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Sequential order of product spaces

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Abstract

We study the sequential order of product spaces. In some classes of sequential spaces we show the product theorems for sequential order. We construct under the continuum hypothesis two Fréchet spaces whose product is sequential and its sequential order is ω_1.

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1. Introduction

There are many papers devoted to studying the Fréchetness of product spaces [1–3,5,6,10–13,18]. Since every Fréchet space is a sequential space with the sequential order one, it seems natural to pose the following two problems instead of asking the Fréchetness of product spaces:

Problem 1.1. When is the product of Fréchet spaces sequential?

Problem 1.2. If the product of Fréchet spaces is sequential, then are there any relationships between the sequential order of the product space and that of factor spaces?

We just note here that Problem 1 is considered as a special case of the general problem, namely when the product of sequential spaces is sequential which has been widely studied by many authors [3–6,8,18]. We only concentrate on studying Problem 2 in this paper.

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According to Noble [9], the product of two countably compact Fréchet spaces is sequential. One can easily show that in this case the sequential order of the product space is at most two (Corollary 2.3). Hence the sequential order of the product of compact Fréchet spaces constructed by Simon [17] is actually two. In Section 2 we show that if the product of two Fréchet spaces with point-countable k-networks is sequential, then the sequential order of the space is at most two. The class of Fréchet spaces with point-countable k-networks is subtler than the class of countably compact Fréchet spaces because if we do not assume the sequentiality of the product then we have an example of two Fréchet spaces with point-countable k-networks whose product is not sequential, and the sequential order of the sequential coreflection of the product space is three.

In Section 3 we will construct using the continuum hypothesis (CH) two Fréchet spaces whose product is sequential and its sequential order is ω_1.

All spaces in this paper are assumed to be Hausdorff topological spaces.

2. Product theorems for sequential order

A space X is *sequential* if whenever $A \subset X$ and A is not closed, there is a sequence from A converging to a point outside the set A, and X is *Fréchet* (or *Fréchet–Urysohn*) if whenever $x \in \overline{A}$, there is a sequence from A converging to x.

If A is a subset of a space X, then $[A]^{seq}$ denotes the sequential closure of A, i.e., the set of limits of convergent sequences consisting of points of A. Obviously $A \subseteq [A]^{seq}$. We define $[A]_\alpha$ by induction on $\alpha \in \omega_1 + 1$ as follows: $[A]_0 = A$, $[A]_{\alpha+1} = [[A]_\alpha]^{seq}$ and $[A]_\alpha = \bigcup\{[A]_\beta \mid \beta < \alpha\}$ for a limit α. One can easily see that $[A]_{\omega_1+1} = [A]_{\omega_1}$, and that a space X is sequential if and only if $\overline{A} = [A]_{\omega_1}$ for every $A \subset X$. For a sequential space X we define $so(X)$, the sequential order of X, by $so(X) = \min\{\alpha \in \omega_1 + 1 \mid \overline{A} = [A]_\alpha$ for every $A \subset X\}$. Note that a Fréchet space is a sequential space with the sequential order 1 [6].

For a space X we introduce a new topology on X in such a way that a subset A of X is closed if $A \cap K$ is closed for any compact metric (or equivalently any convergent sequence together with the limit point) subset K of X. We call this new space the *sequential coreflection* of X and denote by $S[X]$. Clearly if X is a sequential space then both X and $S[X]$ have the same topology. A collection γ of subsets of X is called *point-countable* if every point $x \in X$ belongs to at most countably many $\xi \in \gamma$. Recall that a collection γ of subset of X is a *k-network* if for every compact $K \subset X$ and any open $U \supseteq K$ there is a finite $\gamma_K \subseteq \gamma$ such that $K \subseteq \cup\gamma_K \subseteq U$. A space X is a *k-space* if whenever $A \subseteq X$ and A is not closed, there is a compact subset K of X such that $A \cap K$ is not closed in K. If we can choose countably many such compact subsets from X, the resulting space is called a *k_ω-space*. It is easy to see that every sequential space is a k-space and it was shown that every k-space with a point-countable k-network is sequential [7].

Lemma 2.1. *Let X be a sequential space and Y be a countably compact sequential space. If $x \in [\pi_X(A)]_\alpha$ for a subset A of $X \times Y$, then there is a point $z \in \pi_X^{-1}(x)$ such that $z \in [A]_\alpha$, where $\pi_X : X \times Y \to X$ is a projection.*

Proof. First we show the case when $\pi_X(A)$ is a sequence converging to x. If $\pi_X^{-1}(x) \cap \overline{A} = \emptyset$, then $\pi_X(\overline{A}) = \pi_X(A) \not\ni x$. This is a contradiction because π_X is a closed map. Now we show the general case. The proof will be done by induction on α. Assume the assertion of the lemma is true for any $\beta < \alpha$. If α is a limit ordinal, then we have nothing to do. Assume $\alpha = \beta + 1$ and $x \in [\pi_X(A)]_{\beta+1} \backslash [\pi_X(A)]_\beta$. Then we can choose a sequence from $[\pi_X(A)]_\beta$ converging to x. Hence this case is reduced to the above one. The proof is completed. □

Theorem 2.2. *Let X be a sequential space and let Y be a regular (locally) countably compact sequential space. Then $X \times Y$ is sequential and $so(X \times Y) \leqslant so(X) + so(Y)$.*

Proof. It is well known that the product $X \times Y$ is sequential [8]. We will determine the sequential order of it.

Let $so(X) \leqslant \alpha$ and $so(Y) \leqslant \beta$. Let $A \subset X \times Y$ and $(x_0, y_0) \in \overline{A} \backslash A$. We show $(x_0, y_0) \in [[A]_\alpha]_\beta$. If $(x_0, y_0) \in \overline{A \cap (\{x_0\} \times Y)}$, then clearly $(x_0, y_0) \in [A]_\beta$. We may assume without loss of generality $A \cap \{x_0\} \times Y = \emptyset$. For any basic neighborhood $W = U \times V$ of (x_0, y_0), choose a point

$$z(W) = \big(x_0, y(W)\big) \in U \times V$$

such that $z(W) \in \overline{A \cap W}$. Then by Lemma 2.1 $z(W) \in [A]_\alpha$. Since the closure of the set $\{z(W) \mid W \text{ ranges all the neighborhood of } z(W)\}$ contains the point (x_0, y_0), we complete the proof. □

Corollary 2.3. *Let X and Y be countably compact Fréchet spaces. Then $so(X \times Y) \leqslant 2$.*

Remark. Shakhmatov [15] posed a problem if the product of two countably compact Fréchet groups is Fréchet. Here is another problem whether the sequential order of a topological group is ω_1 if it is not Fréchet [14]. In view of the above theorem the following problem arises naturally.

