Explicit Solutions for Second Order Operator Differential Equations with Two Boundary Value Conditions*

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ABSTRACT

Boundary value problems for second order operator differential equations with two boundary value conditions are studied. Explicit expressions of the solutions in terms of data problems are given. By means of the application of algebraic techniques, analogous expressions to the ones known for the scalar case are obtained.

1. INTRODUCTION

Infinite systems of linear differential equations occur frequently in the theory of stochastic processes, the degradation of polymers, infinite ladder network theory in engineering [1, 20], denumerable Markov chains [14], and moment problems [24]. In mathematical formulation of physical problems and their solutions, infinite matrices arise more naturally than finite matrices. Infinite dimensional systems of differential equations have been studied in several papers and with several techniques [6, 12, 8, 16, 15, 19, 20, 18, 24].

In a recent paper [13], we studied boundary value problems and Cauchy problems for the equation

$$X^{(2)} + A_1X^{(1)} + A_0X = 0, \tag{1.1}$$

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where A_i, for $i = 0, 1$, are bounded linear operators on a complex separable Hilbert space H. Explicit expressions for solutions of Cauchy problems and boundary value problems with a boundary condition are given in terms of data problems and a solution of the algebraic operator equation

$$X^2 + A_1 X + A_0 = 0. \tag{1.2}$$

See [13] for details. The algebraic equation (1.2) is solvable if the polynomial operator $L(z) = z^2 + A_1 z + A_0$ is linearly factorizable. Several recent characterizations about the problem of the factorization of $L(z)$ may be found in [2], [9], [17], and [22]. A methodology for solving the algebraic operator equation (1.2) is given in [13], by means of the application of annihilating analytic functions of operators. It is easy to show that the equation (1.2) can be unsolvable; for instance, if $A_1 = 0$ and $-A_0$ is an unilateral weighted shift operator on H, then $-A_0$ has no square roots [23, p. 63].

This paper may be regarded as a continuation of [13]. By means of the introduction of the concept of fundamental set of solutions for the equation (1.1), boundary value problems for this equation, with two boundary value conditions, are studied in Section 2. In an analogous way to the scalar case, different fundamental sets of solutions for the equation (1.1) are obtained in terms of solutions of the corresponding algebraic equation (1.2). Depending on the existence of a single root or a double root of (1.2), different explicit expressions for the solutions of the boundary value problem are obtained. In Section 3 we study a nonhomogeneous boundary value problem for the equation

$$X^{(2)} + A_1 X^{(1)} + A_0 X = F(t) \tag{1.3}$$

with two boundary value conditions, and an explicit expression for the solution in terms of a Green operator function analogous to the one obtained for the scalar case is given.

Throughout this paper H will denote a separable, complex Hilbert space, finite or infinite dimensional, and $L(H)$ will denote the algebra of all bounded linear operators on H with the operator norm. If T lies in $L(H)$, we represent its spectrum by $\sigma(T)$.

2. BOUNDARY VALUE PROBLEMS: THE HOMOGENEOUS CASE

We begin this section by considering the problem of generating any solution of the operator differential equation (1.1) when the characteristic algebraic equation (1.2) is solvable. If we consider the algebra $L(H)$ with the strong operator topology, we obtain a topological vector space, which will be denoted by $L_S(H)$. In either one of the two spaces $L_S(H)$ or $L(H)$ we can look at the operator differential equation (1.1).

DEFINITION 2.1. A solution X of the equation (1.1) on an interval J of the real line is a $L(H)$ valued function such that at each point t of J, there exist the strong derivatives $X^{(i)}(t)$ for $i=1,2$, and the equation (1.1) is satisfied for all t of J.

DEFINITION 2.2. Consider a pair $\{U_1, U_2\}$ of solutions of the equation (1.1). Then we say that $\{U_i,\ i=1,2\}$ is a fundamental set of solutions of (1.1) on the interval J if U_1 and U_2 are solutions of (1.1) and any solution X of this equation can be expressed in the form

$$X(t) = U_1(t)T_1 + U_2(t)T_2 \tag{2.1}$$

for some operators T_i, $i=1,2$, uniquely determined by X, and for every t in the interval J.

Notice that for the scalar case this definition coincides with the usual concept of a fundamental set of solutions. In an analogous way to the scalar case, we are interested in finding fundamental sets of solutions for (1.1) in terms of solutions of the algebraic operator equation (1.2). The following result contains different fundamental sets of solutions of (1.1) in terms of a double root of the equation (1.2), or in terms of two different roots satisfying a certain additional condition.

