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Frobenius manifolds: natural submanifolds and induced bi-Hamiltonian structures

I.A.B. Strachan

Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK

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Abstract

Submanifolds of Frobenius manifolds are studied. In particular, so-called natural submanifolds are defined and, for semi-simple Frobenius manifolds, classified. These carry the structure of a Frobenius algebra on each tangent space, but will, in general, be curved. The induced curvature is studied, a main result being that these natural submanifolds carry a induced pencil of compatible metrics. It is then shown how one may constrain the bi-Hamiltonian hierarchies associated to a Frobenius manifold to live on these natural submanifolds whilst retaining their, now non-local, bi-Hamiltonian structure.

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1. Introduction

The study of the structures induced on a submanifold and their relationship to the ambient manifold is one of the oldest problems in differential geometry. The aim of this paper is to study the properties of submanifolds of Frobenius manifolds. Frobenius manifolds have a particularly rich structure, and have their origin, as well as applications, in a wide range of seemingly disparate areas of mathematics such as:

- topological quantum field theory;
- algebraic/enumerative geometry and quantum cohomology;
- singularity theory;
- integrable systems.

E-mail address: i.strachan@maths.gla.ac.uk (I.A.B. Strachan).

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The emphasis in this paper will be on the purely geometric properties of submanifolds and their application within the theory of integrable systems. This will draw on ideas from singularity theory, in particular properties of discriminants and caustics. Whether or not the ideas are of relevance to the other areas is an open question.

The key property of a Frobenius manifold is the existence of a Frobenius algebra on each tangent space to the manifold:

Definition 1.1. An algebra $(\mathcal{A}, \circ, \langle , \rangle)$ over $\mathbb{C}$ is a Frobenius algebra if:

- the algebra $\{\mathcal{A}, \circ\}$ is commutative, associative with unity e;
- the multiplication is compatible with a $\mathbb{C}$-valued bilinear, symmetric, nondegenerate inner product
$$\langle , \rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$$
in the sense that
$$\langle a \circ b, c\rangle = \langle a, b \circ c\rangle$$
for all $a, b, c \in \mathcal{A}$.

With this structure one may defined a Frobenius manifold [4]:

Definition 1.2. $(M, \circ, e, \langle , \rangle, E)$ is a Frobenius manifold if each tangent space T_pM is a Frobenius algebra varying smoothly over M with the additional properties:

- the inner product is a flat metric on M (the term 'metric' will denote a complex-valued quadratic form on M);
- $\nabla e = 0$, where ∇ is the Levi-Civita connection of the metric;
- the tensor $(\nabla_W c)(X, Y, Z)$ is totally symmetric for all vectors $W, X, Y, Z \in TM$;
- a vector field E must be determined such that
$$\nabla(\nabla E) = 0$$
and that the corresponding one-parameter group of diffeomorphisms acts by conformal transformations of the metric and by rescalings on the Frobenius algebras T_pM.

It is immediately apparent that an arbitrary submanifold $N \subset M$ of a Frobenius manifold will not be a Frobenius manifold, as the induced metric will in general be curved. Moreover, the induced multiplication on the subtangent space $T_pN \subset T_pM$, $p \in N$ will not be, in general, associative. Rather than develop a full structural theory for submanifolds—which could easily be done—only so-called natural submanifolds will be studied. On such submanifolds the induced multiplication is associative and compatible with the induced metric. For semi-simple Frobenius manifolds such natural submanifolds may be classified. The simplest example comes from the Frobenius manifold constructed from the Coxeter group A_3. Here the natural submanifolds are the swallow-tail discriminant, the cylinder over the semi-cubical caustic and the planar Maxwell set.

The motivation for studying submanifolds came from two main examples, more detail of which are given below. One of the best understood classes of Frobenius manifolds come from the unfolding of the

A_n singularity [4], $z^{n+1} \mapsto z^{n+1} + a_1 z^{n-1} + \cdots + a_n = p(z)$. This derivation assumes that the roots of $p'(z) = 0$ are distinct. However without this assumption, i.e., with multiple roots, much of the structure of a Frobenius manifold remains—one has a semi-simple Frobenius algebra on each tangent space, compatible with an Euler vector field, and having a covariantly constant identity vector field. However the metric is no longer, in general, flat (a similar question was raised in [13, III.7.1]). This manifold of multiple roots should be thought of a submanifold (in fact a caustic) of the original Frobenius manifold. The details of this constitute Main Example A below. The second motivation came from studying systems of hydrodynamic type associated with Toda/Benney hierarchies [9,16], the simplest being

$$u_T = u v_X,$$
$$v_T = v u_X$$

which is just the familiar dispersionless Toda equation (and hence related to the quantum cohomology of $\mathbb{C}P^1$). An obvious reduction of this system is to constrain the system to the submanifold $u - v = 0$, reducing the system to the Riemann equation $u_T = u u_X$. Such submanifolds are clearly very special and should be thought off a submanifold (in fact a discriminant) of the original Frobenius manifold. The details of this constitute Main Example B below.

Main Example A [4]. Consider the space M of complex polynomials

$$p(z) = z^{m+1} + a_1 z^{m-1} + \cdots + a_m.$$

Such a space carries the structure of a Frobenius manifold, associated with the Coxeter group A_m. Tangent vector to M take the form

$$\dot p(z) = \dot a_1 z^{m-1} + \cdots + \dot a_m,$$

and the algebra on the tangent space is

$$A_p = \mathbf{C}[z]/p'(z)$$

and the inner product is

$$\langle f, g\rangle_p = \operatorname*{res}_{z=\infty} \left\{ \frac{f(z)g(z)}{p'(z)} \right\}. \tag{1.1}$$

In terms of canonical coordinates $u^i = p(\alpha_i)$ where the α_i are (distinct) roots of $p'(z) = 0$ the metric becomes diagonal and Egoroff:

$$g = \sum \frac{1}{p''(\alpha_i)} \left(du^i\right)^2, \tag{1.2}$$

$$\frac{1}{p''(\alpha_i)} = \frac{1}{m+1} \frac{\partial a_1}{\partial u^i}. \tag{1.3}$$

Note that this all assumes that the roots of $p'(z) = 0$ are distinct. In what follows no such assumption will be assumed. The generic features will remain—the metric will remain diagonal and Egoroff. The result will be incorporated in a more general scheme which will be developed over the subsequent sections. One may regard the manifold N with repeated roots as a submanifold (in fact a caustic) in M.

Suppose that

$$p'(z)=(m+1)\prod_{i=1}^{n}(z-\alpha_i)^{k_i}$$

where $k_i\geqslant 1$, $\sum_{i=1}^n k_i=m$, $\sum_{i=1}^n k_i\alpha_i=0$. Coordinates on N are defined by $\tau^i=p(\alpha_i)$. It follows immediately that

$$\delta_{ij}=\left.\frac{\partial p}{\partial\tau^j}\right|_{z=\alpha_i},$$
$$0=\left.\frac{d^k}{dz^k}\frac{\partial p}{\partial\tau^j}\right|_{z=\alpha_i},\quad 1\leqslant k\leqslant k_i-1.$$

A simple parameter counts yields m equations for the m unknowns in $\frac{\partial p}{\partial\tau^j}$. This gives two different forms for $\frac{\partial p}{\partial\tau^j}$:

$$\frac{\partial p}{\partial\tau^j}=p^{(j)}(z)\prod_{i\neq j}(z-\alpha_i)^{k_i}\tag{1.4}$$
$$=1+\sum_{k=k_j}^{m-1}\frac{1}{k!}\frac{d^k}{dz^k}\left.\left(\frac{\partial p}{\partial\tau^j}\right)\right|_{z=\alpha_j}.(z-\alpha_j)^k,\tag{1.5}$$

where p_j is a polynomial of degree k_j-1 with $p_j(\alpha_j)\neq 0$. The metric on N is given by

$$g_{ij}=\operatorname*{res}_{z=\infty}\left\{\frac{\frac{\partial p}{\partial\tau^i}\frac{\partial p}{\partial\tau^j}}{p'}\,dz\right\}$$

and using (1.4) gives immediately that the metric is diagonal and, on using (1.5), that

$$g_{ii}=\operatorname*{res}_{z=\alpha_i}\left\{\frac{1}{p'(z)}\right\}.$$

In the case of simple poles this reduces to (1.2). An immediate corollary of this is that

$$\sum_{i=1}^{n}g_{ii}=\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{dz}{p'(z)}=0,\tag{1.6}$$

where $\mathcal{C}$ is a large contour containing all of the α_i. To show that this metric is Egoroff is considerable more involved, even though the final result is simple. A more geometric proof will be given below, here it will be derived by direct calculation.

Let

$$h^{(i)}=\prod_{r\neq i}(z-\alpha_r)^{k_r}=\sum_{s=0}^{m-k_i}h_s^{(i)}\frac{(z-\alpha_i)^s}{s!}$$

and define the coefficients $h_s^{(i,-1)}$ by the inverse series

$$\left(h^{(i)}\right)^{-1}=\sum_{s=0}^{\infty}h_s^{(i,-1)}\frac{(z-\alpha_i)^s}{s!}.$$

Since $h_0^{(i)} = \prod_{r\neq i}(\alpha_i - \alpha_r)^{k_r} \neq 0$ these coefficients are uniquely defined. To proceed further one requires the following:

Lemma 1.3. *Consider the expansion of $p^{(i)}(z)$ around $z = \alpha_i$, and let*

$$p_r^{(i)} = \left.\frac{d^r p^{(i)}}{dz^r}\right|_{z=\alpha_i}$$

then $p_r^{(i)} = h_r^{(i,-1)}$ for $r = 0, \dots, k_i - 1$.

This is proved by showing that the linear equations for the $h_r^{(i,-1)}$ and the $p_r^{(i)}$ are identical. In the case of simple zeros this result is immediate. Note that the explicit form of these coefficients is not required, just their equality. Then

$$\begin{aligned}
g_{ii} &= \operatorname*{res}_{z=\alpha_i} \frac{1}{p'(z)} \\
&= \frac{1}{m+1} \operatorname*{res}_{z=\alpha_i} \frac{1}{(z-\alpha_i)^{k_i}} \big(h^{(i)}\big)^{-1} \\
&= \frac{1}{m+1} \operatorname*{res}_{z=\alpha_i} \frac{1}{(z-\alpha_i)^{k_i}} \sum_{s=0}^{\infty} h_s^{(i,-1)} \frac{(z-\alpha_i)^s}{s!} \\
&= \frac{1}{m+1} \frac{h_{k_i-1}^{(i,-1)}}{(k_i-1)!} \\
&= \frac{1}{m+1} \frac{p_{k_i-1}^{(i)}}{(k_i-1)!} \\
&= \frac{1}{m+1} \quad \text{coefficient of } z^{m-1} \text{ in expansion of } \frac{\partial p}{\partial \tau^i} \\
&= \frac{1}{m+1} \frac{\partial a_1}{\partial \tau^i}.
\end{aligned}$$

Hence the metric is Egoroff. Other properties may be similarly derived; N carries an Euler vector field and a covariantly constant unity vector field e (this following from (1.6)).