Problem 3. Is there any ordinal number α, $1 \leqslant \alpha \leqslant \omega_1$ such that $so(G) \neq \alpha$ for any countably compact topological group G? If G is not Fréchet then is $so(G)$ limit?

Another problem is:

Problem 4. Is there a (countably) compact Fréchet space X such that $so(X^n) > 2$ for some $n > 3$?

It is well known that the product of a sequential space and a first countable (or even metrizable) space need not be sequential [6,5,18]. So in this case it seems natural to consider the sequential order of the sequential coreflection of the product.

Theorem 2.4. *Let X be a sequential space and Y be a first countable space. Then $so(S[X \times Y]) \leqslant so(X) + 1$.*

Proof. Let us prove the following fact by induction on α:

$PR(F,U,z,\alpha)$: Let $F \subseteq X \times Y$, $z \in [F]_\alpha$, $\pi_Y(z) \in U \subseteq Y$, U is open in Y. Then there is $P \subseteq F$ with the following properties:

(1_P) $z \in \overline{P} \subseteq X \times U$,

(2_P) if $x \in [\pi_X(P)]_\gamma$ then there is $z' \in \pi_X^{-1}(x)$ such that $z' \in [P]_\gamma$.

Suppose we have proved $PR(F,U,z,\beta)$ for all the quadruplets (F,U,z,β), where $\beta < \alpha$. Choose $\{U_i\}_{i\in\omega}$ a base of open neighborhoods at the point $\pi_Y(z)$ so that $U_0 = U$, $U_{i+1} \subseteq U_i$. We may assume without loss of generality that $\alpha > 0$ so there is a sequence $\{z_i\}_{i\in\omega}$ such that $z_i \in [F]_{\beta_i}$, $\beta_i < \alpha$, $\pi_Y(z_i) \in U_i$, $z_i \to z$ as $i \to \infty$. Now since $\beta_i < \alpha$ by $PR(F,U_i,z_i,\beta_i)$ find $P_i \subseteq F$ with properties below:

(1_i) $z_i \in \overline{P_i} \subseteq X \times U_i$,

(2_i) if $x \in [\pi_X(P_i)]_\gamma$ then there is $z' \in \pi_X^{-1}(x)$ such that $z' \in [P_i]_\gamma$.

Now put $P = \bigcup_{i\in\omega} P_i$. Let $z' \in \overline{P}$. If $\pi_Y(z') \neq \pi_Y(z)$ then there is $i \in \omega$ such that $\pi_Y(z') \notin \overline{U_i}$ and thus

$$z' \in \bigcup_{j\leqslant i} \overline{P_j} \subseteq X \times U_0 = X \times U.$$

Since $\overline{P} \supseteq \overline{P_i} \ni z_i$ and $z_i \to z$ as $i \to \infty$, then $z \in \overline{P}$. Thus (1_P) holds. Let us now use induction on γ mentioned in (2_P). Suppose the property is satisfied for all $\delta < \gamma$. Choose a sequence $\{x_i\}_{i\in\omega}$ such that $x_i \to x$ as $i \to \infty$, $x_i \in [\pi_X(P)]_{\sigma_i}$, $\sigma_i < \gamma$. For every $i \in \omega$ by (2_i) there is $y_i \in Y$ such that $(x_i, y_i) \in \pi_X^{-1}(x_i)$ and $(x_i, y_i) \in [P]_{\sigma_i}$. Suppose $y_i \neq \pi_Y(z)$. Then there is $j \in \omega$ such that $y_i \notin \overline{U_j}$. It follows that

$$(x_i, y_i) \in \bigcup_{l\leqslant j} \overline{P_l}.$$

Thus it may be assumed without loss of generality that there are three cases:

(1) all $y_i = \pi_Y(z)$,

(2) there is $n \in \omega$ such that $(x_i, y_i) \in [P_n]_{\sigma_i}$,

(3) for every $i \in \omega$ there is $j_i \in \omega$ such that $(x_i, y_i) \in [P_{j_i}]_{\sigma_i}$ and $j_{i+1} > j_i$.

Consider case (2). Since $x_i \to x$ as $i \to \infty$ and $(x_i, y_i) \in [P_n]_{\sigma_i}$ one has that $x \in [\pi_X(P_n)]_\gamma$. By (2_n) one has that there is $z' \in \pi_X^{-1}(x)$ such that $z' \in [P_n]_\gamma \subseteq [P]_\gamma$. Now consider case (3). In this case $y_i \in \pi_Y(\overline{P_{j_i}}) \subseteq U_{j_i}$. Since $\{U_{j_i}\}_{i\in\omega}$ is a base of neighborhoods by the choice of U_i's $y_i \to \pi_Y(z)$ as $i \to \infty$ and now $(x_i, y_i) \to (x, \pi_Y(z))$ as $i \to \infty$. Since $(x_i, y_i) \in [P_{j_i}]_{\sigma_i} \subseteq [P]_{\sigma_i}$ we conclude that $z' = (x, \pi_Y(z)) \in [P]_\gamma$. Case (1) is considered analogously. Thus (2_P) takes place.

