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A new approach for shape preserving interpolating curves☆

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ABSTRACT

In this paper, we present an new approach to construct the so-called shape preserving interpolation curves. The basic idea is first to approximate the set of interpolated points with a class of MQ quasi-interpolation operators and then pass through the set with the use of multivariate interpolation by using compactly supported radial basis functions. This approach possesses the advantages of certain shape preserving and good approximation behaviors. The proposed algorithm is easy to implement.

1. Introduction

The problem of scattered data interpolation consists of constructing a function that interpolates data values which are known at some scattered points. However, we often have some additional information that we wish to confine to interpolation. For example, we know the quantity from which the data is sampled, is positive, monotonic or convex. Thus, it is important to construct a function which satisfies the underlying constraints. Many papers have been written on the problem of shape preserving interpolation by Schumaker, Gregory, Goodman, Manni, Costantini et al.. They mainly used polynomial interpolation or spline interpolation with a certain continuity to solve it, e.g. [1–5].

However, there has been little work done on the imposition of constraints for these meshless interpolation methods by using radial basis functions. For RBF, in the special case of thin-plate splines for 2D data, Utreras [6] has shown how positivity can be imposed as a constraint. Xiao and Woodbury [7] look at several meshless methods for constrained scattered data interpolation for 3D data. For Shepard, Brodile et al. [8] discussed the modified quadratic Shepard method, which interpolates scattered data of any dimensionality, can be constrained to preserve positivity.

A radial basis function (RBF) is a relatively simple multivariate function generated by a univariate function. Due to its simple form and good approximation behavior, the radial basis function approach has become an effective tool for multivariate scattered data interpolation during the last two decades [9–14].

For any given scattered data $(X_j, f_j) \in \mathbb{R}^n \times \mathbb{R}, j = 1, \ldots, N$, where points $X_1, \ldots, X_N \in \mathbb{R}^n$ are pairwise distinct. By $\|X - X_j\|$ we usually denote the Euclid distance between two points X and X_i, the so-called radial basis function interpolation is to use a function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ to construct the following interpolant:

$$s(X) = \sum_{j=1}^{N} \lambda_j \phi(\|X - X_j\|), \quad X = (x_1, x_2, \ldots, x_n) \tag{1}$$

satisfying the interpolation conditions

$$s(X_i) = \sum_{j=1}^{N} \lambda_j \phi(\|X_i - X_j\|) = f_i, \quad i = 1, \ldots, N.$$

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Table 1
Some of Wendland's CS-RBFs $\phi_{d,k} \in PD_d \cap C^{2k}$.

$\phi_{1,0} = (1-r)_+$,	$PD_1 \cap C^0$
$\phi_{3,0} = (1-r)_+^2$,	$PD_3 \cap C^0$
$\phi_{3,1} = (1-r)_+^4(4r+1)$,	$PD_3 \cap C^2$

The positive definiteness of ϕ guarantees that the interpolation problem possesses a unique solution and refers to ϕ as a radial basis function.

Compactly supported radial basis functions (CS-RBFs for short) have only recently been constructed. Wu first constructed a broad variety of CS-RBFs [15]. Very recently, Wendland constructed these functions such that they possess the lowest degree among all CS-RBFs which are positive definite for a given space dimension and prescribed order of smoothness [16]. They are radial basis functions which are positive definite on $\mathbb{R}^d$ for a given space dimension d (PD_d), belong to a prescribed smoothness class(C^{2k}), are compactly supported and easy to evaluate. Some examples of such radial basis functions are given in Table 1.

We find this a useful property in practice and thus provide a good selection of Wendland's functions with respect to the order of continuity and the dimension of space. Nowadays, CS-RBFs have become a popular tool for multivariate interpolation of large scattered data, implicit surface reconstruction and so on. Many researchers (Floater, Lazzaro, Morse, Turk, Ohtake et al.) have done a lot of significant work on the above problems [17–21].

In order to adapt the interpolation to scattered data of different densities, it is necessary to be able to scale the support of ϕ. So from now on we assume the radius α of support of ϕ is one and replace ϕ by

$$\phi_\alpha(\cdot) = \phi(\cdot/\alpha), \quad \text{for } \alpha > 0.$$

Meanwhile, in order to achieve both the best possible approximation behavior and best possible stability with respect to the support of CS-RBFs, we adopt two possible strategies for choosing α in [17].

- fix $\gamma > 0$ and set $\alpha = Q/\gamma$, or
- fix $\delta > 0$ and set $\alpha = q/\delta$,

where

$$Q = \max_{x \in \Omega} \min_{1 \le i \le N} \|X - X_i\|, \qquad q = \min_{1 \le j < i \le N} \|X_j - X_i\|/2,$$

and Ω is the compact region in $\mathbb{R}^n$ containing the original point set $\{X_j\}_{j=1}^N$.

