# On the continuity of separately continuous bihomomorphisms

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ABSTRACT

Separately continuous bihomomorphisms on a product of convergence or topological groups occur with great frequency. Of course, in general, these need not be jointly continuous. In this paper, we exhibit some results of Banach–Steinhaus type and use these to derive joint continuity from separate continuity. The setting of convergence groups offers two advantages. First, the continuous convergence structure is a powerful tool in many duality arguments. Second, local compactness and first countability, the usual requirements for joint continuity, are available in much greater abundance for convergence groups.

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## 1. Introduction

Let $G$, $H$ and $L$ be topological groups and $u : G \times H \to L$ a separately continuous bihomomorphism. One need look no further than the evaluation mapping $\omega : \widehat{G} \times G \to \mathbb{T}$ to see that such bihomomorphisms need not be jointly continuous [13]. The problem of determining conditions on $G$, $H$ and $L$ so that $u$ is jointly continuous is a difficult one and has a long history in the literature. In fact the problem has been studied extensively in the larger context of topological spaces. Over several decades, many results have appeared (see e.g. [15–18,12]) guaranteeing points of joint (quasi-)continuity for various combinations of topological spaces $G$, $H$ and $L$. Results involving topological groups can be found in [8] and [11]. Recurring themes were the notions of compactness and countability, the latter usually appearing in some form of the Baire property. The topological group case simplifies the problem considerably since joint (quasi-)continuity of a bihomomorphism at one point implies joint continuity.

In this paper we address this problem in the context of convergence groups. Apart from providing greater generality than topological groups, this permits the use of continuous convergence in duality arguments. With the aid of the notion of a $g$-barrelled convergence group, we establish theorems of Banach–Steinhaus type and use these together with duality arguments to establish the joint continuity of bihomomorphisms. The main result is the following: If $G$ is a $g$-barrelled convergence group, $H$ a locally compact convergence group and $L$ a Hausdorff locally quasi-convex topological group, every separately continuous bihomomorphism $u : G \times H \to L$ is jointly continuous. The generality of this result can be seen as various special cases recover many of the results in the literature.

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Let $X$ be a set and suppose that to each $x$ in $X$ is associated a collection $\lambda(x)$ of filters on $X$ satisfying for all $x \in X$:

(i) the ultrafilter $\dot{x} := \{A \subseteq X \colon x \in A\} \in \lambda(x)$,
(ii) if $\mathcal{F} \in \lambda(x)$ and $\mathcal{G} \in \lambda(x)$, then $\mathcal{F} \cap \mathcal{G} \in \lambda(x)$,
(iii) if $\mathcal{F} \in \lambda(x)$, then $\mathcal{G} \in \lambda(x)$ for all filters $\mathcal{G} \supseteq \mathcal{F}$.

The totality $\lambda$ of filters $\lambda(x)$ for $x$ in $X$ is called a *convergence structure* for $X$, the pair $(X, \lambda)$ a *convergence space* and filters $\mathcal{F}$ in $\lambda(x)$ *convergent* to $x$. A convergence space $(X, \lambda)$ will usually be denoted by $X$ if no confusion arises. We write $\mathcal{F} \to x$ instead of $\mathcal{F} \in \lambda(x)$. A mapping $f \colon X \to Y$ between the two convergence spaces $X$ and $Y$ is *continuous* if $f(\mathcal{F}) \to f(x)$ in $Y$ whenever $\mathcal{F} \to x$ in $X$.

Let $G$ be an Abelian group (all groups will be assumed to be Abelian) and assume $\lambda$ is a convergence structure on $G$. The pair $(G, \lambda)$ is a *convergence group* if $\lambda$ is compatible with the group operations, i.e., if the mapping

$$- \colon G \times G \to G, \quad (x, y) \mapsto x - y$$

is continuous. This means that if $\mathcal{F} \to x$ and $\mathcal{G} \to y$ in $G$, then the filter $\mathcal{F} - \mathcal{G}$ generated by $\{A - B \mid A \in \mathcal{F}, \ B \in \mathcal{G}\}$ converges to $x - y$ in $G$.

Every topological space is a convergence space, the convergent filters at any point being precisely those finer than the neighbourhood filter. Likewise, every topological group is a convergence group. The converse statements fail. Convergence groups need not be topological (see e.g. [4, 3.1] and [4, 4.3.9]).

A convergence space $X$ is called *Hausdorff* if limits are unique, i.e., if $\mathcal{F} \to p$ and $\mathcal{F} \to q$ in $X$, then $p = q$. It is called *compact* if each ultrafilter converges and *locally compact* if it is Hausdorff and each convergent filter contains a compact set.