Let now

$$z \in \overline{F}^{S[X\times Y]}.$$

Then $z \in [F]_\alpha$ for some $\alpha < \omega_1$. Choose $\{U_i\}_{i\in\omega}$ a neighborhood base at $\pi_Y(z)$. For every $i \in \omega$ using $PR(F,U_i,z,\alpha)$ find $P_i \subseteq F$ with properties (1_{P_i}), (2_{P_i}). By (1_{P_i}) $z \in \overline{P_i}$ so $\pi_X(z) \in [\pi_X(P_i)]_{so(X)}$. Now by (2_{P_i}) there is $z_i \in \pi_X^{-1}(\pi_X(z))$ such that $z_i \in [P_i]_{so(X)}$. By (1_{P_i}) $\pi_Y(z_i) \in U_i$. Thus $z_i \to z$ as $i \to \infty$. We obtain that $z \in [\bigcup_{i\in\omega} P_i]_{so(X)+1} \subseteq [F]_{so(X)+1}$. $\square$

Let $t:\omega^2 \to \Gamma \subseteq X$ be a bijection. Then we say that a pair (t,Γ) is *a table in* X. A set $\theta \subseteq \omega^2$ will be called *thin* (*thick*) if $|\theta| \geqslant \aleph_0$ and for any $n_0 \in \omega$ the following holds: $|\{m \in \omega \mid (n_0,m) \in \theta\}| < \aleph_0$ ($|\{\, n \in \omega \,|\, |\{n\} \times \omega \cap \theta| \geqslant \aleph_0 \,\}| \geqslant \aleph_0$ respectively). We say that a set $\Gamma_\theta = t(\theta) \subseteq X$ is thin (thick) if $\theta \subseteq \omega^2$ is a thin (thick) set.

The following lemma is proved in [16] but we include its proof for the reader's convenience.

Lemma 2.5. *Let (t,Γ) be a table in X and X have a countable k-network. Then there exists a thin set $\Gamma_{\theta'} = \Gamma \backslash \Gamma_\theta$ for some $\Gamma_\theta \subseteq \Gamma$ such that for any thin set $\kappa \subseteq \theta$ there is a thick set σ such that for any open neighborhood U of a nonempty set of cluster points of an arbitrary thin set $\Gamma_{\kappa'}$ where $\kappa' \subseteq \sigma$, we have $U \cap \Gamma_{\kappa\backslash f} \neq \emptyset$, where $f \subseteq \omega^2$ is finite.*

Proof. Let γ be a countable k-network for Γ. Without loss of generality we may assume that γ is closed under finite unions and intersections and that $\Gamma\backslash f \in \gamma$ for any finite $f \subseteq \Gamma$. Let $\Xi = \{\xi_i \mid i \in \omega\} \subseteq \gamma$ be the set of all such $\xi_i \in \gamma$ that the set $\Gamma\backslash(\xi_i \cup \Gamma_{\{0,\ldots,n_i\}\times\omega})$ is thin for some $n_i \in \omega$. Without loss of generality we may assume that $n_0 = 0$ and that $n_{i+1} > n_i$. We let

$$\Gamma_\theta = \bigcup_{i\in\omega}\left(\left(\bigcap_{k\leqslant i}\xi_k\right) \cap \Gamma_{\{n_i+1,\ldots,n_{i+1}\}\times\omega}\right).$$

Then $\Gamma\backslash\Gamma_\theta$ is a thin set by the way ξ_i's and n_i's were chosen. Let now $\kappa \subseteq \theta$ be an arbitrary thin set. Let $M = \{\mu_i \mid i \in \omega\} \subseteq \gamma$ be the set of all $\mu_i \in \gamma$ such that $\mu_i \cap \Gamma_\kappa = \emptyset$. Then note that $\mu_i \notin \Xi$. We will show that M is closed under finite unions. Let $M' \subseteq M$ be finite. Then $\mu = \cup M' \in \gamma$, $\mu \cap \Gamma_\kappa = \emptyset$. Let us prove that $\mu \notin \Xi$. Suppose not, then $\mu = \xi_k$ for some $k \in \omega$. Since κ is infinite and thin, there exists $n' \geqslant n_k$ such that $(n',m) \in \kappa$ for some $m \in \omega$. Since $\kappa \subseteq \theta$,

$$t(n',m) \in \bigcap_{i\leqslant k}\xi_i \subseteq \xi_k = \mu.$$

But $\mu \cap \Gamma_\kappa = \emptyset$, a contradiction. Hence $\mu \notin \Xi$ and $\mu \in M$. For every $i \in \omega$ choose $m_i \in \omega$ such that

$$\left|t(\{m_i\}\times\omega)\backslash\bigcup_{j\leqslant i}\mu_j\right| \geqslant \aleph_0, \quad m_{i+1} > m_i.$$

It is possible since otherwise there is $j \in \omega$ such that

$$\Gamma\backslash\left(\bigcup_{i\leqslant j}\mu_i \cap \left(\Gamma\backslash\Gamma_{\{0,\ldots,n_j\}\times\omega}\right)\right)$$

is thin for some $n_i \in \omega$. Then $\bigcup_{i\leqslant j}\mu_i \in \Xi$ contradicting $\bigcup_{i\leqslant j}\mu_i \in M$ and $M \cap \Xi = \emptyset$. We let

$$\Gamma_\sigma = \bigcup_{i\in\omega}\left(t(\{m_i\}\times\omega)\backslash\bigcup_{j\leqslant i}\mu_i\right).$$

By the way μ_i's were chosen σ is a thick set. Let $\kappa' \subseteq \sigma$ be a thin set, B be the set of all cluster points of $\Gamma_{\kappa'}$. Suppose that there is an open neighborhood $U \supseteq B$ such that $\overline{U} \cap \Gamma_{\kappa \setminus f} = \emptyset$ for some finite $f \subseteq \omega^2$. Then $T = (U \cap \Gamma_{\kappa'}) \setminus \{x\}$, $x \in B$ is not closed and (since $\overline{T} \subseteq U$) $\overline{T} \subseteq X \setminus \overline{\Gamma_{\kappa \setminus f}}$. Then there exists $\xi \in \gamma$ such that $\xi \cap \overline{\Gamma_{\kappa \setminus f}} = \emptyset$ and $|\xi \cap T| \geqslant \aleph_0$. Since $\xi \setminus \Gamma_f \in \gamma$ we may assume that $\xi \cap \overline{\Gamma_\kappa} = \emptyset$. Then there are two possibilities:

(1) $\xi \notin M$. Then it follows from $\xi \in \gamma$, $\xi \cap \Gamma_\kappa = \emptyset$ that $\xi \in \Xi$. But then $\xi \cap \Gamma_\kappa \neq \emptyset$. A contradiction.

