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Mixed n-step MIR inequalities: Facets for the n-mixing set

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ABSTRACT

Günlük and Pochet [O. Günlük, Y. Pochet, Mixing mixed integer inequalities. Mathematical Programming 90 (2001) 429–457] proposed a procedure to *mix* mixed integer rounding (MIR) inequalities. The mixed MIR inequalities define the convex hull of the mixing set $\{(y^1, \ldots, y^m, v) \in \mathbb{Z}^m \times \mathbb{R}_+ : \alpha_1 y^i + v \geq \beta_i, i = 1, \ldots, m\}$ and can also be used to generate valid inequalities for general as well as several special mixed integer programs (MIPs). In another direction, Kianfar and Fathi [K. Kianfar, Y. Fathi, Generalized mixed integer rounding inequalities: facets for infinite group polyhedra. Mathematical Programming 120 (2009) 313–346] introduced the n-step MIR inequalities for the mixed integer knapsack set through a generalization of MIR. In this paper, we generalize the mixing procedure to the n-step MIR inequalities and introduce the mixed n-step MIR inequalities. We prove that these inequalities define facets for a generalization of the mixing set with n integer variables in each row (which we refer to as the n-mixing set), i.e. $\{(y^1, \ldots, y^m, v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+ : \sum_{j=1}^n \alpha_j y_j^i + v \geq \beta_i, i = 1, \ldots, m\}$. The mixed MIR inequalities are simply the special case of $n = 1$. We also show that mixed n-step MIR can generate valid inequalities based on multiple constraints for general MIPs. Moreover, we introduce generalizations of the capacitated lot-sizing and facility location problems, which we refer to as the multi-module problems, and show that mixed n-step MIR can be used to generate valid inequalities for these generalizations. Our computational results on small MIPLIB instances as well as a set of multi-module lot-sizing instances justify the effectiveness of the mixed n-step MIR inequalities.

1. Introduction

Understanding the polyhedral structures of simple mixed integer sets and using them in developing valid inequalities for general mixed integer programs (MIPs) have been a successful approach. In this paper, we consider a generalization of the well-known mixing set [1], which we refer to as the *n-mixing set*. This set is defined as follows:

$$Q^{m,n} = \left\{ (y^1, \ldots, y^m, v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+ : \sum_{j=1}^{n} \alpha_j y_j^i + v \geq \beta_i, i = 1, \ldots, m \right\},$$

where $\alpha_j \in \mathbb{R}, \alpha_j > 0, j = 1, \ldots, n$ and $\beta_i \in \mathbb{R}, i = 1, \ldots, m$. The mixing set studied by Günlük and Pochet [1] is the special case of $Q^{m,1}$. They showed that the mixed integer rounding (MIR) inequalities [2,3] (called *1-step* MIR inequalities in this paper) based on individual constraints of $Q^{m,1}$ can be *mixed* in a particular way to generate valid inequalities for $Q^{m,1}$, which also define the convex hull of this set. The mixed 1-step MIR inequalities can also be used to generate valid inequalities for

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general mixed integer sets. Moreover, they generate valid inequalities for special structure MIPs such as constant-capacity lot-sizing, facility location, and network design problems [1]. Variations of the mixing set $Q^{m,1}$ have also been studied: an important variation is the mixing set with different capacities, i.e. the set

$$\widetilde{Q}^m = \left\{(y^1, \dots, y^m, v) \in \mathbb{Z}^m \times \mathbb{R}_+ : \alpha_1^i y^i + v \geq \beta_i, i = 1, \dots, m\right\},$$

where $\alpha_1^i, \beta_i \in \mathbb{R}$ and $\alpha_1^i > 0, i = 1, \dots, m$. The set $\widetilde{Q}^2$ with divisible capacities, i.e. when $\alpha_1^2 | \alpha_1^1$, was studied in [4], and the set $\widetilde{Q}^2$ where capacities are not necessarily divisible was studied in [5]. The set $\widetilde{Q}^m$ with divisible capacities, i.e. when $\alpha_1^m | \alpha_1^{m-1} | \cdots | \alpha_1^1$, was studied in [6,7]. A simple algorithm for linear optimization over $\widetilde{Q}^m$ with divisible capacities along with a compact extended formulation for this set was devised in [8]. Other variants of the mixing set $Q^{m,1}$ include the continuous mixing set [9,10], the mixing set with flows [11] and the mixing set linked by bidirected paths [12]. The mixing inequalities of [1] for general mixed integer sets were studied from a group-theoretic perspective in [13] and bounds on their MIR rank were proposed in [14,15].

In another direction, Kianfar and Fathi [16] presented the n-step MIR inequalities for the general mixed integer knapsack set through a generalization of MIR. These inequalities are facet-defining for the mixed integer knapsack set under certain conditions [17]. Although their theoretical derivation is rather involved, the n-step MIR inequalities are easily generated by applying the so-called n-step MIR functions on a general mixed integer constraint. The n-step MIR functions also define extreme inequalities for the infinite group polyhedra and can be used to generate facets for finite cyclic group polyhedra [18]. A variant of the n-step MIR inequalities are the n-step mingling inequalities, which utilize the bounds on integer variables to generate stronger inequalities for general MIPs, which are facet-defining in many cases [17].

In this paper, we show that the idea of mixing can be generalized to n-step MIR inequalities. Through this generalization, we develop the *type I and type II mixed n-step MIR inequalities* for the n-mixing set $Q^{m,n}$ under the condition that for each constraint i of $Q^{m,n}$ used in the mixing, α_j's and β_i satisfy the same conditions required for validity of the n-step MIR inequality, i.e. $\alpha_j \left\lceil \beta_i^{(j-1)}/\alpha_j \right\rceil \leq \alpha_{j-1}, j = 2, \dots, n$ (Section 3). The mixed MIR inequalities of [1] simply correspond to the special case of $n = 1$. We then demonstrate the strength of the mixed n-step MIR inequalities by showing that the type I mixed n-step MIR inequalities define facets for the convex hull of $Q^{m,n}$, denoted by $conv(Q^{m,n})$, and type II mixed n-step MIR inequalities define faces of dimension at least $n(m-1)$ for $conv(Q^{m,n})$ and are facet-defining for this set if some additional conditions are satisfied (Section 4).

We then show how the mixed n-step MIR inequalities for $Q^{m,n}$ can be used to generate mixed n-step MIR inequalities for the general multi-constraint mixed integer set

$$Y_m = \left\{(x_1, \dots, x_N, s) \in \mathbb{Z}_+^N \times \mathbb{R}_+^m : \sum_{t \in T} a_{it} x_t + s_i \geq b_i, i = 1, \dots, m\right\},$$

where $T = \{1, \dots, N\}$ and $a_{it}, b_i \in \mathbb{R}$ for all i and t (Section 5). Note that any set defined by m mixed integer constraints can be relaxed to a set of the form Y_m (see Section 5). As a result, for a general MIP, the mixed n-step MIR generates valid inequalities that are based on multiple constraints. A mixed n-step MIR inequality for Y_m has n positive parameters, namely $\alpha_1, \dots, \alpha_n$, which must satisfy the n-step MIR conditions, i.e. $\alpha_j \left\lceil b_i^{(j-1)}/\alpha_j \right\rceil \leq \alpha_{j-1}, j = 2, \dots, n$, for any constraint i of Y_m that is used in generating the inequality. Any set of values for the parameters $\alpha_1, \dots, \alpha_n$ that satisfy these conditions give a corresponding mixed n-step MIR inequality for Y_m. Notice that for the validity of the mixed n-step MIR inequality for Y_m, no conditions on the coefficients a_{it} in Y_m are required. In other words, the restriction of n-step MIR conditions is only on the parameters of the cut, i.e. $\alpha_1, \dots, \alpha_n$, and as we will see in Section 5, there are always infinitely many choices for these parameters that satisfy the n-step MIR conditions.

