JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 103 (1999) 251–261

Numerical solution of one-dimensional Burgers equation: explicit and exact-explicit finite difference methods

S. Kutluay *, A.R. Bahadir, A. Özdeş

Department of Mathematics, Faculty of Arts and Science, İnönü University, 44100-Malatya, Turkey

Received 16 April 1998; received in revised form 13 October 1998

Abstract

This paper presents finite-difference solution and analytical solution of the finite-difference approximations based on the standard explicit method to the one-dimensional Burgers equation which arises frequently in the mathematical modelling used to solve problems in fluid dynamics. Results obtained by these ways for some modest values of viscosity have been compared with the exact (Fourier) one. It is shown that they are in good agreement with each other. © 1999 Elsevier Science B.V. All rights reserved.

AMS classification: 65N06

Keywords: Burgers equation; Exact-explicit finite difference; Explicit finite difference

1. Introduction

We consider the one-dimensional quasi-linear parabolic partial differential equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}, \quad a<x<b, \ t>0, \tag{1}$$

with the initial condition

$$u(x,0) = \phi(x), \quad a<x<b, \tag{2}$$

and the boundary conditions

$$u(a,t) = f(t) \quad \text{and} \quad u(b,t) = g(t), \ t>0, \tag{3}$$

where $v>0$ is the coefficient of kinematic viscosity and ϕ, f and g are the prescribed functions of the variables. Historically, Eq. (1) was first introduced by Bateman [3] who gave its steady solutions.

* Corresponding author.

PII: S0377-0427(98)00261-1

It was later treated by Burgers [5] as a mathematical model for turbulence and after whom such an equation is widely referred to as Burgers equation.

Many problems can be modelled by the Burgers equation [9]. For example, the Burgers equation can be considered as an approach to the Navier–Stokes equations [2, 12] since both contain nonlinear terms of the type: unknown functions multiplied by a first derivative and both contain higher-order terms multiplied by a small parameter.

The Burgers equation is one of the very few nonlinear partial differential equation which can be solved exactly for a restricted set of initial function $\phi(x)$, only. In the context of gas dynamic, Hopf [11] and Cole [7] independently showed that this equation can be transformed to the linear diffusion equation and solved exactly for an arbitrary initial condition (2). The study of the general properties of the Burgers equation has motivated considerable attention due to its applications in field as diverse as number theory, gas dynamics, heat conduction, elasticity, etc.

The exact solutions of the one-dimensional Burgers equation have been surveyed by Berton and Platzman [4]. Many other authors [1, 6, 8, 10, 13–15, 17] have used a variety of numerical techniques based on finite-difference, finite-element and boundary element methods in attempting to solve the equation particularly for small values of the kinematic viscosity v which correspond to steep fronts in the propagation of dynamic waveforms.

2. Statement of the problem

Consider the Burgers equation (1) with the initial condition

$$u(x,0) = \sin(\pi x), \quad 0 < x < 1, \tag{4}$$

and the homogeneous boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0. \tag{5}$$

By the Hopf–Cole transformation [13]

$$u(x,t) = -2v\frac{\theta_x}{\theta}, \tag{6}$$

the Burgers equation transforms to the linear heat equation

$$\frac{\partial\theta}{\partial t} = v\frac{\partial^2\theta}{\partial x^2}, \quad 0 < x < 1, \; t > 0, \tag{7}$$

with the initial condition

$$\theta(x,0) = \exp\{-(2\pi v)^{-1}[1 - \cos(\pi x)]\}, \quad 0 < x < 1, \tag{8}$$

and the boundary conditions

$$\theta_x(0,t) = \theta_x(1,t) = 0, \quad t > 0. \tag{9}$$

This means that if $\theta(x,t)$ is any solution of the heat equation (7) subject to the conditions (8) and (9), then the transformation (6) is a solution of the Burgers equation (1) with the conditions (4) and (5).