THEOREM 2.1.

(i) *Let* X_0 *be a double root of the equation* (1.2), *that is, a solution of* (1.2) *such that*

$$2X_0 + A_1 = 0. \tag{2.2}$$

Then the operator functions $U_1(t) = \exp(tX_0)$ *and* $U_2(t) = t\exp(tX_0)$ *define a fundamental set of solutions of* (1.1) *on the real line.*

(ii) *If* X_0, X_1 *are two solutions of the equation* (1.2) *such that* $X_1 - X_0$ *is an invertible operator in* $L(H)$, *then* $U_1(t) = \exp(tX_0)$ *and* $U_2(t) = \exp(tX_1)$ *define a fundamental set of solutions of* (1.1) *on the real line.*

Proof. (i): It is easy to show that under the hypothesis (2.2), $U_1(t) = \exp(tX_0)$ and $U_2(t) = t\exp(tX_0)$ are solutions of the equation (1.1). Let X be a solution of the equation (1.1) in the real line, and let $C_0 = X(0)$ and $C_1 = X^{(1)}(0)$. Now, we consider the operator function $U(t) = \exp(tX_0)T_1 + t\exp(tX_0)T_2 = \exp(tX_0)(T_1 + tT_2)$, where T_1 and T_2 are unknown operators in $L(H)$. In order to satisfy the Cauchy problem

$$\begin{aligned} &Y^{(2)} + A_1 Y^{(1)} + A_0 Y = 0, \\ &Y(0) = C_0, \qquad Y^{(1)}(0) = C_1, \end{aligned} \tag{2.3}$$

the operators T_1 and T_2 must verify the following conditions:

$$U(0) = T_1 = C_0, \qquad U^{(1)}(0) = T_2 + X_0 T_1. \tag{2.4}$$

The system (2.4) is equivalent to the system

$$\begin{bmatrix} I & 0 \\ X_0 & I \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}. \tag{2.5}$$

Thus, the system (2.4) is uniquely solvable if the operator matrix

$$S = \begin{bmatrix} I & 0 \\ X_0 & I \end{bmatrix}$$

is an invertible operator in $L(H \oplus H)$, and it is clear that

$$S^{-1} = \begin{bmatrix} I & 0 \\ -X_0 & I \end{bmatrix}. \tag{2.6}$$

From this and (2.5) it follows that

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = S^{-1} \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 - X_0 C_0 \end{bmatrix}. \tag{2.7}$$

(ii): Let X be a solution of the equation (1.1) with $X(0) = C_0$ and $X^{(1)}(0) = C_1$. Considering the operator function $U(t) = \exp(tX_0)T_1 + \exp(tX_1)T_2$, where T_1 and T_2 are unknown operators in $L(H)$, the Cauchy problem (2.3) is satisfied if T_1 and T_2 verify the conditions

$$\begin{aligned} U(0) &= T_1 + T_2 = C_0, \\ U^{(1)}(0) &= X_0 T_1 + X_1 T_2 = C_1. \end{aligned} \tag{2.8}$$

The system (2.8) is equivalent to the system

$$\begin{bmatrix} I & I \\ X_0 & X_1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}. \tag{2.9}$$

Thus (2.9) is uniquely solvable if the operator matrix

$$R = \begin{bmatrix} I & I \\ X_0 & X_1 \end{bmatrix}$$

is an invertible operator in $L(H \oplus H)$. Taking into account the decomposition

$$R = \begin{bmatrix} I & 0 \\ X_0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & X_1 - X_0 \end{bmatrix}\begin{bmatrix} I & I \\ 0 & I \end{bmatrix},$$

it follows that R is invertible if and only if the operator $V = X_1 - X_0$ is invertible in $L(H)$. From the hypothesis, V is invertible, and computing, one gets

$$R^{-1} = \begin{bmatrix} I + V^{-1}X_0 & -V^{-1} \\ -V^{-1}X_0 & V^{-1} \end{bmatrix}. \tag{2.10}$$

From (2.9), it follows that

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = R^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} (I + V^{-1}X_0)C_0 - V^{-1}C_1 \\ -V^{-1}X_0C_0 + V^{-1}C_1 \end{bmatrix}. \tag{2.11}$$