Next, we introduce a generalization of the capacitated lot-sizing problem, which we refer to as the multi-module lot-sizing problem (MML), and show that the mixed n-step MIR inequalities can be used to generate valid inequalities for this problem. In MML, the total capacity in each period is the summation of integer multiples of several modules of different capacities. The mixed n-step MIR inequalities for MML generalize the (k, l, S, I) inequalities for the constant-capacity lot-sizing problem (CCL) [1,19]. Similarly, we also introduce a generalization of the capacitated facility location problem, which we refer to as the multi-module facility location problem (MMF), and show that the mixed n-step MIR inequalities can be used to generate valid inequalities for this problem. The mixed n-step MIR inequalities for MMF generalize the mixed MIR inequalities for the constant-capacity facility location problem (CCF) [1,20,21] (Section 6).

Finally, we provide our preliminary computational results on using the mixed n-step MIR inequalities in solving small MIPLIB instances [22] as well as a set of MML instances (Section 7). These results justify the effectiveness of the mixed n-step MIR inequalities.

We also note that in the special case where the parameters $\alpha_j, j = 1, \dots, n$, in $Q^{m,n}$ are *divisible*, i.e. $\alpha_n | \alpha_{n-1} | \cdots | \alpha_1$, the n-step MIR validity conditions are always satisfied. Consequently, all results in this paper are always true for the special case of divisible parameters (as we will see in Section 6, in the case of MML and MMF, the parameters $\alpha_j, j = 1, \dots, n$, are the capacities of modules).

First, we briefly review the necessary concepts related to the mixed MIR and the n-step MIR inequalities in Section 2.

2. Necessary background

In this section, we briefly review the n-step MIR inequalities [16] and the mixed MIR inequalities [1]. Kianfar and Fathi [16] studied the single-constraint set

$$Q^{1,n} = \left\{ (y_1, \ldots, y_n, v) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+ : \sum_{j=1}^{n} \alpha_j y_j + v \geq \beta \right\}$$

(recall that $\alpha_j \in \mathbb{R}, \alpha_j > 0, j = 1, \ldots, n$, and $\beta \in \mathbb{R}$). They developed the n-step MIR inequality for this set. To present this inequality we define the following notation: for $\beta \in \mathbb{R}$ define the recursive remainders $\beta^{(j)} := \beta^{(j-1)} - \alpha_j \lfloor \beta^{(j-1)}/\alpha_j \rfloor$, where $\beta^{(0)} := \beta$. Note that $0 \leq \beta^{(j)} < \alpha_j$ for $j = 1, \ldots, n$. We also assume that $\sum_a^b(\cdot) = 0$ and $\prod_a^b(\cdot) = 1$ whenever $a > b$. Kianfar and Fathi [16] showed that if the n-step MIR conditions are satisfied, i.e.

$$\alpha_j \lceil \beta^{(j-1)}/\alpha_j \rceil \leq \alpha_{j-1}, \quad j = 2, \ldots, n, \tag{1}$$

then the n-step MIR inequality

$$\beta^{(n)} \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil y_j + v \geq \beta^{(n)} \prod_{l=1}^{n} \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil \tag{2}$$

is valid for $Q^{1,n}$ and defines a facet for $conv(Q^{1,n})$. Note that for (2) to be non-trivial, it is also assumed $\beta^{(j-1)}/\alpha_j \notin \mathbb{Z}$, $j = 1, \ldots, n$. An intermediate result from [16], which will be useful in this paper, is that the inequalities

$$\alpha_j \left(\sum_{i=1}^{j} \prod_{l=i+1}^{j} \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil y_i - \prod_{l=1}^{j} \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil + \left\lceil \frac{\beta^{(j-1)}}{\alpha_j} \right\rceil \right) + \sum_{i=j+1}^{n} \alpha_i y_i + v \geq \beta^{(j-1)}; \quad j = 1, \ldots, n \tag{3}$$

are also valid for $Q^{1,n}$ if conditions (1) are satisfied.

Moreover, through a procedure that uses the facet-defining property of inequality (2) for $Q^{1,n}$, Kianfar and Fathi [16] also developed the n-step MIR inequality for the general mixed integer knapsack set $Y_1 = \{(x_1, \ldots, x_N, s) \in \mathbb{Z}_+^N \times \mathbb{R}_+ : \sum_{j \in T} a_t x_t + s \geq b\}$, where the coefficients a_t, $t \in T$, and the right-hand side b are real numbers that do not need to satisfy any conditions. To generate an n-step MIR inequality for Y_1, a parameter vector $\alpha = (\alpha_1, \ldots, \alpha_n) > 0$ is chosen. These parameters must satisfy the n-step MIR conditions, i.e. $\alpha_j \lceil b^{(j-1)}/\alpha_j \rceil \leq \alpha_{j-1}, j = 2, \ldots, n$. The n-step MIR inequality for Y_1 is then

$$\sum_{t \in T} \mu^n_{\alpha,b}(a_t) x_t + s \geq \mu^n_{\alpha,b}(b), \tag{4}$$

where $\mu^n_{\alpha,b}$ is the so-called n-step MIR function. Each instance of the function $\mu^n_{\alpha,b}$ is completely defined by the parameter vector α and the right-hand side b. The formulation and properties of this function are presented in [16,17]. According to (4), the n-step MIR inequality is obtained by applying the n-step MIR function on a_t's and b. It is also interesting to note that $Q^{1,n}$ is a special case of Y_1. So we can use (4) to generate a valid inequality for $Q^{1,n}$. If we do so, we get exactly (2).

In another direction, Günlük and Pochet [1] presented the mixed MIR inequalities for the 1-mixing set $Q^{m,1}$. These inequalities are generated by "*mixing*" the 1-step MIR inequalities written for the individual constraints of $Q^{m,1}$ as follows. Let $M = \{1, \ldots, m\}$. The 1-step MIR inequality [1,3,23] for the inequality i in $Q^{m,1}$ can be written as

$$v \geq \beta_i^{(1)} \left(\lceil \beta_i/\alpha_1 \rceil - y^i \right) \tag{5}$$

(inequality (5) can be obtained from inequality (2) by setting $n = 1$ and $\beta = \beta_i$). Consider a non-empty $K \subseteq M$. To simplify the notation and without loss of generality we assume $K = \{1, \ldots, k\}$ and $\beta_{i-1}^{(1)} \leq \beta_i^{(1)}, i = 2, \ldots, k$. By *mixing* the 1-step MIR inequalities (5) for $i \in K$, Günlük and Pochet [1] presented the following inequalities for $Q^{m,1}$:

$$v \geq \sum_{i=1}^{k} \left(\beta_i^{(1)} - \beta_{i-1}^{(1)} \right) \left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil - y^i \right) \tag{6}$$

$$v \geq \sum_{i=1}^{k} \left(\beta_i^{(1)} - \beta_{i-1}^{(1)} \right) \left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil - y^i \right) + \left(\alpha_1 - \beta_k^{(1)} \right) \left(\left\lceil \frac{\beta_1}{\alpha_1} \right\rceil - y^1 - 1 \right), \tag{7}$$

where $\beta_0^{(1)} = 0$ by definition. We refer to (6) and (7) as the *type I* and *type II* mixed MIR inequalities generated by K, respectively. It is shown in [1] that the convex hull of $Q^{m,1}$ is completely described by inequalities of the form (6) and (7) generated by all possible subsets K of M.

