Computers and Mathematics with Applications 59 (2010) 1318–1325

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On almost automorphic mild solutions for fractional semilinear initial value problems[☆]

Anping Chen[a,b], Fulai Chen[b,*], Siqing Deng[b]

[a] School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411005, China
[b] Department of Mathematics, Xiangnan University, Chenzhou 423000, China

ARTICLE INFO

Keywords:
Fractional differential equation
Almost automorphic function
Mild solution

ABSTRACT

This paper investigates almost automorphic mild solutions of the fractional semilinear equation $D^{\alpha}x(t) = Ax(t) + f(t, x(t))$, $0 < \alpha < 1$, considered in a Banach space X, where A is a linear operator of sectorial type $\omega < 0$. Some sufficient conditions are given for the existence, uniqueness and uniform stability of almost automorphic mild solutions to this semilinear equation.

1. Introduction

Fractional differential equation has received increasing attention during recent years since fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [1]. Recently, there has been a significant development in some basic theories, including the existence and uniqueness of solutions, for the initial value problems of fractional differential equations involving Riemann–Liouville differential operator (see, for example, [2–6] and the references therein).

In this paper, we consider the existence, uniqueness and uniform stability of almost automorphic mild solutions to the following fractional semilinear initial value problem involving Riemann–Liouville differential operator

$$\begin{cases} D^{\alpha}x(t) = Ax(t) + f(t, x(t)), & t \in J = [0, T], 0 < \alpha < 1 \\ [D^{\alpha-1}x(t)]_{t=0} = x_0, \end{cases} \tag{1}$$

where $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X, $f : J \times X \to X$ is an almost automorphic function in t for each $x \in X$.

Definition 1.1. A continuous function $f : R \to X$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in N}$ there exists a subsequence $(s_n)_{n\in N} \subset (s'_n)_{n\in N}$ such that $g(t) := \lim_{n\to\infty} f(t + s_n)$ is well defined for each $t \in R$, and $f(t) = \lim_{n\to\infty} g(t - s_n)$ for each $t \in R$.

Almost automorphic functions constitute a Banach space $AA(X)$ endowed with the supnorm given by $\|f\|_{\infty} := \sup_{t\in J} \|f(t)\|$. For more details on almost automorphicity, see [7].

The study of almost automorphic mild solutions of problem (1) in the borderline case $\alpha = 1$ was well studied in [8,9]. In the case $1 < \alpha < 2$, Araya and Lizama [10] proved that the existence and uniqueness of mild solutions of problem

☆ This work was supported by the Scientific Research Foundation of Hunan Provincial Education Department (05A057), the Aid Program for Science and Technology Innovative Research Team in Higher Educational Institutions of Hunan Province, and the Construct Program of the Key Discipline in Hunan Province.

* Corresponding author.
E-mail addresses: chenap@263.net (A. Chen), cflmath@yahoo.com.cn (F. Chen), dengsq66@163.com (S. Deng).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.07.001

(1) satisfies a global Lipschitz condition and takes values on X; Cuevas and Lizama [7] discussed almost automorphic mild solutions of the semilinear fractional equation $D^\alpha x(t) = Ax(t) + D^{\alpha-1} f(\cdot, x)$ in a Banach space X, where A is a linear operator of sectorial type $\omega < 0$. In the case $0 < \alpha < 1$, Jaradat et al. [11] investigated the existence of mild solutions for fractional semilinear initial value problems, however, the definition of the mild solution in [11] is inappropriate.

The paper has been organized as follows. In Section 2, motivated by [7,10], we study the existence and uniqueness of almost automorphic mild solutions for problem (1) with different conditions of f by means of the Banach contraction principle and the Schauder fixed point theorem. In Section 3, we discuss the uniform stability of almost automorphic mild solutions for problem (1).

2. Existence and uniqueness

We first consider the linear version for problem (1), that is

$$\begin{cases} D^\alpha x(t) = Ax(t) + f(t), & t \in J, 0 < \alpha < 1, \\ [D^{\alpha-1} x(t)]_{t=0} = x_0. \end{cases} \tag{2}$$

Similar to Example 4.3 in [1], the solution of the initial value problem (2) can be obtained with the help of the Laplace transform, we have the solution of problem (2)

$$x(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) x_0 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^\alpha) f(\tau) \mathrm{d}\tau, \tag{3}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function.