Notice that the Cauchy problem (2.3) has only one solution, because this problem is equivalent to the extended Cauchy problem

$$\frac{d}{dt}\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix}\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}, \qquad \begin{bmatrix} Y_1(0) \\ Y_2(0) \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix},$$

and this problem has only one solution [12]. As the operator function $U(t)$ defined by $\exp(tX_0)T_1 + t\exp(tX_0)T_2$ in (i), where T_1, T_2 are given by (2.7), and by $\exp(tX_0)T_1 + \exp(tX_1)T_2$ in (ii), where T_1, T_2 are given by (2.11), satisfy the Cauchy problem (2.3), from the uniqueness [12] it turns out that U coincides with X on the real line. ∎

REMARK 1. Unlike the scalar case, two different characteristic roots of the algebraic equation (1.2) do not necessarily define a fundamental set of solutions of (1.1), taking the corresponding exponential functions. In fact, if we consider the differential equation $X^{(2)} - X^{(1)} = 0$, then the characteristic equation $X^2 - X = 0$ is satisfied by any projection on H, but it is clear that if P and Q are projections in $L(H)$ such that $P - Q$ is not invertible, then (2.9) is not uniquely solvable and thus T_1 and T_2 are not determined by the initial conditions C_0 and C_1. In fact, if we consider the finite dimensional Cauchy problem

$$X^{(2)} - X^{(1)} = 0, \qquad X(0) = X^{(1)}(0) = 0$$

and P, Q are two proper projections in $L(H)$ such that $\{0\} \subsetneq \operatorname{ran}(P) \subsetneq \operatorname{ran}(Q) \neq H$, then $Q - P$ is not invertible. From the above comments, the operator

$$R = \begin{bmatrix} I & I \\ P & Q \end{bmatrix}$$

is not invertible in $L(H \oplus H)$. Let N be the orthogonal subspace of $\operatorname{ran}(R)$ in $H \oplus H$, and let N_1 and N_2 be the subspaces of $H \oplus H$ defined by $N_1 = N \cap (H \oplus \{0\})$, $N_2 = N \cap (\{0\} \oplus H)$. Then if we consider the projections T_1 and T_2 in $L(H)$ such that $\operatorname{ran}(T_1) = N_1$ and $\operatorname{ran}(T_2) = N_2$, then it follows that

$$R\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus one gets two different representations of the null function of the type $0 = \exp(tQ)0 + \exp(tP)0 = \exp(tQ)T_1 + \exp(tP)T_2$, for all t on the real line. In consequence the pair $\{P, Q\}$ does not yield a fundamental set of solutions of (1.1) taking the corresponding exponential functions.

When double eigenvalues exist, it is well known that for the scalar case, the pair $\exp(tx_0)$ and $t\exp(tx_0)$ defines a fundamental set of solutions of the equation (1.1) if x_0 is a double root of the equation (1.2). From Theorem 2.1(i), the result is also verified for the operator case.

The following result yields an explicit expression for the solution of a boundary value problem of the type

$$X^{(2)} + A_1 X^{(1)} + A_0 X = 0,$$

$$X(b) - X(0) = E, \quad X^{(1)}(b) - X^{(1)}(0) = F, \qquad b > 0 \qquad (2.12)$$

where E, F, A_0, and A_1 are operators in $L(H)$.

Before the statement of the next theorem, let us consider the following decomposition of an operator matrix $S=(S_{ij})$, for $1=i$, $j=2$, and S_{11} invertible. One can write S in the following way:

$$S=\begin{bmatrix} I & 0 \\ S_{21}S_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} S_{11} & 0 \\ 0 & L \end{bmatrix}\begin{bmatrix} I & S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix}, \tag{2.13}$$

where $L=S_{22}-S_{21}S_{11}^{-1}S_{12}$. Thus, as the third and the first operator matrix on the right hand side of (2.13) are invertible operators in $L(H\oplus H)$, it turns out that S is invertible if and only if L is invertible in $L(H)$.

THEOREM 2.2.