3. Mixed *n*-step MIR inequalities for the *n*-mixing set

In this section, we show that mixing can be generalized to the n-step MIR inequalities. In other words, one can mix the n-step MIR inequalities written for the individual constraints of the n-mixing set $Q^{m,n}$ and get a valid inequality based on multiple constraints (called the mixed n-step MIR inequality) for this set. Any subset of constraints of $Q^{m,n}$ can be chosen to be mixed. Let $K \subseteq M$ denote the index set of the chosen constraints. To simplify the notation and without loss of generality throughout the paper we assume $K = \{1, \ldots, k\}$ and $\beta_{i-1}^{(n)} \leq \beta_i^{(n)}, i = 2, \ldots, k$. Also note that according to (1), for the n-step MIR inequality to be valid for each *base* constraint i, $i \in K$, the conditions

$$\alpha_j \left\lceil \beta_i^{(j-1)}/\alpha_j \right\rceil \leq \alpha_{j-1}, \quad j = 2, \ldots, n, i \in K \tag{8}$$

must be satisfied (as mentioned, the assumptions $\beta_i^{(j-1)}/\alpha_j \notin \mathbb{Z}, j = 1, \ldots, n, i \in K$ are also required to avoid trivial inequalities). Now assuming (8) holds, the n-step MIR inequality (2) written for constraint i of $Q^{m,n}$, $i \in K$, is valid for $Q^{m,n}$ and can be written as

$$v \geq \beta_i^{(n)} \left(\prod_{l=1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil y_j^i \right). \tag{9}$$

To simplify notation in the rest of the paper, we define the function $\phi^i : \mathbb{Z}^n \to \mathbb{Z}$ to denote the integer-valued expression inside the parentheses in (9) and refer to it as the n-mixing function, i.e.

$$\phi^i(y^i) := \prod_{l=1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil y_j^i \quad \text{for } i \in K. \tag{10}$$

Note that ϕ^i is a function of variables $y^i = (y_1^i, \ldots, y_n^i)$ which depends on parameters α and β_i. Now the n-step MIR inequality (9) can be written as

$$v \geq \beta_i^{(n)} \phi^i(y^i). \tag{11}$$

We show that inequalities (11), $i \in K$, can be mixed to obtain the following valid inequalities for $Q^{m,n}$:

$$v \geq \sum_{i=1}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i), \tag{12}$$

$$v \geq \sum_{i=1}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) + \left(\alpha_n - \beta_k^{(n)} \right) \left(\phi^1(y^1) - 1 \right), \tag{13}$$

where $\beta_0^{(n)} = 0$ by definition. We refer to (12) and (13) as the *type I and type II mixed n-step MIR inequalities*, respectively. The validity of (12) and (13) can be proved using an argument similar to the one used in [1] for validity of (6) and (7) but requires an additional lemma.

Lemma 1. *For $i \in K$, the inequality*

$$v \geq \beta_i^{(n)} + \alpha_n \left(\phi^i(y^i) - 1 \right) \tag{14}$$

is valid for $Q^{m,n}$.

Proof. For $i \in K$, since (8) holds, inequality (3) written for the constraint i of $Q^{m,n}$ and $j = n$, i.e.

$$\alpha_n \left(\sum_{i=1}^{n} \prod_{l=i+1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil y_i - \prod_{l=1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil + \left\lceil \frac{\beta_i^{(n-1)}}{\alpha_n} \right\rceil \right) + v \geq \beta_i^{(n-1)} \tag{15}$$

is valid for $Q^{m,n}$. By subtracting $\alpha_n \left\lfloor \beta_i^{(n-1)}/\alpha_n \right\rfloor$ from both sides and re-arranging the terms we get (14). □

Theorem 2. *If conditions (8) hold, the type I and type II mixed n-step MIR inequalities (12) and (13) are valid for $Q^{m,n}$.*

Proof. To prove the validity of (12), consider a fixed point $(\hat{y}^1, \ldots, \hat{y}^m, \hat{v}) \in Q^{m,n}$. Define $\lambda := \max_{i \in K} \phi^i(\hat{y}^i)$ and $p := \max\{i \in K : \phi^i(\hat{y}^i) = \lambda\}$. If $\lambda \leq 0$, then it is trivial that (12) is satisfied because $\hat{v} \geq 0$, and by the assumed ordering of indices in K, $\beta_i^{(n)} - \beta_{i-1}^{(n)} \geq 0$, $i \in K$. If $\lambda \geq 1$, then since $\phi^i(\hat{y}^i)$ is an integer, we can write

$$\begin{aligned}
\sum_{i=1}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)}\right) \phi^i(\hat{y}^i) &\leq \sum_{i=1}^{p} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)}\right) \lambda + \sum_{i=p+1}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)}\right) (\lambda - 1) \\
&= \beta_p^{(n)}(\lambda) + \left(\beta_k^{(n)} - \beta_p^{(n)}\right)(\lambda - 1) \\
&= \beta_p^{(n)} + \beta_k^{(n)}(\lambda - 1) \\
&\leq \beta_p^{(n)} + \alpha_n(\lambda - 1) \\
&= \beta_p^{(n)} + \alpha_n(\phi^p(\hat{y}^p) - 1) \\
&\leq \hat{v}.
\end{aligned}$$

The last inequality follows from Lemma 1. This proves the validity of (12). The validity of (13) can be proved very similarly. □

Note that for $n = 1$ this proof reduces to the proof of validity of the mixed 1-step MIR inequalities in [1], where Lemma 1 was not required because for $n = 1$ inequality (14) simply reduces to the base inequality $\alpha_1 y_1^i + v \geq \beta_i$.

Consider the following generalization of $Q^{m,n}$ which has different continuous variables in each row:

$$\widehat{Q}^{m,n} = \left\{ (y^1, \ldots, y^m, v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^m : \sum_{j=1}^{n} \alpha_j y_j^i + v_i \geq \beta_i, i = 1, \ldots, m \right\}.$$

Let the variable $\overline{v} \in \mathbb{R}_+$ be such that $\overline{v} \geq v_i$ for all $i \in K$. Then as a direct result of Theorem 2, we have the following.

