A Counterexample to Kippenhahn's Conjecture on Hermitian Pencils

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ABSTRACT

A counterexample is constructed to a conjecture of Kippenhahn (Math. Nachr. 6:193–228 (1951–52)). A pair of Hermitian 8×8 matrices $H, K$ is found such that (1) $H, K$ generate $M_8(\mathbb{C})$ and (2) the minimal polynomial of the pencil $xH + yK$ has degree 4. Recent work of H. Shapiro [e.g. Linear Algebra Appl. 43:201–221 (1982)] has established the conjecture for $n \times n$ matrices $n < 5$.

Let $H, K$ be $n \times n$ (complex) Hermitian matrices, and let $f(x, y, z) = \det(xH + yK - zI) \in \mathbb{C}[x, y, z]$ be the characteristic polynomial of the pencil $xH + yK$. Kippenhahn [2] has conjectured that if $f(x, y, z)$ has a repeated factor in the polynomial ring $\mathbb{C}[x, y, z]$, then $H, K$ are (simultaneously) unitarily similar to direct sums $H_1 \oplus H_2, K_1 \oplus K_2$ with $H_i, K_i \in M_{n_i}(\mathbb{C})$ for some $n_i$ with $1 \leq n_i < n$, $i = 1, 2$ [or, equivalently, using Burnside's theorem, $H, K$ do not generate $M_n(\mathbb{C})$]. Kippenhahn verified the conjecture whenever the degree of the minimal polynomial of $xH + yK$ has degree 1 or 2. In recent work [4, 5], Shapiro has obtained a number of results which support the conjecture. In particular, she shows that it holds if $n \leq 5$. In this note, however, we show that the conjecture fails in general, by constructing a counterexample with $n = 8$. Using different methods, Waterhouse [6] has independently found counterexamples to the conjecture.

Our construction is based on the following observation: A matrix $A$ is nonderogatory if and only if the only matrices which commute with $A$ are the polynomials in $A$. If $H, K$ are real symmetric matrices such that $xH + yK$ commutes with a nonsymmetric matrix in the matrix ring $M_n(\mathbb{C}[x, y])$, then $xH + yK$ is derogatory as a matrix in $M_n(F)$, where $F$ is the field $\mathbb{C}(x, y)$. Hence (using Gauss's lemma), $f(x, y, z)$ has a repeated factor in $\mathbb{C}[x, y, z]$, so the pair $(H, K)$ satisfies the hypotheses of Kippenhahn's conjecture.


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Example. Let

\[
A = \begin{bmatrix}
P & X \\
-X' & Q
\end{bmatrix}, \quad B = \begin{bmatrix}
U & 0 \\
0 & U
\end{bmatrix},
\]

where

\[
P = \begin{bmatrix}
0 & -1 & 3 & -6 \\
1 & 0 & -6 & -3 \\
-3 & 6 & 0 & 1 \\
6 & 3 & -1 & 0
\end{bmatrix}, \quad Q = \begin{bmatrix}
0 & -2 & -2 & 6 \\
2 & 0 & 6 & 2 \\
2 & -6 & 0 & 2 \\
-6 & -2 & -2 & 0
\end{bmatrix},
\]

\[
X = \begin{bmatrix}
-1 & -1 & 5 & -7 \\
-1 & 1 & -7 & -5 \\
-1 & 13 & 1 & -1 \\
13 & 1 & -1 & -1
\end{bmatrix}, \quad U = \begin{bmatrix}
0 & -2 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 \\
0 & 0 & 2 & 0
\end{bmatrix},
\]

(where ' denotes transpose).

Let \( H = A^2 \) and \( K = AB + BA \).

Note first that \( U^2 = -4I \), so \( B^2 = -4I \). Hence \( xA + yB \) commutes with \( xH + yK \). Note that \( A, B \) are (real) skew-symmetric matrices and that \( H, K \) are symmetric. Hence as a matrix in \( M_8(\mathbb{C}(x, y)) \), \( xH + yK \) is derogatory, so the pair \( H, K \) satisfies the hypotheses of Kippenhahn’s conjecture. Thus in order to invalidate the conjecture, it is sufficient to show that the algebra \( \mathbb{C} \) generated (over \( \mathbb{C} \)) by \( H, K \) is \( M_8(\mathbb{C}) \).