(2) $\xi \in M$. Then $\xi = \mu_i$ for some $i \in \omega$. Since $\xi \cap \Gamma_{\kappa'}$ is infinite and thin, there is $n \in \omega$, $n \geqslant i$ such that $t(m_n, k) \in \xi = \mu_i$ and $(m_n, k) \in \kappa'$ for some $k \in \omega$. But

$$t(m_n, k) \in t(\{m_n\} \times \omega) \setminus \bigcup_{j \leqslant n} \mu_j \subseteq X \setminus \mu_i.$$

A contradiction. □

If $f : X \to Y$ is a map then by $f|A$ we denote the restriction of f on $A \subseteq X$.

Lemma 2.6. *Let X be a Fréchet space with a point-countable k-network. Let $\Gamma = \{x(i,j) \mid i, j \in \omega\} \subset X$ be such that:*

(1) $t : \omega^2 \to \Gamma$ is a bijection where $t(i,j) = x(i,j)$,

(2) $x(i,j) \to x(i)$ as $j \to \infty$ for some $x(i) \in X$,

(3) $x(n) \neq x(m) \neq x(i,j)$ for any $n \neq m$, $i, j \in \omega$,

(4) $x(i) \to x \in X$ as $i \to \infty$.

Then there is a thin set $\theta' \subset \omega^2$ such that $\Gamma' = \overline{\{x(i,j) \mid (i,j) \notin \theta'\}}$ is compact with every point $x(i,j)$ isolated in Γ'.

Proof. We may assume without loss of generality that $X = \Gamma \cup \{x(i) \mid i \in \omega\} \cup \{x\}$ and thus X has a countable k-network. Strengthen the topology of X by declaring every point $x(i,j)$ isolated and the points x and $x(i)$ having their original neighborhoods. It is easy to see that in such topology to be denoted as τ X remains a Fréchet space with a countable k-network. First prove that the lemma is valid for X equipped with such topology. Choose θ as in Lemma 2.5. Let us prove that in the new topology $\Gamma(\tau)' = \overline{\{x(i,j) \mid (i,j) \in \theta\}}^{\tau}$ is compact. Suppose not. Then there exists a thin set $\kappa \subset \omega^2 \setminus \theta' = \theta$ such that Γ_κ is discrete and closed in Γ' (we use the fact that every $x(i,j)$ is isolated in the new topology). Find a thick set $\sigma \subset \theta$ as in Lemma 2.5. Since σ is thick we have that $x \in \overline{\{x(i,j) \mid (i,j) \in \sigma\}}^{\tau}$. Now Γ is Fréchet in τ so there exists a thin set $\kappa' \subset \sigma$ such that $S = \{x(i,j) \mid (i,j) \in \kappa'\} \cup \{x\}$ is homeomorphic to a countable compact with the unique nonisolated point x. Now $U = \Gamma \setminus \Gamma_\kappa$ is a neighborhood of x in the new topology τ and x is the unique cluster point of $S = \Gamma_{\kappa'}$. But $U \cap \Gamma_\kappa = \emptyset$ contradicting to Lemma 2.5. Now $\Gamma(\tau)'$ is compact in a stronger topology τ. So $\Gamma(\tau)' = \Gamma'$. □

Lemma 2.7. *Let X and Y be Fréchet spaces with point-countable k-networks. Suppose that*

$$\Gamma = \{z(i,j) \mid i, j \in \omega\} \subset X \times Y$$

is such that $z(i,j) \to z(i)$ *as* $j \to \infty$, $z(i) \to z$ *as* $i \to \infty$, $z(i_1,j_1) \neq z(i_2,j_2)$ *if* $(i_1,j_1) \neq (i_2,j_2)$, $z(i) \neq z(j)$ *if* $i \neq j$, $z \neq z(i) \neq z(m,n) \neq z$ *for any* $i,m,n \in \omega$. *Suppose also that there is no sequence in* Γ *converging to* z. *Then there exist infinite set* $\sigma \subset \omega$ *and a projection* $\pi \in \{\pi_X, \pi_Y\}$ *such that* $\pi(z(i)) = \pi(z(j))$ *for any* $i,j \in \sigma$.

Proof. Suppose that there are no such π, σ. Then it is easy to see that there exists $\sigma' \subset \omega$ such that $\pi_X|\{z(i) \mid i \in \sigma'\}$ and $\pi_Y|\{z(i) \mid i \in \sigma'\}$ are injections. We may assume without loss of generality that $\sigma' = \omega$. Then it may be easily shown that there exists a thick set $\theta \subset \omega^2$ such that either $\pi_Y|\{z(i,j) \mid (i,j) \in \theta\}$ and $\pi_X|\{z(i,j) \mid (i,j) \in \theta\}$ are injections or $\pi_Z|\{z(i,j) \mid (i,j) \in \theta\}$ is an injection and

$$\pi_{\{X,Y\}\setminus Z}\big(\{z(i) \mid (i,j) \in \theta\}\big) = \pi_{\{X,Y\}\setminus Z}\big(\{z(i,j) \mid (i,j) \in \theta\}\big)$$

where $Z \in \{X,Y\}$. Consider the first case. We may assume without loss of generality that $\theta = \omega^2$. So we have that $\pi_X|\Gamma \cup \{z(i) \mid i \in \omega\} \cup \{z\}$ and $\pi_Y|\Gamma \cup \{z(i) \mid i \in \omega\} \cup \{z\}$ are injections. It follows from Lemma 2.6 that there exists a thin set $\theta' \subset \omega^2$ such that

$$\pi_X\big(\{z(i,j) \mid (i,j) \notin \theta'\} \cup \{z\} \cup \{z(i) \mid i \in \omega\}\big)$$

and

$$\pi_Y\big(\{z(i,j) \mid (i,j) \notin \theta'\} \cup \{z\} \cup \{z(i) \mid i \in \omega\}\big)$$

are compact and thus metrizable subsets. But then

$$\{z(i,j) \mid (i,j) \notin \theta'\} \cup \{z\} \cup \{z(i) \mid i \in \omega\}$$

is a Fréchet subspace of $X \times Y$ which contradicts the fact that there is no sequence in Γ converging to z. $\square$

We recall the definition of *sequential fan* S_ω. S_ω is obtained from disjoint sum of countably many nontrivial convergent sequences by identifying the limit points to one point, endowed with the quotient topology. Let S_i be a convergent sequence with the limit point x_i and let $S_0 = \{y_i \mid i = 1,2,\dots\} \cup \{x_0\}$. Identifying points y_i and x_i to z_i for $i = 1,2,\dots$ and equipping the resulting space with quotient topology one get another canonical space S_2.