Alternatively, one could even compromise the two strategies, and set

$$\alpha = \lambda Q/\gamma + (1-\lambda)q/\gamma, \quad \lambda \in (0, 1).$$

Our proposed method is interesting and flexible from numerical experimental results rather than from the theoretic point of view. We describe the following four possible properties of the so-called shape preserving interpolating curves.

Monotonicity: The interpolating curve is increasing (respectively decreasing) if x_i is increasing (respectively decreasing).

Convexity: The interpolating curve is convex (concave) if the piecewise linear interpolant is convex (concave).

Smoothness: The interpolating curve belongs to C^k when the smoothness of CS-RBFs is chosen to be C^k.

Fairness: The interpolating curve is pleasing to the eye. It is important in CAGD which several criteria can be suggested to be relevant, such as magnitude, rate of change or monotonicity of the curvature.

In this paper, we present an algorithm to construct a kind of so-called shape preserving interpolating curve with the use of CS-RBFs and a class of MQ quasi-interpolation operator. The basic idea is first to approximate the point set with a class of MQ quasi-interpolation operators and then pass through the set of interpolated points with the use of multivariate interpolation by compactly supported radial basis functions. This approach possesses the above mentioned four possible properties. The proposed algorithm is relatively simple and easy to implement. It is useful in practical fields.

2. MQ quasi-interpolation

Quasi-interpolation of a univariate function $f : [a, b] \to \mathbb{R}$ with MQ basis at the scattered points

$$a = x_0 < x_1 < \cdots < x_N = b$$

has the form

$$\mathcal{L}(f) := \sum_{i=0}^{N} f(x_i)\psi_i(x),$$

where each function $\psi_i(x)$ is the linear combination of the MQ basis

$$\phi_i(x) = \sqrt{(x - x_i)^2 + c^2}$$

and c is the shape parameter. The MQ function was first proposed by Hardy in 1971 [22]. In the summarized paper [23], Franke pointed out that MQ interpolation was best among 29 scattered data interpolation methods in terms of timing, storage, accuracy, visual pleasantness of surface, and ease of implementation. Beaton and Powell [24] proposed a univariate quasi-interpolation formula which is the linear combination of Hardy's and lower order polynomials. Since their formula requires the derivative values at the endpoints, it is not convenient for practical use. Later, Wu and Schaback [25] gave another quasi-interpolation formula without using the derivative values at the endpoints.

Wu–Schaback's MQ quasi-interpolation formula is given by:

$$(\mathcal{L}_{\mathcal{D}}f)(x) = \sum_{i=0}^{N} f(x_i)\psi_i(x), \tag{2}$$

where

$$\begin{aligned}
\psi_0(x) &= \frac{1}{2} + \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)},\\
\psi_1(x) &= \frac{\phi_2(x) - \phi_1(x)}{2(x_2 - x_1)} - \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)},\\
\psi_i(x) &= \frac{\phi_{i+1}(x) - \phi_i}{2(x_{i+1} - x_i)} - \frac{\phi_i(x) - \phi_{i-1}}{2(x_i - x_{i-1})}, \quad i = 2, \ldots, N-2,\\
\psi_{N-1}(x) &= \frac{(x_N - x) - \phi_{N-1}(x)}{2(x_N - x_{N-1})} - \frac{\phi_{N-1}(x) - \phi_{N-2}(x)}{2(x_{N-1} - x_{N-2})},\\
\psi_N(x) &= \frac{1}{2} + \frac{\phi_{N-1}(x) - (x_N - x)}{2(x_N - x_{N-1})}.
\end{aligned}$$

Theorem 2.1 (*[25]*). *The MQ quasi-interpolation operator* (2) *preserves linear reproduction, monotonicity, convexity and variation-diminishing.*

Remark 2.1. How to select a good value for the parameter c in multiquadric interpolation is well studied by Carlson and Foley [26].

3. Shape preserving interpolation of scattered data

Our discussed problem in this paper can be addressed as

Problem 3.1. For given a set of scattered data $PS = \{(x_i, y_i)\}_{i=0}^N$, where $x_0, x_1, \ldots, x_N$ are distinct, the implicit interpolation is to find a function $f(x, y)$ such that its zero level-set $\mathcal{C} := \{(x, y)|f(x, y) = 0\}$ passes through the set PS.