In fact, taking into account the initial condition of problem (2), the Laplace transform of equation $D^\alpha x(t) = Ax(t) + f(t)$ $(0 < \alpha < 1)$ yields

$$s^\alpha Y(s) = AY(s) + F(s) + x_0,$$

where

$$F(s) = L\{f(t); s\} = \int_0^\infty \mathrm{e}^{-st} f(t) \mathrm{d}t$$

is called the Laplace transform of the function $f(t)$.

Then, from

$$Y(s) = \frac{F(s)}{s^\alpha - A} + \frac{x_0}{s^\alpha - A},$$

and the inverse Laplace transform of the following transform

$$\int_0^\infty \mathrm{e}^{-pt} t^{\beta-1} E_{\alpha,\beta}(at^\alpha) \mathrm{d}t = \frac{p^{\alpha-\beta}}{p^\alpha - a}, \quad (\mathrm{Re}(p) > |a|^{\frac{1}{\alpha}}),$$

we can obtain Eq. (3).

Let

$$S_\alpha(t) := t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha), \quad 0 < \alpha < 1,$$

then the solution of problem (2) can be written as

$$x(t) = S_\alpha(t) x_0 + \int_0^t S_\alpha(t-\tau) f(\tau) \mathrm{d}\tau.$$

For $S_\alpha(t)$, Cuesta's result [12] proves that if A is a sectorial operator of type $\omega < 0$ for some $M > 0$ and $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$, then there exists $C > 0$ such that

$$\|S_\alpha(t)\| \leq \frac{CM}{1 + |\omega| t^\alpha} \leq CM, \quad \text{for } t \in J. \tag{4}$$

The above consideration motivates the following definition.

Definition 2.1. A function $x : J \to X$ is said to be an almost automorphic mild solution to problem (1) if the function $\tau \to S_\alpha f(\tau, x(\tau))$ is integrable on $(0, t)$ for each $t \in J$ and

$$x(t) = S_\alpha(t) x_0 + \int_0^t S_\alpha(t-\tau) f(\tau, x(\tau)) \mathrm{d}\tau, \tag{5}$$

for each $t \in J$.

Remark 2.1. In [11], a continuous solution $x(t)$ of integral equation $x(t) = T(t-t_0)x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}T(t-s)f(s,x(s),Gx(s),Sx(s))\mathrm{d}s$ is called an almost automorphic mild solution of the initial value problem

$$\begin{cases} D^{\alpha}x(t) = Ax(t) + f(t,x(t),Gx(t),Su(t)), & t \geq t_0, \\ x(t_0) = x_0. \end{cases}$$

From the above discussion, we may find that this definition is inappropriate in that it is only a simple extension of the integer-order mild solution.

Theorem 2.1. *Assume that A is sectorial of type $\omega < 0$. Let $f : J \times X \to X$ be almost automorphic in t for each $x \in X$ and satisfies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|, \quad \text{for all } x, y \in X, t \in J. \tag{6}$$

Then problem (1) has a unique almost automorphic mild solution, provided that there exists $L^* = \sup_{t\in J}\int_0^t \frac{L(\tau)}{1+|\omega|(t-\tau)^{\alpha}}\mathrm{d}\tau$ such that $CML^* < 1$.

Proof. Define the operator P on $AA(X)$ by

$$Px(t) = S_{\alpha}(t)x_0 + \int_0^t S_{\alpha}(t-\tau)f(\tau,x(\tau))\mathrm{d}\tau. \tag{7}$$

Then by using the dominated convergence theorem, we can see that P maps $AA(X)$ in $AA(X)$. Next, we shall use the Banach contraction principle to prove that the P defined by (7) has a fixed point.