Corollary 3. *If conditions (8) hold, the mixed n-step MIR inequalities*

$$\overline{v} \geq \sum_{i=1}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)}\right) \phi^i(y^i) \tag{16}$$

$$\overline{v} \geq \sum_{i=1}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)}\right) \phi^i(y^i) + \left(\alpha_n - \beta_k^{(n)}\right) \left(\phi^1(y^1) - 1\right) \tag{17}$$

are valid for $\widehat{Q}^{m,n}$. □

Remark 1 (*Divisible Coefficients*). An interesting special case of the n-mixing set $Q^{m,n}$ is when the coefficients are divisible, i.e. $\alpha_j | \alpha_{j-1}$, $j = 2, \ldots, n$. Note that in this case for any $i \in K$ and $j \in \{2, \ldots, n\}$, by definition of $\beta_i^{(j-1)}$, we have $\alpha_{j-1}/\alpha_j \geq \beta_i^{(j-1)}/\alpha_j$, which implies $\alpha_{j-1}/\alpha_j \geq \left\lceil \beta_i^{(j-1)}/\alpha_j \right\rceil$ because α_{j-1}/α_j is an integer. That means in this case conditions (8) are automatically satisfied. Consequently, all results in this paper are always true for the case where the elements of the parameter vector α are divisible, i.e. $\alpha_j | \alpha_{j-1}$, $j = 2, \ldots, n$. □

4. Facets defined by mixed *n*-step MIR inequalities

In this section, we prove that the type I mixed n-step MIR inequalities define facets for $conv(Q^{m,n})$. We also show that the type II inequalities define faces of dimension at least $n(m-1)$ for $conv(Q^{m,n})$ and define facets for this set if some additional conditions on parameters are satisfied. These results demonstrate the strength and importance of these inequalities. Note that $conv(Q^{m,n})$ is non-empty and full-dimensional (is of dimension $mn + 1$). That is because a point $P = (y^1, \ldots, y^m, v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}^+$ with sufficiently large coordinates is feasible to $Q^{m,n}$ (since $\alpha_j > 0$, $j = 1, \ldots, n$) and $P + e \in Q^{m,n}$ for all unit vectors $e \in \mathbb{R}^{mn+1}$.

To prove the facet-defining property of the type I mixed n-step MIR inequality, we need to define some points and prove some properties for them first.

Definition 4. For $i \in M$, $t = 1, \ldots, n$, define the points $p^{i,t} = (p_1^{i,t}, \ldots, p_n^{i,t}) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$ such that

$$p_j^{i,t} = \begin{cases} \left\lfloor \beta_i^{(j-1)}/\alpha_j \right\rfloor & \text{for } j = 1, \ldots, t-1 \\ \left\lceil \beta_i^{(j-1)}/\alpha_j \right\rceil & \text{for } j = t \\ 0 & \text{for } j = t+1, \ldots, n, \end{cases}$$

and for $i \in K$, $t = 1, \ldots, n$, define the points $q^{i,t} = (q_1^{i,t}, \ldots, q_n^{i,t}) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$ such that

$$q_j^{i,t} = \begin{cases} \left\lfloor \beta_i^{(j-1)}/\alpha_j \right\rfloor & \text{for } j = 1, \ldots, t \\ 0 & \text{for } j = t+1, \ldots, n. \end{cases}$$

Lemma 5. *The point $P = (\hat{y}^1, \ldots, \hat{y}^m, \hat{v}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$ satisfies constraint i of $Q^{m,n}$ if any of the following is true:*

(a) *$i \in M$ and $\hat{y}^i = p^{i,t}$ for some $t \in \{1, \ldots, n\}$,*
(b) *$i \in K$ and $\hat{y}^i = q^{i,t}$ for some $t \in \{1, \ldots, n\}$ and $\hat{v} \geq \beta_i^{(t)}$.*

Proof. See Appendix. □

Lemma 6. *For $i \in M$, $\phi^i(p^{i,t}) = 0$, $t = 1, \ldots, n$, and for $i \in K$, $\phi^i(q^{i,n}) = 1$.*

Proof. See Appendix. □

Recall that without loss of generality we have assumed that the set of indices of inequalities used in mixing are $K = \{1, \ldots, k\}$, where $\beta_{i-1}^{(n)} \leq \beta_i^{(n)}$, $i = 2, \ldots, k$.

Theorem 7. *If conditions (8) hold, the type I mixed n-step MIR inequality (12) defines a facet for $conv(Q^{m,n})$.*

Proof. Consider the support hyperplane of inequality (12), i.e.

$$v = \sum_{i=1}^{k} (\beta_i^{(n)} - \beta_{i-1}^{(n)}) \phi^i(y^i) \tag{18}$$

and the face defined by it, i.e. $F_1 = \{(y^1, \ldots, y^m, v) \in conv(Q^{m,n}) : (18)\}$. We prove that any generic hyperplane

$$\lambda_0 v + \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \lambda_j^i y_j^i \right) = \theta \tag{19}$$

that passes through F_1 has to be a scalar multiple of (18). For this, consider the point $P^1 = (p^{1,1}, \ldots, p^{m,1}, 0) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$. By Lemma 5(a), $P^1 \in Q^{m,n}$ and by Lemma 6, P^1 satisfies (18) so $P^1 \in F_1$, and hence must satisfy (19) too. This means

$$\sum_{i=1}^{m} \lambda_1^i \left\lceil \frac{\beta_i}{\alpha_1} \right\rceil = \theta. \tag{20}$$

Based on (20), hyperplane (19) reduces to

$$\lambda_0 v = \sum_{i=1}^{m} \left(\lambda_1^i \left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil - y_1^i \right) - \sum_{j=2}^{n} \lambda_j^i y_j^i \right). \tag{21}$$

For $i \in M$, consider the point $P^{i,2} = (p^{1,1}, \ldots, p^{i-1,1}, p^{i,2}, p^{i+1,1}, \ldots, p^{m,1}, 0) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$. Again by Lemmas 5 and 6, $P^{i,2} \in F_1$, and hence must satisfy (21) too. Substituting $P^{i,2}$, $i \in M$, in (21) gives

$$\lambda_1^i = \lambda_2^i \left\lceil \frac{\beta_i^{(1)}}{\alpha_2} \right\rceil, \quad i \in M. \tag{22}$$

Based on (22), hyperplane (21) reduces to

$$\lambda_0 v = \sum_{i=1}^{m} \left(\lambda_2^i \left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil \left\lceil \frac{\beta_i^{(1)}}{\alpha_2} \right\rceil - \left\lceil \frac{\beta_i^{(1)}}{\alpha_2} \right\rceil y_1^i - y_2^i \right) - \sum_{j=3}^{n} \lambda_j^i y_j^i \right). \tag{23}$$

Starting with (23), and for each $i \in M$, repeating the same argument using the points $P^{i,3}, P^{i,4}, \ldots, P^{i,n} \in F_1$ one after the other, where $P^{i,t} = (p^{1,1}, \ldots, p^{i-1,1}, p^{i,t}, p^{i+1,1}, \ldots, p^{m,1}, 0)$ for $t = 1, \ldots, n$, we get the identities

$$\lambda_{t-1}^i = \lambda_t^i \left\lceil \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rceil, \quad t = 2, \ldots, n, i \in M. \tag{24}$$