Let $G$, $H$ be convergence groups and $\Gamma(G, H)$ the space of continuous group homomorphisms from $G$ to $H$. The *continuous convergence* structure $\lambda_c$ on $\Gamma(G, H)$ is the coarsest convergence structure on $\Gamma(G, H)$ making the evaluation mapping

$$\omega \colon \Gamma(G, H) \times G \to H, \quad (\varphi, x) \mapsto \varphi(x)$$

continuous. A filter $\Phi \to \varphi$ in $(\Gamma(G, H), \lambda_c)$ if, whenever $\mathcal{F} \to x$ in $G$, the filter $\omega(\Phi \times \mathcal{F})$ converges to $\omega(\varphi, x) = \varphi(x)$ in $H$. The continuous convergence structure is compatible with the group operations on $\Gamma(G, H)$ and the resulting convergence group is denoted $\Gamma_c(G, H)$. If $H = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, then $\Gamma G = \Gamma(G, H)$ is called the *character group* of $G$ and $\Gamma_c G = \Gamma_c(G, H)$ is called the *continuous dual* of $G$. Note that, when $G$ is a topological group, the continuous dual $\Gamma_c G$ is a locally compact convergence group. In general it is not topological, but this is so if $G$ is locally compact. In this case the continuous convergence structure coincides with the compact-open topology.

The *weak convergence structure* on $\Gamma(G, H)$ is the initial convergence structure induced by the family of mappings $(\varphi \mapsto \varphi(x))_{x \in G}$. The resulting convergence group is denoted by $\Gamma_s(G, H)$. As above, when $H = \mathbb{T}$, this becomes $\Gamma_s G$.

If $u \colon G \to H$ is a continuous homomorphism between convergence groups, then $u^* \colon \Gamma H \to \Gamma G$ is defined by $u^*(\psi) = \psi \circ u$. It is continuous if both character groups are endowed with either the continuous convergence structure or the weak convergence structure. In this way $\Gamma_c$ becomes a functor which has strong categorical properties. It is a left adjoint and takes final structures to initial structures, in particular quotients to embeddings and direct limits to inverse limits.

If $G$ is any convergence group, then the canonical mapping $\kappa_G \colon G \to \Gamma_c \Gamma_c G$ defined by

$$\kappa_G(x)(\varphi) = \varphi(x) \quad \text{for all } x \in G \text{ and all } \varphi \in \Gamma G$$

is always continuous. A convergence group $G$ is called *embedded* if $\kappa_G$ is an isomorphism onto its range and *reflexive* if $\kappa_G$ is an isomorphism.

A convergence space is called *first countable* if, to each convergent filter, there is a coarser filter with a countable base converging to the same point. Clearly, a mapping between convergence spaces is called *sequentially continuous* if each converging sequence is mapped to a converging sequence. It is a consequence of [4, 1.6.14] that, if $X$ is first countable and $Y$ is topological, then every sequentially continuous mapping $f \colon X \to Y$ is continuous.

Finally, $\rho \colon \mathbb{R} \to \mathbb{T}$ denotes the canonical projection and we set

$$\mathbb{T}_+ = \rho\big([-1/4, 1/4]\big).$$

If $\mathbb{T}$ is realized as the unit circle, this is the right half of it.

Further information on convergence spaces and, in particular, convergence groups can be found in [6] and [4].

## 2. *g*-barrelled convergence groups

In a linear setting, topological vector spaces and convergence vector spaces, the Banach–Steinhaus theorem relates pointwise bounded and equicontinuous sets as well as pointwise and continuously convergent sequences (see, e.g., [7,4,5]). Whereas the notion of equicontinuity generalizes very naturally to the setting of convergence groups, the notion of (pointwise) boundedness is usually not available and must be replaced.

**Definition 2.1.** Let $G$, $H$ be convergence groups. A set $M \subseteq \Gamma(G, H)$ is called *equicontinuous* if and only if, for all filters $\mathcal{F}$ which converge to 0 in $G$, the filter $M(\mathcal{F})$ converges to 0 in $H$. Here $M(\mathcal{F})$ denotes the filter generated by $\{M(F): F \in \mathcal{F}\} = \{\omega(M \times F): F \in \mathcal{F}\}$.

It is clear that, when $G$ and $H$ are topological groups, this coincides with the usual definition of equicontinuity.
As the next proposition shows, equicontinuity is preserved as $G$ and $H$ pass to final and initial structures respectively.

**Proposition 2.2.** *Let $G$ and $H$ be convergence groups. If $G$ carries the final group convergence structure with respect to a family of homomorphisms $(u_i : G_i \to G)_{i \in I}$ and $H$ carries the initial group convergence structure with respect to a family of homomorphisms $(v_j : H \to H_j)_{j \in J}$, then a set $M \subseteq \Gamma(G, H)$ is equicontinuous if and only if for all $i \in I$ and $j \in J$ the set*

$$v_j \circ M \circ u_i = \{v_j \circ w \circ u_i : w \in M\}$$

*is an equicontinuous subset of $\Gamma(G_i, H_j)$.*

**Proof.** An easy argument shows that $v_j \circ M \circ u_i$ is equicontinuous for all $i \in I$ and $j \in J$ if $M$ is equicontinuous. To show the converse, assume that $\mathcal{F} \to 0 \in G$. Since $G$ carries the final group convergence structure with respect to $(u_i)$ there are $i_1, \ldots, i_n \in I$ and filters $\mathcal{F}_k \to 0 \in G_{i_k}$ such that

$$\mathcal{F} \supseteq u_{i_1}(\mathcal{F}_1) + \cdots + u_{i_n}(\mathcal{F}_n).$$

