace of $L_{f_K}(1+|\not{D}|)^{-n}$.

Lemma 6.4. *The trace norm*

$$\big\|L_{f_K}\big(L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big)^s-L_{f_K}\big(1+|\not{D}|\big)^{-ns}\big\|_1 \tag{6.5}$$

is a bounded function of s, for $1\leqslant s\leqslant 2$.

Proof. Write $s=:1+\varepsilon$, with $0<\varepsilon\leqslant 1$. We use the following spectral representation, generalizing (4.7), for fractional powers of a positive operator A:

$$A^\varepsilon=\frac{\sin\pi\varepsilon}{\pi}\int_0^\infty A(1+\lambda A)^{-1}\lambda^{-\varepsilon}\,d\lambda.$$

Since L_{e_K} is an orthogonal projector and $L_{f_K}L_{e_K}=L_{f_K}$, we can write

$$\begin{aligned}L_{f_K}\big(L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big)^s&=L_{f_K}L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big(L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big)^\varepsilon\\&=L_{f_K}\big(1+|\not{D}|\big)^{-n}\big(L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big)^\varepsilon.\end{aligned}$$

Hence,

$$\begin{aligned}&L_{f_K}\big(L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big)^s-L_{f_K}\big(1+|\not{D}|\big)^{-ns}\\&\quad=L_{f_K}\big(1+|\not{D}|\big)^{-n}\frac{\sin\pi\varepsilon}{\pi}\\&\qquad\times\int_0^\infty\bigg(\frac{L_{e_K}(1+|\not{D}|)^{-n}L_{e_K}}{1+\lambda L_{e_K}(1+|\not{D}|)^{-n}L_{e_K}}-\frac{(1+|\not{D}|)^{-n}}{1+\lambda(1+|\not{D}|)^{-n}}\bigg)\lambda^{-\varepsilon}\,d\lambda.\end{aligned} \tag{6.6}$$

The first fraction in parenthesis may be rewritten as

$$\big(\big(1+|\not{D}|\big)^n+\lambda T_K\big)^{-1}T_K,$$

where

$$T_K := \left(1+|\not{D}|\right)^n L_{e_K}\left(1+|\not{D}|\right)^{-n} L_{e_K}.$$

Since L_{e_K} is a projector, we get

$$\begin{aligned} T_K &= L_{e_K}^2 + \left[\left(1+|\not{D}|\right)^n, L_{e_K}\right]\left(1+|\not{D}|\right)^{-n} L_{e_K} \\ &= L_{e_K} + \sum_{0\leqslant k<r\leqslant n} \binom{n}{r} |\not{D}|^k \left[|\not{D}|, L_{e_K}\right] |\not{D}|^{r-k-1}\left(1+|\not{D}|\right)^{-n} L_{e_K} \\ &=: L_{e_K} + \sum_{0\leqslant k<r\leqslant n} A_{rk}. \end{aligned} \tag{6.7}$$

The crucial issue in showing the difference (6.6) to be ε-uniformly trace-class is that, excepting the first summand in (6.7) which is merely bounded, all the other summands A_{rk} are compact. More precisely, using Proposition 4.9 (plus routine commutations), we can check that each $A_{rk} \in \mathcal{L}^p(\mathcal{H})$ for all $p > n$.

Following the procedure of Rennie [41, Theorem 12], we reduce the difference of fractions in (6.6) as follows:

$$\begin{aligned} &\frac{L_{e_K}(1+|\not{D}|)^{-n}L_{e_K}}{1+\lambda L_{e_K}(1+|\not{D}|)^{-n}L_{e_K}} - \frac{(1+|\not{D}|)^{-n}}{1+\lambda(1+|\not{D}|)^{-n}} \\ &\quad= \left(\left(1+|\not{D}|\right)^n + \lambda T_K\right)^{-1} T_K - \left(\left(1+|\not{D}|\right)^n + \lambda\right)^{-1} \\ &\quad= \left(\left(\left(1+|\not{D}|\right)^n + \lambda T_K\right)^{-1} - \left(\left(1+|\not{D}|\right)^n + \lambda\right)^{-1}\right) T_K + \left(\left(1+|\not{D}|\right)^n + \lambda\right)^{-1}(T_K - 1) \\ &\quad= \left(\left(1+|\not{D}|\right)^n + \lambda\right)^{-1}(\lambda - \lambda T_K)\left(\left(1+|\not{D}|\right)^n + \lambda T_K\right)^{-1} T_K + \left(\left(1+|\not{D}|\right)^n + \lambda\right)^{-1}(T_K - 1) \\ &\quad= \left(\left(1+|\not{D}|\right)^n + \lambda\right)^{-1}(T_K - 1)\left(1 - \left(\left(1+|\not{D}|\right)^n + \lambda T_K\right)^{-1}\lambda T_K\right) \\ &\quad= \left(\left(1+|\not{D}|\right)^n + \lambda\right)^{-1}(T_K - 1)\left(1 + \lambda L_{e_K}\left(1+|\not{D}|\right)^{-n} L_{e_K}\right)^{-1}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &L_{f_K}\left(L_{e_K}\left(1+|\not{D}|\right)^{-n} L_{e_K}\right)^s - L_{f_K}\left(1+|\not{D}|\right)^{-ns} \\ &\quad= L_{f_K}\left(1+|\not{D}|\right)^{-n} \frac{\sin\pi\varepsilon}{\pi} \int_0^\infty \frac{1}{(1+|\not{D}|)^n+\lambda}(T_K - 1)\frac{1}{1+\lambda L_{e_K}(1+|\not{D}|)^{-n}L_{e_K}} \lambda^{-\varepsilon}\, d\lambda \\ &\quad= L_{f_K}\left(1+|\not{D}|\right)^{-n} \frac{\sin\pi\varepsilon}{\pi} \int_0^\infty \frac{1}{(1+|\not{D}|)^n+\lambda} L_{e_K}(T_K - 1)\frac{1}{1+\lambda L_{e_K}(1+|\not{D}|)^{-n}L_{e_K}} \lambda^{-\varepsilon} d\lambda \\ &\qquad+ L_{f_K}\frac{\sin\pi\varepsilon}{\pi} \int_0^\infty \left[L_{e_K}, \frac{(1+|\not{D}|)^{-n}}{(1+|\not{D}|)^n+\lambda}\right](T_K - 1)\frac{1}{1+\lambda L_{e_K}(1+|\not{D}|)^{-n}L_{e_K}} \lambda^{-\varepsilon}\, d\lambda. \end{aligned}$$

We now show that the second term on the right-hand side is uniformly bounded in trace norm. We write

$$\begin{aligned}\big[L_{e_K}, \big(1+|\not{D}|\big)^{-n}\big(\big(1+|\not{D}|\big)^n+\lambda\big)^{-1}\big] &= \big[L_{e_K}, \big(1+|\not{D}|\big)^{-n}\big]\big(\big(1+|\not{D}|\big)^n+\lambda\big)^{-1} \\ &\quad + \big(1+|\not{D}|\big)^{-n}\big[L_{e_K}, \big(\big(1+|\not{D}|\big)^n+\lambda\big)^{-1}\big],\end{aligned}$$

and the first of these summands yields the trace norm estimate:

$$\begin{aligned}&\Bigg\| L_{f_K}\big[L_{e_K}, \big(1+|\not{D}|\big)^{-n}\big]\frac{\sin\pi\varepsilon}{\pi} \\ &\quad \times \int_0^\infty \frac{1}{(1+|\not{D}|)^n+\lambda} L_{e_K}(T_K-1)\frac{\lambda^{-\varepsilon}}{1+\lambda L_{e_K}(1+|\not{D}|)^{-n}L_{e_K}}\,d\lambda\Bigg\|_1 \\ &\leqslant \big\|L_{f_K}\big[L_{e_K}, \big(1+|\not{D}|\big)^{-n}\big]\big\|_1 \frac{\sin\pi\varepsilon}{\pi}\int_0^\infty \big\|\big(\big(1+|\not{D}|\big)^n+\lambda\big)^{-1}\big\|\,\big\|L_{e_K}(T_K-1)\big\| \\ &\quad \times \big\|\big(1+\lambda L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big)^{-1}\big\|\lambda^{-\varepsilon}d\lambda \\ &\leqslant \big\|L_{e_K}(T_K-1)\big\|\,\big\|L_{f_K}\big[L_{e_K}, \big(1+|\not{D}|\big)^{-n}\big]\big\|_1\frac{\sin\pi\varepsilon}{\pi}\int_0^\infty \frac{\lambda^{-\varepsilon}}{1+\lambda}\,d\lambda \\ &= \big\|L_{e_K}(T_K-1)\big\|\,\big\|L_{f_K}\big[L_{e_K}, \big(1+|\not{D}|\big)^{-n}\big]\big\|_1 =: C_1.\end{aligned}$$