Main Example B. The multicomponent Toda hierarchy is defined in terms of a Lax function

$$\mathcal{L}(z) = z^{M-1} + \sum_{i=-1}^{M-2} z^i S^i(X, \mathbf{T}), \qquad \mathbf{T} = \{T_1, T_2, \dots\}$$

by the Lax equation

$$\frac{\partial \mathcal{L}}{\partial T_n} = \left\{ \left(\mathcal{L}^{\frac{n}{M-1}}\right)_+, \mathcal{L} \right\}. \tag{1.7}$$

Here the bracket is defined by the formula

$$\{f, g\} = z \frac{\partial f}{\partial z} \frac{\partial g}{\partial X} - z \frac{\partial f}{\partial X} \frac{\partial g}{\partial z}$$

and $(\mathcal{O})_+$ denotes the projection of the function $\mathcal{O}$ onto non-negative powers of p. For example, one obtains from the Lax equation (1.7) with $M=2, n=1$ the system (where $\mathcal{L}=z+S+Pz^{-1}$)

$$\begin{aligned} S_T &= P_X, \\ P_T &= P S_X \end{aligned} \tag{1.8}$$

and, with $M=3, n=1$ the system (where $\mathcal{L}=z^2+Sz+P+Qz^{-1}$)

$$\begin{aligned} S_T &= P_X - \frac{1}{2} S S_X, \\ P_T &= Q_X, \\ Q_T &= \frac{1}{2} Q S_X. \end{aligned} \tag{1.9}$$

A change of dependent variables from the $\{S^i(X,\mathbf{T})\}$ to so-called modified variables $\{v^i(X,\mathbf{T})\}$ defined by a factorization of the Lax equation

$$\mathcal{L} = \frac{1}{z}\prod_{i=1}^{N}\bigl[z+v_i(X,\mathbf{T})\bigr]$$

provides an extremely useful computational tool in the study of the Toda hierarchy [9,16]. Quantities such as

$$Q^{(n)} = \frac{1}{2\pi i}\oint \mathcal{L}^{\frac{n}{m-1}}\frac{dz}{z}$$

which are conserved with respect to the evolutions defined by the Lax equation (1.7) may be evaluated for all values of M and n in terms of a simple combinatorial formula

$$Q^{(n)} = \sum_{\{r_i:\, \sum_{i=1}^{M} r_i = n\}} \left\{ \prod_{i=1}^{M} \binom{\frac{n}{M-1}}{r_i} v_i^{r_i} \right\},$$

and similar formulae exist for the evolution equations themselves. Thus the modified variables enables one to perform the general calculations with an arbitrary numbers of fields with little increase in complexity. The geometrical significance of these variables is that they are basically the flat coordinates for the intersection form of the underlying Frobenius manifold, or equivalently, flat coordinates for the second Hamiltonian structure. This Hamiltonian structure is defined by the manifestly flat metric

$$\mathbf{g} = \sum_{i\neq j} \frac{dv_i}{v_i}\frac{dv_j}{v_j}$$

(so the actual flat coordinates are $\tilde{v}_i = \log v_i$).

Consider (1.8) written in terms of these modified variables

$$\begin{aligned} u_T &= u v_X, \\ v_T &= v u_X \end{aligned}$$

where $S = u + v$ and $P = uv$. Similarly the 3-component system (1.9) transforms to

$$\begin{aligned} u_T &= u(-u_X + v_X + w_X), \\ v_T &= v(+u_X - v_X + w_X), \\ w_T &= w(+u_X + v_X - w_X), \end{aligned}$$

where $S = u + v + w$, $P = uv + vw + wu$ and $Q = uvw$. It is clear from the symmetric form of these equations that one possible reduction is to constrain the systems onto the surface given by the constraint $u - v = 0$. In terms of the original variables this corresponds to the constraint $S^2 - 4P = 0$ and in the second this corresponds to the constraint

$$4P^3 + 27Q^2 - 18PQS - P^2S^2 + 4QS^3 = 0,$$

these being the condition for the corresponding polynomial equation $\mathcal{L}(z) = 0$ to have a double root. Clearly these ideas generalize to an arbitrary number of fields and arbitrary multiple roots. Geometrically one is constraining an M dimensional system onto a N dimensional submanifold. Note that these reductions are far easier to study using these modified variables. All these results generalize to rational Lax equations in an entirely analogous fashion.

These two examples provided the motivation for the study of submanifolds. The key property possessed by the submanifolds in both these examples is the commutative, associative and quasihomogeneous multiplication on the subspace's tangent bundle. Submanifolds with such induced structures will be referred to as 'natural' submanifolds. These examples also have an induced identity vector field on the submanifolds, and hence one has a Frobenius algebra on each tangent space of the submanifold. This paper is a much extended version of the paper [18].

2. F-manifolds and their natural submanifolds

The definition of a Frobenius manifold consists of a large number of intermeshing parts, and it is perhaps difficult to see which components are the most important. One point of view, coming via singularity theory, is that it is the multiplication and the Euler vector field which are the central elements; the existence of a compatible flat metric being, for example, derived results. This point of view is encapsulated in the weaker notion of an F-manifold [10,11,13]. Here one has a commutative, associative multiplication with a single additional property, automatically satisfied in the case of Frobenius manifolds. Starting with an F-manifold one may gradually add additional structures and compatibility conditions until one obtains a Frobenius manifold. This has the advantage that one may see on what structures the various compatibility conditions depend.

A similar approach will be taken here for submanifolds. One may defined a natural submanifold N of an F-manifold M by requiring that $TN \circ TN \subset TN$. As one adds various structures and compatibility conditions onto M one can also study the induced structures on N and the failure, of otherwise, of the associated compatibility conditions. The various results in this section are formulated with this approach in mind, even though the main aim is to study natural submanifolds of Frobenius manifolds. The results in this section are mainly algebraic; curvature properties being studied in Section 3.

Definition 2.1 [11,13]. An F-manifold is a pair $(M,\circ)$ where M is a manifold and $\circ$ is a commutative, associative multiplication $\circ : TM\times TM\to TM$ satisfying the following condition:

$$Lie_{X\circ Y}(\circ)=X\circ Lie_Y(\circ)+Y\circ Lie_X(\circ),\quad \forall X,Y\in TM. \tag{2.1}$$

Expanding the definition yields the equivalent condition

$$\begin{aligned}&[X\circ Y,Z\circ W]-[X\circ Y,Z]\circ W-[X\circ Y,W]\circ Z\\&\quad -X\circ[Y,Z\circ W]+X\circ[Y,Z]\circ W+X\circ[Y,W]\circ Z\\&\quad -Y\circ[X,Z\circ W]+Y\circ[X,Z]\circ W+Y\circ[X,W]\circ Z=0\end{aligned}$$

for all $W,X,Y,Z\in TM$. To such a manifold one may add various structures, demanding that they are compatible with the multiplication.

Definition 2.2. (a) An F_E manifold is an F-manifold with an Euler field of weight d. This is a global vector field satisfying the condition

$$Lie_E(\circ)=d\cdot\circ. \tag{2.2}$$

(b) An F_g manifold is an F-manifold with a metric $\langle,\rangle$ compatible with the multiplication:

$$\langle X\circ Y,Z\rangle=\langle X,Y\circ Z\rangle,\quad X,Y,Z\in TM. \tag{2.3}$$

(c) An $\mathcal{F}$ manifold is both an F_E and an F_g manifold, with the E and g related by the relation

$$Lie_E\langle,\rangle=D\langle,\rangle \tag{2.4}$$

for some constant D.

Expanding definition (2.2) yields the equivalent condition

$$[E,X\circ Y]-[E,X]\circ Y-X\circ[E,Y]-d\cdot X\circ Y=0$$

for all $X,Y\in TM$, and (2.4) yields the equivalent condition

$$E\langle X,Y\rangle-\big\langle[E,X],Y\big\rangle-\big\langle X,[E,Y]\big\rangle=D\langle X,Y\rangle$$

for all $X,Y\in TM$.

The following definition of a natural submanifold will play a central role in this paper.

Definition 2.3. A natural submanifold N of an F_E manifold $(M,\circ,E)$ is a submanifold $N\subset M$ such that:

(a) $TN\circ TN\subset TN$,
(b) $E_x\in TN$ for all $x\in N$.

One could clearly define the notion of a natural submanifold of an F-manifold by ignoring the second condition. An immediate consequence of this definition is the following basic result, the proof of which follows from the fact that if $X,Y\in TN$ then $[X,Y]\in TN$:

Lemma 2.4. *All natural submanifolds of an F_E manifold are F_E manifolds with respect to the natural induced structures.*

Example 2.5 (*Massive F-manifolds*). Given a semi-simple F_E manifold $(M, \circ, E)$ the tangent space T_pM at a generic point decomposes into one-dimensional algebras with

$$\delta_i \circ \delta_j = \delta_{ij}\delta_i$$

(so δ_i are the idempotents of the algebra on T_pM). The F manifold condition ensures that these vector fields commute $[\delta_i, \delta_j] = 0$ and hence provide a canonical coordinate system $\{u^i\}$ with

$$\partial_i = \frac{\partial}{\partial u^i}.$$

In this basis one has then:

$$\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \delta_{ij}\frac{\partial}{\partial u^i}, \quad i, j = 1, \ldots, m = \dim M,$$
$$E = \sum_{i=1}^{\dim M} u^i \frac{\partial}{\partial u^i}.$$

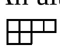
Then the submanifolds defined by the level sets

$$\underbrace{\{u^i = 0,\ i \in \mathcal{D}\}}_{\text{discriminant hypersurfaces}} \cap \underbrace{\{u^i - u^j = 0,\ (i, j) \in \mathcal{C}\}}_{\text{caustic hypersurfaces}}$$

are natural F_E manifolds. Here $\mathcal{D}$ and $\mathcal{C}$ are arbitrary subsets of I and $I \times I$ where $I = \{1, \ldots, \dim M\}$. This example will turn out to be canonical for natural submanifolds of semi-simple Frobenius manifolds.

Relabeling the coordinates gives the following parametrization of a natural submanifold:

$$\left(u^1, \ldots, u^m\right) = \big(\underbrace{\tau^1, \ldots, \tau^1}_{k_1}; \ldots; \underbrace{\tau^n, \ldots, \tau^n}_{k_n}; \underbrace{0, \ldots, 0}_{m-(k_1+\cdots+k_n)}\big).$$

In what follows the notation $(k_1, \ldots, k_n, 0)$ will be used to denote a particular submanifold, so the original manifold would just be $(1, 1, \ldots, 1)$. An alternative notation is to use a Young tableau, so, for example, the $(4, 2)$ caustic would be denoted . The terms pure discriminant will refer to a submanifold where $k_i = 1$ and a pure caustic will refer to a submanifold where $\sum k_i = m$. For $\dim M = 2$ the only possibilities are (the notation $M \to N$ means that N is a natural codimension one submanifold of M):

$$\begin{tikzcd} & \{1,1\} \arrow[dl] \arrow[dr] & \\ \{1,0\} & & \{2\} \end{tikzcd}$$

For $\dim M = 3$ one obtains the following strata of nested submanifolds:

$$\begin{tikzcd}
\{1,1,1\} \arrow[d] \arrow[dr] & & \\
\{1,1,0\} \arrow[d] \arrow[dr] & \{2,1\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{1,0,0\} & \{2,0\} & \{3\}
\end{tikzcd}$$

For $\dim M > 3$ such diagrams become considerably more complicated, the number of such submanifolds being $\sum_{n=1}^{m} \mu(n)$, where $\mu(n)$ is the number of partitions of n.

Suppose now one has an F_g-manifold. Then on any (non-null) submanifold N one may define an induced metric g_N and also an induced product $\star : TN \times TN \to TN$ where $\star$ is defined by

$$X \star Y = pr(X \circ Y) \quad \forall X, Y \in T_x N \subset T_x M,$$

where *pr* denotes the projection (using the original metric g on M) of $u \circ v \in T_x M$ onto $T_x N$ (Fig. 1). This induced multiplication may have very different algebraic properties than those of its progenitor. However the induced metric and multiplication remain compatible.