Let x_1, x_2 be in $AA(X)$, we have

$$\begin{aligned} \|(Px_1)(t) - (Px_2)(t)\| &= \left\| \int_0^t S_{\alpha}(t-\tau)[f(\tau,x_1(\tau)) - f(\tau,x_2(\tau))]\mathrm{d}\tau \right\| \\ &\leq CM\int_0^t \frac{L(\tau)}{1+|\omega|(t-\tau)^{\alpha}}\|x_1(\tau)-x_2(\tau)\|\mathrm{d}\tau \\ &\leq CM\|x_1-x_2\|_{\infty}\int_0^t \frac{L(\tau)}{1+|\omega|(t-\tau)^{\alpha}}\mathrm{d}\tau \\ &\leq CML^*\|x_1-x_2\|_{\infty}. \end{aligned}$$

Thus

$$\|Px_1 - Px_2\|_{\infty} \leq CML^*\|x_1-x_2\|_{\infty}.$$

Since $CML^* < 1$, it follows that P is a strict contraction. As a consequence of the Banach fixed point theorem, we deduce that there exists a unique fixed point which is a unique almost automorphic mild solution of problem (1) on J. The proof is complete. □

Corollary 2.1. *Assume that A is sectorial of type $\omega < 0$. Let $f : J \times X \to X$ be almost automorphic in t for each $x \in X$ and satisfies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \quad \text{for all } x, y \in X, t \in J. \tag{8}$$

Then problem (1) has a unique almost automorphic mild solution, provided that the constant L satisfies that $CMLT < 1$.

Proof. Let $L(t) \equiv L$ in (6), $L^* = \sup_{t\in J}\int_0^t \frac{L}{1+|\omega|(t-\tau)^{\alpha}}\mathrm{d}\tau \leq LT$, then $CMLT < 1$ implies that $CML^* < 1$. According to Theorem 2.1, there exists a unique mild solution of problem (1) on J. The proof is complete. □

Theorem 2.2. *Assume that A is sectorial of type $\omega < 0$. Suppose that $f : J \times X \to X$ is Lebesgue measurable and almost automorphic with respect to t and is continuous with respect to x. For each $k \in N$, there exists a function $h_k : J \to R^+$ and a constant $\gamma \geq 0$ such that*

(i) $\sup_{\|x\|\leq k}\|f(t,x)\| \leq h_k(t)$;
(ii) *the function* $\tau \to \frac{1}{1+|\omega|\tau^{\alpha}}h_k(t-\tau)$ *belongs to* $L^1(J,R^+)$ *such that*

$$\lim_{k\to+\infty}\inf\frac{1}{k}\int_0^t \frac{1}{1+|\omega|(t_k-\tau)^{\alpha}}h_k(\tau)\mathrm{d}\tau = \gamma < +\infty.$$

Then problem (1) at least has an almost automorphic mild solution on J whenever $CM\gamma < 1$.

Proof. Let P be the function defined by (7). We shall use the Schauder fixed point theorem to prove that P has a fixed point. The proof will be given in several steps.

Step 1. P is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ on J. This, together with the continuity of $f(t, x)$ with respect to x, implies that $f(\tau, x_n(\tau)) \to f(\tau, x(\tau))$ on J as $n \to \infty$.

For each $t \in J$

$$\begin{aligned}
\|(Px_n)(t) - (Px)(t)\| &= \left\| \int_0^t S_\alpha(t-\tau)[f(\tau, x_n(\tau)) - f(\tau, x(\tau))]\mathrm{d}\tau \right\| \\
&\leq \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\| \int_0^t \frac{CM}{1+|\omega|(t-\tau)^\alpha}\mathrm{d}\tau \\
&\leq \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\| \int_0^t CM\mathrm{d}\tau \\
&\leq CMT\|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|.
\end{aligned}$$

Since $f(\cdot, x_n(\cdot))$ is convergent to $f(\cdot, x(\cdot))$ as $n \to \infty$, we have

$$\|Px_n - Px\| \to 0 \quad \text{as } n \to \infty,$$

this follows that P is continuous.

Step 2. P maps bounded sets into bounded sets on J.