By assumption, $(v_j \circ M \circ u_{i_k})(\mathcal{F}_k)$ converges to 0 in $H_j$ for all $j \in J$ and $k \in \{1, \ldots, n\}$ and therefore $v_j(M(\mathcal{F})) = (v_j \circ M)(\mathcal{F})$ converges to 0 for all $j$. Since $H$ carries the initial group convergence structure with respect to $(v_j)$, we get $M(\mathcal{F}) \to 0 \in H$ as desired. $\square$

What makes equicontinuous sets valuable for our purposes is the following result (see [4, 2.4.2] for a general formulation).

**Proposition 2.3.** *Let $G$ and $H$ be convergence groups and let $M \subseteq \Gamma(G, H)$ be an equicontinuous set. Then the weak convergence structure and the continuous convergence structure coincide on $M$.*

The following notion was defined for topological groups in [10].

**Definition 2.4.** A convergence group $G$ is called *g-barrelled* if the compact subsets of $\Gamma_s G$ are equicontinuous.

The standard examples of $g$-barrelled topological groups are countably Čech-complete topological groups so, in particular, complete metrizable or locally compact ones. Also separable Baire or metrizable hereditarily Baire groups are $g$-barrelled [15,20,10]. Finally, the additive group of a barrelled topological vector space is $g$-barrelled [14]. To obtain an example of a non-topological $g$-barrelled convergence group we recall that a topological group $G$ is said to *respect compactness* if each $\sigma(G, \Gamma G)$-compact subset of $G$ is compact [19]. Here $\sigma(G, \Gamma G)$ denotes the initial topology on $G$ with respect to $\Gamma G$.

**Proposition 2.5.** *If $G$ is a reflexive topological group that respects compactness, then $\Gamma_c G$ is g-barrelled.*

**Proof.** Since $G$ is reflexive, the natural mapping $\kappa_G : G \to \Gamma_c \Gamma_c G$ is an isomorphism and therefore $\kappa_G : (G, \sigma(G, \Gamma G)) \to \Gamma_s \Gamma_c G$ is an isomorphism. If $M \subseteq \Gamma_s \Gamma_c G$ is compact, then $\kappa_G^{-1}(M)$ is a $\sigma(G, \Gamma G)$-compact subset of $G$ and therefore compact. This implies that $M = \kappa_G(\kappa_G^{-1}(M))$ is compact in $\Gamma_c \Gamma_c G$. So it is equicontinuous by the Arzelà–Ascoli theorem [4, 2.5.8]. $\square$

**Corollary 2.6.** *If $G$ is a nuclear group, then $\Gamma_c G$ is g-barrelled.*

**Proof.** If $G$ is a complete nuclear group, it is reflexive by [4, 8.4.19] and it respects compactness by [3]. Therefore $\Gamma_c G$ is $g$-barrelled by Proposition 2.5. If $G$ is an arbitrary nuclear group, then its completion $\widetilde{G}$ is nuclear by [1, 21.4]. Also $\Gamma_c G = \Gamma_c \widetilde{G}$ by [4, 8.4.4] and so the result follows. $\square$

It should be noted that the reflexive locally convex topological vector spaces which respect compactness are precisely the Montel spaces [19, Theorem 1.4].
The next several propositions derive permanence properties of $g$-barrelled convergence groups.

**Proposition 2.7.**

(i) *Let $G$ and $G'$ be convergence groups with the same underlying group such that $\Gamma G = \Gamma G'$. If the identity mapping $id : G \to G'$ is continuous, then $G$ is g-barrelled if $G'$ is.*

(ii) *A convergence group which carries the final group convergence structure with respect to a family of group homomorphisms from g-barrelled convergence groups is g-barrelled.*

(iii) *A topological group which carries the final group topology with respect to a family of group homomorphisms from g-barrelled topological groups is g-barrelled.*

**Proof.** (i) Evidently $\Gamma_s G = \Gamma_s G'$, so if $M \subseteq \Gamma_s G$ is compact, then $M$ is compact in $\Gamma_s G'$ and therefore equicontinuous. If $\mathcal{F}$ converges to 0 in $G$ then it converges to 0 in $G'$ and therefore $M(\mathcal{F})$ converges to 0. So $M$ is an equicontinuous subset of $\Gamma G$.