Based on (24), we get the identities

$$\lambda_t^i = \lambda_n^i \prod_{j=t+1}^{n} \left\lceil \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rceil, \quad t = 1, \ldots, n-1, i \in M, \tag{25}$$

which reduce hyperplane (23) to

$$\lambda_0 v = \sum_{i=1}^{m} \lambda_n^i \left(\prod_{l=1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil y_j^i \right),$$

or

$$\lambda_0 v = \sum_{i=1}^{m} \lambda_n^i \phi^i(y^i). \tag{26}$$

Now for $i \in K$, consider the point $S^i = (q^{1,n}, \ldots, q^{i,n}, p^{i+1,1}, \ldots, p^{m,1}, \beta_i^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$. Since $\beta_t^{(n)} \le \beta_i^{(n)}$ for $t = 1, \ldots, i$, by Lemma 5, $S^i \in Q^{m,n}$. By Lemma 6, S^i satisfies (18) so $S^i \in F_1$, and hence must satisfy (26). Substituting in (26) gives

$$\lambda_0 \beta_i^{(n)} = \sum_{t=1}^{i} \lambda_n^t, \quad i \in K,$$

which implies

$$\lambda_n^i = \lambda_0 \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right), \quad i \in K. \tag{27}$$

Identities (27) reduce hyperplane (26) to

$$\lambda_0 v = \lambda_0 \sum_{i=1}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) + \sum_{i=k+1}^{m} \lambda_n^i \phi^i(y^i). \tag{28}$$

Now for $i = k+1, \ldots, m$, consider the point $G^i = (p^{1,1}, \ldots, p^{i-1,1}, g^i, p^{i+1,1}, \ldots, p^{m,1}, 0) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$, where $g^i \in \mathbb{Z} \times \mathbb{Z}_+^{m-1}$, $\phi^i(g^i) \ne 0$, and g^i has sufficiently large coordinates for $(g^i, 0)$ to satisfy constraint i in $Q^{m,n}$ (clearly such g^i exists because $\alpha_j > 0, j = 1, \ldots, n$). Therefore using Lemma 5, $G^i \in Q^{m,n}$. Also, based on Lemma 6, G^i satisfies (18), so $G^i \in F_1$, and hence must satisfy (28). Substituting G^i in (28), based on Lemma 6 and since $\phi^i(g^i) \ne 0$, we get $\lambda_n^i = 0$. Therefore, $\lambda_n^i = 0, i = k+1, \ldots, m$, so (28) reduces to $\lambda_0 v = \lambda_0 \sum_{i=1}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i)$, which is λ_0 times (18). This completes the proof. □

Next we address the type II mixed n-step MIR inequality. We will show that the face defined by a type II mixed n-step MIR inequality for $conv(Q^{m,n})$ has always a dimension of at least $n(m-1)$, and moreover, is a facet if some additional conditions on $(\alpha_1, \ldots, \alpha_n)$, β_1, and β_k are satisfied. To prove this result first we define some more points and establish some properties for them.

Definition 8. Assuming $\left\lfloor \beta_1^{(j-1)}/\alpha_j \right\rfloor \ge 1, j = 2, \ldots, n$, define the points $r^t = (r_1^t, \ldots, r_n^t) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}, t = 2, \ldots, n$, such that

$$r_j^t = \begin{cases} \left\lfloor \beta_1^{(j-1)}/\alpha_j \right\rfloor & \text{for } j = 1, \ldots, t-2 \\ \left\lfloor \beta_1^{(j-1)}/\alpha_j \right\rfloor - 1 & \text{for } j = t-1 \\ 2\left\lfloor \beta_1^{(j-1)}/\alpha_j \right\rfloor + 1 & \text{for } j = t \\ \left\lfloor \beta_1^{(j-1)}/\alpha_j \right\rfloor & \text{for } j = t+1, \ldots, n \end{cases}$$

and the point $s = (s_1, \ldots, s_n) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$ such that $s = q^{1,n} - e_n$, where $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$.

Lemma 9. *The point $P = (\hat{y}^1, \ldots, \hat{y}^m, \hat{v}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$ satisfies constraint 1 of $Q^{m,n}$ if any of the following is true:*

(a) $\hat{y}^1 = r^t$ *for some* $t \in \{2, \ldots, n\}$ *and* $\hat{v} \ge \beta_1^{(n)} + \alpha_{t-1} - \alpha_t \left\lceil \beta_1^{(t-1)}/\alpha_t \right\rceil$,

(b) $\hat{y}^1 = s$ *and* $\hat{v} \ge \alpha_n + \beta_1^{(n)}$.

Proof. See Appendix. $\square$

Lemma 10. *$\phi^1(r^t) = 1$ for $t = 2, \ldots, n$, and $\phi^1(s) = 2$.*

Proof. See Appendix. $\square$

Theorem 11. *If conditions (8) hold, the type II mixed n-step MIR inequality defines a face of dimension at least $n(m-1)$ for $conv(Q^{m,n})$. Moreover, this inequality defines a facet for $conv(Q^{m,n})$ if the following additional conditions are satisfied:*

(a) $\left\lfloor \beta_1^{(j-1)}/\alpha_j \right\rfloor \geq 1, j = 2, \ldots, n,$

(b) $\beta_k^{(n)} - \beta_1^{(n)} \geq \max \left\{ \alpha_{j-1} - \alpha_j \left\lceil \beta_1^{(j-1)}/\alpha_j \right\rceil, j = 2, \ldots, n \right\}.$

Proof. Consider the support hyperplane of inequality (13), i.e.

$$v = \sum_{i=1}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)}\right) \phi^i(y^i) + \left(\alpha_n - \beta_k^{(n)}\right) \left(\phi^1(y^1) - 1\right), \tag{29}$$

and the face defined by it, i.e. $F_2 = \{(y^1, \ldots, y^m, v) \in conv(Q^{m,n}) : (29)\}$. We prove that any generic hyperplane defined by $(\lambda^1, \ldots, \lambda^m, \lambda_0, \theta) \in \mathbb{R}^{mn+2}$, i.e.

$$\lambda_0 v + \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \lambda_j^i y_j^i \right) = \theta, \tag{30}$$

that passes through F_2 is the linear combination of at most $n+1$ linearly independent hyperplanes, making F_2 a face of dimension at least $mn + 1 - (n+1) = n(m-1)$.

Consider the point $S^1 = (q^{1,n}, p^{2,1}, \ldots, p^{m,1}, \beta_1^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$. As argued in the proof of Theorem 7, $S^1 \in Q^{m,n}$. Moreover, using Lemma 6, it is easy to verify that S^1 satisfies (29). So $S^1 \in F_2$ and hence must satisfy (30). Substituting into (30) gives

$$\lambda_0 \beta_1^{(n)} + \sum_{j=1}^{n} \lambda_j^1 \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor + \sum_{i=2}^{m} \lambda_1^i \left\lceil \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rceil = \theta. \tag{31}$$

Based on (31), hyperplane (30) reduces to

$$\lambda_0 \left(v - \beta_1^{(n)}\right) + \sum_{j=1}^{n} \lambda_j^1 \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor \right) = \sum_{i=2}^{m} \left(\lambda_1^i \left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil - y_1^i \right) - \sum_{j=2}^{n} \lambda_j^i y_j^i \right). \tag{32}$$

Consider the points $R^{i,t} = (q^{1,n}, p^{2,1}, \ldots, p^{i-1,1}, p^{i,t}, p^{i+1,1}, \ldots, p^{m,1}, \beta_1^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+, i = 2, \ldots, m, t = 2, \ldots, n$. By Lemma 5, these points belong to $Q^{m,n}$, and by Lemma 6, they satisfy (29). Therefore $R^{i,t} \in F_2, i = 2, \ldots, m, t = 2, \ldots, n$. Starting with hyperplane (32), and for each $i \in \{2, \ldots, m\}$, substituting the points $R^{i,2}, \ldots, R^{i,n}$ in the hyperplane, one after the other, we get

$$\lambda_{t-1}^i = \lambda_t^i \left\lceil \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rceil, \quad t = 2, \ldots, n, i = 2, \ldots, m. \tag{33}$$