For each $k \in N$, let

$$B_k = \{x \in AA(X) : \|x(t)\| \leq k, t \in J\}.$$

We show that there exists a $k^* \in N$ such that $PB_{k^*} \subset B_{k^*}$. If it is not true, then for each $k \in N$, there would exist $x_k \in B_k$ and $t_k \in R$ such that $\|(PB_k)(t)\| > k$. This, together with the assumption of f, yields

$$\begin{aligned}
k < \|(PB_k)(t)\| &= \left\| S_\alpha(t)x_0 + \int_0^t S_\alpha(t-\tau)f(\tau, x_k(\tau))\mathrm{d}\tau \right\| \\
&\leq \|S_\alpha(t)x_0\| + \left\| \int_0^t S_\alpha(t-\tau)f(\tau, x_k(\tau))\mathrm{d}\tau \right\| \\
&\leq CM\|x_0\| + CM\int_0^t \frac{1}{1+|\omega|(t-\tau)^\alpha}\|f(\tau, x_k(\tau))\|\mathrm{d}\tau \\
&\leq CM\|x_0\| + CM\int_0^t \frac{1}{1+|\omega|(t-\tau)^\alpha}h_k(\tau)\mathrm{d}\tau. \qquad (9)
\end{aligned}$$

Dividing both sides of (9) by k, we obtain

$$1 < \frac{1}{k}CM\|x_0\| + CM\frac{1}{k}\int_0^t \frac{1}{1+|\omega|(t-\tau)^\alpha}h_k(\tau)\mathrm{d}\tau. \qquad (10)$$

Taking the lower limit on both sides of (10) as $k \to \infty$, assumption (ii) follows that

$$1 \leq CM\gamma.$$

This is a contradiction with $CM\gamma < 1$. Then $PB_{k^*} \subset B_{k^*}$ for some $k^* \in N$.

Step 3. P maps bounded sets into equicontinuous sets on J.

Let $t_1, t_2 \in J$, $t_1 < t_2$, B_{k^*} be a bounded set on J as in Step 2, and $x \in B_{k^*}$. Then

$$\begin{aligned}
\|(Px)(t_2) - (Px)(t_1)\| &\leq \|S_\alpha(t_2)x_0 - S_\alpha(t_1)x_0\| + \left\| \int_0^{t_2} S_\alpha(t_2-\tau)f(\tau, x(\tau))\mathrm{d}\tau - \int_0^{t_1} S_\alpha(t_1-\tau)f(\tau, x(\tau))\mathrm{d}\tau \right\| \\
&\leq \|t_2^{\alpha-1}E_{\alpha,\alpha}(At_2^\alpha) - t_1^{\alpha-1}E_{\alpha,\alpha}(At_1^\alpha)\|\|x_0\| + \left\| \int_0^{t_2} S_\alpha(\tau)f(t_2-\tau, x(t_2-\tau))\mathrm{d}\tau \right. \\
&\quad \left. - \int_0^{t_1} S_\alpha(\tau)f(t_1-\tau, x(t_1-\tau))\mathrm{d}\tau \right\| \\
&\leq t_2^{\alpha-1}\left\| E_{\alpha,\alpha}(At_2^\alpha) - \left(\frac{t_1}{t_2}\right)^{\alpha-1}E_{\alpha,\alpha}(At_1^\alpha) \right\| \|x_0\| + \left\| \int_0^{t_1} S_\alpha(\tau)[f(t_2-\tau, x(t_2-\tau)) \right. \\
&\quad \left. - f(t_1-\tau, x(t_1-\tau))]\mathrm{d}\tau \right\| + \left\| \int_{t_1}^{t_2} S_\alpha(\tau)f(t_2-\tau, x(t_2-\tau))\mathrm{d}\tau \right\|
\end{aligned}$$

$$
\begin{aligned}
&\le t_2^{\alpha-1}\|E_{\alpha,\alpha}(At_2^{\alpha})-E_{\alpha,\alpha}(At_1^{\alpha})\|\|x_0\|+CM\int_0^{t_1}\|f(t_2-\tau,x(t_2-\tau))\\
&\quad -f(t_1-\tau,x(t_1-\tau))\|\mathrm{d}\tau+CM\int_{t_1}^{t_2}\frac{1}{1+|\omega|\tau^{\alpha}}h_{k^*}(t_2-\tau)\mathrm{d}\tau\\
&\le t_2^{\alpha-1}\left\|\frac{A}{\alpha}E'_{\alpha,\alpha}(At_2^{\alpha})\right\|\|x_0\|(t_2^{\alpha}-t_1^{\alpha})+CM\int_0^{t_1}\|f(t_2-\tau,x(t_2-\tau))\\
&\quad -f(t_1-\tau,x(t_1-\tau))\|\mathrm{d}\tau+CM\int_{t_1}^{t_2}\frac{1}{1+|\omega|\tau^{\alpha}}h_{k^*}(t_2-\tau)\mathrm{d}\tau\\
&:= A_1+A_2+A_3.
\end{aligned}
$$