This constant C_1 is finite (and independent of ε) since

$$L_{f_K}\big[L_{e_K}, (1+|\not{D}|)^{-n}\big] = L_{f_K}\sum_{0\leqslant k<r\leqslant n}\binom{n}{r}\frac{|\not{D}|^k}{(1+|\not{D}|)^n}\big[|\not{D}|, L_{e_K}\big]\frac{|\not{D}|^{r-k-1}}{(1+|\not{D}|)^n}, \tag{6.8}$$

and each term of the sum lies in $\mathcal{L}^1(\mathcal{H})$, using Proposition 4.9 and the Hölder inequality. Analogously, one can show that the trace norm of

$$\begin{aligned}&L_{f_K}\big(1+|\not{D}|\big)^{-n}\frac{\sin\pi\varepsilon}{\pi} \\ &\quad\times\int_0^\infty\big[L_{e_K}, \big(\big(1+|\not{D}|\big)^n+\lambda\big)^{-1}\big]L_{e_K}(T_K-1)\frac{\lambda^{-\varepsilon}}{1+\lambda L_{e_K}(1+|\not{D}|)^{-n}L_{e_K}}\,d\lambda\end{aligned}$$

is bounded by the constant $C_2 := \|L_{f_K}(1+|\not{D}|)^{-n}\|\|[L_{e_K}, (1+|\not{D}|)^{-n}]\|_1$, independent of ε.

Using expansion (6.7) of T_K, we finally obtain

$$
\begin{aligned}
&\big\| L_{f_K}\big(L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big)^s - L_{f_K}\big(1+|\not{D}|\big)^{-ns}\big\|_1 \\
&\quad \leqslant C_1 + C_2 + \sum_{0\leqslant k<r\leqslant n}\Bigg\| L_{f_K}\big(1+|\not{D}|\big)^{-n}\frac{\sin\pi\varepsilon}{\pi} \\
&\qquad \times \int_0^\infty \frac{1}{(1+|\not{D}|)^n+\lambda} L_{e_K}A_{rk}\frac{\lambda^{-\varepsilon}}{1+\lambda L_{e_K}(1+|\not{D}|)^{-n}L_{e_K}}\,d\lambda\Bigg\|_1 \\
&\quad \leqslant C_1 + C_2 + \sum_{0\leqslant k<r\leqslant n}\big\| L_{f_K}\big(1+|\not{D}|\big)^{-n}\big\|_{p/(p-1)}\|L_{e_K}A_{rk}\|_p\frac{\sin\pi\varepsilon}{\pi}\int_0^\infty\frac{\lambda^{-\varepsilon}}{1+\lambda}\,d\lambda \\
&\quad = C_1 + C_2 + \sum_{0\leqslant k<r\leqslant n}\big\| L_{f_K}\big(1+|\not{D}|\big)^{-n}\big\|_{p/(p-1)}\|L_{e_K}A_{rk}\|_p,
\end{aligned}
$$

which is finite for $p > n$. □

Proof of Theorem 6.1. For $1 < s \leqslant 2$, the operator $L_{f_K}(1+|\not{D}|)^{-ns}$ appearing in (6.5) is trace-class, since it equals the product of $L_{f_K}(1+|\not{D}|)^{-n} \in \mathcal{L}^{1,\infty}(\mathcal{H})$ by $L_{e_K}(1+|\not{D}|)^{-n(s-1)} \in \mathcal{L}^p(\mathcal{H})$ for $p > 1/(s-1)$, plus a commutator of trace class. The difference of traces