(i) *Let X_0 be a double root of the algebraic equation* (1.2), *that is, such that* $2X_0+A_1=0$, *and such that*

$$\text{for all } z \text{ in } \sigma(X_0), \qquad z \neq \frac{2k\pi i}{b}, \quad k \text{ integer}. \tag{2.14}$$

Then the boundary value problem (2.12) *has only one solution in* $[0,b]$, *given by the expression* $X(t)=\exp(tX_0)(T_1+tT_2)$, *where* T_1 *and* T_2 *are given by*

$$\begin{aligned} T_1 &= [\exp(bX_0)-I]^{-1}E \\ &\quad + b\exp(bX_0)[\exp(bX_0)-I]^{-2}(X_0E-F), \\ T_2 &= [\exp(bX_0)-I]^{-1}(F-X_0E). \end{aligned} \tag{2.15}$$

(ii) *If* $\{X_0, X_1\}$ *is a pair of solutions of the equation* (1.2) *such that* X_1-X_0 *is invertible in* $L(H)$ *and such that both satisfy the condition* (2.14), *then the boundary value problem* (2.12) *has only one solution in* $[0,b]$, *given by the expression*

$$\begin{aligned} T_1 &= [\exp(bX_0)-I]^{-1}[I+(X_1-X_0)^{-1}X_0]E \\ &\quad - [\exp(bX_0)-I]^{-1}(X_1-X_0)^{-1}F, \\ T_2 &= -[\exp(bX_1)-I]^{-1}(X_1-X_0)^{-1}X_0E \\ &\quad + [\exp(bX_1)-I]^{-1}(X_1-X_0)^{-1}F. \end{aligned} \tag{2.16}$$

Proof. (i): From the spectral mapping theorem [7] and from the hypothesis (2.14) the operator $\exp(bX_0)-I$ is invertible in $L(H)$. From Theorem 2.1(i), the general solution of the operator differential equation arising in (2.12) can be expressed in the form $X(t)=\exp(tX_0)(T_1+tT_2)$. In order to determine the operators T_1 and T_2, considering the boundary value conditions of (2.12), it follows that T_1 and T_2 must verify the following system:

$$\begin{aligned} \exp(bX_0)(bT_2+T_1)-T_1&=E,\\ X_0[\exp(bX_0)-I]T_1+[\exp(bX_0)(bX_0+I)-I]T_2&=F. \end{aligned} \tag{2.17}$$

Taking into account that $\exp(bX_0)(bX_0+I)-I=[\exp(bX_0)-I]+bX_0\exp(bX_0)$, and premultiplying each equation of the system (2.17) by $[\exp(bX_0)-I]^{-1}$, this system may be written in the form

$$\begin{aligned} &T_1+b[\exp(bX_0)][\exp(bX_0)-I]^{-1}T_2\\ &\quad=[\exp(bX_0)-I]^{-1}E,\\ &X_0T_1+\{I+bX_0\exp(bX_0)[\exp(bX_0)-I]^{-1}\}T_2\\ &\quad=[\exp(bX_0)-I]^{-1}F. \end{aligned} \tag{2.18}$$

The system (2.18) may be expressed in the following compact form:

$$\begin{bmatrix} I & b\exp(bX_0)[\exp(bX_0)-I]^{-1}\\ X_0 & I+bX_0\exp(bX_0)[\exp(bX_0)-I]^{-1} \end{bmatrix}\begin{bmatrix} T_1\\ T_2\end{bmatrix} = \begin{bmatrix} [\exp(bX_0)-I]^{-1}E\\ [\exp(bX_0)-I]^{-1}F\end{bmatrix}. \tag{2.19}$$

If we denote by S the coefficient operator matrix of the system (2.19), from (2.13) it follows that S is invertible if and only if $L=S_{22}-S_{21}S_{11}^{-1}S_{12}$ is invertible. In this case one has

$$\begin{aligned} L&=I+bX_0\exp(bX_0)[\exp(bX_0)-I]^{-1}\\ &\quad-bX_0\exp(bX_0)[\exp(bX_0)-I]^{-1}=I. \end{aligned}$$

Thus S is invertible, and from (2.19) one gets

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = S^{-1} \begin{bmatrix} [\exp(bX_0) - I]^{-1}E \\ [\exp(bX_0) - I]^{-1}F \end{bmatrix}. \tag{2.20}$$

An easy computation shows that the inverse of the operator matrix S is given by

$$S^{-1} = \begin{bmatrix} S_{11}^{-1}S_{12}L^{-1}S_{21}S_{11}^{-1} + S_{11}^{-1} & -S_{11}^{-1}S_{12}L^{-1} \\ -L^{-1}S_{21}S_{11}^{-1} & L^{-1} \end{bmatrix}. \tag{2.21}$$

In this case we have

$$S^{-1} = \begin{bmatrix} I + bX_0 \exp(bX_0)[\exp(bX_0) - I]^{-1} & -b\exp(bX_0)[\exp(bX_0) - I]^{-1} \\ -X_0 & I \end{bmatrix}.$$