The following lemma was proved in [4, Lemma 4].

Lemma 2.8. *Let* X *be a k-space with a closed point-countable k-network and let* $X \times S_\omega$ *be a k-space. Then* X *has a point-countable k-network consisting of compact sets.*

Lemma 2.9. *Let* X *be a Fréchet space with a point-countable k-network. If* X *is not first-countable then it contains a closed subspace homeomorphic to* S_ω.

Proof. Let $x \in X$ be a strongly Fréchet point. Declaring every point $x' \in X$, $x' \neq x$ isolated and x having old neighborhoods we get a regular strongly Fréchet topology τ on X and X has a point-countable k-network in this topology. Now by [7, Corollary 3.6] X is first-countable at x in the topology τ and thus in its original topology.

Let now X be nonfirst-countable at $x \in X$. By the fact proved above x is not a strongly Fréchet point. So there exists

$$\Gamma = \{x(i,j) \mid i,j \in \omega\} \subset X$$

such that $x(i,j) \to x$ as $j \to \infty$ and there is no sequence $x(i_k, j_k)$ such that $x(i_k, j_k) \to x$ as $k \to \infty$ and $i_{k+1} > i_k$ for every $k \in \omega$. Let γ be a point-countable k-network in X. Let

$$\gamma' = \{\xi_n \mid n \in \omega\} = \{\xi \in \gamma \mid \xi \cap \Gamma \neq \emptyset,\ x \notin \overline{\xi}\}.$$

Now it is easy to choose for every $i \in \omega$ a number $n_i \in \omega$ such that

$$x(i,j) \notin \bigcup_{k \leqslant i} \overline{\xi_k}$$

whenever $j \geqslant n_i$. It is easy to see that $\Gamma' = \{x(i,j) \mid j \geqslant n_i\} \cup \{x\}$ is homeomorphic to S_ω. Let us prove that Γ' is closed in X. Suppose not. Then there exists a sequence $x(i_k, j_k)$ such that $x(i_k, j_k) \to x' \neq x$ as $k \to \infty$, $x(i_k, j_k) \in \Gamma'$ and $i_{k+1} > i_k$. So there exists $\xi_p \in \gamma'$ such that $\xi_p \cap \{x(i_k, j_k) \mid k \in \omega\}$ is infinite. But it is easy to check that $x(i_k, j_k) \notin \xi_p$ when $i_k > p$, a contradiction. □

Lemma 2.10. *Let X, Y be Fréchet spaces with point-countable k-networks. Let $x \in X$, $y(i) \to y$ as $i \to \infty$, $y(i), y \in Y$, $y(i) \neq y(j)$ if $i \neq j$. Let $\Gamma = \{z(i,j) \mid i,j \in \omega\} \subset X \times Y$ be such that:*

(1) $z(i,j) \to (x, y(i))$ as $j \to \infty$,

(2) there is no sequence in Γ converging to (x,y).

Then there exists $S' \subset \Gamma$ such that $S = \pi_X(S') \cup \{x\}$ is homeomorphic to S_ω, closed in X and so that x is the unique nonisolated point in S.

Proof. Using Lemma 2.5. we can choose a thick set $\theta \subset \omega^2$ such that

$$K = \pi_Y(\Gamma) \cup \{(x, y(i)) \mid i \in \omega\} \cup \{(x,y)\}$$

is compact. To simplify the notations we assume that $\theta = \omega^2$. By induction on $i \in \omega$ and Hausdorffness of X we can choose a thick set $\theta' \subset \omega^2$ such that every point $\pi_X(z(i,j))$ where $(i,j) \in \theta$ is isolated in

$$X' = \pi_X(\{z(i,j) \mid (i,j) \in \omega^2\}).$$

Now it is easy to see that $X' \times K$ is not Fréchet where K is a compact metrizable set. Thus X' is not first-countable and, by Lemma 2.10, contains a closed subset homeomorphic to S_ω. Using the fact that x is the unique nonisolated point in X' one can easily choose the set $S' \subset \Gamma$ having all necessary properties. □

Using the method similar to that of the proof of Lemma 2.9 one can prove:

Lemma 2.11. *Let X be a Fréchet space with a point-countable k-network and $S(i)$ be a closed subspace of X homeomorphic to S_ω with the unique nonisolated (in $S(i)$) point*

$x(i)$. Let also $x(i) \to x$ as $i \to \infty$. Then there exists a sequence of open sets $U(i) \ni x(i)$ such that $\{x\} \cup \bigcup_{i\in\omega}(S(i) \cap U(i))$ is closed in X.

Theorem 2.12. *Let X and Y be Fréchet spaces with point-countable k-networks. Suppose that $X \times Y$ is sequential. Then $so(X \times Y) \leqslant 2$.*

Proof. Suppose the contrary. Then there exists a set

$$\Gamma\Gamma = \{z(i,j,k) \mid i,j,k \in \omega\} \subset Z = X \times Y$$

such that $z(i,j,k) \to z(i,j)$ as $k \to \infty$, $z(i,j) \to z(i)$ as $j \to \infty$, $z(i) \to z$ as $i \to \infty$ and hold:

(1) there is no sequence in $\Gamma\Gamma$ converging to t, $z(i)$,

(2) there is no sequence in $\Gamma = \{z(i,j) \mid i,j \in \omega\}$ converging to t.