Using the method of separation of variables the (exact) Fourier series solution to the above linearized problem defined by Eqs. (7)–(9) can be obtained easily as

$$\theta(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2 vt)\cos(n\pi x), \tag{10}$$

with the Fourier coefficients

$$a_0 = \int_0^1 \exp\{-(2\pi v)^{-1}[1-\cos(\pi x)]\}\,\mathrm{d}x, \tag{11}$$

$$a_n = 2\int_0^1 \exp\{-(2\pi v)^{-1}[1-\cos(\pi x)]\}\cos(n\pi x)\,\mathrm{d}x \quad (n=1,2,3,\ldots). \tag{12}$$

Thus, using the Hopf–Cole transformation given by Eq. (6), the (exact) Fourier solution to the problem given by Eqs. (1), (4) and (5) is obtained as

$$u(x,t) = 2\pi v \frac{\sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2 vt) n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2 vt)\cos(n\pi x)}, \tag{13}$$

where a_0 and a_n $(n = 1,2,3,\ldots)$ are defined by Eqs. (11) and (12), respectively.

3. Methods of solution

The solution domain $\{(x,t): x \in [0,1],\ t \in [0,\infty)\}$ is discretized into cells described by the node set (x_i, t_j) in which $x_j = ih$, $t_j = jk$ $(i = 0(1)N;\ j = 0(1)J,\ Nh = 1$ and $Jk = t_f)$ $h \equiv \Delta x$ is a spatial mesh size, $k \equiv \Delta t$ is the time step and t_f is the final time.

3.1. An explicit finite-difference solution

A standard explicit finite-difference approximation to Eq. (7) with the boundary conditions (9) is given by

$$\theta_{i,j+1} = (1-2r)\theta_{i,j} + 2r\theta_{i+1,j}, \quad i = 0, \tag{14a}$$

$$\theta_{i,j+1} = r\theta_{i-1,j} + (1-2r)\theta_{i,j} + r\theta_{i+1,j}, \quad i = 1(1)N-1, \tag{14b}$$

$$\theta_{i,j+1} = 2r\theta_{i-1,j} + (1-2r)\theta_{i,j}, \quad i = N, \tag{14c}$$

for $j = 0(1)J$ with a truncation error of $O(k) + O(h^2)$ (see, e.g., [16, Section 2]). In the above equations, $r = kv/h^2$ and $\theta_{i,j}$ denotes the finite-difference approximation to the exact solution $\theta(x_i, t_j)$ at the point (x_i, t_j). For stability analysis it is convenient to use Von Neumann's approach (see, e.g., [16, Section 2]) to obtain the bound on the size of the time step k. It can be obtained as $k \leqslant h^2/2v$.

Thus, using the Hopf–Cole transformation given by Eq. (6), the explicit finite difference solution to the non-linear problem by Eqs. (1), (4) and (6) is obtained as

$$u(x_i, t_j) = -\frac{v}{h}\left(\frac{\theta_{i+1,j} - \theta_{i-1,j}}{\theta_{i,j}}\right), \quad i = 1(1)N-1,\ j = 0(1)J. \tag{15}$$

3.2. An exact-explicit finite-difference solution

Assume that the finite-difference equation (14b) has product solutions of the form

$$\theta_{i,j} = f_i g_j \tag{16}$$

where f_i depends on i (or x) only and g_j depends on j (or t) only (see, e.g., [16, Section 3]).

Substitution of Eq. (16) into Eq. (14b) we obtain

$$\frac{g_{j+1}}{g_j} = \frac{rf_{i-1} + (1-2r)f_i + rf_{i+1}}{f_i}. \tag{17}$$

Since the left member of Eq. (17) is independent of i and the right member is independent of j, the two equal expressions in Eq. (17) must both be equal to a constant c. Setting each member of Eq. (17) equal to this constant c gives the two homogeneous difference equations for f_i and g_j, namely

$$g_{j+1} - cg_j = 0 \tag{18}$$

and

$$rf_{i-1} + (1-2r-c)f_i + rf_{i+1} = 0. \tag{19}$$

The general solution of Eq. (18) is

$$g_j = Ac^j, \tag{20}$$

where A is an arbitrary constant.

Since the solution of Eq. (7) is periodic in x, the solution of Eq. (19) is periodic in i. Thus,

$$f_i = B\cos(i\alpha) + C\sin(i\alpha), \tag{21}$$

where B and C denote arbitrary constants, and

$$\cos\alpha = \frac{2r + c - 1}{2r}. \tag{22}$$

The boundary conditions (9) at $x=0$ and $x=1$, in terms of central differences, lead to