$$
\operatorname{Tr}\big(L_{f_K}\big(L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big)^s\big) - \operatorname{Tr}\big(L_{f_K}\big(1+|\not{D}|\big)^{-ns}\big)
$$

is therefore a bounded function of s, for $1 \leqslant s \leqslant 2$. Thus,

$$
\lim_{s\downarrow 1}(s-1)\operatorname{Tr}\big(L_{f_K}\big(L_{e_K}\big(1+|\not{D}|\big)^{-n}L_{e_K}\big)^s\big) = \lim_{s\downarrow 1}(s-1)\operatorname{Tr}\big(L_{f_K}\big(1+|\not{D}|\big)^{-ns}\big). \tag{6.9}
$$

Moreover,

$$
\lim_{s\downarrow 1}(s-1)\operatorname{Tr}\big(L_{f_K}\big(1+|\not{D}|\big)^{-ns}\big) = \lim_{s\downarrow 1}(s-1)\operatorname{Tr}\big(L_{f_K}\big(1+\not{D}^2\big)^{-ns/2}\big). \tag{6.10}
$$

Indeed, for $1 \leqslant s \leqslant 2$, the following operator inequalities hold:

$$
\begin{aligned}
0 &\leqslant \big(1+\not{D}^2\big)^{-ns/2} - \big(1+|\not{D}|\big)^{-ns} = \big(1+|\not{D}|\big)^{-ns}\left(\left(1+\frac{2|\not{D}|}{1+\not{D}^2}\right)^{ns/2}-1\right) \\
&\leqslant \big(1+|\not{D}|\big)^{-n}\left(\left(1+\frac{2|\not{D}|}{1+\not{D}^2}\right)^{n}-1\right) = \big(1+|\not{D}|\big)^{-n}\sum_{k=1}^n\binom{n}{k}\left(\frac{2|\not{D}|}{1+\not{D}^2}\right)^k,
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\big|\operatorname{Tr}\big(L_{f_K}\big(\big(1+\not{D}^2\big)^{-ns/2} - \big(1+|\not{D}|\big)^{-ns}\big)\big)\big| \\
&\quad = \big|\operatorname{Tr}\big(L_{f_K}L_{e_K}\big(\big(1+\not{D}^2\big)^{-ns/2} - \big(1+|\not{D}|\big)^{-ns}\big)L_{e_K}\big)\big| \\
&\quad \leqslant \|L_{f_K}\|\operatorname{Tr}\big(L_{e_K}\big(\big(1+\not{D}^2\big)^{-ns/2} - \big(1+|\not{D}|\big)^{-ns}\big)L_{e_K}\big)
\end{aligned}
$$

$$\leqslant \|L_{f_K}\| \sum_{k=1}^{n} \binom{n}{k} \operatorname{Tr}\left(L_{e_K} \left(1 + |\not{D}|\right)^{-n} \left(\frac{2|\not{D}|}{1+\not{D}^2} \right)^k L_{e_K} \right)$$

$$\leqslant \|L_{f_K}\| \sum_{k=1}^{n} \binom{n}{k} \left\| L_{e_K} \left(1 + |\not{D}|\right)^{-n} \right\|_{p/(p-1)} \left\| \left(\frac{2|\not{D}|}{1+\not{D}^2} \right)^k L_{e_K} \right\|_p,$$

which is finite for $p > n$.