Using Lemma 2.7 we can assume without loss of generality that $\pi_X(z(i)) = \pi_X(z)$ for every $i \in \omega$. Applying Lemma 2.7 to every set

$$\Gamma_i = \{z(i,j,k) \mid j,k \in \omega\} \cup \{z(i,j) \mid j \in \omega\} \cup \{z(i)\}$$

we have that we may assume without loss of generality that either $\pi_Y(z(i,j)) = \pi_Y(z(i))$ or $\pi_X(z(i,j)) = \pi_X(z(i)) = \pi(z)$ for all $i,j \in \omega$. But in the last case the set $\Gamma \cup \{z\}$ lies in $\pi_X^{-1}(z)$ – the Fréchet subspace of Z and $\overline{\Gamma} \ni z$ which contradicts to condition (2). So for every $i \in \omega$, $\pi_Y(z(i,j)) = \pi_Y(z(i))$. Now using Lemma 2.10 we have that $\pi_Y(\{z(i,j,k) \mid j,k \in \omega\} \cup \{z(i)\})$ contains a subset, say $S(i)$, homeomorphic to S_ω and such that $y(i) = \pi_Y(z(i))$ is the unique nonisolated point in $S(i)$. By Lemma 2.11 we may assume that the set

$$T = \bigcup_{i\in\omega} S(i) \cup \{\pi_Y(z)\}$$

is closed in Y. Using Lemma 2.10, condition (2) and the fact that $\pi_X(z(i)) = \pi_X(z)$ it is easy to show that X contains a closed subset homeomorphic to S_ω. Now it may be easily checked that T is a regular closed countable subspace of Y. So $T \times S_\omega$ is a k-space. Applying Lemma 2.8 we get that T has a countable k-network consisting of compact sets. Let $\gamma = \{\xi_i \mid i \in \omega\}$ be such a network for T. Since every ξ_i is compact one can choose for every $i \in \omega$ a sequence $Sq(i) \subset S(i)$ converging to $y(i)$ such that

$$Sq(i) \cap \bigcup_{k\leqslant i} \xi_i = \emptyset.$$

Then

$$\overline{\bigcup_{i\in\omega} Sq(i)} \ni \pi_Y(z).$$

But there is no sequence in $\bigcup_{i\in\omega} Sq(i)$ converging to $\pi_Y(z)$. Otherwise such a sequence would have an infinite intersection with some ξ_i which contradicts to the fact that

$$Sq(i) \cap \bigcup_{k\leqslant i} \xi_i = \emptyset$$

for every $i \in \omega$. We obtain a contradiction to the Fréchetness of Y. $\square$

In view of Theorems 2.2 and 2.4, one may expect that the sequentiality of products in the above theorem may be omitted by replacing $so(X \times Y)$ by $so(S[X \times Y])$. The following example shows that this is not the case.

Example 2.13. There are Fréchet spaces X and Y with point-countable k-networks such that $so(S[X \times Y]) = 3$.

Proof. Let $X = S_\omega$. Let S_i, $i \in \omega$ be sequential fans. Let D be a discrete sum of S_i's and $Y = D \cup \{t\}$, where t is a point and $t \notin D$. Every point $y \in S_i \subseteq Y$ has its usual neighborhoods. The point t has a neighborhood base consisting of all the sets of the form $U = \{t\} \cup \{D\backslash$ finitely many S_i's$\}$. Obviously Y is Fréchet and has a countable closed k-network. We prove $so(S[X \times Y]) \geqslant 3$. To show it let us put

$$X = S_\omega = \{s\} \cup \{x(n,m) \mid n, m \in \omega\},$$

$x(n,m) \to s$ as $m \to \infty$ and

$$S_i = \{s_i\} \cup \{x_i(n,m) \mid n, m \in \omega\},$$

$x_i(n,m) \to s_i$ as $m \to \infty$.

Let us put

$$A = \bigcup\bigcup\Big\{\big\{\{(x(k,n), x_k(n,m)) \mid m \in \omega\} \mid n \in \omega\big\} \mid k \in \omega\Big\}.$$

Then it is easy to see that:

$$[A]_1 = A \cup \bigcup\big\{\{(x(k,m), s_k) \mid m \in \omega\} \mid k \in \omega\big\},$$
$$[A]_2 = [A]_1 \cup \{(s, s_k) \mid k \in \omega\},$$
$$[A]_3 = [A]_2 \cup \{(s,t)\}.$$

The proof is completed. $\square$

Problem 5. Let X and Y be k-spaces with point-countable k-networks and $X \times Y$ be sequential. Then is $so(X \times Y) \leqslant \max\{so(X) + so(Y), so(Y) + so(X)\}$?

Remark. Note that $+$ in the above theorems and problem means the *sum of ordinals*, so it is not commutative. If we assume that X in the above problem is Fréchet and $so(Y)$ is finite, then we can solve the above problem similarly. But in general case we don't know the answer even if we assume X is Fréchet.

3. $so(X \times Y)$ can be arbitrarily large for Fréchet X and Y

Lemma 3.1. *Let X be a space with topology τ. Let $\{x_n \mid n \in \omega\}$ be a closed discrete subset of X and x be a point of X. Then there is the strongest topology σ weaker than τ such that D is a sequence converging to the point x with respect to σ.*

Proof. Let γ_n, $n \in \omega$, and γ be neighborhood bases in τ at x_n, $n \in \omega$, and x, respectively. We only change the neighborhood base γ at x by the new neighborhood base γ':

$$\gamma' = \Big\{U \cup W \mid U \in \gamma,\ W = \bigcup\{V_n \in \gamma_n \mid n \geqslant i \text{ for some } i \in \omega\}\Big\}.$$

It is easy to show that our new topology satisfies all the requirements. □

Before mentioning the following lemma, we would like to note that a countable k_ω-space is sequential.

Lemma 3.2 (CH). *Let X be a countable k_ω-space. Then there exist two topologies τ_1 and τ_2 on X weaker than the original topology of X such that X is embedded into X^2 equipped with topology $\tau_1 \times \tau_2$ as a closed subset and τ_1 and τ_2 are strongly Fréchet, $\tau_1 \times \tau_2$ is sequential.*

Proof. Let X be a countable k_ω-space. Let us fix $\{U_n\}_{n\in\omega}$—a countable number of open subsets of X such that for any $x_1, x_2 \in X$ such that $x_1 \neq x_2$ there are $U_{n_1} \ni x_1$ and $U_{n_2} \ni x_2$ such that $U_{n_1} \cap U_{n_2} = \emptyset$.