From (33) we get

$$\lambda_t^i = \lambda_n^i \prod_{j=t+1}^{n} \left\lceil \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rceil, \quad t = 2, \ldots, n, i = 2, \ldots, m, \tag{34}$$

which reduces (32) to

$$\lambda_0 \left(v - \beta_1^{(n)}\right) + \sum_{j=1}^{n} \lambda_j^1 \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor \right) = \sum_{i=2}^{m} \lambda_n^i \phi^i(y^i). \tag{35}$$

Now consider the points $S^i = (q^{1,n}, \ldots, q^{i,n}, p^{i+1,1}, \ldots, p^{m,1}, \beta_i^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+, i = 2, \ldots, k$, that were used in the proof of Theorem 7. We argued that these points belong to $Q^{m,n}$. Moreover, using Lemma 6, it can be easily verified that they satisfy (29), so $S^i \in F_2, i = 2, \ldots, k$. Therefore, they must satisfy (35). Substituting $S^i, i = 2, \ldots, k$, in (35), we get

$$\lambda_0 \left(\beta_i^{(n)} - \beta_1^{(n)}\right) = \sum_{t=2}^{i} \lambda_n^t, \quad i = 2, \ldots, k,$$

which implies

$$\lambda_n^i = \lambda_0 \left(\beta_i^{(n)} - \beta_{i-1}^{(n)}\right), \quad i = 2, \dots, k. \tag{36}$$

Identities (36) reduce hyperplane (35) to

$$\lambda_0 \left(v - \sum_{i=2}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)}\right) \phi^i(y^i) - \beta_1^{(n)}\right) + \sum_{j=1}^{n} \lambda_j^1 \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor\right) = \sum_{i=k+1}^{m} \lambda_n^i \phi^i(y^i). \tag{37}$$

Now for $i = k+1, \dots, m$, consider the points $H^i = (q^{1,n}, p^{2,1}, \dots, p^{i-1,1}, h^i, p^{i+1,1}, \dots, p^{m,1}, \beta_1^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$, where $h^i \in \mathbb{Z} \times \mathbb{Z}_+^{m-1}$, $\phi^i(h^i) \neq 0$, and h^i has sufficiently large coordinates for $(h^i, \beta_1^{(n)})$ to satisfy constraint i in $Q^{m,n}$ (clearly such h^i exists because $\alpha_j > 0, j = 1, \dots, n$). Therefore using Lemma 5, $H^i \in Q^{m,n}$. Also, based on Lemma 6, H^i satisfies (29), so $H^i \in F_2$, and hence must satisfy (37). Substituting H^i in (37), based on Lemma 6 and since $\phi^i(h^i) \neq 0$, we get $\lambda_n^i = 0$. Therefore, $\lambda_n^i = 0, i = k+1, \dots, m$, so (37) reduces to

$$\lambda_0 \left(v - \sum_{i=2}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)}\right) \phi^i(y^i) - \beta_1^{(n)}\right) + \sum_{j=1}^{n} \lambda_j^1 \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor\right) = 0. \tag{38}$$

So we have shown that in the generic hyperplane (30) defined by $(\lambda^1, \dots, \lambda^m, \lambda_0, \theta) \in \mathbb{R}^{mn+2}$, at most $(\lambda^1, \lambda_0) \in \mathbb{R}^{n+1}$ are independent. That means the generic hyperplane can be the linear combination of at most $n+1$ linearly independent hyperplanes. This proves that F_2 is a face of dimension at least $n(m-1)$.

To prove the second part of the theorem, assume that the additional conditions (a) and (b) are satisfied. Notice that (29) can also be written as

$$v - \sum_{i=2}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)}\right) \phi^i(y^i) - \beta_1^{(n)} = \left(\alpha_n + \beta_1^{(n)} - \beta_k^{(n)}\right) \left(\phi^1(y^1) - 1\right). \tag{39}$$

Any point on F_2 satisfies both (38) and (39). These two identities together imply that the identity

$$\lambda_0 \left(\alpha_n + \beta_1^{(n)} - \beta_k^{(n)}\right) \left(\phi^1(y^1) - 1\right) + \sum_{j=1}^{n} \lambda_j^1 \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor\right) = 0 \tag{40}$$

holds for any point on F_2. Replacing for $\phi^1(y^1)$ from (A.1) in the proof of Lemma 10 in Appendix, identity (40) can be written as

$$\sum_{j=1}^{n} c_j \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor\right) = 0 \tag{41}$$

where $c_j = \lambda_j^1 - \lambda_0 \left(\alpha_n + \beta_1^{(n)} - \beta_k^{(n)}\right) \prod_{l=j+1}^{n} \left\lceil \beta_1^{(l-1)}/\alpha_l \right\rceil$. Now, consider the point $U = (s, q^{2,n}, \dots, q^{k,n}, p^{k+1,1}, \dots, p^{m,1}, \alpha_n + \beta_1^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$ (condition (a) guarantees that $s \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$). By Lemma 9(b), U satisfies constraint 1 of $Q^{m,n}$, and by Lemma 5, it satisfies constraints $2, \dots, m$ of $Q^{m,n}$; therefore $U \in Q^{m,n}$. Also using Lemmas 6 and 10, it is easy to verify that U lies on (29). Therefore $U \in F_2$ and must satisfy (41). Similarly, for $t = 2, \dots, n$ consider the point $V^t = (r^t, q^{2,n}, \dots, q^{k,n}, p^{k+1,1}, \dots, p^{m,1}, \beta_k^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$ (condition (a) guarantees that $r^t \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$). By Lemma 9 and condition (b) of this theorem, V^t satisfies the first constraint of $Q^{m,n}$, and by Lemma 5, it satisfies constraints $2, \dots, m$ of $Q^{m,n}$. Therefore $V^t \in Q^{m,n}$, $t = 2, \dots, n$. Moreover, using Lemmas 6 and 10, it can be easily verified that the points V^t, $t = 2, \dots, n$, lie on hyperplane (29) and so $V^t \in F_2$, $t = 2, \dots, n$, and must satisfy (41). Starting with identity (41), and substituting in it the points $U, V^n, V^{n-1}, \dots, V^2$ one by one in that order, we get $c_n = 0, c_{n-1} = 0, \dots, c_1 = 0$, respectively. Therefore

$$\lambda_j^1 = \lambda_0 \left(\alpha_n + \beta_1^{(n)} - \beta_k^{(n)}\right) \prod_{l=j+1}^{n} \left\lceil \beta_1^{(l-1)}/\alpha_l \right\rceil, \quad j = 1, \dots, n. \tag{42}$$

Identities (42) reduce hyperplane (38) to

$$\lambda_0 \left(v - \sum_{i=2}^{k} \left(\beta_i^{(n)} - \beta_{i-1}^{(n)}\right) \phi^i(y^i) - \beta_1^{(n)} + \left(\alpha_n + \beta_1^{(n)} - \beta_k^{(n)}\right) \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor\right)\right) = 0. \tag{43}$$