From this and (2.20) it follows that

$$\begin{aligned} T_1 &= [\exp(bX_0) - I]^{-1}E + bX_0 \exp(bX_0)[\exp(bX_0) - I]^{-2} \\ &\quad - b\exp(bX_0)[\exp(bX_0) - I]^{-2}F \\ &= [\exp(bX_0) - I]^{-1}E + b\exp(bX_0)[\exp(bX_0) - I]^{-2}(X_0E - F) \\ T_2 &= -X_0[\exp(bX_0) - I]^{-1}E + [\exp(bX_0) - I]^{-1}F \\ &= [\exp(bX_0) - I]^{-1}(F - X_0E). \end{aligned}$$

Thus (2.15) is established and (i) is proved.

(ii): From Theorem 2.1(ii) and the hypothesis, the general solution of the operator differential equation of (2.12) is given by the expression $X(t) = \exp(tX_0)T_1 + \exp(tX_1)T_2$. If we require $X(t)$ to satisfy the boundary value conditions of (2.12), the operators T_1 and T_2 must verify

$$\begin{aligned} [\exp(bX_0) - I]T_1 + [\exp(bX_1) - I]T_2 &= E, \\ [\exp(bX_0) - I]X_0T_1 + [\exp(bX_1) - I]X_1T_2 &= F. \end{aligned} \tag{2.22}$$

This system is equivalent to the following

$$\begin{bmatrix} \exp(bX_0)-I & \exp(bX_1)-I \\ [\exp(bX_0)-I]X_0 & [\exp(bX_1)-I]X_1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} E \\ F \end{bmatrix}. \quad (2.23)$$

From the invertibility of the operator $\exp(bX_0)-I$ and (2.13) it follows that the coefficient operator matrix S of the system (2.23) is invertible if and only if the operator $L = [\exp(bX_1)-I]X_1 - [\exp(bX_0)-I]X_0[\exp(bX_0)-I]^{-1}[\exp(bX_1)-I] = X_1[\exp(bX_1)-I] - X_0[\exp(bX_1)-I] = (X_1-X_0)[\exp(bX_1)-I]$ is. From the invertibility of $X_1 - X_0$ and $\exp(bX_1)-I$, it follows that L is invertible. So, solving (2.23) and taking into account that

$$S^{-1} = \begin{bmatrix} [\exp(bX_0)-I][I+(X_1-X_0)^{-1}X_0] & -[\exp(bX_0)-I]^{-1}(X_1-X_0)^{-1} \\ -[\exp(bX_1)-I]^{-1}(X_1-X_0)^{-1}X_0 & [\exp(bX_1)-I](X_1-X_0)^{-1} \end{bmatrix},$$

one gets

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = S^{-1}\begin{bmatrix} E \\ F \end{bmatrix},$$

$$T_1 = [\exp(bX_0)-I]^{-1}[I+(X_1-X_0)^{-1}X_0]E - [\exp(bX_0)-I]^{-1}(X_1-X_0)^{-1}F,$$

$$T_2 = -[\exp(bX_1)-I]^{-1}(X_1-X_0)^{-1}X_0E + [\exp(bX_1)-I]^{-1}(X_1-X_0)^{-1}F.$$

Hence the result is established. ∎

3. BOUNDARY VALUE PROBLEMS: THE NONHOMOGENEOUS CASE

Consider the boundary value problem

$$y^{(2)} + A_1y^{(1)} + A_0y = f(t),$$

$$E_1y(0) + F_1y(b) = 0, \quad E_2y^{(1)}(0) + F_2y^{(1)}(b) = 0, \qquad b > 0, \quad (3.1)$$

where the given function f and the unknown y are vector functions with values in $\mathbf{C}^m$ (where $\mathbf{C}$ denotes the complex plane) and the coefficients A_0, A_1, E_i, and F_i for $i = 1,2$ are $m \times m$ matrices. The boundary value conditions arising in (3.1) are said to be well set if the correspondent homogeneous equation has the trivial solution only, and in this case the solution of the problem (3.1) may be written $y(t)=\int_0^b G(t,s)f(s)\,ds$, where $G(t,s)$ is the Green's function of (3.1); see [4, Chapter 7]. The following results are concerned with an analogous operator case.