(ii) Assume that $G$ carries the final group convergence structure with respect to a family of group homomorphisms $(G_i \to G)_{i \in I}$ such that all $G_i$ are g-barrelled. If $\mathcal{F}$ converges to 0 in $G$, there are finitely many $i_1, \ldots, i_n \in I$ and filters $\mathcal{F}_j$ converging to zero in $G_{i_j}$ such that

$$\mathcal{F} \supseteq u_{i_1}(\mathcal{F}_1) + \cdots + u_{i_n}(\mathcal{F}_n).$$

Take any compact subset $M$ of $\Gamma_s G$. Then $u_i^*(M)$ is compact in $\Gamma_s G_i$ for all $i$ and therefore equicontinuous. So $M(u_{i_j}(\mathcal{F}_j)) = u_{i_j}^*(M)(\mathcal{F}_j)$ converges to 0 in $\mathbb{T}$ and so does

$$M(u_{i_1}(\mathcal{F}_1)) + \cdots + M(u_{i_n}(\mathcal{F}_n)).$$

The claim now follows from

$$M(\mathcal{F}) \supseteq M(u_{i_1}(\mathcal{F}_1) + \cdots + u_{i_n}(\mathcal{F}_n)) \supseteq M(u_{i_1}(\mathcal{F}_1)) + \cdots + M(u_{i_n}(\mathcal{F}_n)).$$

(iii) This is [10, 1.9].    □

**Lemma 2.8.** *Let $(G_i)_{i \in I}$ be a family of convergence groups, $G = \prod_{i \in I} G_i$ their product and let $e_i : G_i \to G$ be the natural injections. If $M \subseteq \Gamma_s G$ is compact, then there is a finite subset $I_0 \subseteq I$ such that $\varphi \circ e_i = 0$ for all $\varphi \in M$ and all $i \in I \setminus I_0$.*

**Proof.** Since the mapping

$$\Gamma_c G \longrightarrow \bigoplus_{i \in I} \Gamma_c G_i, \quad \varphi \mapsto (\varphi \circ e_i)$$

is an isomorphism by [4, 8.1.18], we will regard the elements of $\Gamma G$ as elements in $\bigoplus \Gamma G_i$. The claim then is that there is a finite set $I_0 \subseteq I$ such that $\varphi_i = 0$ for all $\varphi \in M$ and all $i \in I \setminus I_0$. So assume that this is not true. For shorter reference, for all $\varphi \in \Gamma G$ we set $Supp(\varphi) = \{i \in I : \varphi_i \neq 0\}$. These sets are all finite. Define inductively sequences $(\varphi_n)$ in $M$ and $(i_n)$ in $I$ in the following way:

Choose any $\varphi_1 \in M$, $\varphi_1 \neq 0$ and any $i_1 \in I$ such that $\varphi_{1,i_1} \neq 0$. (Here and in what follows $\varphi_{n,i}$ denotes the $i$th component of $\varphi_n$.) Assume that $\varphi_1, \ldots, \varphi_{n-1}$ and $i_1, \ldots, i_{n-1}$ have been chosen. Then there is a $\varphi_n \in M$ such that $Supp(\varphi_n) \nsubseteq Supp(\varphi_1) \cup \cdots \cup Supp(\varphi_{n-1})$. Choose any $i_n \in Supp(\varphi_n) \setminus (Supp(\varphi_1) \cup \cdots \cup Supp(\varphi_{n-1}))$. Then $\varphi_{n,i_n} \neq 0$.

Note that, by construction, we have

$$\varphi_{j,i_n} = 0 \quad \text{for all } j < n$$

and therefore, in particular, $i_j \neq i_n$ for all $j < n$.

Now choose any $x = (x_i) \in G$ such that $x_i = 0$ if $i \notin \{i_n : n \in \mathbb{N}\}$. Then for all $r \in \mathbb{N}$ we get:

$$\varphi_r(x) = \sum_{i \in I} \varphi_{r,i}(x_i) = \sum_{n \in \mathbb{N}} \varphi_{r,i_n}(x_{i_n}) = \sum_{r \geqslant n} \varphi_{r,i_n}(x_{i_n}) = \sum_{n < r} \varphi_{r,i_n}(x_{i_n}) + \varphi_{r,i_r}(x_{i_r}).$$

Set $\mathbb{T}_0 = \rho([-1/16, 1/16])$. We show that for each finite set $J \subseteq I$ there is an element $x \in G$ such that $x_i = 0$ for all $i \in J$ and $\varphi_n(x) \notin \mathbb{T}_0$ for all but finitely many $n$.

Choose a finite set $J \subseteq I$ and a $k \in \mathbb{N}$ such that $i_n \notin J$ for all $n \geqslant k$. Now define $x \in G$ in the following way: $x_i = 0$ if $i \notin \{i_n : n \in \mathbb{N}\}$ and also $x_i = 0$ for all $i \in \{i_n : n < k\}$. Then $x_i = 0$ if $i \in J$. Define $x_{i_n}$ for all $n \geqslant k$ inductively as follows: One has

$$\varphi_k(x) = \varphi_{k,i_k}(x_{i_k})$$

and since $\varphi_{k,i_k} \neq 0$ there is some $x_{i_k} \in G_{i_k}$ such that $\varphi_{k,i_k}(x_{i_k}) \notin \mathbb{T}_0$.