A simple method for solving Problem 3.1 is to construct a function $g(x) = \sum_{i=0}^N c_i\phi_i(x)$ satisfying the interpolation conditions $g(x_i) = y_i, i = 0, 1, \ldots, N$, where $\phi_i(x)$ can be chosen to be polynomial bases, B-spline bases, radial basis functions and so on. Then the curve $\mathcal{C} := \{(x, y)|y - g(x) = 0\}$ certainly passes through the set PS. However, this method does not consider the additional information that confine to implicit interpolation of scattered data (such as convexity, monotonicity).

For example, if the data $\{y_i\}_{i=0}^N$ is increasing (respectively decreasing), then the interpolating curve is naturally required to increase (respectively decrease). In other words, our aim is to construct a function such that its zero level-set $\mathcal{C}$ cannot only interpolate the point set PS but also possess the above mentioned four possible properties.

In this section, we propose a flexible and convenient algorithm to solve shape preserving interpolation with the combined use of CS-RBFs and the MQ quasi-interpolation operator $(\mathcal{L}_{\mathcal{D}}f)(x)$. The basic idea is first to approximate the point set PS with MQ quasi-interpolation operator $(\mathcal{L}_{\mathcal{D}}f)(x)$ which possesses good shape preserving

$$(\mathcal{L}_{\mathcal{D}}f)(x) = \sum_{j=0}^{N} y_j\psi_j(x).$$

Meanwhile, we define an error function

$$e(x, y) = y - (\mathcal{L}_{\mathcal{D}}f)(x) \tag{3}$$

and obtain the error values

$$e_i = e(x_i, y_i), \quad i = 0, 1, \ldots, N. \tag{4}$$

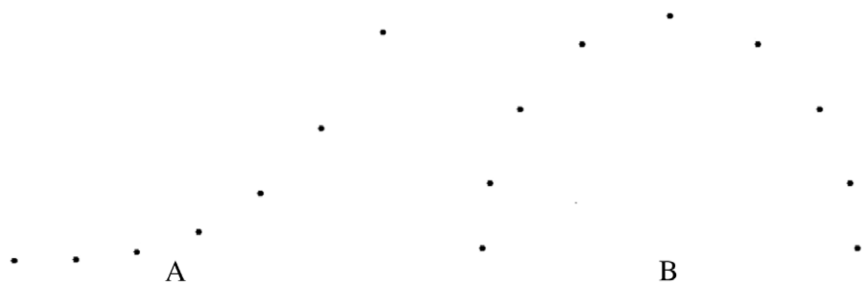

Fig. 1. Two sets of scattered data (A) and (B).

Secondly, we construct the following interpolant

$$\varepsilon(x, y) = \sum_{j=0}^{N} \lambda_j \phi_\alpha(r_j), \quad r_j = \sqrt{(x - x_j)^2 + (y - y_j)^2} \tag{5}$$

satisfying the interpolation conditions

$$\varepsilon(x_i, y_i) = e_i, \quad i = 0, 1, \dots, N. \tag{6}$$

Here, $\phi_\alpha(r)$ is a CS-RBF with the support of radius α.

Finally, we obtain a formula in the form:

$$y = (\mathcal{L}_\mathcal{D} f)(x) + \varepsilon(x, y). \tag{7}$$

The first term of the right-hand side of (7) can be considered as a base approximation in order to possess certain shape preserving properties. While, the second term represents local detail in order to have the property of interpolation.

Hence, the curve $\mathcal{C} := \left\{(x, y) | y = (\mathcal{L}_\mathcal{D} f)(x) + \varepsilon(x, y)\right\}$ is the required interpolating curve which possesses the above mentioned four possible properties.

The main algorithm of the so-called shape preserving interpolation for scattered data is outlined as follows:

Algorithm 3.1. Shape preserving interpolating curve for scattered data.

Input: A set of scattered data $PS = \{(x_i, y_i)\}_{i=0}^N$.
Output: A curve $\mathcal{C}$ passes through PS with certain shape preserving properties.
Step 1. Construct the MQ quasi-interpolation operator $(\mathcal{L}_\mathcal{D} f)(x)$ with scattered data PS.
Step 2. Define the error function

$$e(x, y) = y - (\mathcal{L}_\mathcal{D} f)(x)$$

and compute the error values

$$e_i = e(x_i, y_i), \quad i = 0, 1, \dots, N.$$

Step 3. Construct the function $\varepsilon(x, y)$ satisfying $\varepsilon(x_i, y_i) = e_i,\ i = 0, 1, \dots, N$ based on multivariate interpolation by using CS-RBFs.

Therefore, $\mathcal{C} := \left\{(x, y) | y = (\mathcal{L}_\mathcal{D} f)(x) + \varepsilon(x, y)\right\}$ is the required shape preserving interpolating curve.