Lemma 2.6. *The induced structures satisfy the condition*

$$\langle X \star Y, Z \rangle = \langle X, Y \star Z \rangle \quad \forall X, Y, Z, \in T_x N.$$

The proof following immediately from the definitions. Putting these results together gives the following:

Proposition 2.7. *Any natural submanifold of an $\mathcal{F}$-manifold is an $\mathcal{F}$-manifold with respect to the naturally induced structures.*

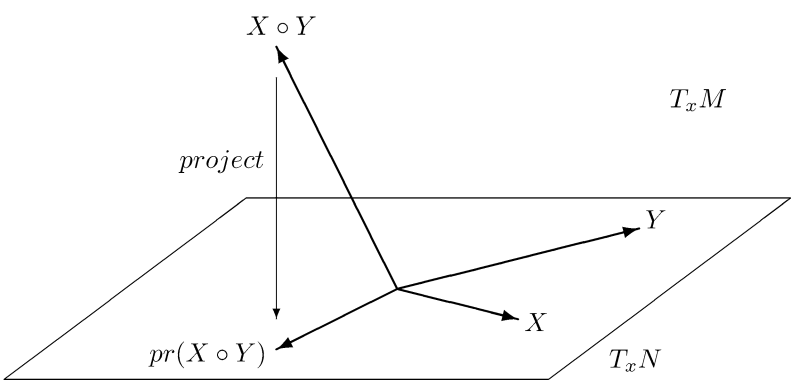


Fig. 1. The definition of the induced multiplication.

One note of caution though: this is a formal result—it may be the case that the induced metric is not defined or is degenerate on a specific natural submanifold.

The definition of a natural submanifold just uses the multiplication and the Euler vector field. If one has, in addition, an identity vector field then a natural submanifold will inherit an induced identity field:

Lemma 2.8. *Let $(M, \circ, g)$ be an F_g manifold with a unity vector field e and let N be a natural submanifold of M. Then N possesses an induced identity vector field.*

Proof. (Note, this lemma only uses part (a) of the definition of a natural submanifold.) Using the metric, one has an orthogonal decomposition (assuming the induced metric on N is not degenerate) of the tangent space T_xM (at points $x \in N$):

$$T_xM \cong (T_xN) \oplus (T_xN)^{\perp}, \quad x \in N, \tag{2.5}$$

so e decomposes as $e = e^{\top} + e^{\perp}$. Hence

$$X \circ e^{\perp} = X - X \circ e^{\top} \in T_xN.$$

Clearly $\langle X \circ e^{\perp}, n\rangle = 0$ for all $n \in (T_xN)^{\perp}$ and

$$\bigl\langle X \circ e^{\perp}, Y\bigr\rangle = \bigl\langle X \circ Y, e^{\perp}\bigr\rangle = 0$$

for all $Y \in T_xN$, using the invariance property of the multiplication. Thus $X \circ e^{\perp} = 0$ and hence $X \circ e^{\top} = X$ for all $X \in T_xN$. $\square$

An immediate corollary of this is:

Corollary 2.9. *Let M be a Frobenius manifold and let $N \subset M$ be a natural submanifold. Then each tangent space T_xN carries the structure of a Frobenius algebra with respect to the induced structures.*

For a semi-simple $\mathcal{F}$-manifold one may classify all natural submanifolds, at least formally. The idea is to describe an arbitrary submanifold as the intersection of level sets, $N = \bigcap\{\phi^{\tilde{\alpha}} = 0\}$, the geometric conditions on N to be a natural submanifold then reduce to a simple set of overdetermined partial differential equations for the functions $\phi^{\tilde{\alpha}}$ which may be solved.

Theorem 2.10. *Let $\{M, \circ, E, g\}$ be a semi-simple $\mathcal{F}$ manifold. Then*:

(a) *the only natural submanifolds are those given in the above example*;
(b) *the identity field is tangential to a natural submanifold if and only if it is a pure caustic.*

Proof. (a) Let $\imath : N \to M$ be the inclusion of a submanifold N in the manifold M. Vector fields on N may be pushed-forward to vector fields on M. Adopting a parametrization of the submanifold N, so $u^i = u^i(\tau^{\alpha})$, where $i = 1, \ldots, m$, $\alpha = 1, \ldots, n$, one obtains

$$\imath_{\star} : TN \to TM,$$
$$\imath_{\star}\left(\frac{\partial}{\partial \tau^{\alpha}}\right) = \frac{\partial u^i}{\partial \tau^{\alpha}} \frac{\partial}{\partial u^i}.$$

Similarly [17], using the orthogonal decomposition (2.5) (assuming the induced metric on N is not degenerate):

$$\frac{\partial}{\partial u^i} = A_i^\alpha \frac{\partial}{\partial \tau^\alpha} + n_i^{\tilde\alpha} \frac{\partial}{\partial \nu^{\tilde\alpha}} \tag{2.6}$$

where $\operatorname{span}(\partial_\nu) = (T_x N)^\perp$.

Consider now

$$\frac{\partial}{\partial \tau^\alpha} \circ \frac{\partial}{\partial \tau^\beta} = \frac{\partial u^i}{\partial \tau^\alpha}\frac{\partial u^j}{\partial \tau^\beta}\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \sum_{i=1}^m \frac{\partial u^i}{\partial \tau^\alpha}\frac{\partial u^i}{\partial \tau^\beta}\left(A_i^\alpha \frac{\partial}{\partial \tau^\alpha} + n_i^{\tilde\alpha} \frac{\partial}{\partial \nu^{\tilde\alpha}}\right),$$

on using the canonical multiplication. To ensure that $TN \circ TN \subset TN$ one must have

$$\Xi_{\alpha\beta}^{\tilde\alpha} = 0 \tag{2.7}$$

where

$$\Xi_{\alpha\beta}^{\tilde\alpha} = \sum_{i=1}^m \frac{\partial u^i}{\partial \tau^\alpha}\frac{\partial u^i}{\partial \tau^\beta} n_i^{\tilde\alpha}.$$

To proceed further one adopts a Monge parametrization of N so

$$u^i = \tau^i, \quad i = 1, \dots, n,$$
$$u^{n+\tilde\alpha} = h^{\tilde\alpha}(\tau^\alpha), \quad \tilde\alpha = 1, \dots, m-n.$$

With this N may be described as the intersection of level sets

$$N = \bigcap_{\tilde\alpha=1}^{m-n} \{\phi^{\tilde\alpha} = 0\}$$

where $\phi^{\tilde\alpha} = h^{\tilde\alpha} - u^{n+\tilde\alpha}$. This may be used to find the normal vectors $n_i^{\tilde\alpha}$ with which the condition $\Xi_{\alpha\beta}^{\tilde\alpha} = 0$ become

$$\delta_{\alpha\beta}\frac{\partial h^{\tilde\alpha}}{\partial \tau^\alpha} = \frac{\partial h^{\tilde\alpha}}{\partial \tau^\alpha}\frac{\partial h^{\tilde\alpha}}{\partial \tau^\beta}, \quad \alpha, \beta = 1, \dots, n,\ \tilde\alpha = 1, \dots, m-n.$$

If $\alpha = \beta$ then $h_\alpha^{\tilde\alpha} = 0$ or 1. But if $\alpha \neq \beta$ then $h_\alpha^{\tilde\alpha} h_\beta^{\tilde\alpha} = 0$ which implies that $h_\alpha^{\tilde\alpha} = 0$ except, possibly, for one values of $\alpha \in \{1, \dots, n\}$. Such a value will be denoted $\pi(\tilde\alpha)$. Hence there are two possibilities:

$$h^{\tilde\alpha} = a^{\tilde\alpha}, \quad h^{\tilde\alpha} = u^{\pi(\tilde\alpha)} + b^{\tilde\alpha}$$

for arbitrary constants $a^{\tilde\alpha}, b^{\tilde\alpha}$. Note, if one was to consider semi-simple F_g manifolds, then the classification of natural submanifolds would stop here.

For N to be a natural submanifold requires the further condition $E_x \in T_x N$,

$$E_x = \sum_{i=1}^m u^i \frac{\partial}{\partial u^i} = \sum_{i=1}^m u^i \left(A_i^\alpha \frac{\partial}{\partial \tau^\alpha} + n_i^{\tilde\alpha} \frac{\partial}{\partial \nu^{\tilde\alpha}}\right).$$

Thus $(E_x)^\perp = 0$ implies, using this parametrization, that

$$\sum_{i=1}^m u^i \frac{\partial}{\partial u^i} h^{\tilde\alpha} = h^{\tilde\alpha},$$

so the $h^{\tilde{\alpha}}$ must be homogeneous functions of degree 1. Hence $a^{\tilde{\alpha}} = b^{\tilde{\alpha}} = 0$. Thus

$$h^{\tilde{\alpha}} = 0, \quad h^{\tilde{\alpha}} = u^{\pi(\tilde{\alpha})}. \tag{2.8}$$

On renaming the coordinates one arrives at the examples described above.

(b) Note that, as a consequence of semi-simplicity, there exists a unity vector field

$$e = \sum_{i=1}^{m} \frac{\partial}{\partial u^i}$$

with the property that $e \circ X = X$ for all $X \in TM$. Similarly

$$e_N = \sum_{\alpha=1}^{n} \frac{\partial}{\partial \tau^\alpha}$$

will be a unity vector field on TN. Consider now the restriction of e to a natural submanifold. Using the above formulae, and in particular (2.8), it is straightforward to show that

$$\sum_{i=1}^{m} \frac{\partial}{\partial u^i} = \sum_{\alpha=1}^{n} \frac{\partial}{\partial \tau^\alpha} - \sum_{\tilde{\alpha}=1}^{m-n} \left\{ 1 - \sum_{j=1}^{n} \frac{\partial h^{\tilde{\alpha}}}{\partial \tau^i} \right\} \frac{\partial}{\partial \nu^{\tilde{\alpha}}}.$$

Hence $e^\perp = 0$ if and only if

$$\sum_{j=1}^{n} \frac{\partial h^{\tilde{\alpha}}}{\partial \tau^i} = 1 \quad \forall \tilde{\alpha} = 1, \ldots, m-n,$$

that is, using (2.8), if and only if N is a pure caustic. □

In the next section curvature properties of natural submanifolds will be examined.

3. Frobenius manifolds and the curvature properties of natural submanifolds

Given an $\mathcal{F}$-manifold one may define the following tensors, $c(X, Y, Z) = g(X \circ Y, Z)$, which, from (2.3), is a totally symmetric $(3, 0)$ tensor, $\nabla\circ$ and ∇c. The following theorem is due to Hertling [10]:

Theorem 3.1. *Let* $(M, \circ, \nabla)$ *be a manifold* M *with a commutative associative multiplication* $\circ$ *on* TM *and with a torsion free connection* ∇. *By definition,* $\nabla \circ (X, Y, Z)$ *is symmetric in* Y *and* Z. *If the* $(3, 1)$-*tensor* $\nabla\circ$ *is symmetric in all three arguments, then the multiplication satisfies for any local vector fields* X *and* Y

$$Lie_{X\circ Y}(\circ) = X \circ Lie_Y(\circ) + Y \circ Lie_X(\circ).$$

The converse, however, is false; one requires the properties of a unity vector field. So far little mention has been made of the possibility of having a unity vector field e on M, i.e., $e \in TM$ such that $e \circ X = X, \forall X \in TM$. Such fields play an important role in Frobenius and $\mathcal{F}$ manifolds, as it connects the metric and the multiplication since $\langle X, Y \rangle = c(X, Y, e)$. With this field one may prove the following, again due to Hertling [10]:

Theorem 3.2. *Let $(M, \circ, e, g)$ be a manifold with a commutative and associative multiplication $\circ$ on TM, a unit field e, and a metric $\langle , \rangle$ on TM which is multiplication invariant* (2.3). *∇ denotes the Levi-Civita connection to the metric. The coidentity ε is the 1-form defined by $\varepsilon(X) = \langle X, e\rangle$. The following conditions are equivalent*:

(i) *$(M, \circ, e)$ is an F manifold and ε is closed*;
(ii) *the $(4, 0)$ tensor ∇c is totally symmetric*;
(iii) *the $(3, 1)$ tensor $\nabla\circ$ is totally symmetric.*

The property that ∇c is a totally symmetric $(4, 0)$ tensor is sometimes referred to as quasi-potentially conditions since, if the metric is flat, one may integrate the equations to give c in terms of derivatives of a prepotential F, i.e.,

$$c(X, Y, Z) = XYZ(F).$$

Definition 3.3. A Frobenius manifold is an $\mathcal{F}$ manifold $(M, \circ, E, g)$ endowed with a unity vector field e and satisfying the following conditions:

(a) g is flat;
(b) $d\varepsilon = 0$ where $\varepsilon(\cdot) = \langle e, \cdot\rangle$;
(c) $Lie_e\langle , \rangle = 0$;
(d) $\Delta(U, V) = 0$,

where $\Delta(U, V) = \nabla_U\nabla_V E - \nabla_{\nabla_U V} E$, and ∇ is the Levi-Civita connection.