If $x_{i_k}, \ldots, x_{i_{r-1}}$ have been constructed, we get

$$\varphi_r(x) = \sum_{n < r} \varphi_{r,i_n}(x_{i_n}) + \varphi_{r,i_r}(x_{i_r}).$$

If $\sum_{n<r} \varphi_{r,i_n}(x_{i_n}) \notin \mathbb{T}_0$, then set $x_{i_r} = 0$ otherwise there is some $x_{i_r} \in G_{i_r}$ such that $\varphi_{r,i_r}(x_{i_r}) \notin \mathbb{T}_+$ and then $\varphi_r(x) \notin \mathbb{T}_0$.

Assume now that the sequence $(\varphi_n)$ has a cluster point $\psi \in \Gamma_s G$. Then there is a finite set $J \subseteq I$ such that $\psi_i = 0$ for all $i \in I \setminus J$. Choose $x$ as above. Then $\psi(x) = 0$ and so there must be infinitely many $n$ such that $\varphi_n(x) = \varphi_n(x) - \psi(x) \in \mathbb{T}_0$, contradicting the construction of $x$. □

**Proposition 2.9.** *Let $(G_i)_{i \in I}$ be a family of g-barrelled convergence groups. Then $\prod_{i \in I} G_i$ is g-barrelled.*

**Proof.** Set $G = \prod_{i \in I} G_i$ and let $M \subseteq \Gamma_s G$ be a compact set. Since $e_i^* : \Gamma_s G \to \Gamma_s G_i$ is continuous for all $i$, also $e_i^*(M)$ is compact in $\Gamma_s G_i$ and therefore equicontinuous. By Lemma 2.8, there are elements $i_1, \ldots, i_n \in I$ such that $\varphi \circ e_i = 0$ for all $\varphi \in M$ and all $i \neq i_1, \ldots, i_n$. Take any filter $\mathcal{F}$ which converges to 0 in $G$. Then $p_i(\mathcal{F})$ converges to 0 in $G_i$, where $p_i$ denotes the projection, and therefore $M(e_i(p_i(\mathcal{F}))) = e_i^*(M)(p_i(\mathcal{F}))$ converges to 0 for all $i$. Choose a zero neighbourhood $U$ in $\mathbb{T}$. Then there is a zero neighbourhood $V$ in $\mathbb{T}$ such that $nV = V + \cdots + V \subseteq U$. Then there is a set $F \in \mathcal{F}$ such that

$$M\big(e_{i_1}\big(p_{i_1}(F)\big)\big) + \cdots + M\big(e_{i_n}\big(p_{i_n}(F)\big)\big) \subseteq nV \subseteq U.$$

Take now any $\varphi \in M$ and $x \in F$. Then we have

$$\varphi(x) = \sum_{i \in I} e_i^*(\varphi)(x_i) = \sum_{i \in I} \varphi \circ e_i(x_i) = \sum_{j=1}^{n} \varphi \circ e_{i_j}(x_{i_j}) = \sum_{j=1}^{n} \varphi\big(e_{i_j}\big(p_{i_j}(x)\big)\big) \in U$$

and so $M(F) \subseteq U$. □

Locally quasi-convex topological groups were studied in [2]. Since they will be of importance in the sequel, we introduce them here as well as the locally quasi-convex modification.

A subset $A$ of a topological group $G$ is called *quasi-convex* if for each $x \in G \setminus A$ there is a character $\varphi \in \Gamma G$ such that $\varphi(A) \subseteq \mathbb{T}_+$ while $\varphi(x) \notin \mathbb{T}_+$. Furthermore, $G$ is called *locally quasi-convex* if it has a zero neighbourhood base consisting of quasi-convex sets. As it turns out each Hausdorff topological group $G$ is locally quasi-convex if and only if it is embedded (see [4, 8.4.7]).

If $G$ is a convergence group, then the finest locally quasi-convex topology on $G$ which is coarser than the convergence structure of $G$ is called the *locally quasi-convex modification* of $G$ and the resulting topological group is denoted by $\tau(G)$. In order to give an explicit description thereof, for subsets $A \subseteq G$ and $H \subseteq \Gamma G$ we define

$$A^\circ = \big\{\varphi \in \Gamma G : \varphi(A) \subseteq \mathbb{T}_+\big\}$$

and

$$H^\diamond = \big\{x \in G : H(x) \subseteq \mathbb{T}_+\big\}.$$

In this terminology $A$ is quasi-convex if and only if $A = A^{\circ\diamond}$.

**Theorem 2.10.** *Let $G$ be a convergence group. Then*

$$\mathcal{B} := \big\{H^\diamond : H \subseteq \Gamma G \text{ equicontinuous}\big\}$$