Let us count all the quadruplets $\{(x, F, S, G)\} = T$ where $x \in X$, $F \subset X$ is closed,

$$S = \bigcup_{p\in\omega} S_p, \quad S_p = \{x(p,n) \mid n \in \omega\},$$

$x(p,n) \neq x(q,k)$ if $(p,n) \neq (q,k)$, $G \subset X^2$ is closed. By CH

$$T = \{(x_\alpha, F_\alpha, S_\alpha, G_\alpha)\}_{\alpha<\omega_1}, \quad S_\alpha = \bigcup_{p\in\omega} S_p^\alpha.$$

For each $\alpha < \omega_1$ we shall construct two discrete subsets $D_1^\alpha \subset X$ and $D_2^\alpha \subset X$, two topologies τ_1^α and τ_2^α on X and four subsets $x_\alpha \in O_\alpha^i \subset X$ and $F_\alpha \subset V_\alpha^i \subset X$, $i \in \{1,2\}$ and two families of subsets $\{L_k^\alpha \mid k \in \omega\}$, $\{R_k^\alpha \mid k \in \omega\}$ such that:

(1) If

$$x_\beta \notin \overline{F_\beta}^{\tau_i^\beta},$$

then $O_\beta^i \cap V_\beta^i = \emptyset$, O_β^i, V_β^i are open in τ_i^β for $\beta \leqslant \alpha$, $i \in \{1,2\}$. Otherwise $O_\beta^i = V_\beta^i = X$.

(2) D_i^α is a closed discrete subset of X in topology τ_i^β for $\beta < \alpha$, $D_i^\alpha \subset S_\alpha$.

(3a) If $x(p,n) \to x(p)$ as $n \to \infty$ in some $\tau_i^{\beta_p}$, $\beta_p < \alpha$ and $x(p) \to x_\alpha$ as $p \to \infty$ in some τ_i^β, $\beta < \alpha$ and no sequence from S_α converges to x_α, then D_i^α is infinite and $D_i^\alpha \cap S_p$ is finite, $i \in \{1,2\}$.

(3b) Otherwise D_i^α is empty.

(4) $D_i^{\beta_i} \cap D_{2/i}^{\beta_{2/i}}$ is finite for any β_i, $\beta_{2i} \leqslant \alpha$, where $2/i = 1$ if $i = 2$ and $= 2$ if $i = 1$.

(5) L_k^α's are open in τ_1^α, R_k^α's are open in τ_2^α.

(6) If G_α is closed in X^2 in topology $\tau_1^\alpha \times \tau_2^\alpha$ then every point $z \in X^2 \backslash G_\alpha$ has a neighborhood of the form $L_k^\alpha \times R_k^\alpha \ni z$ such that $L_k^\alpha \times R_k^\alpha \cap G_\alpha = \emptyset$.

(7) U_n's are open in τ_i^β, $\beta < \alpha$.

(8) τ_i^α is the strongest topology such that τ_i^α is weaker than the original topology on X and $D_i^\beta \cup \{x_\beta\}$ (see (3)) is homeomorphic to a countable compact space with the limit point x_β, $\beta \leqslant \alpha$.

Now it may be easily shown that provided (1)–(8) hold the following conditions hold:

(9) Every τ_i^α is a k_ω-topology (follows from (8)).

(10) Every τ_i^α is Hausdorff (follows from (7) and the choice of $\{U_n\}$) and thus (by (9) and Lemma 3.2 below) normal.

(11) Let $y(k) \to y$ as $k \to \infty$ in topology τ_i^β for $i \in \{1,2\}$, $\beta < \alpha$. Then $D_i^\alpha \cap \{y(k) \mid k \in \omega\}$ is finite (follows from (8) and (2)).

Let now τ_1 be the strongest topology on X such that τ_1 is weaker than τ_1^α for any $\alpha < \omega_1$, τ_2 may be defined analogously. It is possible to prove the following facts about τ_i:

(12) τ_i is the strongest topology on X such that τ_i is weaker than the original topology on X and $D_i^\alpha \cup \{x_\alpha\}$ (see (3) and Lemma 3.1) is homeomorphic to a countable compact with the limit point x_α for every $\alpha < \omega_1$.

(13) The sets $\{U_n\}$, O_β^i and V_β^i (see (1)), $\{L_k^\alpha\}$and $\{R_k^\alpha\}$ (see (5)) are open in τ_i for $\beta < \omega_1$.

Let us prove that τ_i is regular. First let us note that it follows from (13) and the construction of $\{U_n\}$ that τ_i is Hausdorff. Suppose that $F \subset X$ is closed in τ_i and $x \in X$, $x \notin F$. Then F is closed in the original topology of X by (12) and thus $F = F_\alpha$, $x = x_\alpha$ for some $\alpha < \omega_1$. Now $x_\alpha \in O_\alpha^i$, $F_\alpha \subset V_\alpha^i$ and $O_\alpha^i \cap V_\alpha^i = \emptyset$ by (1). By (13) O_α^i and V_α^i are open in τ_i. We conclude that τ_i is regular and thus Lindelöf.

Let us prove that τ_i is strongly Fréchet. Suppose not. Then there exist a set $\{x(p,n) \mid p,n \in \omega\}$ such that $x(p,n) \to x(p)$ as $n \to \infty$ in τ_i and $x(p) \to x$ as $p \to \infty$ in τ_i, $x(p,n) \neq x(q,n)$ if $(p,n) \neq (q,k)$ and there is no sequence in $S = \{x(p,n) \mid p,n \in \omega\}$ converging to x. It easily follows from the definition of topology τ_i that there exists $\alpha < \omega_1$ such that $S = S_\alpha$ and (3a) holds. But then D_i^α is a sequence in S converging to $x = x_\alpha$ in topology τ_i^α and thus in τ_i.

Now $\Delta = \{(x,x) \mid x \in X\} \subset X^2$ is closed in X^2 in topology $\tau_1 \times \tau_2$ since it is the diagonal of X^2 where X is equipped with a weaker than both τ_1 and τ_2 Hausdorff topology τ generated by $\{U_n\}$.

Δ is homeomorphic to X in its original topology. Indeed otherwise there would exist a sequence $\{x(k) \mid k \in \omega\} \subset \Delta$ such that $\pi_i(x(k)) \to x \in X$ as $k \to \infty$ in τ_i and $\pi_i(x(k)) \not\to x$ in the original topology of X. By (12) we can assume that

$$\{\pi_i(x(k)) \mid k \in \omega\} \subset D_i^{\alpha_i}$$

for some $\alpha_i < \omega_1$. But $D_i^{\alpha_i} \cap D_{2/i}^{\alpha_{2/i}}$ is finite by (4) which contradicts the fact that $\pi_1(x(k)) = \pi_2(x(k))$ and thus $D_i^{\alpha_i} \cap D_{2/i}^{\alpha_{2/i}} = \{\pi_i(x(k)) \mid k \in \omega\}$ is infinite.