This definition differs somewhat from the conventional one given above. Conditions (b) and (c) together imply that $\nabla e = 0$ By the above theorem, condition (b) implies that ∇c is totally symmetric, and condition (a) then implies that there exists a prepotential. Condition (d) perhaps needs a little explanation. If U and V are flat vector fields the $\Delta(U, V) = \nabla_U\nabla_V E$. Alternatively, $\Delta^k_{ij} = \nabla_i\nabla_j E^k$ in terms of the flat coordinate system.

To proceed further in the study of these natural submanifolds one must study the curvature of the induced metric and its relation with the various other structures.

A powerful in the study of Frobenius manifolds is the extended connection $\widetilde{\nabla}$ on $M \times \mathbf{P}^1$ defined by

$$\widetilde{\nabla}_U = \nabla_U + zU\circ,$$
$$\widetilde{\nabla}_{z\frac{d}{dz}} = z\frac{d}{dz} + zE\circ - \nu,$$

where $\nu(U) = \frac{D}{2}U - \nabla_U E$ (so $\langle X, \nu(Y)\rangle + \langle \nu(X), Y\rangle = 0$). The vanishing of the curvature of this extended connection is then equivalent to the above definition of a Frobenius manifold.

Since the pull-back of any totally symmetric $(r, 0)$ tensor from M to a submanifold N remains totally symmetric, any natural submanifold of a Frobenius manifold retains the quasi-potentiality condition, even if it is not flat (note, however, that there is no reason for an arbitrary $\mathcal{F}$ manifold to be quasi-potential). Similarly, one may restrict the extended connection on $M \times \mathbf{P}^1$ to $N \times \mathbf{P}^1$. This new connection will not be flat, but it still has special curvature properties.

Proposition 3.4. *The curvature of the restriction of the extended connection of a Frobenius manifold to a natural submanifold is independent of z.*

Proof. Consider

$$\begin{aligned}\widetilde{R}(U,V)W &= \bigl(\widetilde{\nabla}_U\widetilde{\nabla}_V - \widetilde{\nabla}_V\widetilde{\nabla}_U - \widetilde{\nabla}_{[U,V]}\bigr)W\\ &= \bigl\{R(U,V)W\bigr\}\\ &\quad + z\bigl\{U\circ\nabla_V W - V\circ\nabla_U W - [U,V]\circ W + \nabla_U(V\circ W) - \nabla_V(U\circ W)\bigr\}\\ &\quad + z^2\bigl\{U\circ(V\circ W) - V\circ(U\circ W)\bigr\}.\end{aligned}$$

The last term vanishes by the commutativity and associativity of the $\circ$ product. The middle term vanishes by a result of Hertling (using the definition (2.1) of an F manifold). The first term is just the curvature of the manifold N. The only other curvature to calculate is

$$\begin{aligned}\widetilde{R}\left(z\frac{d}{dz},U\right)V &= \bigl(\widetilde{\nabla}_{z\frac{d}{dz}}\widetilde{\nabla}_U - \widetilde{\nabla}_U\widetilde{\nabla}_{z\frac{d}{dz}}\bigr)V\\ &= \nabla_U\mu(V) - \mu(\nabla_U V)\\ &= -\nabla_U\nabla_V E + \nabla_{\nabla_U V}E\\ &= -\Delta(U,V),\end{aligned}$$

the other terms vanishing for similar reasons as above. Thus the only non-zero terms in the curvature of the extended connection is the curvature $R(U,V)W$ of the manifold N and $\Delta(U,V)$. However since

$$\begin{aligned}\Delta(U,V) - \Delta(V,U) &= \nabla_U\nabla_V E - \nabla_V\nabla_U E - \nabla_{\nabla_U V - \nabla_V U}E\\ &= \nabla_U\nabla_V E - \nabla_V\nabla_U E - \nabla_{[U,V]}\\ &= R(U,V)E\end{aligned}$$

the independent non-zero terms are curvature and the symmetric part $\Delta^s(U,V)$ of $\Delta(U,V)$. □

The following result relates Δ to the curvature, $\Delta(U,V) = R(U,E)V$. This properties may be proved using submanifold theory. The following theorem is standard, and is included here only to fix notation.

Theorem 3.5. *Let M be a manifold with Levi-Civita connection $\overline{\nabla}$ and let N be an arbitrary submanifold. Then for all $W,X,Y,Z\in TN$ and normal vectors $\xi,\eta\in TN^{\perp}$:*

- *Gauss formula*:

$$\overline{\nabla}_X Y = \underbrace{\nabla_X Y}_{TN} + \underbrace{\alpha(X,Y)}_{TN^{\perp}};$$

- *Weingarten formula*:

$$\overline{\nabla}_X\xi = \underbrace{-A_\xi X}_{TN} + \underbrace{\nabla^{\perp}_X\xi}_{TN^{\perp}};$$

- *Gauss equation*:

$$\bigl\langle\overline{R}(X,Y)Z,W\bigr\rangle = \bigl\langle R(X,Y)Z,W\bigr\rangle - \bigl\langle\alpha(Y,Z),\alpha(X,W)\bigr\rangle + \bigl\langle\alpha(X,Z),\alpha(Y,W)\bigr\rangle;$$

- *Codazzi equation*:

$$\left(\overline{R}(X,Y),Z\right)^{\perp}=\left(\nabla^{\perp}_{X}\alpha\right)(Y,Z)-\left(\nabla^{\perp}_{Y}\alpha\right)(X,Z);$$

- *Ricci equation*:

$$\left\langle\overline{R}(X,Y)\xi,\eta\right\rangle=\left\langle R^{\perp}(X,Y)\xi,\eta\right\rangle-\left\langle[A_{\xi},A_{\eta}]X,Y\right\rangle.$$

Here α is the second fundamental form and A is the shape operator, which are related by

$$\left\langle\alpha(X,Y),\xi\right\rangle=\langle A_{\xi}X,Y\rangle\quad\forall X,Y\in TN,\ \xi\in TM^{\perp}.$$

Proposition 3.6. *Let N be a natural submanifold of a Frobenius manifold M. Then N (with the naturally induced structures) is an $\mathcal{F}$ manifold with, for all $U,V\in TN$:*

$$\Delta(U,V)=R(U,E)V,$$
$$Lie_{E}\alpha=0,$$
$$\left\langle\nabla_{U}e^{\top},V\right\rangle=\left\langle\alpha(U,V),e^{\perp}\right\rangle.$$

Proof. The proof is a simple exercise in submanifold theory. Recall that for a natural submanifold $E^{\perp}=0$ so $E\in TN$ and that $\alpha(U,V)\in TN^{\perp}$ for all $U,V\in TN$.

$$\begin{aligned}\overline{\Delta}(U,V)&=\overline{\nabla}_{U}\overline{\nabla}_{V}E-\overline{\nabla}_{\overline{\nabla}_{U}V}E\\&=\Delta(U,V)-A_{\alpha(V,E)}U+A_{\alpha(U,V)}E\\&\quad+\alpha(U,\nabla_{V}E)-\alpha(\nabla_{U}V,E)-\left[\alpha(U,V),E\right]+\nabla^{\perp}_{U}\alpha(V,E)-\nabla^{\perp}_{E}\alpha(U,V),\end{aligned}\tag{3.1}$$

where the torsion free condition has been used to calculate the term $\overline{\nabla}_{\alpha(U,V)}E$. Taking the tangential component of (3.1) yields (since $\overline{\Delta}=0$)

$$\Delta(U,V)=A_{\alpha(V,E)}U-A_{\alpha(U,V)}E+\left[\alpha(U,V),E\right]^{\top}$$

and taking the inner product of this with $W\in TN$ gives

$$\left\langle\Delta(U,V)W\right\rangle=\left\langle\alpha(V,E),\alpha(U,W)\right\rangle-\left\langle\alpha(U,V),\alpha(E,W)\right\rangle+\left\langle\left[\alpha(U,V),E\right]^{\top},W\right\rangle.$$

Using (2.4) gives $\langle[\alpha(U,V),E],W\rangle=0$ so

$$\left[E,\alpha(U,V)\right]^{\top}=0.\tag{3.2}$$

Hence, by the Gauss equation

$$\left\langle\Delta(U,V),W\right\rangle=\left\langle R(U,E)V,W\right\rangle=R(U,E,V,W)$$

or $\Delta(U,V)=R(U,E)V$.

Taking the perpendicular component of (3.1) yields

$$0=\nabla^{\perp}_{U}\alpha(V,E)-\nabla^{\perp}_{E}\alpha(U,V)+\alpha(U,\nabla_{V}E)-\alpha(\nabla_{U}V,E)-\left[\alpha(U,V),E\right]^{\perp}.$$

Using the Codazzi equation and the torsion free property of the induced connection gives

$$(Lie_{E}\alpha)(U,V)=\left[Lie_{E}\alpha(U,V)\right]^{\top}=0$$

by (3.2), so $Lie_{E}\alpha=0$.

To obtain the last part of the proposition, recall that the identity field e satisfies the relation $\overline{\overline{\nabla}} e = 0$ and decomposes as $e = e^{\top} + e^{\perp}$ on TN. Thus

$$\begin{aligned}
0 &= \overline{\overline{\nabla}}_U e \\
&= \overline{\overline{\nabla}}_U e^{\top} + \overline{\overline{\nabla}}_U e^{\perp} \\
&= \bigl(\nabla_U e^{\top} - A_{e^{\perp}} U\bigr) + \bigl(\nabla^{\perp}_U e^{\perp} + \alpha(U, e^{\top})\bigr).
\end{aligned}$$

Decomposing this into tangential and perpendicular components gives

$$\begin{aligned}
&\bigl\langle \nabla_U e^{\top}, V\bigr\rangle = \langle A_{e^{\top}} U, V\rangle = \bigl\langle \alpha(U, V), e^{\perp}\bigr\rangle, \\
&\nabla^{\perp}_U e^{\perp} = -\alpha(U, e^{\top}).
\end{aligned}$$

[Note that $\langle \nabla_U e^{\top}, V\rangle - \langle \nabla_V e^{\top}, U\rangle = 0$ so the 1-form $\varepsilon_N(\cdot) = \langle e^{\top}, \cdot\rangle$ is closed, as it must, since $d(\imath^{\star}\varepsilon) = 0$.] $\square$

An immediate corollary of this proposition is the following:

Corollary 3.7. *Any flat caustic of a semi-simple Frobenius manifold is itself a Frobenius manifolds, i.e., it is a Frobenius submanifold. All two dimensional caustics are Frobenius submanifolds.*

Proof. The only thing to note is that for natural submanifold of a semi-simple Frobenius manifold

$$\{e^{\perp} = 0\} \quad \Leftrightarrow \quad \{N \text{ is a caustic}\}.$$

Hence by the above proposition $\nabla_U e = 0$. If the caustic is flat then all obstruction vanish. Note that on a general (non-flat) caustic $\alpha(e, U) = 0$ so from the Gauss equation

$$R(W, X, Y, Z) = 0 \quad \text{if any of the vector fields } W, X, Y, Z = e. \tag{3.3}$$