*is a zero neighbourhood base of the locally quasi-convex modification of $G$.*

**Proof.** Clearly $\mathcal{B}$ is a filter basis consisting of symmetric sets and, if $H$ is an equicontinuous subset of $\Gamma G$ containing 0, then $H + H$ is also equicontinuous and

$$(H + H)^\diamond + (H + H)^\diamond \subseteq H^\diamond.$$

Therefore $\mathcal{B}$ is the zero neighbourhood basis of a locally quasi-convex topology $\tau$ on $G$. If $\mathcal{F}$ converges to 0 in $G$, then $H(\mathcal{F})$ converges to 0 in $\mathbb{T}$ and so there is some $F \in \mathcal{F}$ such that $H(F) \subseteq \mathbb{T}_+$. This gives $F \subseteq H^\diamond$ and so the zero neighbourhood filter of $\tau$ is contained in $\mathcal{F}$ which gives the continuity of the identity mapping $id : G \to (G, \tau)$. Finally, if $\mu$ is any locally quasi-convex topology on $G$ coarser than that of $G$ and $V$ is any quasi-convex zero neighbourhood in $(G, \mu)$, then $V^\circ$ is an equicontinuous subset of $\Gamma(G, \mu)$, and therefore of $\Gamma G$. Consequently, $V = V^{\circ\diamond} \in \mathcal{B}$ and so $id : (G, \tau) \to (G, \mu)$ is continuous. □

**Proposition 2.11.** *If $G$ is a convergence group then $\Gamma G = \Gamma \tau(G)$ and both character groups share the same equicontinuous subsets.*

**Proof.** Clearly each equicontinuous subset of $\Gamma \tau(G)$ is equicontinuous in $\Gamma G$. On the other hand, if $H$ is an equicontinuous subset of $\Gamma G$, then $H^\diamond$ is a zero neighbourhood of $\tau(G)$ and so $H^{\diamond\circ}$ is an equicontinuous subset of $\Gamma \tau(G)$ containing $H$. □

**Corollary 2.12.** *A convergence group $G$ is g-barrelled if and only if $\tau(G)$ is.*

**Proof.** It follows from Proposition 2.11 that $\Gamma_s G = \Gamma_s \tau(G)$ and so the result follows from the second statement in Proposition 2.11. □

**Corollary 2.13.** *A topological group which carries the final locally quasi-convex group topology with respect to a family of group homomorphisms from g-barrelled topological groups is g-barrelled.*

**Proof.** An easy argument shows that the final locally quasi-convex group topology is the locally quasi-convex modification of the final group convergence structure and so the result follows from Proposition 2.7(ii) and Corollary 2.12. □

The concept of $g$-barrelledness allows us to relate the compact subsets of $\Gamma_s(G, H)$ and the equicontinuous subsets of $\Gamma(G, H)$. The following two theorems can be thought of as theorems of Banach–Steinhaus type.

**Theorem 2.14.** *Let $G$ and $H$ be convergence groups. If $G$ is g-barrelled and $H$ is locally compact, then each compact subset of $\Gamma_s(G, \Gamma_c H)$ is equicontinuous.*

**Proof.** Let $M$ be a compact subset of $\Gamma_s(G, \Gamma_c H)$ and assume that $\mathcal{F} \to 0$ in $G$. We have to show that $M(\mathcal{F}) \to 0$ in $\Gamma_c H$. So let $\mathcal{H} \to z$ in $H$. Since $H$ is locally compact, $\mathcal{H}$ contains a compact set $K$ and $M(\mathcal{F})(\mathcal{H})$ is finer than $M(\mathcal{F})(K)$. We claim that $M(\mathcal{F})(K)$ converges to 0 in $\mathbb{T}$ which will give the desired result.

Consider the mapping

$$T : \Gamma_s(G, \Gamma_c H) \times H \to \Gamma_s G$$

given by $T(u, y)(x) = u(x)(y)$ for all $(u, y) \in \Gamma(G, \Gamma_c H) \times H$ and all $x \in G$. An easy calculation shows that $T$ is continuous and so $T(M \times K)$ is compact in $\Gamma_s G$. Since $G$ is g-barrelled, $T(M \times K)$ is equicontinuous in $\Gamma G$. Hence $M(\mathcal{F})(K) = T(M \times K)(\mathcal{F}) \to 0$ in $\mathbb{T}$ as required. □

**Theorem 2.15.** *Let $G$ be a g-barrelled convergence group and $L$ a Hausdorff locally quasi-convex topological group. Then the compact subsets of $\Gamma_s(G, L)$ are equicontinuous.*

**Proof.** Since $L$ is a Hausdorff locally quasi-convex topological group, it is an embedded convergence group, and therefore isomorphic to a subgroup of $\Gamma_c \Gamma_c L$. Set $H = \Gamma_c L$. Then $H$ is locally compact. So if $M$ is a compact subset of $\Gamma_s(G, L)$, it can be considered a compact subset of $\Gamma_s(G, \Gamma_c H)$ and is therefore equicontinuous by Theorem 2.14. Clearly $M$ is then equicontinuous in $\Gamma(G, L)$. □

Since the weak convergence structure and the continuous convergence structure coincide on equicontinuous sets by Proposition 2.3, Theorem 2.15 gives conditions under which each compact subset of $\Gamma_s(G, L)$ is even compact in $\Gamma_c(G, L)$.