$$\theta_{1,j} = \theta_{-1,j} \quad \text{(for all } j) \tag{23}$$

and

$$\theta_{N+1,j} = \theta_{N-1,j} \quad \text{(for all } j), \tag{24}$$

respectively. Applying Eqs. (23) and (24) to Eq. (16) and utilising Eq. (21) we obtain $C\sin\alpha = 0$ and $B\sin(N\alpha)\sin\alpha - C\sin\alpha\cos(N\alpha) = 0$, respectively. Since we are interested in the non-trivial solution of the problem it follows that $\sin(N\alpha) = 0$ giving $\alpha = s\pi/N$, $s = 0, 1, 2, \ldots$. Therefore, the difference equation (21) takes the form

$$f_i = B\cos\left(\frac{is\pi}{N}\right), \tag{25}$$

and from Eq. (22) we obtain

$$c = 1 - 4r\sin^2\left(\frac{s\pi}{2N}\right). \tag{26}$$

Substitution of Eqs. (20), (25) and (26) in Eq. (16) we obtain

$$\theta_{i,j} = Dc^j \cos\left(\frac{is\pi}{N}\right) \quad (s = 0, 1, 2, \ldots), \tag{27}$$

where $D = AB$.

Since Eq. (14b) is linear in $\theta_{i,j}$, the sum of different solutions is a solution of Eq. (14b). Thus, we form the series

$$\theta_{i,j} = \sum_{s=0}^{\infty} D_s c^j \cos\left(\frac{is\pi}{N}\right), \quad i = 1(1)N-1, \ j = 0(1)J. \tag{28}$$

Applying the initial condition (8) to Eq. (28) yields $\theta_{i,0} = \sum_{s=0}^{\infty} D_s \cos(is\pi/N)$ in which

$$D_0 = \int_0^1 \theta(x,0)\,\mathrm{d}x \quad \text{and} \quad D_s = 2\int_0^1 \theta(x,0)\cos(s\pi x)\,\mathrm{d}x \quad (s = 1, 2, 3, \ldots).$$

Thus, using the Hopf–Cole transformation given by Eq. (6), the exact-explicit finite-difference solution to the nonlinear problem is easily obtained as

$$u(x_i, t_j) = 2\pi v \frac{\sum_{s=1}^{\infty} D_s(1 - 4r\sin^2 s\pi/2N)^j s \sin(s\pi x_i)}{D_0 + \sum_{s=1}^{\infty} D_s(1 - 4r\sin^2 s\pi/2N)^j s\cos(s\pi x_i)}, \quad i = 1(1)N-1, \ j = 0(1)J. \tag{29}$$

It can be shown that when $r = kv/h^2$ the exact-explicit finite-difference solution (29) converges to the Fourier solution as the mesh size tends to zero, for finite values of time t.

4. Numerical results and conclusions

All calculations were performed in double-precision arithmetic on a Pentium 166 processor using Microsoft FORTRAN Compiler. All results are obtained when the coeffiecients of both series are equal to or less than 0.1×10^{-9}.

Tables 1 and 2 display results obtained by explicit solution (15) and exact-explicit finite difference solution (29) of the problem, respectively. It is observed that both results are reasonably in good agreement with the exact solution (13), and exhibit the expected convergence as the mesh size is refined. Table 3 presents the values obtained by applying Richardson's extrapolation to the value of the weighted 1-norm defined by

$$\|e\|_1 = \frac{1}{N}\sum_{i=1}^{N-1}\left|1 - \frac{u_{i,j}}{u(x_i, t_j)}\right|, \quad e = [e_1 \cdots e_{N-1}]^{\mathrm{T}}$$

which gives an approximate rate of convergence of 1.9594 for the explicit method and 1.9536 for the exact-explicit method. Both are consistent with the theoretical expectation of $O(h^2)$. It is also clear from Table 3 that the error in both solutions decreases as N increases.

Table 1
Comparison of the explicit finite-difference solutions with exact solution at $t_f = 0.1$ for $v = 1$ and $\Delta t = 0.00001$

x	Numerical solution				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.10863	0.10931	0.10948	0.10952	0.10954
0.2	0.20805	0.20935	0.20967	0.20975	0.20979
0.3	0.28946	0.29128	0.29173	0.29184	0.29190
0.4	0.34501	0.34719	0.34773	0.34786	0.34792
0.5	0.36845	0.37079	0.37137	0.37151	0.37158
0.6	0.35601	0.35828	0.35884	0.35898	0.35905
0.7	0.30728	0.30924	0.30973	0.30985	0.30991
0.8	0.22588	0.22733	0.22769	0.22778	0.22782
0.9	0.11966	0.12043	0.12062	0.12067	0.12069