Hence all two-dimensional caustics are Frobenius submanifolds. $\square$

3.1. Semi-simple $\mathcal{F}$-manifolds

In this subsection the curvature properties of semi-simple $\mathcal{F}$-manifolds will be studied. Again the approach will stress those properties intrinsic to an $\mathcal{F}$-manifold as defined above, and those which a natural submanifold of a Frobenius manifold possesses. From the semi-simplicity and the compatibility of the multiplication with the metric:

$$\begin{aligned}
\eta_{ij} &= \langle \partial_i, \partial_j\rangle \\
&= \langle e, \partial_i \circ \partial_j\rangle \\
&= \delta_{ij}\langle e, \partial_i\rangle
\end{aligned}$$

and hence the metric is diagonal. Curvature calculations for diagonal metrics are standard. For completeness and to fix notation:

Proposition 3.8. *Let* $\eta_{ii} = H_i^2$, $\beta_{ij} = \frac{\partial_i H_j}{H_i}$. *Then*:

$$\Gamma^i_{jk} = 0, \quad \text{where } i, j, k \text{ are distinct};$$
$$\Gamma^i_{ik} = \frac{H_k}{H_i}\beta_{ki};$$
$$\Gamma^i_{jj} = -\frac{H_j}{H_i}\beta_{ij}, \quad \text{where } i \neq j.$$

Similar formulae hold for Γ^{ij}_k, *where* $\Gamma^{ij}_k = -g^{is}\Gamma^j_{sk}$. *Moreover*: $R^{ij}_{kl} = 0$ *if* i, j, k, l *are distinct*, $R^{ii}_{kl} = R^{ij}_{kk} = 0$, $R^{ij}_{il} = -R^{ij}_{li} = R^{ji}_{li} = -R^{ji}_{il}$, *and, for* $i \neq j, i \neq l$,

$$R^{ij}_{il} = \frac{1}{H_i H_j}\{\partial_l \beta_{ji} - \beta_{jl}\beta_{li}\},$$
$$R^{ij}_{ij} = \frac{1}{H_i H_j}\left\{\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{p \neq i,j} \beta_{pj}\beta_{pi}\right\}.$$

Here $R^{ij}_{kl} = \eta^{is} R^j_{skl}$ *and* $R(\partial_i, \partial_j)\partial_k = R^r_{kij}\partial_r$.

In addition to the metric and multiplication on $\mathcal{F}$ one has unity and Euler vector fields:

$$e = \sum_{i=1}^n \partial_i, \qquad E = \sum_{i=1}^n u^i \partial_i$$

and the homogeneity condition (2.4) becomes $E(\eta_{ii}) = (D-2)\eta_{ii}$, or $E(\beta_{ij}) = -\beta_{ij}$. No more can be said about the curvature properties of a general $\mathcal{F}$ manifold.

Consider now a semi-simple Frobenius manifold. Condition (b) in its definition implies:

$$(\text{b}') \quad \Rightarrow \quad \{\text{metric is Egoroff, i.e., } \beta_{ij} = \beta_{ji}\};$$

and hence the induced metric on the natural submanifold is Egoroff, as condition (b) hold for any submanifold (and it is clear that the induced metric on a natural submanifold remains diagonal). Thus natural submanifolds of semi-simple Frobenius manifolds are Egoroff (one may also prove this directly from the definition of a natural submanifold).

The following proposition may be derived by direct computation in diagonal coordinates, so no proof will be given.

Proposition 3.9. *Consider a semi-simple Egoroff* $\mathcal{F}$ *manifold in canonical coordinates. Then*:

$$\nabla_i \nabla_j E^k = 0 \quad \Leftrightarrow \quad \partial_k \beta_{ij} - \beta_{ik}\beta_{kj} = 0, \quad i, j, k \text{ distinct},$$
$$\nabla_i \nabla_j E^i = 0 \quad \Leftrightarrow \quad e(\beta_{ij}) = 0, \quad i \neq j.$$

Hence

$$\Delta = 0 \quad \Leftrightarrow \quad R = 0$$

and so there is only one obstruction to the extended connection on $N \times \mathbf{P}^1$ *being zero, namely* $\Delta(U, V)$.

For an Egoroff metric, the curvature components may be combined to give

$$R^{ij}_{ij} + \sum_{p\neq i,j} R^{ij}_{ip} = \frac{1}{H_i H_j} e(\beta_{ij}), \quad i \neq j.$$

This formula enables one to consider the curvature of pure caustics and pure discriminants. On a pure caustic $e(H_i) = 0 \Rightarrow e(\beta_{ij}) = 0$ and this right hand side vanishes (in accordance with (3.3)). On a pure discriminant $\partial_k \beta_{ij} - \beta_{ik}\beta_{kj} = 0, i, j, k$ distinct (this will be shown in the next section), so the only non-zero curvatures are $R^{ij}_{ij} = \frac{1}{H_i H_j} e(\beta_{ij})$. Pure discriminants also have the property of having flat normal bundles, $R^{\perp} = 0$, the conditions $\{u^i = 0,\ i \in \mathcal{D}\}$ being a holonomic nets of lines of curvature [7].

3.2. The induced intersection form and pencils of compatible metrics

An important feature of a Frobenius manifold M is the existence of a second flat metric, the intersection form, defined by

$$^{(2)}g^{ij} = E\big(dt^i \circ dt^j\big)$$

(in what follows the original metric g will be denoted $^{(1)}g$). Here $\{t^i\}$ are the flat coordinates on M. One important feature of this metric is that

$$Lie_e\, {}^{(2)}g^{ij} = {}^{(1)}g^{ij}$$

and it follows from this that the pencil of metric defined by $^{(\Lambda)}g^{ij} = {}^{(2)}g^{ij} + \Lambda\, {}^{(1)}g^{ij}$ is flat for all values of Λ.

A second metric, and hence a pencil of (inverse)-metrics, may also be defined on a natural submanifold $N \subset M$ since, by definition, $TN \circ TN \subset TN$ and $E_x \in TN\ \forall x \in TN$. This is not immediately obvious for a discriminant submanifold since an equivalent definition of canonical coordinates for a Frobenius manifold is as solutions of the polynomial equation

$$\det\big[{}^{(2)}g^{ij} - u\, {}^{(1)}g^{ij}\big] = 0, \tag{3.4}$$

and hence on a submanifold with $u^i = 0$, $\det[{}^{(2)}g^{ij}] = 0$ so the metric is non-invertible. However the problem lies in the orthogonal component to N. One may use the orthogonal decomposition $TM \cong TN \oplus TN^{\perp}$, and consider

$$g^{ij} \frac{\partial}{\partial t^i} \otimes_s \frac{\partial}{\partial t^j} \in TM \otimes_s TM \cong (TN \otimes_s TN) \oplus \big(TN \otimes_s TN^{\perp}\big) \oplus \big(TN^{\perp} \otimes_s TN^{\perp}\big).$$

The cross term vanishes and hence on obtains a symmetric bilinear from on $TN \otimes_s TN$. In canonical coordinates (and hence for a semi-simple Frobenius manifold)

$$^{(2)}g_{ij} = \sum_{i=1}^{m} \frac{\eta_{ii}}{u^i} \big(du^i\big)^2,$$

and similar looking formulae hold for the induced metrics on a natural submanifold. Although both induced metric will no longer, in general, be flat, certain special curvature properties remain.

Lemma 3.10. *Consider two diagonal metrics* $(i=1,2)$

$$ {}^{(i)}g=\sum_{r=1}^{n} {}^{(i)}g_{rr}\left(du^r\right)^2 $$

with rotation coefficients ${}^{(r)}\beta_{ij}$ *and* ${}^{(i)}H_r=\sqrt{{}^{(i)}g_{rr}}$. *Let*

$$ {}^{(2)}g_{rr}=\frac{{}^{(1)}g_{rr}}{u^r} $$

and

$$ {}^{(\Lambda)}g_{rr}={}^{(2)}g_{rr}+\Lambda\,{}^{(1)}g_{rr}. \tag{3.5} $$

Then

$$ \begin{aligned} {}^{(\Lambda)}\Gamma_k^{ij}&={}^{(2)}\Gamma_k^{ij}+\Lambda\,{}^{(1)}\Gamma_k^{ij},\\ {}^{(\Lambda)}R_{rs}^{ij}&={}^{(2)}R_{rs}^{ij}+\Lambda\,{}^{(1)}R_{rs}^{ij} \end{aligned} $$

where ${}^{(\Lambda)}\Gamma_k^{ij}$ *and* ${}^{(\Lambda)}R_{rs}^{ij}$ *are the appropriate Christoffel and curvatures of the pencil of metric* (3.5).

Thus any semi-simple $\mathcal{F}$ manifold carries such a pencil, and in particular, so does any natural submanifold of a Frobenius manifold. In the terminology of [14], on such (sub)-manifolds one has a pencil of compatible metrics. It is this result that will be behind the study of induced bi-Hamiltonian structures on natural submanifolds that will be given in the next section. Pure discriminant submanifold have further special properties:

Theorem 3.11 [4]. *Let* M *be a Frobenius manifold and let* N *be a pure discriminant submanifold. Then the metric on* N *induced from* ${}^{(2)}g$ *is flat.*

Thus on a pure discriminant one has a distinguished coordinate system being the flat coordinates for the second induced metric. Examples will be given in the next section (see also Main Example B). Since

$$ {}^{(2)}\beta_{ij}=\sqrt{\frac{u_i}{u_j}}\left({}^{(1)}\beta_{ij}\right) $$

(and note that the second metric will not be Egoroff) it follows from

$$ \partial_k\,{}^{(2)}\beta_{ij}-{}^{(2)}\beta_{ik}\,{}^{(2)}\beta_{kj}=\sqrt{\frac{u_i}{u_j}}\left(\partial_k\,{}^{(1)}\beta_{ij}-{}^{(1)}\beta_{ik}\,{}^{(1)}\beta_{kj}\right) $$

that ${}^{(1)}R_{ik}^{ij}=0$, as stated at the end of the last section.

3.3. Tangent vectors to a natural submanifold

The intersection form of a Frobenius manifold M enables one to construct natural vector fields tangent to a natural submanifold. In a flat coordinate system $\{t^i\}$ where $e=\partial_1$, ${}^{(2)}g^{mi}=E^i$, so the components of the last row/column of the intersection form are the components of the Euler vector field which, by definition, is tangent to a natural submanifold. Consider the vector fields defined on N by

$$V^{(\alpha)} = \left(\frac{\partial t^i}{\partial \tau^\alpha} {}^{(1)}g_{ij} {}^{(2)}g^{jk} \right) \bigg|_N \frac{\partial}{\partial t^k}$$

which a priori lie in TM. Using (2.6) and the fact that

$$E^k\big|_N = E_N^\alpha \frac{\partial t^k}{\partial \tau^\alpha}$$

it follows that the component of $V^{(\alpha)}$ in $TN^\perp$ is

$$\left(\Xi^{\tilde\alpha}_{\alpha\beta} E_N^\beta\right) \frac{\partial}{\partial \nu^{\tilde\alpha}}$$

and hence is zero.

Using techniques identical to the above, one may show that if

$$X \circ Y \in TN \quad \forall X \in TN,\ Y \in TM \tag{3.6}$$

then the m vector fields

$$V^{(i)} = {}^{(2)}g^{ij} \frac{\partial}{\partial t^j}$$

are all tangential to the submanifold. In the semi-simple case one may show, using their explicit parametrization, that the only natural submanifolds which satisfy the above condition are pure discriminants. For Frobenius manifolds based on Coxeter groups this result is already known [1,15] and more generally [4,10]. For codimension one discriminants a simple proof that (3.6) holds may be given using the decomposition of the unity vector field $e = e^\top + e^\perp$. Since, if N is a natural submanifold, $X \circ e^\perp = 0, \forall X \in TN$ then because any vector in $TN^\perp$ must be a multiple of $e^\perp$ the result follows.