## 3. Joint continuity of bihomomorphisms

In this section we make use of the results of the previous section to extract the joint continuity of separately continuous bihomomorphisms in several special cases. A key observation here is the following:

**Proposition 3.1.** *Let $G$, $H$ and $L$ be convergence groups and $u : G \times H \to L$ be a separately continuous bihomomorphism. Then the mapping*

$$u_H : H \longrightarrow \Gamma_s(G, L)$$

*defined by $u_H(y)(x) = u(x, y)$ is continuous. Furthermore, $u$ is jointly continuous if and only if*

$$u_H : H \longrightarrow \Gamma_c(G, L)$$

*is continuous.*

**Proof.** The first part is clear. Now from the universal property of the continuous convergence structure, $u_H$ is continuous if and only if the mapping

$$\omega \circ (id_G \times u_H) : G \times H \to L$$

is continuous. But evidently $\omega \circ (id_G \times u_H) = u$ and so the claim follows. □

**Proposition 3.2.** *Let $G$, $H$ and $L$ be convergence groups such that the compact subsets of $\Gamma_s(G, L)$ are equicontinuous. Assume further that $u : G \times H \to L$ is a separately continuous bihomomorphism. Then $u$ is jointly continuous in either of the following two cases*:

(i) *$H$ is locally compact.*
(ii) *$G$ and $H$ are first countable and $L$ is topological.*

**Proof.** (i) By Proposition 3.1 it is sufficient to show that $u_H : H \to \Gamma_c(G, L)$ is continuous. By the same proposition, $u_H : H \to \Gamma_s(G, L)$ is continuous. Let $\mathcal{F} \to y_0$ in $H$. Then $u_H(\mathcal{F})$ converges to $u_H(y_0)$ in $\Gamma_s(G, L)$. Since $\mathcal{F}$ contains a compact set, $u_H(\mathcal{F})$ contains a compact subset of $\Gamma_s(G, L)$. By assumption, this set is equicontinuous and so $u_H(\mathcal{F})$ converges to $u_H(y_0)$ in $\Gamma_c(G, L)$ by Proposition 2.3.

(ii) We first show that $u_H$ is sequentially continuous: If $(y_n)$ is a sequence which converges to $y_0$ in $H$, then $B = \{y_n : n \in \mathbb{N}\} \cup \{y_0\}$ is a compact subset of $H$ and therefore $u_H(B)$ is a compact subset of $\Gamma_s(G, L)$ and hence equicontinuous. Again, this implies that $(u_H(y_n))$ converges to $u_H(y_0)$ in $\Gamma_c(G, L)$, and so $u_H$ is sequentially continuous.

Next we show that $u$ is sequentially continuous. Let $(x_n, y_n)$ be a sequence in $G \times H$ which converges to $(x_0, y_0)$ in $G \times H$. Since $u_H$ is sequentially continuous, $(u_H(y_n))$ converges to $u_H(y_0)$ in $\Gamma_c(G, L)$ and so $(u(x_n, y_n)) = (u_H(y_n)(x_n))$ converges to $u(x_0, y_0) = u_H(y_0)(x_0)$ in $L$. Thus $u$ is sequentially continuous.

Since $G$ and $H$ are first countable so is $G \times H$ [4, 1.6.8] and, since $L$ is topological, $u$ is continuous. $\quad \square$

From Propositions 3.1, 3.2 and Theorem 2.15 we get the main result of this section:

**Theorem 3.3.** *Let $G$ and $H$ be convergence groups, $G$ g-barrelled, and let $L$ be a Hausdorff locally quasi-convex topological group. Then every separately continuous bihomomorphism $u : G \times H \to L$ is jointly continuous in either of the following cases*:

(i) *$H$ is locally compact.*
(ii) *$G$ and $H$ are first countable.*

Part (ii) of the above theorem yields joint continuity results for first countable convergence groups. It uses duality arguments. One can also obtain joint continuity in first countable situations using standard Baire category techniques.

**Proposition 3.4.** *Let $G$ and $L$ be topological groups, $G$ Baire, and let $H$ be a first countable convergence group. Then each separately continuous bihomomorphism $u : G \times H \to L$ is jointly continuous.*

**Proof.** Since $u$ is separately continuous, it suffices to show that $u$ is continuous at $(0, 0)$. So assume that $\mathcal{F}$ converges to $0$ in $H$. Since $H$ is first countable, there is a filter $\mathcal{V} \subseteq \mathcal{F}$ with a countable base $(V_n)$ which also converges to $0$. Take a closed zero neighbourhood $W$ in $L$. For all $n \in \mathbb{N}$ consider the set

$$A_n = \big\{ x \in G \colon u(x \times V_n) \subseteq W \big\}.$$

We first claim that $\bigcup A_n = G$: For any $x \in G$, $u(x, 0) = 0$. Since $u(x, \cdot)$ is continuous, there is an $n \in \mathbb{N}$ such that $u(x \times V_n) \subseteq W$. This means $x \in A_n$.

Next, we show that each $A_n$ is closed. For all $y \in V_n$ we have $u(A_n \times y) \subseteq W$. Since $u$ is separately continuous, we have

$$u(\overline{A_n} \times y) \subseteq \overline{u(A_n \times y)} \subseteq \overline{W} = W$$

and so $u(x, y) \in W$ for all $x \in \overline{A_n}$ and all $y \in V_n$. This gives $u(x \times V_n) \subseteq W$ for all $x \in \overline{A_n}$ and therefore $\overline{A_n} \subseteq A_n$.