Using (A.1), hyperplane (43) can be written as

$$\lambda_0\left(v-\sum_{i=2}^{k}\left(\beta_i^{(n)}-\beta_{i-1}^{(n)}\right)\phi^i(y^i)-\beta_1^{(n)}-\left(\alpha_n+\beta_1^{(n)}-\beta_k^{(n)}\right)\left(\phi^1(y^1)-1\right)\right)=0,$$

or

$$\lambda_0\left(v-\sum_{i=1}^{k}\left(\beta_i^{(n)}-\beta_{i-1}^{(n)}\right)\phi^i(y^i)-\left(\alpha_n-\beta_k^{(n)}\right)\left(\phi^1(y^1)-1\right)\right)=0,$$

which is simply λ_0 times (29). This proves that F_2 defines a facet for $conv(Q^{m,n})$. □

Example 1. Consider the 3-mixing set with 2 rows $Q^{2,3}=\{(y^1,y^2,v)\in(\mathbb{Z}\times\mathbb{Z}_+^2)^2\times\mathbb{R}_+ : 31y_1^1+10y_2^1+3y_3^1+v\geq 89;\ 31y_1^2+10y_2^2+3y_3^2+v\geq 59\}$. Therefore $\alpha=(\alpha_1,\alpha_2,\alpha_3)=(31,10,3)$, $\beta_1=89$, $\beta_2=59$, and we have $\beta_1^{(1)}=27$, $\beta_1^{(2)}=7$, $\beta_1^{(3)}=1$, $\beta_2^{(1)}=28$, $\beta_2^{(2)}=8$, and $\beta_2^{(3)}=2$. So $\lceil\beta_1^{(1)}/\alpha_2\rceil=\lceil\beta_1^{(2)}/\alpha_3\rceil=\lceil\beta_2^{(1)}/\alpha_2\rceil=\lceil\beta_2^{(2)}/\alpha_3\rceil=3$ and it is easily verified that conditions (8) are satisfied. Therefore, based on (12) and (13), the type I and type II mixed 3-step MIR inequalities obtained from the two defining inequalities of $Q^{2,3}$ are as follows (note that $\beta_1^{(3)}<\beta_2^{(3)}$):

$$v\geq(27-9y_1^1-3y_2^1-y_3^1)+(18-9y_1^2-3y_2^2-y_3^2),\tag{44}$$

$$v\geq(27-9y_1^1-3y_2^1-y_3^1)+(18-9y_1^2-3y_2^2-y_3^2)+(27-9y_1^1-3y_2^1-y_3^1-1).\tag{45}$$

Based on Theorem 7, inequality (44) defines a facet for $conv(Q^{2,3})$. The additional conditions (a) and (b) of Theorem 11 are also satisfied, i.e. (a) $\lfloor\beta_1^{(1)}/\alpha_2\rfloor=\lfloor\beta_1^{(2)}/\alpha_3\rfloor=2>1$, and (b) $\beta_2^{(3)}-\beta_1^{(3)}=1\geq 1=\max\{\alpha_1-\alpha_2\lceil\beta_1^{(1)}/\alpha_2\rceil,\alpha_2-\alpha_3\lceil\beta_1^{(2)}/\alpha_3\rceil\}$. Therefore, based on Theorem 11, inequality (45) also defines a facet for $conv(Q^{2,3})$.

Similarly, consider the 2-mixing set $Q^{2,2}=\{(y^1,y^2,v)\in(\mathbb{Z}\times\mathbb{Z}_+)^2\times\mathbb{R}_+ : 31y_1^1+10y_2^1+v\geq 89;\ 31y_1^2+10y_2^2+v\geq 59\}$. It is easy to see that conditions (8) as well as conditions (a) and (b) of Theorem 11 are satisfied as $\alpha_1,\alpha_2,\beta_1$, and β_2 have the same values as above. Therefore, the type I and type II mixed 2-step MIR inequalities

$$v\geq 7(9-3y_1^1-y_2^1)+(6-3y_1^2-y_2^2)$$

$$v\geq 7(9-3y_1^1-y_2^1)+(6-3y_1^2-y_2^2)+2(9-3y_1^1-y_2^1-1)$$

are facet-defining for $conv(Q^{2,2})$ based on Theorems 7 and 11, respectively. □

5. Mixed *n*-step MIR inequalities for general MIP

As mentioned in Section 2, n-step MIR can be used to generate valid inequalities for the general single-constraint mixed integer knapsack set Y_1 [16]. In this section, we show that the mixed n-step MIR inequality for the set $Q^{m,n}$ can be used to generate mixed n-step MIR inequalities for the general multi-constraint mixed integer set Y_m. This implies that mixed n-step MIR can generate valid inequalities based on multiple constraints for a general MIP because the feasible set of a general MIP with m constraints can be relaxed to a set of the form Y_m as follows. Define the feasible set of a general MIP as $\{(x,w)\in\mathbb{Z}_+^N\times\mathbb{R}_+^{|C|} : \sum_{t\in T}a_{it}x_t+\sum_{t\in C}c_{it}w_t=b_i, i=1,\ldots,m\}$, where C is the index set of the continuous variables w, and $b_i,a_{it},c_{it}\in\mathbb{R}$ for all i and j. This set can be relaxed to $\{(x,w)\in\mathbb{Z}_+^N\times\mathbb{R}_+^{|C|} : \sum_{t\in T}a_{it}x_t+\sum_{t\in C:c_{it}>0}c_{it}w_t\geq b_i, i=1,\ldots,m\}$. Representing $\sum_{t\in C:c_{it}>0}c_{it}w_t$ by s_i, we get the set Y_m.

Any subset of the m rows in Y_m can be used to generate a mixed n-step MIR inequality for this set. Like before without loss of generality, we assume that this subset of rows is $K=\{1,\ldots,k\}$, where $k\leq m$. A set of n parameters must be chosen to generate the mixed n-step MIR inequality. We denote the vector of these parameters by $\alpha=(\alpha_1,\ldots,\alpha_n)$, where $\alpha\in\mathbb{R}^n$ and $\alpha>0$. As we will see, these parameters must satisfy the n-step MIR conditions for all rows in K, i.e.

$$\alpha_j\left\lceil b_i^{(j-1)}/\alpha_j\right\rceil\leq\alpha_{j-1},\quad j=1,\ldots,n, i\in K\tag{46}$$

(like before we also assume $b_i^{(j-1)}/\alpha_j\notin\mathbb{Z}, j=1,\ldots,n, i\in K$, to avoid trivial inequalities). Notice that conditions (46) are on the parameters α_j chosen by the user and no conditions on coefficients a_{it} in Y_m are required. Without loss of generality, we also assume that the rows are indexed such that $b_{i-1}^{(n)}\leq b_i^{(n)}, i=2,\ldots,k$. Here we present the type I mixed n-step MIR inequality for Y_m. The type II can be generated in a similar fashion.

Let $a_t=(a_{1t},a_{2t},\ldots,a_{kt})$ and $b=(b_1,\ldots,b_k)$ and let $\pi:\mathbb{R}^k\to\{0,\ldots,n\}^k$ be a mapping. For $i\in K$ and $p=0,\ldots,n$, let $T_p^i:=\{t\in T:\pi(a_t)_i=p\}$, where $\pi(a_t)_i$ is the ith component of $\pi(a_t)$.