Table 2
Comparison of the exact-explicit finite-difference solutions with exact solution at $t_f = 0.1$ for $v = 1$ and $\Delta t = 0.00001$

x	Numerical solution				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.11048	0.10977	0.10959	0.10955	0.10954
0.2	0.21159	0.21023	0.20989	0.20981	0.20979
0.3	0.29435	0.29250	0.29204	0.29192	0.29190
0.4	0.35080	0.34863	0.34809	0.34795	0.34792
0.5	0.37458	0.37232	0.37175	0.37161	0.37158
0.6	0.36189	0.35974	0.35921	0.35907	0.35905
0.7	0.31231	0.31050	0.31004	0.30993	0.30991
0.8	0.22955	0.22825	0.22792	0.22783	0.22782
0.9	0.12160	0.12091	0.12074	0.12070	0.12069

Tables 4 and 5 display explicit and exact-explicit finite-difference solutions for $v = 0.1$ and $v = 0.01$ with $\Delta t = 0.001$ at different times. It is clearly observed that both numerical predictions are reasonably in good agreement with the exact solution. It is seen that for small values of v, one must consider a large value of N to obtain proper solution. To achieve a better accuracy, large values of N and t must be taken since the (exact) Fourier solution fails for small values of v and t [13].

In order to show how good the numerical solutions exhibit the correct physical characteristic of the problem we only give the graph in Fig. 1 which shows the numerical solutions at different times for $v = 1.0$, $h = 0.025$, $k = 0.0001$. The exact solution given by Eq. (13) also is drawn on the same figure, but the graphs can not be distinguished due to the closeness of the numerical solutions to the exact one.

It is also possible to solve Burgers-like problems with different initial and boundary conditions by the above approach so-called the exact-explicit finite-difference method. For example, for the Burgers

Table 3
Values of $\|e\|_1$ for numerical predictions shown in Tables 1 and 2

N	$\|e\|_1$	
	Explicit	Exact-explicit
10	0.007571	0.007278
20	0.002025	0.001885
40	0.000555	0.000448
80	0.000177	0.000077

Table 4
Comparison of the numerical solutions with exact solution at different times for $v=0.1$, $\Delta x=0.025$ and $\Delta t=0.001$

x	t_f	Numerical solution		Exact solution
		Explicit	Exact-explicit	
0.25	0.4	0.30834	0.30891	0.30889
	0.6	0.24039	0.24075	0.24074
	0.8	0.19543	0.19568	0.19568
	1.0	0.16238	0.16257	0.16256
	3.0	0.02718	0.02720	0.02720
0.50	0.4	0.56911	0.56964	0.56963
	0.6	0.44676	0.44721	0.44721
	0.8	0.35888	0.35924	0.35924
	1.0	0.29162	0.29192	0.29192
	3.0	0.04017	0.04021	0.04021
0.75	0.4	0.62555	0.62542	0.62544
	0.6	0.48701	0.48721	0.48721
	0.8	0.37366	0.37392	0.37392
	1.0	0.28723	0.28748	0.28747
	3.0	0.02974	0.02977	0.02977

equation (1) with the boundary conditions (5) and the initial condition

$$u(x,0)=4x(1-x), \quad 0<x<1$$

the exact-explicit finite-difference solution can be easily obtained in the similar way to the previous problem. Obviously, the only marked difference is the initial conditions which is

$$\theta(x,0)=\exp\{-(3-2x)x^2/3v\}, \quad 0<x<1.$$

Explicit and exact-explicit finite difference solutions obtained to the problem for $v=1.0$ (with $\Delta t=0.0001$) and $v=0.01$ (with $\Delta t=0.001$) at different times are displayed in Tables 6 and 7. It is clearly seen that the obtained numerical results are in good agreement with the exact solution. Fig. 2 shows the numerical solutions for $v=0.1$, $h=0.025$, $k=0.01$ at different times which exhibit