As $t_2\to t_1$, $t_2^{\alpha}\to t_1^{\alpha}$ follows that A_1 tends to 0, the Lebesgue measurability of f with respect to t implies that A_2 tends to 0, the integrability of $\frac{1}{1+|\omega|\tau^{\alpha}}h_{k^*}(t_2-\tau)$ with respect to τ implies that A_3 tends to 0. Hence P is equicontinuous.

It remains to prove that $V(t)=\{(Px)(t):x\in B_{k^*}\}$ is relatively compact in X. $S_{\alpha}(\cdot)$ is compact in X since it is generated by the dense operator A. Then $V(0)=S_{\alpha}(0)x_0$ is relatively compact in X. Fix $t\in(0,T]$, for each $\epsilon\in(0,t)$ and $x\in B_{k^*}$, define $(P_{\epsilon}x)(t)$ as follows

$$
(P_{\epsilon}x)(t)=S_{\alpha}(t)x_0+\int_0^{t-\epsilon}S_{\alpha}(t-\tau)f(\tau)\mathrm{d}\tau. \tag{11}
$$

Then the sets $V_{\epsilon}(t)=\{(P_{\epsilon}x)(t):x\in B_{k^*}\}$ are relatively compact in X. Moreover, for each $x\in B_{k^*}$, we have

$$
\begin{aligned}
\|(Px)(t)-(P_{\epsilon}x)(t)\| &\le \left\|\int_{t-\epsilon}^{t}S_{\alpha}(t-\tau)f(\tau)\mathrm{d}\tau\right\|\\
&\le \int_{t-\epsilon}^{t}\|S_{\alpha}(t-\tau)f(\tau)\|\mathrm{d}\tau\\
&\le CM\int_{t-\epsilon}^{t}\frac{h_{k^*}}{1+|\omega|(t-\tau)^{\alpha}}\mathrm{d}\tau.
\end{aligned} \tag{12}
$$

which, combining condition (ii), follows that there are relatively compact sets arbitrarily close to $V(t)$ and hence $V(t)$ is also relatively compact in X. Thus, the Arzela–Ascoli theorem implies that PB_{k^*} is relatively compact, P is completely continuous on B_{k^*}.

As a consequence of Steps 1 to 3 together with the Schauder fixed point theorem, we deduce that P has a fixed point in B_{k^*} which is an almost automorphic mild solution of problem (1). The proof is complete. □

Theorem 2.3. *Assume that A is sectorial of type $\omega<0$. Suppose that $f:J\times X\to X$ is Lebesgue measurable and almost automorphic with respect to t and is continuous with respect to x. There exists a function $h:J\to R^+$ such that*

(i)′ $\sup\|f(t,x)\|\le h(t)$;
(ii)′ *the integral* $\int_0^t\frac{1}{1+|\omega|\tau^{\alpha}}h(t-\tau)\mathrm{d}\tau$ *exists for all* $t\in J$.

Then problem (1) at least has an almost automorphic mild solution on J.

Proof. Let P be the function defined by (7). We shall also apply the Schauder fixed point theorem to prove this theorem. The proof of Step 1 in this theorem is the same as the proof of Step 1 in Theorem 2.2 and so is omitted. In the following, we start our proof from Step 2.

Step 2. Let $B=\{x\in AA(X):\|x(t)\|\le d,t\in J\}$, where $d=CM(\|x_0\|+h^*)$, and $h^*=\sup_{t\in J}\int_0^t\frac{1}{1+|\omega|\tau^{\alpha}}h(t-\tau)\mathrm{d}\tau$.