4. Numerical examples

In this section, several experimental results are provided to show the interpolating curves have the above mentioned four possible properties.

Throughout this section, we choose the parameter c in the MQ function as $c = 0.05$, Wendland's CS-RBFs as $\phi(r) = (1 - r)_+^4(4r + 1)$.

Example 4.1. Given two sets of scattered data $\{(x_i, y_i)\}_{i=0}^N$ (see Fig. 1). For convenience, the first set of scattered data is chosen from the function $y = x^3 + 2,\ x \in [0, 1]$, and the second set of scattered data is chosen from the function $y = \sqrt{1 - 4x^2},\ x \in [-1/2, 1/2]$, respectively.

The ordinary interpolating curves by using CS-RBFs, and the shape preserving interpolating curves by using our proposed method for scattered data (A) and (B) are simultaneously shown in Figs. 2 and 3, respectively.

Meanwhile, we choose 100 arbitrary points as a testing set. Their maximum errors and variances for Figs. 2 and 3 are listed in Table 2.

Example 4.2. Given two sets of scattered data (see Fig. 4). Our shape preserving interpolation for scattered data (C) and (D) are shown in Fig. 5, respectively.

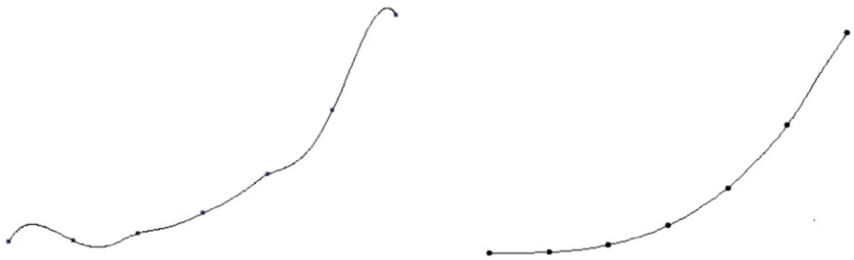

Fig. 2. Ordinary interpolating curve (left) and our shape preserving interpolating curve (right) for scattered data (A).

Fig. 3. Ordinary interpolating curve (left) and our shape preserving interpolating curve (right) for scattered data (B).

Table 2
Max errors and variances.

	Fig. 2 (left)	Fig. 2 (right)	Fig. 3 (left)	Fig. 3 (right)
Max error	1.06(−1)	1.65(−2)	8.44(−2)	2.52(−2)
Variance	7.58(−4)	1.79(−5)	6.52(−4)	5.22(−5)

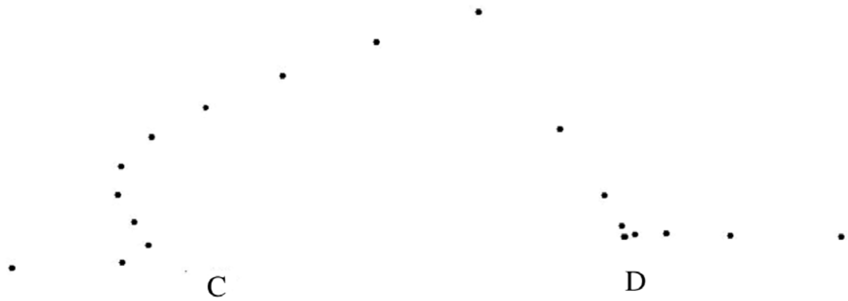

Fig. 4. Two sets of scattered data (C) and (D).

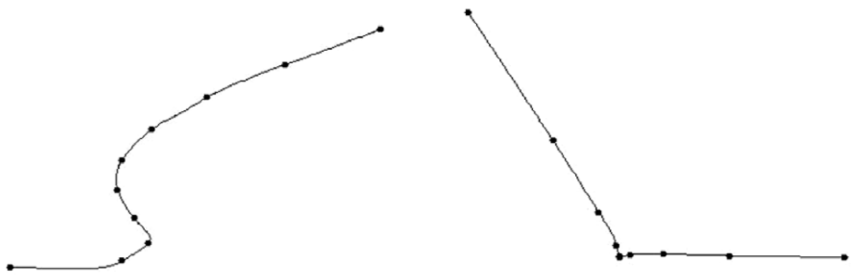

Fig. 5. Our shape preserving interpolating curves for scattered data (C) and (D).

5. Conclusion

A kind of so-called shape preserving interpolating curves by using CS-RBFs and a class of MQ quasi-interpolation operator is proposed. The constructed interpolating curves in this paper might possess the above mentioned four possible properties from numerous experimental results. In the future we hope to extend this work to solve 3D scattered data shape preserving interpolation or approximation.

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