Since $G$ is a Baire space, some $A_k$ has an interior point $x_0$ and so there is a zero neighbourhood $U$ in $G$ such that $x_0 + U \subseteq A_k$. This gives

$$u(x_0 + x, y) \in W \quad \text{for all } x \in U, \ y \in V_k.$$

Furthermore, there is a zero neighbourhood $V$ in $H$ such that $u(x_0 \times V) \subseteq W$ and so

$$u(x_0, y) \in W \quad \text{for all } y \in V.$$

Finally we get, for all $x \in U$ and $y \in V_k \cap V$,

$$u(x, y) = u(x_0 + x, y) - u(x_0, y) \in W - W$$

which shows that $u$ is continuous at $(0, 0)$. $\quad \square$

One factor which makes the results of Theorem 3.3 strong is the size of the class of $g$-barrelled convergence groups. Even if one restricts oneself to topological groups, this class remains large. As seen in the previous section, it includes all countably Čech-complete topological groups and is closed under the formation of arbitrary products and inductive limits.

**Examples 3.5.** (i) Let $G$ be an inductive limit of locally compact topological groups, $H$ a locally compact convergence group and $L$ a Hausdorff locally quasi-convex topological group. Then every separately continuous bihomomorphism $u : G \times H \to L$ is jointly continuous.

(ii) Let $G$ be a convergence inductive limit of complete metrizable topological groups, $H$ a metrizable topological group and $L$ a Hausdorff locally quasi-convex topological group. Then every separately continuous bihomomorphism $u : G \times H \to L$ is jointly continuous.

It should be mentioned that, in general, inductive limits depend very heavily on the setting in which they are taken. It is a consequence of Proposition 2.7 and Corollary 2.12, however, that the convergence group (or topological group or locally quasi-convex group) inductive limit of $g$-barrelled groups is also $g$-barrelled. Thus in Example 3.5(i), $G$ may be any appropriate inductive limit.

Another factor which adds scope to the results of Theorem 3.3 and Proposition 3.4 is the fact that, for convergence groups, the notions of local compactness and first countability are not as restrictive as for topological groups. Convergence group inductive limits preserve both properties. Also, for any topological group $G$, the continuous character group $\Gamma_c G$ is locally compact.

**Examples 3.6.** (i) Let $G$ be a convergence inductive limit of Baire groups, $H$ a convergence inductive limit of metrizable topological groups and $L$ any topological group. Then every separately continuous bihomomorphism $u : G \times H \to L$ is jointly continuous.

(ii) Let $G$ be a convergence inductive limit of complete metrizable topological groups, $H$ and $L$ topological groups, $L$ Hausdorff and locally quasi-convex. Then every separately continuous bihomomorphism $u : G \times \Gamma_c H \to L$ is jointly continuous.

(iii) Let $G$ and $H$ be separable metrizable topological groups, $G$ complete and $L$ any topological group. Then any separately continuous bihomomorphism $u : G \times \Gamma_c H \to L$ is jointly continuous.

(iv) Let $G$, $H$ and $L$ be topological groups, $G$ nuclear and $L$ Hausdorff and locally quasi-convex. Then every separately continuous bihomomorphism $u : \Gamma_c G \times \Gamma_c H \to L$ is jointly continuous.

**Remarks 3.7.** Results on joint continuity can be viewed as results on triples $(G, H, L)$ of convergence or topological groups. In such situations relaxing restrictions on one variable often requires tightening them on another. Consider the following:

(i) In [12] it is shown that separate continuity implies joint continuity if $G$ and $H$ are both countably Čech complete and $L$ is metrizable. This is a relaxation of the condition of the local compactness of $H$ in Theorem 3.3(i) but is much more restrictive on $G$ and $L$.

(ii) One can easily generalize the notion of sequential barrelledness defined in [10] and [14] to convergence groups. A convergence group $G$ is sequentially barrelled if every convergent sequence in $\Gamma_s G$ is equicontinuous. This is a large class of groups which includes all $g$-barrelled convergence groups and all Baire groups. It is possible to imitate the proof of Theorem 3.3 to obtain joint continuity for $G$ first countable and sequentially barrelled, $H$ first countable and $L$ a second countable Hausdorff locally quasi-convex topological group. This is a relaxation of the conditions on $G$ but is much more restrictive on $L$.

(iii) If $G$ is assumed only to be $g$-barrelled, it does not appear that one can relax the condition of local compactness on $H$ very far. If $G$ is a complete metrizable topological group which is Pontryagin reflexive but not locally compact, and $H = \widehat{G}$ is its Pontryagin dual, then $G$ is $g$-barrelled and $H$ is a $k$-space and $k$-group [9], but the evaluation mapping $\omega : G \times H \to \mathbb{T}$ is not jointly continuous [13].

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