To carry this out we first perform the calculations

\[
K = \begin{bmatrix}
-4I_2 & 0 & 0 & 0 \\
0 & 4I_2 & 0 & 0 \\
0 & 0 & -8I_2 & 0 \\
0 & 0 & 0 & 8I_2
\end{bmatrix},
\]

\[
H = \begin{bmatrix}
H_1 & H_2 \\
H_2' & H_3
\end{bmatrix},
\]

where

\[
H_1 = \begin{bmatrix}
-122 & 0 & 12 & 18 \\
0 & -122 & -6 & -12 \\
12 & -6 & -218 & 0 \\
18 & -12 & 0 & -218
\end{bmatrix}.
\]
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\[
H_2 = \begin{bmatrix}
-30 & 18 & 26 & 10 \\
-16 & -28 & 20 & -16 \\
44 & 8 & 24 & 12 \\
-2 & -34 & -10 & 22 \\
\end{bmatrix}
\]

\[
H_3 = \begin{bmatrix}
216 & 0 & -12 & -8 \\
0 & -216 & -8 & 36 \\
-12 & -8 & -120 & 0 \\
-8 & 36 & 0 & -120 \\
\end{bmatrix}
\]

For \( Y \in M_8(\mathbb{C}) \), write \( Y = (Y_{ij}) \) as a block matrix where \( Y_{ij} \in M_2(\mathbb{C}) \). We write \( \hat{Y}_{ij} \) for the matrix whose \((i, j)\) block is \( Y_{ij} \) and all whose other blocks are zero. Since \(-4, 4, -8, 8\) are distinct, we can find polynomials \( f_i \) \((i = 1, 2, 3, 4)\) such that \( f_i(K) \) has an \( I_2 \) in the \( i \)th diagonal block and zeros everywhere else. Since \( K \in \mathcal{O} \), we thus see that if \( Y \in \mathcal{O} \), so does \( \hat{Y}_{ij} = f_i(K)Yf_j(K) \) \((i, j = 1, 2, 3, 4)\). Note that

\[
(H_{12}H_{21})_{11} = \begin{pmatrix} 468 & -288 \\ -288 & 180 \end{pmatrix} = M, \quad \text{say,}
\]

and that

\[
(H_2H_2')_{11} = \begin{pmatrix} 2000 & 336 \\ 336 & 1696 \end{pmatrix} = N, \quad \text{say.}
\]

Since \( M, N \) are real symmetric and do not commute, they generate \( M_2(\mathbb{C}) \). So \( \mathcal{O} \) contains \( \hat{X}_{11} \) \((X_{11} \text{ arbitrary})\). Next,

\[
(H_{21}H_{12})_{22} = \begin{pmatrix} 180 & 288 \\ 288 & 468 \end{pmatrix} \quad \text{and} \quad (H_2H_2')_{22} = \begin{pmatrix} 2720 & -336 \\ -336 & 1744 \end{pmatrix},
\]

so, repeating the argument, we find that \( \mathcal{O} \) contains \( \hat{X}_{22} \) \((X_{22} \text{ arbitrary})\). Noting now that \( H_{12}, H_{21} \) have none of their entries zero, we conclude that \( \mathcal{O} \) contains

\[
\left\{ \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} \mid Y \in M_4(\mathbb{C}) \right\}.
\]

On considering

\[
\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}H \quad \text{and} \quad H\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}
\]

we see that \( \mathcal{O} \) contains

\[
\begin{pmatrix} 0 & YH_2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ H_2Y & 0 \end{pmatrix}
\]
respectively, for all \( Y \in M_4(\mathbb{C}) \). On multiplying these, we see that \( \mathcal{C} \) contains

\[
\begin{bmatrix}
0 & 0 \\
0 & H_2^2YH_2
\end{bmatrix}
\]

for all \( Y \in M_4(\mathbb{C}) \). Hence \( \mathcal{C} \) contains

\[
\begin{bmatrix}
M_4(\mathbb{C}) & M_4(\mathbb{C})H_2 \\
H_2'M_4(\mathbb{C}) & H_2'M_4(\mathbb{C})H_2
\end{bmatrix}
\]

But \( H_2 \) is nonsingular, so \( \mathcal{C} = M_8(\mathbb{C}) \), as claimed.

Since for \( x, y \) real, \( xA + yB \) is real skew-symmetric, its eigenvalues occur in complex conjugate pairs \( \pm ia \) (a real), so the eigenvalues of \( (xA + yB)^2 = x^2H + xyK - 4y^2I \) occur with even multiplicity. This implies that the minimal polynomial of \( xH + yK \) has degree at most 4. By [5], it thus follows that it has degree exactly 4.


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