For each $x\in B$, we have

$$
\begin{aligned}
\|(Px)(t)\| &\le \|S_{\alpha}(t)x_0\|+\left\|\int_0^t S_{\alpha}(t-\tau)f(\tau,x(\tau))\mathrm{d}\tau\right\|\\
&\le CM\|x_0\|+CM\int_0^t\frac{1}{1+|\omega|(t-\tau)^{\alpha}}\|f(\tau,x(\tau))\|\mathrm{d}\tau\\
&\le CM\|x_0\|+CM\int_0^t\frac{1}{1+|\omega|\tau^{\alpha}}\|f(t-\tau,x(t-\tau))\|\mathrm{d}\tau\\
&\le CM\|x_0\|+CM\int_0^t\frac{1}{1+|\omega|\tau^{\alpha}}h(t-\tau)\mathrm{d}\tau\\
&\le CM(\|x_0\|+h^*)=d.
\end{aligned}
$$

Therefore, $P:B\to B$.

Step 3. Let $t_1, t_2 \in J, t_1 < t_2$, B be a bounded set on J as in Step 2, and $x \in B$. Then, similar to the proof of Step 3 in Theorem 2.2, we have

$$\begin{aligned}
\|(Px)(t_2) - (Px)(t_1)\| &\le \|S_\alpha(t_2)x_0 - S_\alpha(t_1)x_0\| + \left\| \int_0^{t_2} S_\alpha(t_2-\tau)f(\tau, x(\tau))\mathrm{d}\tau - \int_0^{t_1} S_\alpha(t_1-\tau)f(\tau, x(\tau))\mathrm{d}\tau \right\| \\
&\le \|t_2^{\alpha-1}E_{\alpha,\alpha}(At_2^\alpha) - t_1^{\alpha-1}E_{\alpha,\alpha}(At_1^\alpha)\|\|x_0\| + \left\| \int_0^{t_2} S_\alpha(\tau)f(t_2-\tau, x(t_2-\tau))\mathrm{d}\tau \right. \\
&\quad \left. - \int_0^{t_1} S_\alpha(\tau)f(t_1-\tau, x(t_1-\tau))\mathrm{d}\tau \right\| \\
&\le t_2^{\alpha-1}\left\|E_{\alpha,\alpha}(At_2^\alpha) - \left(\frac{t_1}{t_2}\right)^{\alpha-1} E_{\alpha,\alpha}(At_1^\alpha)\right\|\|x_0\| + \left\| \int_0^{t_1} S_\alpha(\tau)[f(t_2-\tau, x(t_2-\tau)) \right. \\
&\quad \left. - f(t_1-\tau, x(t_1-\tau))]\mathrm{d}\tau \right\| + \left\| \int_{t_1}^{t_2} S_\alpha(\tau)f(t_2-\tau, x(t_2-\tau))\mathrm{d}\tau \right\| \\
&\le t_2^{\alpha-1}\|E_{\alpha,\alpha}(At_2^\alpha) - E_{\alpha,\alpha}(At_1^\alpha)\|\|x_0\| + CM\int_0^{t_1} \|f(t_2-\tau, x(t_2-\tau)) \\
&\quad -f(t_1-\tau, x(t_1-\tau))\|\mathrm{d}\tau + CM\int_{t_1}^{t_2} \frac{1}{1+|\omega|\tau^\alpha} h(t_2-\tau)\mathrm{d}\tau \\
&\le t_2^{\alpha-1}\left\|\frac{A}{\alpha}E'_{\alpha,\alpha}(At_2^\alpha)\right\| \|x_0\|(t_2^\alpha - t_1^\alpha) + CM\int_0^{t_1} \|f(t_2-\tau, x(t_2-\tau)) \\
&\quad -f(t_1-\tau, x(t_1-\tau))\|\mathrm{d}\tau + CM\int_{t_1}^{t_2} \frac{1}{1+|\omega|\tau^\alpha} h(t_2-\tau)\mathrm{d}\tau \\
&:= A_1 + A_2 + A_4.
\end{aligned}$$

As $t_2 \to t_1$, $t_2^\alpha \to t_1^\alpha$ follows that A_1 tends to 0, the Lebesgue measurability of f with respect to t implies that A_2 tends to 0, the integrability of $\frac{1}{1+|\omega|\tau^\alpha}h(t_2-\tau)$ with respect to τ implies that A_4 tends to 0. We can also obtain that P is equicontinuous.