THEOREM 3.1. *Consider the boundary value problem*

$$X^{(2)}(t)+A_1X^{(1)}(t)+A_0X(t)=F(t),$$

$$E_1X(0)+F_1X(b)=0, \qquad b>0,$$

$$E_2X^{(1)}(0)+F_2X^{(1)}(b)=0 \tag{3.2}$$

where E_i, F_i, for $i=1,2$, and A_j, for $j=0,1$, are bounded linear operators in $L(H)$, and $t\to F(t)$ is a $L(H)$ valued continuous function defined in $[0,b]$. If we define the operator matrices

$$E=\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \qquad F=\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \tag{3.3}$$

and

$$A=\begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix}$$

denotes the companion operator of (3.2), *such that*

$$E+F\exp(bA) \text{ is invertible}, \tag{3.4}$$

then the only solution of (3.2) *is given by*

$$X(t)=\int_0^b G(t,s)F(s)\,ds \tag{3.5}$$

where

$$G(t,s)=\begin{cases} C\exp(tA)(I-P)\exp(-sA)B, & 0\leqslant s\leqslant t\leqslant b, \\ -C\exp(At)P\exp(-sA)B, & 0\leqslant t\leqslant s\leqslant b, \end{cases}$$

$$C=[I,\quad 0], \qquad P=\begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad B=[E+F\exp(bA)]^{-1}F\exp(bA).$$

Proof. Let us consider the problem (3.2) and let $U(t)$ be defined by

$$U(t)=\begin{bmatrix} X(t) \\ X^{(1)}(t) \end{bmatrix}.$$

It is easy to show that (3.2) is equivalent to the extended boundary value problem

$$\begin{gathered} U^{(1)}(t)=AU(t)+BF(t), \\ EU(0)+FU(b)=0. \end{gathered} \tag{3.6}$$

From [15, p. 19], for a given initial value $U(0)$, the operator differential equation arising in (3.6) has only one solution, given by

$$U(t)=\exp(tA)U(0)+\int_0^t \exp((t-s)A)BF(s)\,ds.$$

Let us define the operator valued function

$$\begin{aligned} Y(t)=\exp(tA)(I-P)\int_0^t \exp(-As)BF(s)\,ds \\ -\exp(At)P\int_t^b \exp(-As)BF(s)\,ds. \end{aligned} \tag{3.7}$$

It follows that

$$Y(0)=-P\int_0^b \exp(-As)BF(s)\,ds, \tag{3.8}$$

$$Y(b)=\exp(Ab)(I-P)\int_0^b \exp(-As)BF(s)\,ds,$$

$$(I-P)Y(0)+P\exp(-Ab)Y(b)=0. \tag{3.9}$$

From the hypothesis (3.4) and the definition of P it turns out that

$$\begin{aligned} I-P=I-[E+F\exp(Ab)]^{-1}F\exp(Ab) \\ =[E+F\exp(Ab)]^{-1}E+F\exp(Ab)-F\exp(Ab) \\ =[E+F\exp(Ab)]^{-1}E. \end{aligned}$$

Thus, the boundary value condition of (3.6) is equivalent to the boundary value condition (3.9). From the definition of $Y(t)$, given by (3.7), it follows that

$$\begin{aligned} Y^{(1)}(t) &= A\exp(At)(I-P)\int_0^t \exp(-As)BF(s)\,ds \\ &\quad + \exp(At)(I-P)\exp(-At)BF(t) \\ &\quad - A\exp(At)P\int_t^b \exp(-As)BF(s)\,ds \\ &\quad + \exp(At)P\exp(-At)BF(t) \\ &= AY(t)+BF(t). \end{aligned}$$

It follows that Y satisfies the boundary value problem (3.6). From the uniqueness property, the operator function Y, defined by (3.7) is the only solution of the problem (3.6). So $V(t)=CY(t)$ is the only solution of (3.2), and it is given by (3.5). ∎

COROLLARY 3.1. *Consider the boundary value problem* (3.2). *Under each one of the following hypothesis, this problem is uniquely solvable and its solution is given by* (3.5):

(i) *E is an invertible operator and* $\|F\| < \|[E\exp(bA)]^{-1}\|^{-1}$;
(ii) *F is an invertible operator and* $\|E\| < \|[F\exp(bA)]^{-1}\|^{-1}$;
(iii) *E and F are invertible operators.*

Proof. From the spectral mapping theorem [7], the operator $\exp(bA)$ is invertible. From [3, p. 214], the hypotheses (i), (ii), and (iii) imply the invertibility of the operator $E+F\exp(bA)$. Now, the result is a consequence of Theorem 3. ∎

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