3.4. *Examples of Frobenius submanifolds*

Frobenius submanifolds certainly exist—any flat caustic, if they exist, of a semi-simple Frobenius manifold will inherit the structure of a Frobenius manifold. In theory this gives a way to find such submanifolds, though in practice it would be computationally difficult. A more practical way is to look for submanifolds which are hyperplanes (in the flat coordinates $\{t^i\}$, and coordinate hyperplanes in particular).

Example 3.12. Let $I \subset \{1, 2, \ldots, m\}$ and suppose that $\mathcal{N}$ is given by the conditions $t^i = 0$ for $i \notin I$. Then the obstruction reduces to the algebraic condition

$$c_{ij}^k\big|_N = 0, \quad i, j \in I,\ k \notin I.$$

This condition was derived in [19] in the context of Frobenius manifolds constructed from Coxeter groups. Here it is a specialization of the more general condition (2.7).

Example 3.13. [13, Section III.8.7.1].

Table 1
Frobenius submanifolds associated with Coxeter groups

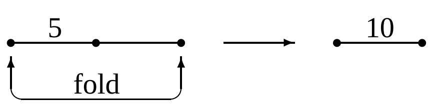
Coxeter group	Coxeter subgroup
A_{2n+1}	B_n
D_{n+1}	B_n
D_5	H_3
E_6	F_4
E_8	H_4
W (arbitrary)	I_2 (Coxeter number of W)

Table 2
Extended affine Frobenius submanifolds

Extended affine Coxeter group	Extended affine Coxeter subgroup
$A^{k=n}_{l=2n-1}$	C_n
D_{n+1}	B_n
E_6	F_4
W (arbitrary)	$A^{k=1}_{l=1}$

It is not clear, but follows from the results above, that these submanifolds are caustics.

Large numbers of examples may be found using these results. For example, for Frobenius manifolds constructed from Coxeter groups one finds that the submanifold associated with another Coxeter group obtained by 'folding' the original Coxter diagram. For example the H_3-Frobenius contains the Frobenius manifold $I_2(10)$ as a submanifold, which corresponds to the folding



The possible foldings, and hence submanifolds, of Coxter groups, are given in Table 1.

These results may also be generalized to Frobenius manifolds constructed from extended affine Coxeter groups [5] (and probably more generally too, since notions of foldings exist more generally in singularity theory). The analogous results are given in Table 2.

However, such submanifolds are just hyperplanes. More interesting examples may be constructed.

Example 3.14. Frobenius submanifolds of A_3 (this is a special case of Main Example A). For the A_3-singularity one takes the polynomial

$$p(z) = z^4 + a_1 z^2 + a_2 z + a_3$$

and constructs the metric via the formula (1.1). This metric is flat, though not in flat coordinates, which are given by

$$\begin{aligned} a_1 &= t_3, \\ a_2 &= t_2, \\ a_3 &= t_1 + \frac{1}{8} t_3^2. \end{aligned}$$

In these coordinates the metric takes the standard antidiagonal form. Having fixed the flat coordinates, so

$$p(z) = x^4 + t_1 + \frac{1}{8}t_3^2 + t_2 x + t_3 x^2 \tag{3.7}$$

the algebra is defined by

$$c_{ijk} = -\operatorname*{res}_{x=\infty} \frac{\partial_{t_i} f\, \partial_{t_j} f\, \partial_{t_k} f}{\partial_z f}\, dz.$$

From this the prepotential may be constructed. The canonical coordinates are now defined as roots of the cubic (3.4). The discriminant and caustics may easily be calculated from this cubic.

Discriminant (the induced structure on the discriminant does not define a Frobenius manifold. The results are given here for use in the next section). If $u^i = 0$ then from (3.4) $\det({}^{(2)}g^{ij}) = 0$, or

$$-t_1^3 + \frac{27}{256}t_2^4 - \frac{9}{16}t_1 t_2^2 t_3 + \frac{1}{8}t_1^2 t_3^2 - \frac{7}{128}t_2^2 t_3^3 + \frac{1}{64}t_1 t_3^4 - \frac{1}{512}t_3^6 = 0. \tag{3.8}$$

This is precisely the condition for the polynomial (3.7) to have a repeated root, i.e., it defines a discriminant hypersurface. Using the fact that such surfaces are ruled one may easily obtain a parametrization of the surface

$$\begin{aligned} t_1 &= +2^{-9}\left(u^4 - 6u^2 v^2 - v^4\right), \\ t_2 &= +2^{-4} u v^2, \\ t_3 &= -2^{-3}\left(u^2 + v^2\right), \end{aligned}$$

where u and v are the flat coordinates for the induced intersection form. Such an explicit parametrization will be used in the next section to construct induced bi-Hamiltonian structures on this discriminant.

Caustics. If $u^i = u^j$ for some $i \neq j$ then the polynomial (3.4) must have a double root, and the condition for this is either (a) $t_2 = 0$ or (b) $27t_2^2 + 8t_3^3 = 0$. These surfaces correspond to the cylinders over the Maxwell strata and caustic of the polynomial (3.7). The induced structures on these surfaces define Frobenius manifolds (corresponding to the Coxeter group $I_2(4)$): the structure on the (a) being studied by Zuber [19] and on (b) by the author [17]. Note how the two parts in the definition of a natural submanifold are used; the induced multiplication on the flat surface $t_2^2 + kt_3^3 = 0$ is associative for all values of k, but the Euler field is tangential only for the two special values of k given above.

Further examples may be obtained by tensoring Frobenius manifolds together and restricting structures to various hyperplanes [13,17].

4. Induced bi-Hamiltonian structure

An equation of hydrodynamic type is, by definition, of the form

$$U_T^i = V_j^i\left(U^k\right)U_X^j. \tag{4.1}$$

It was observed by Riemann that such system transform covariantly with respect to arbitrary changes of dependent variables $\widetilde{U}^i = \widetilde{U}^i(U)$. It is not surprising therefore that geometrical ideas should be used in the study of such equations. Associated with any semi-simple Frobenius manifold is a bi-Hamiltonian hierarchy of such hydrodynamic equations. The aim of this section is to show how one may constrain such systems onto a natural submanifold while retaining its bi-Hamiltonian structure. In this case the system (4.1) may be diagonalised

$$u^i_T = \lambda^i(u)u^i_X, \quad i = 1, \dots, m.$$

(Note that a diagonalised system is defined in terms of Riemann invariants

$$R^i_T = \lambda^i(R)R^i_X$$

and these are only defined up to $R^i \mapsto \widetilde{R}^i(R^i)$ transformation. The canonical coordinates are specific examples of Riemann invariants, and do no have such a freedom.) The λ^i are known as the characteristic speeds of the system. It is immediate from this diagonal form that if the mild finiteness condition

$$\lambda^j\big|_{\lambda^i=0} < \infty \tag{4.2}$$

holds then one may restrict the system to the discriminant $u^i = 0$. To reduce the system to a caustic $u^i - u^j = 0$ requires the much stronger condition

$$\left(\lambda^i - \lambda^j\right)\big|_{u^i - u^j = 0} = 0 \tag{4.3}$$

together with a mild finiteness condition for the remaining λ^i. Such constraints do hold for the systems associated with semi-simple Frobenius manifolds and this will be proved below.

Before this a more general discussion will be given which will associate $\mathcal{F}$ manifolds with certain diagonal sets of equations.

4.1. Semi-Hamiltonian systems and curved $\mathcal{F}$-manifolds

Definition 4.1. A diagonal system of hydrodynamic type

$$u^i_T = \lambda^i(u)u^i_X, \quad i = 1, \dots, m,$$

is semi-Hamiltonian if there exists a diagonal metric

$$g = \sum_{i=1}^m g_{ii}(u)\left(du^i\right)^2$$

satisfying the equations

$$\partial_j \log \sqrt{g_{ii}} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}$$

for $i \neq j$.

On cross differentiating on obtains the identities

$$\partial_k \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \partial_j \frac{\partial_k \lambda^i}{\lambda^k - \lambda^i},$$

for distinct i, j, k. All semi-Hamiltonian systems are integrable, via the generalized hodograph transform. Such systems possess an infinite number of commuting flows

$$u^i_{T'} = w^i(u)u^i_X$$

where the w^i are solutions of the linear system

$$\frac{\partial w^i}{w^j - w^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}, \quad i \neq j,$$

which exists since its integrability follows from the definition of semi-Hamiltonian. If the metric is homogeneous with respect to the vector field E, so

$$E\eta_{ii} = (D-2)\eta_{ii}$$

them one obtains a semi-simple $\mathcal{F}$ manifold. It also follows from the definition of semi-Hamiltonian that the only non-zero curvature components are R^{ij}_{ij}, for $i \neq j$.

Example. Consider the system

$$u^i_T = \left(\sum_{r=1}^m u^r + 2u^i\right) u^i_X, \quad i = 1, \ldots, m, \tag{4.4}$$

which corresponds to the dispersionless limit of the coupled KdV hierarchy [2,8]. Such a system is semi-Hamiltonian with metric

$$g = \sum_{i=1}^m \left\{ \frac{\prod_{r\neq i}(u^r - u^i)}{\phi_i(u^i)} \right\} \left(du^i\right)^2, \quad \phi_i \text{ arbitrary}.$$

The metric ${}^{(1)}g$ is defined as the above metric with $\phi_i = 1$, and is homogeneous with respect to the Euler vector field, and so one obtained semi-simple $\mathcal{F}$ manifold. This metric is not Egoroff, so one does not obtain a Frobenius manifold, nor can it be a natural submanifold of a semi-simple Frobenius manifold. This metric is, however flat; in flat coordinates

$${}^{(1)}g^{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad {}^{(2)}g^{ij} = \begin{pmatrix} 0 & -1 & t_1/2 \\ -1 & 0 & t_2/2 \\ t_1/2 & t_2/1 & t_3 \end{pmatrix}$$

(both flat) and the Euler and identity fields are

$$E = t_1\partial_1 + 2t_2\partial_2 + 3t_3\partial_3,$$
$$e = 3\partial_1 + \frac{1}{2}t_1\partial_2 - \frac{1}{2}t_2\partial_3.$$

The associative multiplication is somewhat complicated when written in the flat coordinate system. Note also that

$$\det\left[{}^{(1)}g\right] = \prod_{i\neq j}\left(u^i - u^j\right)^2,$$

so the metric is degenerate on bifurcation diagrams.

This example may also be used to illustrate how one may reduce hydrodynamic systems to bifurcation diagrams and discriminants, and also some of the problems that may arise. Restricting the system (4.4) to a discriminant is trivial—the form of the equations is unchanged. On a bifurcation diagram $\{k_1, \ldots, k_r\}$

one obtains

$$u^i_T = \left(\sum_{r=1}^n k_r u^r + 2u^i\right) u^i_X, \quad i = 1, \dots, n,$$

which remains semi-Hamiltonian (in fact for all values for the k_i) with metric

$${}^{(1)}g_{\mathcal{C}} = \sum_{i=1}^n \prod_{r \neq i} \left(u^r - u^i\right)^{k_i} \left(du^i\right)^2.$$

Thus one obtains a 'stratified' space where ${}^{(1)}g$ is defined everywhere except on bifurcation diagrams, but with another metric ${}^{(1)}g_{\mathcal{C}}$ on the bifurcation diagram (which in term is defined everywhere on the bifurcation diagram except at sub-bifurcation diagrams etc.).