Note that $M_{f_K}(1+\not{D}^2)^{-ns/2}$ is also trace-class for $s > 1$, on account of the form (6.4) of f_K on $M = V \times \mathbb{R}^l$. Corollary 3.10 now implies that

$$\operatorname{Tr}\left(L_{f_K}\left(1+\not{D}^2\right)^{-ns/2}\right) = \operatorname{Tr}\left(M_{f_K}\left(1+\not{D}^2\right)^{-ns/2}\right).$$

The evaluation of the right-hand side of (6.9) is therefore given by

$$\lim_{s\downarrow 1}(s-1)\operatorname{Tr}\left(L_{f_K}\left(1+|\not{D}|\right)^{-ns}\right) = \lim_{s\downarrow 1}(s-1)\operatorname{Tr}\left(M_{f_K}\left(1+\not{D}^2\right)^{-ns/2}\right). \tag{6.11}$$

The right-hand side may be rewritten as

$$\lim_{s\downarrow 1} \frac{(s-1)}{\Gamma(ns/2)} \int_M f_K(p) \int_0^\infty t^{ns/2-1} e^{-t} K_{e^{-t\not{D}^2}}(p,p)\, dt\, \mu_g(p).$$

Now $\not{D}^2$ is a second-order differential operator of Laplace type by the Lichnerowicz formula [31], and thus $K_{e^{-t\not{D}^2}}(p,p) = 2^{\lfloor n/2 \rfloor}(4\pi t)^{-n/2} + O(t^{-n/2+1})$ when $t \to 0$; the t-integral from ε to ∞, for any $\varepsilon < 1$, gives no contribution thanks to the factor e^{-t}. (See, for instance, [31, Lemma 4.1.4], noting that this estimate for on-diagonal values of the heat kernel does not depend on compactness of the manifold.) Therefore,

$$\begin{aligned} &\lim_{s\downarrow 1}(s-1)\operatorname{Tr}\left(M_{f_K}\left(1+\not{D}^2\right)^{-ns/2}\right) \\ &\quad = \lim_{s\downarrow 1}(s-1)\frac{2^{\lfloor n/2\rfloor}}{(4\pi)^{n/2}\Gamma(n/2)} \int_0^\infty t^{n(s-1)/2-1} e^{-t}\, dt \int_M f_K(p)\mu_g(p) \\ &\quad = C(n) \int_M f_K(p)\mu_g(p). \end{aligned}$$

The proportionality factor $C(n) = 2^{\lfloor n/2\rfloor}/(4\pi)^{n/2}\Gamma(n/2+1)$ is the same as that of Corollary 5.2.

It remains to remove the truncation induced by the projectors L_{e_K}. Notice first that

$$\operatorname{Tr}_\omega\left((L_f - L_{f_K})\left(1+|\not{D}|\right)^{-n}\right) = \operatorname{Tr}_\omega\left((1-L_{e_K})L_f\left(1+|\not{D}|\right)^{-n}\right),$$

since $L_f[L_{e_K}, (1+|\not{D}|)^{-n}]$ is trace-class, as is seen on replacing f_K by f in (6.8), and since L_{e_K} is idempotent. Then, using the factorization property $f = h \star k$ once more, we obtain

$$\left|\operatorname{Tr}_\omega\left((L_f - L_{f_K})\left(1+|\not{D}|\right)^{-n}\right)\right| \leqslant \|L_h - L_{e_K \star h}\| \left|\operatorname{Tr}_\omega\left(L_k\left(1+|\not{D}|\right)^{-n}\right)\right|, \tag{6.12}$$

and the right-hand side vanishes as $K \to \infty$, thanks to estimate (3.3) for the norm of a twisted multiplication operator. On rewriting (6.11) as

$$\operatorname{Tr}_\omega\bigl(L_{f_K}\bigl(1+|\not{D}|\bigr)^{-n}\bigr) = C(n)\int_M f_K(p)\mu_g(p),$$

the left-hand side converges to $\operatorname{Tr}_\omega(L_f(1+|\not{D}|)^{-n})$ as $K \to \infty$. On the right-hand side, the rapid decrease of the coefficients $c_{mn}(\hat{x})$ in (6.4) ensures that $f_K \to f$ in $L^1(M, \mu_g)$. Taking the limit as $K \to \infty$ on both sides of (6.12) therefore yields the desired Dixmier trace evaluation:

$$\operatorname{Tr}_\omega\bigl(L_f\bigl(1+|\not{D}|\bigr)^{-n}\bigr) = C(n)\int_M f(p)\mu_g(p). \qquad \square$$

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