Set $V(t) = \{(Px)(t) : x \in B\}$. Fix $t \in (0, T]$, for each $\epsilon \in (0, t)$ and $x \in B$, let P_ϵ be the function defined by (11), then the sets $V_\epsilon(t) = \{(P_\epsilon x)(t) : x \in B\}$ are relatively compact in X. Meanwhile, (12) implies that $V_\epsilon(t)$ arbitrarily close to $V(t)$ and $V(t)$ is also relatively compact in X. Thus, the Arzela–Ascoli theorem implies that PB is relatively compact, P is completely continuous on B.

According to the conclusions derived in Steps 1 to 3, we can conclude that $P : B \to B$ is continuous and completely continuous. Thus, P has a fixed point in B by using the Schauder fixed point theorem. This implies that problem (1) has at least an almost automorphic mild solution on J. The proof is complete. □

3. Stability

In this section, we study the uniform stability of the almost automorphic mild solution of problem (1).

Theorem 3.1. *Assume that A is sectorial of type $\omega < 0$. Let $f : J \times X \to X$ be almost automorphic in t for each $x \in X$ and satisfies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \le L(t)\|x-y\|, \quad \textit{for all } x, y \in X, t \in J.$$

Then the almost automorphic mild solution of problem (1) is uniformly stable, provided that there exists $L^* = \sup_{t\in J} \int_0^t \frac{L(\tau)}{1+|\omega|(t-\tau)^\alpha}\mathrm{d}\tau$ such that $CML^* < 1$.

Proof. Let $x(t)$ be a solution of

$$x(t) = S_\alpha(t)x_0 + \int_0^t S_\alpha(t-\tau)f(\tau, x(\tau))\mathrm{d}\tau, \tag{13}$$

and let $\tilde{x}(t)$ be a solution of (13) such that $\tilde{x}(0) = \tilde{x}_0$, where $x_0, \tilde{x}_0 \in X$. Then

$$x(t) - \tilde{x}(t) = S_\alpha(t)(x_0 - \tilde{x}_0) + \int_0^t S_\alpha(t-\tau)[f(\tau, x(\tau)) - f(\tau, \widetilde{x(\tau)})]\mathrm{d}\tau$$

$$\|x(t) - \tilde{x}(t)\| \le \|S_\alpha(t)\|\|x_0 - \tilde{x}_0\|_\infty + \int_0^t \|S_\alpha(t-\tau)\|\|f(\tau, x(\tau)) - f(\tau, \widetilde{x(\tau)})\|\mathrm{d}\tau$$

$$\leq CM\|x_0-\tilde{x}_0\|_\infty + CM\int_0^t \frac{L(\tau)}{1+|\omega|(t-\tau)^\alpha}\|x(\tau)-\tilde{x(\tau)}\|\mathrm{d}\tau$$
$$\leq CM\|x_0-\tilde{x}_0\|_\infty + CM\|x-\tilde{x}\|_\infty\int_0^t \frac{L(\tau)}{1+|\omega|(t-\tau)^\alpha}\mathrm{d}\tau$$
$$\leq CM\|x_0-\tilde{x}_0\|_\infty + CML^*\|x-\tilde{x}\|_\infty.$$

Thus

$$\|x-\tilde{x}\|_\infty \leq CM\|x_0-\tilde{x}_0\|_\infty + CML^*\|x-\tilde{x}\|_\infty,$$

$CML^* < 1$ yields

$$\|x-\tilde{x}\|_\infty \leq \frac{CM}{1-CML^*}\|x_0-\tilde{x}_0\|_\infty.$$

Therefore, if $\|x_0-\tilde{x}_0\|_\infty < \delta(\varepsilon)$, then $\|x-\tilde{x}\|_\infty < \varepsilon$, which implies that the almost automorphic mild solution of problem (1) is uniformly stable. The proof is complete. □

Corollary 3.1. *Assume that A is sectorial of type $\omega < 0$. Let $f : J\times X \to X$ be almost automorphic in t for each $x \in X$ and satisfies the Lipschitz condition*

$$\|f(t,x)-f(t,y)\| \leq L\|x-y\|, \quad \text{for all } x,y\in X,\ t\in J.$$

Then the almost automorphic mild solution of problem (1) is uniformly stable, provided that the constant L satisfies that $CMLT < 1$.

4. Example

As the application of our main results, we consider the following example.