Table 5
Comparison of the numerical solutions with exact solution at different times for $\upsilon = 0.01$, $\Delta x = 0.0125$ and $\Delta t = 0.0001$

x	t_f	Numerical solution		Exact solution
		Explicit	Exact-explicit	
0.25	0.4	0.34244	0.34164	0.34191
	0.6	0.26905	0.26890	0.26896
	0.8	0.22145	0.22150	0.22148
	1.0	0.18813	0.18825	0.18819
	3.0	0.07509	0.07515	0.07511
0.50	0.4	0.67152	0.65606	0.66071
	0.6	0.53406	0.52658	0.52942
	0.8	0.44143	0.43743	0.43914
	1.0	0.37568	0.37336	0.37442
	3.0	0.15020	0.15015	0.15018
0.75	0.4	0.94675	0.90111	0.91026
	0.6	0.78474	0.75862	0.76724
	0.8	0.65659	0.64129	0.64740
	1.0	0.56135	0.55187	0.55605
	3.0	0.22502	0.22454	0.22481

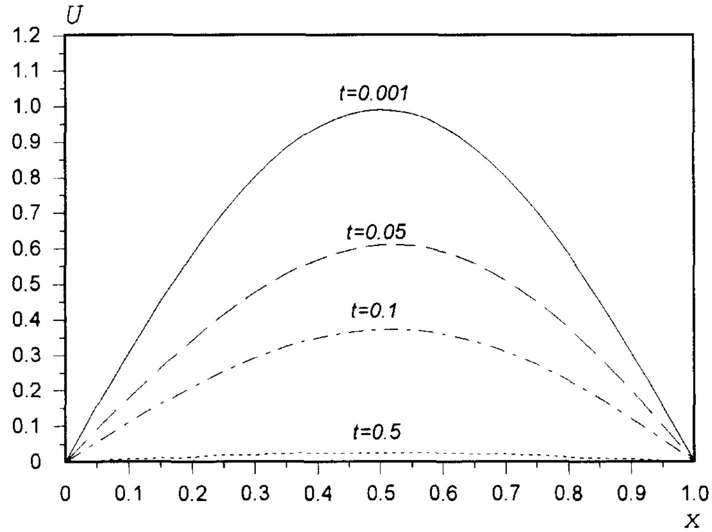

Fig. 1. Solution at different times for $\upsilon = 1.0$, $h = 0.025$, $k = 0.0001$.

the correct physical behaviour of the problem. All solutions are drawn on the same figure since they are very close to each other.

It is observed that in all calculations both solution series of the above Burgers equation with different initial and boundary conditions are used the same number of the terms to get good

Table 6
Comparison of the numerical solutions with exact solution at different times for $v = 1.0$, $\Delta x = 0.025$ and $\Delta t = 0.0001$

x	t_f	Numerical solution		Exact solution
		Explicit	Exact-explicit	
0.25	0.01	0.65915	0.66007	0.66006
	0.05	0.42582	0.42629	0.42629
	0.10	0.26121	0.26149	0.26148
	0.15	0.16132	0.16148	0.16148
	0.25	0.06103	0.06109	0.06109
0.50	0.01	0.91890	0.91972	0.91972
	0.05	0.62745	0.62809	0.62808
	0.10	0.38304	0.38343	0.38342
	1.15	0.23382	0.23406	0.23406
	0.25	0.08715	0.08724	0.08723
0.75	0.01	0.68304	0.68364	0.68364
	0.05	0.46481	0.46526	0.46525
	0.10	0.28129	0.28158	0.28157
	0.15	0.16957	0.16974	0.16974
	0.25	0.06223	0.06229	0.06229

Table 7
Comparison of the numerical solutions with exact solution at different times for $v = 0.01$, $\Delta x = 0.0125$ and $\Delta t = 0.001$

x	t_f	Numerical solution		Exact solution
		Explicit	Exact-explicit	
0.25	0.4	0.36296	0.36185	0.36226
	0.6	0.28217	0.28193	0.28204
	0.8	0.23043	0.23046	0.23045
	1.0	0.19463	0.19474	0.19469
	3.0	0.07611	0.07617	0.07613
0.50	0.4	0.69591	0.67851	0.68368
	0.6	0.55351	0.54508	0.54832
	0.8	0.45625	0.45176	0.45371
	1.0	0.38705	0.38446	0.38568
	3.0	0.15220	0.15215	0.15218
0.75	0.4	0.95925	0.91169	0.92050
	0.6	0.80197	0.77402	0.78299
	0.8	0.67267	0.65617	0.66272
	1.0	0.57501	0.56478	0.56932
	3.0	0.22796	0.22746	0.22774

approximation for $v = 1.0$, $v = 0.1$, and $v = 0.01$. For $v < 0.01$, our solutions show the same behaviour with the exact one of each problem.