4.2. Bi-Hamiltonian hierarchies associated to Frobenius manifolds

Given a Frobenius manifold there exits an associated hierarchy of hydrodynamics type, which, in terms of flat coordinates $\{t^\alpha\}$ are given by

$$\frac{\partial t^\gamma}{\partial T^{(\alpha,p)}} = c_{(\alpha,p)\beta}{}^\gamma \partial_X t^\beta, \quad \begin{cases} \alpha = 1, \dots, m, \\ p = 0, \dots, \infty. \end{cases} \tag{4.5}$$

The primary part of the hierarchy is defined by

$$\frac{\partial t^\gamma}{\partial T^{(\alpha,0)}} = c_{\alpha\beta}{}^\gamma \partial_X t^\beta.$$

From the flatness of the extended connection it follows that there exists functions $h_{(\alpha,p)}$ which satisfy the relation

$$\frac{\partial^2 h_{(\alpha,p)}}{\partial t^\alpha t^\beta} = c_{\alpha\beta}{}^\gamma \frac{\partial h_{(\alpha,p-1)}}{\partial t^\gamma}$$

with the initial condition $h_{(\alpha,0)} = t_\alpha = \eta_{\alpha\beta} t^\beta$. These define Hamiltonians

$$H_{(\alpha,p)} = \int h_{(\alpha,p+1)}\, dX$$

which are conserved with respect to all flows. Here the Hamiltonian structure is defined, by the fundamental theorem of Dubrovin and Novikov, by any flat metric g. For functionals $F = \int f(t, t_X, \dots)\, dX$, $G = \int g(t, t_X, \dots)\, dX$ the Hamiltonian is defined by

$$\{F, G\} = \int \frac{\delta F}{\delta t^i} A^{ij} \frac{\delta G}{\delta t^j}\, dX$$

where

$$A^{ij} = g^{ij}(t) \frac{d}{dX} - g^{is} \Gamma^j_{sk}(t) t^k_X.$$

Here g^{ij} is the (inverse) flat metric and Γ^{jk}_i the corresponding Christoffel symbols. The zero-curvature condition ensures that the bracket satisfies the Jacobi identity.

On a given Frobenius manifold M one has a pencil of flat metrics ${}^{(\Lambda)}g^{ij} ={}^{(2)} g^{ij} + \Lambda^{(1)}g^{ij}$, and this then gives rise to a bi-Hamiltonian structure

$$\{F,G\}_\Lambda = \{F,G\}_2 + \Lambda\{F,G\}_1.$$

It then follows that the hierarchy (4.5) is Hamiltonian with respect to the Poisson bracket defined by $\{.,.\}_2$ in addition to being Hamiltonian with respect to the Poisson bracket defined by $\{.,.\}_1$. In particular, one has the following equation

$$\left\{t^\gamma, H_{(\beta,q-1)}\right\}_2 = \left(q+\mu_\beta+\frac{1}{2}\right)\left\{t^\gamma, H_{(\beta,q)}\right\}_1 + \sum_{k=1}^{q}(R_k)^\sigma_\beta\left\{t^\gamma, H_{(\sigma,q-k)}\right\}_1 \tag{4.6}$$

where the matrices R_k are defined in terms of the monodromy at $z=0$ of the system $\widetilde{\nabla}_{z\frac{d}{dz}}\xi = 0$ (for a precise formulation see [6]). The constants μ_β are defined by the formula for the Euler vector field (in flat-coordinates)

$$E = \sum_\beta\left\{\left(1-\frac{d}{2}-\mu_\beta\right)t^\beta + r_\beta\right\}\frac{\partial}{\partial t^\beta}$$

(and d is defined by $E(F) = (3-d)F +$ quadratic terms where F is the prepotential of the Frobenius manifold).

The following theorems on the restriction of hierarchies to natural submanifolds are formal—they implicitly assume that various functions are finite on the submanifold. For specific classed of Frobenius manifolds one may show that the quantities are finite (see, for example, Example 5.1), but it would seem to be very difficult to say anything about the values of these functions on natural submanifolds of a *general* Frobenius manifold.

Theorem 4.2. *Let M be a semi-simple Frobenius manifold. Then the restriction of the bi-Hamiltonian hierarchy* (4.5) *to a natural submanifold remains bi-Hamiltonian.*

Proof. Given an arbitrary submanifold N of a flat manifold M one may constrain, using the Dirac procedure the corresponding Hamiltonian structure on M to the submanifold N [7]. This results in a Hamiltonian structure of the form

$$\{F,G\} = \int \frac{\delta F}{\delta\tau^\alpha}A^{\alpha\beta}\frac{\delta G}{\delta\tau^\beta}\,dX$$

where

$$A^{\alpha\beta} = g^{\alpha\beta}(\tau)\frac{d}{dX} - g^{\alpha\mu}\Gamma^\beta_{\mu\nu}(\tau)\tau^\nu_X + \sum_{\tilde\alpha} w^\alpha_{\tilde\alpha\mu}\tau^\mu_x(\nabla^\perp)^{-1}w^\beta_{\tilde\alpha\nu}\tau^\nu_x.$$

Here $g^{\alpha\beta}$ is the (inverse) induced metric on N, $\Gamma^\alpha_{\mu\nu}$ the corresponding Christoffel symbols, and $w^\alpha_{\tilde\alpha\beta}$ the Weingarten operators of the submanifold. The operator $\nabla^\perp$ is defined by

$$\nabla^\perp\phi_\alpha = \frac{d}{dX}\phi_\alpha + {\omega_\alpha}^\beta\phi_\beta$$

where ${\omega_\alpha}^\beta$ are the normal connection one-forms.

Recall, from Lemma 3.10, that on a natural submanifold

$$
\begin{aligned}
&{}^{(\Lambda)}\Gamma^{ij}_k = {}^{(2)}\Gamma^{ij}_k + \Lambda\,{}^{(1)}\Gamma^{ij}_k,\\
&{}^{(\Lambda)}R^{ij}_{rs} = {}^{(2)}R^{ij}_{rs} + \Lambda\,{}^{(1)}R^{ij}_{rs}.
\end{aligned}
$$

This then implies that if one restricts the flat bi-Hamiltonian structure on the Frobenius manifold to any natural submanifold one obtains a new bi-Hamiltonian structure with non-local tails, as above. Care has to be taken with the structure of the non-local tail for the curved pencil; it is twice as long as the codimension, each half containing the non-local tail of one of the individual metrics. □

It is not obvious from this result that the resulting system is still local. This may be shown to be case by directly studying the restriction of the hierarchy (4.5) onto a natural submanifold. In order to show this the system will first be rewritten in canonical coordinates.[1] Extensive use will be made of the following formulae, all of which are derived in [4]:

$$
c_{\alpha\beta\gamma} = \sum_{i=1}^{m} \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{1i}},
$$

$$
\frac{\partial t^\alpha}{\partial u^i} = \psi_{i1}\psi_i{}^\alpha, \qquad \frac{\partial u^i}{\partial t^\alpha} = \frac{\psi_{i\alpha}}{\psi_{i1}},
$$

where $\psi_{i1}^2 = \eta_{11}$. Indices on ψ are raised and lowered using $\eta_{\alpha\beta}$, so $\psi_i{}^\alpha = \psi_{i\beta}\eta^{\beta\alpha}$ and

$$
\sum_{i=1}^{m} \psi_{i\alpha}\psi_{i\beta} = \eta_{\alpha\beta}, \qquad \sum_{i=1}^{m} \psi_i{}^\alpha \psi_i{}^\beta = \eta^{\alpha\beta}
$$

and crucially the following:

$$
\left(u^j - u^i\right)\beta_{ij} = \sum_\alpha \left(q_\alpha - \frac{d}{2}\right)\psi_{i\alpha}\psi_j{}^\alpha. \tag{4.7}
$$

In canonical coordinates these become

$$
\nabla_i\nabla_j h_{(\alpha,p)} = \delta_{ij}\frac{\partial h_{(\alpha,p-1)}}{\partial u^j}
$$

and

$$
\begin{aligned}
\lambda^i_{(\alpha,p)} &= \eta^{ii}\nabla_i\nabla_i h_{(\alpha,p)} \qquad (4.8)\\
&= \frac{1}{\eta_{ii}}\frac{\partial h_{(\alpha,p-1)}}{\partial u^i} \qquad (4.9)
\end{aligned}
$$

where ∇ is the Levi-Civita connection of the first metric.

In canonical coordinates the characteristic speeds of the primary part of the hierarchy are given by

$$
\lambda^i_{(\alpha,0)} = \frac{\partial u^i}{\partial t^\alpha}.
$$

[1] N.B. The notation used in the rest of this section differs from that used above. Greek letters denote components of objects in flat coordinates, and Latin letters denote components of objects in canonical coordinates. Also η will refer to the metric ${}^{(1)}g$.

Thus conditions (4.2) and (4.3) become, respectively, the conditions

$$\left.\frac{\partial u^i}{\partial t^\alpha}\right|_{u^j=0} < \infty, \tag{4.10}$$

and

$$\left.\frac{\partial(u^i - u^j)}{\partial t^\alpha}\right|_{u^i-u^j=0} = 0. \tag{4.11}$$

If these conditions hold for the primary part of the hierarchy, then they hold for the entire hierarchy.

Theorem 4.3. *Suppose that condition* (4.11) *holds, and that the set* $\{\mu_\beta\}$ *does not contain negative half-integers. Then*

$$\left(\lambda^i_{(\beta,q+1)} - \lambda^j_{(\beta,q+1)}\right)\Big|_{u^i-u^j=0} = 0$$

and hence the entire hierarchy may be restricted onto the bifurcation diagram $u^i - u^j = 0$.

Proof. Eq. (4.6) implies the following relation

$$\left\{u^i, H_{(\beta,q-1)}\right\}_2 = \left(q + \mu_\beta + \frac{1}{2}\right)\lambda^i_{(\beta,q+2)} + \sum_{k=1}^{q} (R_k)^\sigma_\beta\, \lambda^i_{(\sigma,q-k+2)}.$$

The left-hand-side may be expanded

$$\left\{u^i, H_{(\beta,q-1)}\right\}_2 = \frac{u^i}{\eta_{ii}}\left[\frac{\partial^2}{\partial {u^i}^2} - \sum_r {}^{(2)}\Gamma^r_{ii}\frac{\partial}{\partial u^r}\right] h_{(\beta,q)}$$

and since, by definition,

$$\lambda^i_{(\beta,q+1)} = \frac{1}{\eta_{ii}}\left[\frac{\partial^2}{\partial {u^i}^2} - \sum_r \overset{(1)}{\Gamma}{}^r_{ii}\frac{\partial}{\partial u^r}\right] h_{(\beta,q)},$$

one may eliminate the second derivatives to obtain

$$\left\{u^i, H_{(\beta,q-1)}\right\}_2 = u^i\lambda^i_{(\beta,q+1)} + \frac{u^i}{\eta_{ii}}\left[\sum_r \left({}^{(1)}\Gamma^r_{ii} - {}^{(2)}\Gamma^r_{ii}\right)\frac{\partial h_{(\beta,q)}}{\partial u^r}\right].$$

Expanding the Christoffel symbols (see Proposition 3.8) and using Eqs. (4.7)–(4.9) gives

$$\frac{u^i}{\eta_{ii}}\left[\sum_r \left({}^{(1)}\Gamma^r_{ii} - {}^{(2)}\Gamma^r_{ii}\right)\frac{\partial h_{(\beta,q)}}{\partial u^r}\right] = \sum_{\alpha,\gamma,r\neq i} \eta_{rr}\eta^{\alpha\gamma}\left[\left(q_\alpha - \frac{d}{2}\right)\lambda^r_{(\gamma,0)}\lambda^r_{(\beta,p+1)}\right]\lambda^i_{(\alpha,0)} + \frac{1}{2}\lambda^i_{(\beta,q+1)},$$

since

$$\lambda^i_{(\beta,0)} = \frac{\partial u^i}{\partial t^\beta} = \frac{\psi_{i\beta}}{\psi_{i1}}.$$

Putting these formulae back together yields the following recursion relation between the characteristic speeds:

$$\left(q+\mu_\beta+\frac{1}{2}\right)\lambda^i_{(\beta,q+2)}=\left(\frac{1}{2}+u^i\right)\lambda^i_{(\beta,q+1)}-\sum_{k=1}^{q}(R_k)^\sigma_\beta\lambda^i_{(\sigma,q-k+2)}$$
$$+\sum_{\alpha,\gamma,r\neq i}\eta_{rr}\eta^{\alpha\gamma}\left[\left(q_\alpha-\frac{d}{2}\right)\lambda^r_{(\gamma,0)}\lambda^r_{(\beta,p+1)}\right]\lambda^i_{(\alpha,0)}.$$

This shows two things; firstly that if one can restrict the primary part of the hierarchy onto a discriminant, then one may restrict the entire hierarchy, and also that

$$\left(\lambda^i_{(\beta,p)}-\lambda^j_{(\beta,p)}\right)\big|_{u^i-u^j=0}=0$$

if the results holds for $\alpha=0$. Hence the result. Note that if technical requirement on the set $\{\mu_\beta\}$ is to ensure the coefficient of $\lambda^i_{(\beta,q+2)}$ is always non-zero. □

As already pointed out, these results are formal, depending on the nondegeneracies of various quantities when restricted to the submanifold. An example of what can happen when degeneracies appear will be given in the next section, where the results are illustrated by means of various examples.