Example 4.1. Let $X = L^2[0,\pi]$ and the operator A of problem (1) defined on X by

$$Ax = x'' - \mu x, \quad (\mu > 0)$$

with domain $D(A) = \{x\in L^2[0,\pi] : x''\in L^2[0,\pi], x(0)=x(\pi)=0\}$.

Then $\mu I - A$ is sectorial of type $\omega = -\mu < 0$ because that $\Delta x = x''$ is the generator of an analytic semigroup on $L^2[0,\pi]$ [7].

Let $f(t,x)(s) = \beta b(t)\sin(x(s))$ for all $x\in X$ and $s\in[0,\pi]$, $t\in R$ with $b\in AA(X)$, $\beta\in R$, which implies that $t\to f(t,x)$ is almost automorphic in t for each $x\in X$ and

$$\|f(t,x)-f(t,y)\|_2^2 \leq \int_0^\pi \beta^2|b(t)|^2|\sin(x(s))-\sin(y(s))|^2\mathrm{d}s \leq \beta^2|b(t)|^2\|(x-y)\|_2^2.$$

Therefore, problem (1) has a unique almost automorphic mild solution, provided that there exists $L^* = \sup_{t\in[0,\pi]}\int_0^t \frac{\beta^2|b(\tau)|^2}{1+\mu(t-\tau)^\alpha}\mathrm{d}\tau$ such that $CML^* < 1$ according to Theorem 2.1.

Let $h_k(t) \equiv \beta^2|b(t)|^2$ for $k\in N$, problem (1) at least has an almost automorphic mild solution on J whenever $CM\gamma < 1$ according to Theorem 2.2, where $\gamma = \lim_{k\to+\infty}\inf\frac{1}{k}\int_0^t \frac{1}{1+\mu(t-\tau)^\alpha}h_k(\tau)\mathrm{d}\tau < +\infty$.

Let $h(t) = \beta^2|b(t)|^2$ for $k\in N$, problem (1) at least has an almost automorphic mild solution on J if the integral $\int_0^t \frac{1}{1+\mu\tau^\alpha}h(t-\tau)\mathrm{d}\tau$ exists for all $t\in J$ according to Theorem 2.3.

The almost automorphic mild solution of problem (1) is uniformly stable, provided that there exists $L^* = \sup_{t\in[0,\pi]}\int_0^t \frac{\beta^2|b(\tau)|^2}{1+\mu(t-\tau)^\alpha}\mathrm{d}\tau$ such that $CML^* < 1$ according to Theorem 3.1.

Acknowledgments

The authors thank the referees and the editor for their valuable suggestions and comments.

References

[1] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[2] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, J. Math. Anal. Appl. 265 (2002) 229–248.
[3] C. Yu, G. Gao, On the solution of nonlinear fractional order differential equation, Nonlinear Anal. 63 (2005) e971–e976.
[4] D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, J. Math. Anal. Appl. 204 (1996) 609–625.
[5] S. Zhang, The existence of a positive solution for a nonlinear fractional differential equation, J. Math. Anal. Appl. 252 (2000) 804–812.
[6] V. Daftardar-Gejji, Positive solutions of a system of non-autonomous fractional differential equations, J. Math. Anal. Appl. 302 (2005) 56–64.
[7] C. Cuevas, C. Lizama, Almost automorphic mild solutions to a class of fractional differential equations, Appl. Math. Lett. 21 (2008) 1315–1319.

[8] X. Fu, K. Ezzinbi, Existence of solutions for neutral functional differential evolution equations with nonlocal conditions, Nonlinear Anal. 54 (2003) 215–227.
[9] E. Hernández, Existence results for partial neutral functional differential equations with nonlocal conditions, Cadenos De Math. 02 (2001) 239–250.
[10] D. Araya, C. Lizama, Almost automorphic mild solutions to fractional differential equations, Nonlinear Anal. 69 (2008) 3692–3705.
[11] O.K. Jaradat, A. Al-Omari, S. Momani, Existence of the mild solution for fractional semilinear initial value problems, Nonlinear Anal. 69 (2008) 3153–3159.
[12] E. Cuesta, Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations, Discrete Contin. Dyn. Syst. (Suppl.) (2007) 277–285.