In conclusion, since all the numerical results obtained by the above methods show reasonably good agreement with the exact one for modest values of v, and also exhibit the expected convergence as

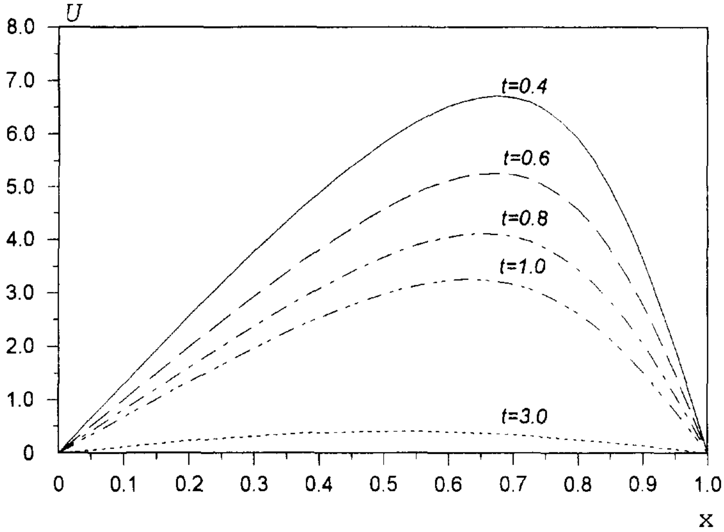

Fig. 2. Solution at different times for $v = 0.1$, $h = 0.025$, $k = 0.001$.

the mesh size is decreased, both methods can therefore be considered to be competitive and worth recommendation. The present solution is an alternative solution to the exact (Fourier) one. But if the initial condition $u_{i,0}$ of a problem is known only at the finite number of the mesh points, for such a problem the present solution method is much more practical than Fourier one.

References

[1] G. Adomian, Explicit solutions of nonlinear partial differential equations, Appl. Math. Comput. 88 (1997) 117–126.
[2] W.F. Ames, Nonlinear Partial Differential Equations in Engineering, Academic Press, New York, 1965.
[3] H. Bateman, Some recent researches on the motion of fluids, Monthly Weather Rev. 43 (1915) 163–170.
[4] E. Benton, G.W. Platzman, A table of solutions of the one-dimensional Burges equations, Quart. Appl. Math. 30 (1972) 195–212.
[5] J.M. Burger, A Mathematical Model Illustrating the Theory of Turbulence, Adv. in Appl. Mech. I, Academic Press, New York, 1948, pp. 171–199.
[6] J. Caldwell, P. Wanless, A.E. Cook, A finite element approach to Burgers' equation, Appl. Math. Modelling 5 (1981) 189–193.
[7] J.D. Cole, On a quaslinear parabolic equations occurring in aerodynamics, Quart. Appl. Math. 9 (1951) 225–236.
[8] D.J. Evans, A.R. Abdullah, The group explicit method for the solution of Burger's equation, Computing 32 (1984) 239–253.
[9] C.A. Fletcher, Burgers equation: a model for all reasons, in: J. Noye (Ed.), Numerical Solutions of Partial Differential Equations, North-Holland, Amsterdam, 1982.
[10] Y.C. Hon, X.Z. Mao, An efficient numerical scheme for Burgers' equation, Appl. Math. Comput. 95 (1998) 37–50.
[11] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math. 3 (1950) 201–230.
[12] V.I. Karpman, Nonlinear Waves in Dispersive Media, Pergamon Press, New York, 1975.
[13] E.L. Miller, Predictor-corrector studies of Burger's model of turbulent flow, M.S. Thesis, University of Delaware, Newark, Delaware, 1966.
[14] R.C. Mittal, P. Sighhal, Numerical Solution of Burger's Equation, Comm. Numer. Meth. Eng. 9 (1993) 397–406.

[15] T. Öziş, A. Özdeş, A direct variational methods applied to Burgers' equation, J. Comput. Appl. Math. 71 (1996) 163–175.

[16] G.D. Smith, Numerical Solution of Partial Differential Equations: Finite Difference Methods, 3rd ed., Clarendon Press, Oxford, 1987.

[17] E. Varoğlu, W.D.L. Finn, Space time finite elements incorporating characteristics for the Burgers' equations, Int. J. Numer. Meth. Eng. 16 (1980) 171–184.