Let us prove that $\tau_1 \times \tau_2$ is sequential. Suppose that $G \subset X^2$ is not closed and there is no sequence in G converging outside G. It is easy to see that $G = G_\alpha$ for some $\alpha < \omega_1$. Since $\tau_1^\alpha \times \tau_2^\alpha$ is a k_ω-topology it is sequential. If G is not closed in $\tau_1^\alpha \times \tau_2^\alpha$ there is a sequence in G converging outside G in topology $\tau_1^\alpha \times \tau_2^\alpha$ and thus in a weaker topology

$\tau_1 \times \tau_2$. So we may assume that G is closed in $\tau_1^\alpha \times \tau_2^\alpha$. Then by (13) and (6) G is closed in $\tau_1 \times \tau_2$, a contradiction.

Let us now construct all the neccessary sets.

The following lemma is well known and easy to prove.

Lemma 3.3. *A k_ω-space is normal.*

Suppose now that all the neccessary sets and τ_i^β are already constructed for every $\beta < \alpha$. Let us prove the following fact.

Fact. *Let X be a countable k_ω-space. Suppose that*

$$S = \bigcup_{p \in \omega} S_p, \quad S_p = \{x(p,n) \mid n \in \omega\},$$

$x(p,n) \neq x(q,k)$ if $(p,n) \neq (q,k)$ and $x(p,n) \to x(p)$ as $n \to \infty$ and $x(p) \to x$ as $p \to \infty$. Let $\{M_n \mid n \in \omega\}$ be a countable number of open subsets of X. If there is no sequence in S converging to x then there is a closed discrete subset D of X such that:

(14) *$D \subset S$.*

(15) *$D \cap S_p$ is finite for every $p \in \omega$.*

(16) *If $x \in M_k$ for some $k \in \omega$ then $D \backslash M_k$ is finite.*

Proof. Without loss of generality we may assume that there are two cases:

Case 1. $x(p) = x$ for all $p \in \omega$.

Case 2. $x(p) \neq x(q)$ if $p \neq q$, $x(p) \neq x$ for any $p \in \omega$ and $\{x(p) \mid p \in \omega\} \cap S = \emptyset$.

Let us now note that a countable k_ω-space is a normal sequential space with a countable closed k-network. Suppose Case 1 holds. Then arguing like in the proof of Lemma 2.9 one can show that, we may assume without loss of generality, $S \cup \{x\}$ is a closed subset of X homeomorphic to sequential fan S_ω. It also may be assumed that $x \in M_n$ for any $n \in \omega$ and $M_{n+1} \subseteq M_n$ for any $n \in \omega$. Now it is easy to choose the points d_i satisfying $d_i \in M_i \cap S_i$. Let $D = \{d_i \mid i \in \omega\}$. One can check that D satisfies (14)–(16).

Now suppose that (15) holds. Shrinking the set $Sq = \{x(p) \mid p \in \omega\} \cup \{x\}$ to a point we obtain a space X/Sq with a countable k-network. Let $\Gamma = S$. Repeating the argument from the proof of Lemma 2.9 beginning with the words "Let γ be a point-countable k-network ..." literally one can show that in X/Sq the set $\Gamma \cup p(Sq)$ may be assumed to be homeomorphic to S_ω and closed where $p: X \to X/Sq$ is the obvious quotient map. It follows that we may assume that $S \cup Sq$ is a closed subset homeomorphic to the space S_2. The rest of the proof is similar to the corresponding part of proof for Case 1. □

Now let

$$\begin{aligned}\{M_n^i \mid n \in \omega\} = \{U_n \mid n \in \omega\} &\cup \{L_k^\beta \mid k \in \omega, \beta < \alpha\} \\ &\cup \{R_k^\beta \mid k \in \omega, \beta < \alpha\} \\ &\cup \{O_\beta^i \mid \beta < \alpha\} \cup \{V_\beta^i \mid \beta < \alpha\}.\end{aligned}$$

Let us construct D_i^α. If S_α is such that (3b) holds, then the construction is obvious. Suppose that S_α satisfies (3a). Let $\{\beta \mid \beta < \alpha\} = \{\beta_k \mid k \in \omega\}$. We may assume without loss of generality that

(*) $S_p^\alpha \cap (\bigcup_{k \leqslant p} D_{2/i}^{\beta_k}) = \emptyset$ (see (11)).

Let $\{M_n \mid n \in \omega\} = \{M_n^i \mid n \in \omega\}$ for $i \in \{1,2\}$, $S = S_\alpha$ and $S_p = S_p^\alpha$. Choose $D \subset S$ satisfying conditions (14)–(16) of the fact. Let $D_i^\alpha = D$. Now let τ_i^α be the strongest topology on X such that τ_i^α is weaker than the original topology of X and $D_i^\beta \cup \{t_\beta\}$ is homeomorphic to a countable compact with the unique nonisolated point t_β. By condition (16) of the fact and condition (7) all U_n's are open in τ_i^α.

Since every U_n is open in τ_i^α, τ_i^α is Hausdorff, and being a k_ω-topology is normal by Lemma 3.3, it is easy to construct O_α^i and V_α^i satisfying (1). Then it is easy to construct $\{L_k^\alpha\}$ and $\{R_k^\alpha\}$ to satisfy (6) using the countability of X^2. By assumption (*) condition (4) is also satisfied. Other conditions are easy to check. So all the neccessary sets are constructed.

Example 3.4 (CH). If we take X being Arhangel'skii–Franklin space [3] we obtain two Fréchet (in fact strongly Fréchet) spaces Y and Z whose product is sequential and $so(Y \times Z) = \omega_1$.

Remark. By Lemma 3.2 one can construct two strongly Fréchet spaces whose product contains a closed nonmetrizable Lašnev subspace. Such an example was constructed in [11] by a different method.

Problem 6. Are there Fréchet spaces $X(\alpha)$ and $Y(\alpha)$ such that $so(X(\alpha) \times Y(\alpha)) = \alpha$ for a given $2 < \alpha < \omega_1$?

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