5. Examples

The Main Examples A and B in the introduction are concrete examples of the general theory developed in this paper. The following examples are self-explanatory. Further examples may be found in [17].

Example 5.1 (*The A_n caustics*). The construction of the Frobenius manifold based on the Coxeter group A_n was given in Main Example A. It follows that

$$\frac{\partial u^i}{\partial t^\alpha}=\frac{\partial p}{\partial t^\alpha}\bigg|_{z=\alpha^i}.$$

On a caustic $u^i=u^j$ implies that $\alpha^i=\alpha^j$ (which is not true on a Maxwell strata), so

$$\frac{\partial u^i}{\partial t^\alpha}\bigg|_{u^i-u^j=0}=\frac{\partial p}{\partial t^\alpha}\bigg|_{z=\alpha^i,\,\alpha^i=\alpha^j}=\frac{\partial u^j}{\partial t^\alpha}\bigg|_{u^i-u^j=0}.$$

Hence one may restrict the A_n hierarchy onto any caustic. The same should be true for restriction onto Maxwell strata.

Example 5.2 (*The A_3 discriminant*). This example is a continuation of Example 3.14. Using the parametrization of the swallowtail discriminant given there, the induced metrics become

$$\eta_N=\left(-u^4+3u^2v^2+v^4\right)du^2+2uv\left(u^2+4v^2\right)du\,dv+v^2\left(7u^2+v^2\right)dv^2,$$
$$g_N=-du^2-dv^2.$$

Note that $\det\eta_N = v^2(v^2-2u^2)^3$. The submanifolds given by $\det\eta_N = 0$ correspond to the components of the subdiscriminant $(1,0,0)$. In terms of the polynomial (3.4) these correspond to further degeneracies amongst the its zeroes.

The induced algebra, Euler vector field and unity vector field are given by:

$$\begin{aligned}
&\partial_u \star \partial_u = \left[u\left(u^2-2v^2\right)/64\right]\partial_u - \left[v\left(u^2+v^2\right)/64\right]\partial_v,\\
&\partial_u \star \partial_v = -\left[v\left(u^2+v^2\right)/64\right]\partial_u - \left[3uv^2/64\right]\partial_v,\\
&\partial_v \star \partial_v = -\left[3uv^2/64\right]\partial_u - \left[v\left(4u^2+v^2\right)/64\right]\partial_v,\\
&E_N = u\,\partial_u + v\,\partial_v,\\
&e_N = [192uv/\Delta]\,\partial_u - \left[64\left(u^2+v^2\right)/\Delta\right]\partial_v,
\end{aligned}$$

where $\Delta = v(2u^2-v^2)^2$.

The $T = T^{(3,1)}$-flow for the A_3 Frobenius manifold may be calculated and then restricted onto the discriminant to yield the hydrodynamic system

$$\begin{aligned}
&u_T = \left(3u^2-3v^2\right)u_x - 6uvv_x,\\
&v_T = -6uvu_x - \left(3u^2+3v^2\right)v_x.
\end{aligned}$$

This may easily be put into Hamiltonian form using the induced intersection form (up to some overall constants)

$$\begin{pmatrix} u \\ v \end{pmatrix}_T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{d}{dX} \begin{pmatrix} \partial_u h \\ \partial_v h \end{pmatrix}$$

where $h = u^4 - 6u^2v^2 - v^4$. This h belongs to a family given by hypergeometric functions

$$h_{(1,r)}(u,v) = u^r\,{}_2F_1\left(-\frac{r}{2}, \frac{1-r}{2}, \frac{3-r}{4}; \frac{1}{2}\left(\frac{v}{u}\right)^2\right),$$

the second family $h_{(2,r)}$ coming the second linearly independent solution of the corresponding hypergeometric equation.

Example 5.3. Consider the Frobenius manifold defined by the prepotential and Euler vector field [12,16]

$$\begin{aligned}
&F = t^1t^2t^3 - 1/2t^2\left(t^3\right)^2 + 1/2t^4\left(t^1-t^3\right)^2 - t^3e^{t^2} + e^{t^4}\left(1+t^3e^{-t^2}\right) + 1/2\left(t^3\right)^2\left(\log t^3 - 3/2\right),\\
&E = t^1\partial_1 + \partial_2 + t^3\partial_3 + 2\partial_4.
\end{aligned}$$

The submanifold corresponding to the limit $t^4 \to -\infty$ is a caustics, since the polynomial (3.4) has a repeated root. However the inverse metric on this caustic is degenerate:

$${}^{(1)}g^{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus the ideas in this paper cannot be directly applied. However, the algebra on this submanifold is still associative, with product

$$\frac{\partial}{\partial t^i} \circ \frac{\partial}{\partial t^j} = \sum_{k=1}^{3} c_{ij}{}^k\big|_{t^4\to-\infty}\frac{\partial}{\partial t^k}$$

so

$$\begin{aligned}
&\partial_i \circ \partial_1 = \partial_i, \\
&\partial_2 \circ \partial_2 = -e^{t^2}\partial_2 - t^3 e^{t^2}(\partial_1 + \partial_3), \\
&\partial_2 \circ \partial_3 = -e^{t^2}(\partial_1 + \partial_3), \\
&\partial_3 \circ \partial_3 = -\partial_3 + 1/t^3 \partial_2.
\end{aligned}$$

One may also reduce the corresponding hydrodynamics systems onto this submanifold, but the systems, while bi-Hamiltonian, have non-trivial Casimirs. Further investigation of such degenerate caustics, similar to the classical limit of quantum cohomology, requires further study.

6. Comments

There are clearly many questions that may be addressed on the structure of submanifolds in general and of submanifolds in particular. Some of the most interesting concern the connection with τ-functions are isomonodromy. For an arbitrary integrable system with conserved densities h_k one may defined the 1-form

$$\omega = \sum_i h_i \, dT_{i+1}$$

which is closed. This then implies the existence of a so-called τ function

$$h_k = \frac{\partial \log \tau}{\partial X \partial T_{k+1}}.$$

This type of definition predates more sophisticated definitions based on Grassmannians and loop groups. On a submanifold $N \subset M$ one may pull back the form, which remains closed, and hence one may define a τ-functions for the submanifold. For the dispersionless integrable systems associated with Frobenius manifolds the central object is the isomonodromic τ-function, denoted τ_I. It would be of interest to see how such an object, and the whole theory of isomonodromy, behaves on a natural submanifold. One problem is that most of the objects are defined on $\mathbb{C}^m \backslash$caustics, so various limiting arguments will have to be used to understand the behaviour of the objects on the caustics themselves. The fact that Frobenius submanifolds lie in such caustics suggests that this may be possible, at least in some cases. The singular nature of the τ_I function on natural submanifolds is also reminiscent of the work of [3], where singularities in τ functions are labeled by Young tableaux. This suggest that a general study of the zero/singular set of τ-functions would be of interest. Intimately connected with the τ_I-function is the whole question of how one may deform these dispersionless hierarchies [6].

Other interesting questions include:

- To what extent, if at all, does a natural submanifold of a Frobenius manifold define a (topological/cohomological) quantum field theory?

The fact that one has a Frobenius algebra on each tangent space indicates that such an interpretation may be possible. Also much of the discussion on cohomological field theories in [13] may be reproduced without the flatness and condition and the existence of a prepotential. A simpler question would be to understand how the field theory corresponding to a Frobenius submanifold is embedded within the larger field theory.

- Do natural submanifolds carry information relevant to enumerative geometry and quantum cohomology?

In the case of Frobenius submanifolds the information they can contain includes certain contracted Gromov–Witten invariants of the ambient manifold [13,17].

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References

[1] V.I. Arnold, Singularities of Caustics and Wave Fronts, Kluwer, Dordrecht, 1990.
[2] M. Antonowicz, A.P. Fordy, Coupled KdV equations with multi-Hamiltonian structures, Phys. D 28 (1987) 345–357.
[3] M. Adler, P. van Moerbeke, Birkhoff strata, Bäcklund transformations, and regularization of isospectral operators, Adv. Math. 108 (1994) 140–204.
[4] B. Dubrovin, Geometry of 2D topological field theories, in: M. Francaviglia, S. Greco (Eds.), Integrable Systems and Quantum Groups, in: Lecture Notes in Mathematics, Vol. 1620, Springer, Berlin, pp. 120–348.
[5] B. Dubrovin, Y. Zhang, Extended affine Weyl groups and Frobenius manifolds, Compositio Math. 111 (1998) 167–219.
[6] B. Dubrovin, Y. Zhang, Bihamiltonian hierarchies in the 2D topological field theory at one-loop approximation, Comm. Math. Phys. 198 (1998) 311–361;
B. Dubrovin, Y. Zhang, Frobenius manifolds and Virasoro constraints, Selecta Math. New Ser. 5 (1999) 423–466;
B. Dubrovin, Y. Zhang, Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants, math.DG/0108160.
[7] E.V. Ferapontov, Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications, in: S.P. Novikov (Ed.), Topic in Topology and Mathematical Physics, in: Amer. Math. Soc. Transl. Ser. 2, Vol. 170, 1995.
[8] E.V. Ferapontov, M.V. Paplov, Quasiclassical limits of couples KdV equations: Riemann invariants, Phys. D 52 (1991) 211–219.
[9] D.B. Fairlie, I.A.B. Strachan, The algebraic and Hamiltonian structure of the dispersionless Benney and Toda hierarchies, Inverse Problems 12 (1996) 885–908.
[10] C. Hertling, Multiplication on the tangent bundle, math.AG/9910116.
[11] C. Hertling, Yu. Manin, Weak Frobenius manifolds, Int. Math. Res. Notices 6 (1999) 277–286.
[12] Y. Kodama, Dispersionless integrable systems and their solutions, in: H.W. Braden, I.M. Krichever (Eds.), Integrability: The Seiberg–Witten and Whitham Equations, Gordon and Breach, 2000.
[13] Y.I. Manin, Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces, in: AMS Colloquium, Vol. 47, 1999.
[14] O.I. Mokhov, Compatible and almost compatible pseudo-Riemannian metrics, Funct. Anal. Appl. 35 (2001) 100–110.
[15] O.P. Shcherbak, Wavefront and reflection groups, Russ. Math. Surv. 43 (3) (1988) 149–194.
[16] I.A.B. Strachan, Degenerate Frobenius manifolds and the bi-Hamiltonian structure of rational Lax equations, J. Math. Phys. 40 (1999) 5058–5079.
[17] I.A.B. Strachan, Frobenius submanifolds, J. Geom. Phys. 38 (2001) 285–307.
[18] I.A.B. Strachan, Frobenius manifolds and bi-Hamiltonian structures on discriminant hypersurfaces, in: M. Guest, R. Miyaoka, Y. Ohnita (Eds.), Integrable Systems in Differential Geometry, in: AMS Contemporary Mathematics Series, Vol. 309, 2002, pp. 251–265.
[19] J.-B. Zuber, On Dubrovin topological field theories, Mod. Phys. Lett. A 9 (1994) 749–760.