Definition 12. The mixed n-step MIR function $\sigma^n_{\alpha,b}: \mathbb{R}^n \to \mathbb{R}$ is defined as follows

$$\sigma^n_{\alpha,b}(d) = \min_{\overline{\pi}\in\{0,\dots,n\}^k} \left\{ \sum_{i=1}^{k} \left(b_i^{(n)} - b_{i-1}^{(n)}\right) \delta^{\pi}_{\alpha,b_i}(d_i) + u^{\pi}(d) : \pi(d) = \overline{\pi} \right\}, \tag{47}$$

where

$$\delta^{\pi}_{\alpha,b_i}(d) = \begin{cases} \sum_{j=1}^{p} \prod_{l=j+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{d_i^{(j-1)}}{\alpha_j} \right\rfloor + \prod_{l=p+2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{d_i^{(p)}}{\alpha_{p+1}} \right\rceil, & \pi(d)_i = p;\ p = 0, 1, \dots, n-1, \\ \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{d_i^{(j-1)}}{\alpha_j} \right\rfloor & \pi(d)_i = n \end{cases}$$

and

$$u^{\pi}(d) := \max\{0, d_i^{(n)} \text{ for all } i \text{ that } \pi(d)_i = n\}. \quad \square$$

Theorem 13. *Given a positive parameter vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ which satisfies conditions (46), the mixed n-step MIR inequality*

$$\sum_{t\in T} \sigma^n_{\alpha,b}(a_t)x_t + \overline{s} \geq \sigma^n_{\alpha,b}(b) \tag{48}$$

is valid for Y_m, where $\overline{s} \in \mathbb{R}_+$ is a variable such that $\overline{s} \geq s_i$ for all $i \in K$.

Proof. Given a mapping π, each constraint of Y_m can be relaxed in the same way as the defining constraint of Y_1 is relaxed in [16]. In other words, for $i \in K$, constraint i of Y_m can be relaxed to

$$\sum_{p=0}^{n-1} \sum_{t\in T_p^i} \left(\sum_{j=1}^{p} \alpha_j \left\lfloor \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rfloor + \alpha_{p+1} \left\lceil \frac{a_{it}^{(p)}}{\alpha_{p+1}} \right\rceil \right) x_t + \sum_{t\in T_n^i} \left(\sum_{j=1}^{n} \alpha_j \left\lfloor \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rfloor + a_{it}^{(n)} \right) x_t \geq b_i. \tag{49}$$

Notice that this is a relaxation because for any $p \in \{0, 1, \dots, n\}$

$$a_{it} = \sum_{j=1}^{p} \alpha_j \left\lfloor a_{it}^{(j-1)}/\alpha_j \right\rfloor + a_{it}^{(p)} \tag{50}$$

and so

$$a_{it} \leq \sum_{j=1}^{p} \alpha_j \left\lfloor a_{it}^{(j-1)}/\alpha_j \right\rfloor + \alpha_{p+1} \left\lceil a_{it}^{(p)}/\alpha_{p+1} \right\rceil. \tag{51}$$

In other words, to get (49), the coefficient a_{it} in every row $i \in K$ of Y_m is relaxed to the right-hand side of (51) for $t \in T_p^i$, $p = 0, 1, \dots, n-1$, and is replaced with the right-hand side of identity (50) for $t \in T_n^i$. Rearranging the terms of (49), we get

$$\sum_{j=1}^{n} \alpha_j \left(\sum_{t\in T_{j-1}^i} \left\lceil \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rceil x_t + \sum_{p=j}^{n} \sum_{t\in T_p^i} \left\lfloor \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rfloor x_t \right) + \sum_{t\in T_n^i} a_{it}^{(n)} x_t + s_i \geq b_i, \quad i \in K. \tag{52}$$

Now for $i \in K$ and $j = 1, \dots, n$, the expression $\sum_{t\in T_{j-1}^i} \left\lceil a_{it}^{(j-1)}/\alpha_j \right\rceil x_t + \sum_{p=j}^{n} \sum_{t\in T_p^i} \left\lfloor a_{it}^{(j-1)}/\alpha_j \right\rfloor x_t$ in (52) is an integer (note that for $j = 2, \dots, n$ it is also nonnegative) and can be treated as y_j^i in $\widehat{Q}^{m,n}$. Also for $i \in K$, the expression $\sum_{t\in T_n^i} a_{it}^{(n)} x_t + s_i$ is nonnegative and can be treated as v_i in $\widehat{Q}^{m,n}$. We choose the upper bound variable $\overline{v}$ in (16) to be $\sum_{t\in T} u^{\pi}(a_t)x_t + \overline{s}$. Since by assumption conditions (46) hold, according to Corollary 3, the type I mixed n-step MIR inequality for $\widehat{Q}^{m,n}$ (inequality (16)), when y_j^i and $\overline{v}$ are replaced with their aforementioned corresponding expressions, is valid for Y_m. That is

$$\sum_{t\in T} u^{\pi}(a_t)x_t + \overline{s} \geq \sum_{i=1}^{k} \left(b_i^{(n)} - b_{i-1}^{(n)}\right) \left(\prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\sum_{t\in T_{j-1}^i} \left\lceil \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rceil x_t + \sum_{p=j}^{n} \sum_{t\in T_p^i} \left\lfloor \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rfloor x_t \right) \right). \tag{53}$$

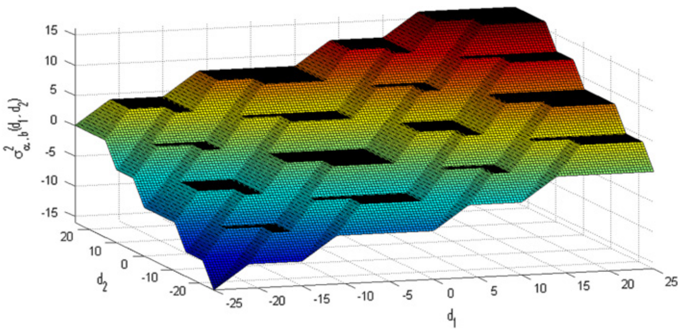

Fig. 1. $\sigma^2_{\alpha,b}(d_1, d_2)$ over $[-25, 25]^2$ with $\alpha = (25, 10)$ and $b = (39, 18)$.

Putting all multiples of x_t in (53) together for each $t \in T$, we can write it as

$$\sum_{t\in T}\left(\sum_{i=1}^{k}\left(b_i^{(n)} - b_{i-1}^{(n)}\right)\delta^{\pi}_{\alpha,b_i}(a_j) + u^{\pi}(a_t)\right)x_t + \bar{s} \geq \sum_{i=1}^{n}\left(b_i^{(n)} - b_{i-1}^{(n)}\right)\prod_{l=1}^{n}\left\lceil\frac{b_i^{(l-1)}}{\alpha_l}\right\rceil. \tag{54}$$

We would like to choose $\pi(a_t)$ such that we get the strongest inequality, i.e. the coefficient of x_t in (54) is minimized. Therefore the smallest coefficient for x_t will be obtained by $\sigma^n_{\alpha,b}(a_t)$. Also, $\sigma^n_{\alpha,b}(b) = \sum_{i=1}^{n}\left(b_i^{(n)} - b_{i-1}^{(n)}\right)\prod_{l=1}^{n}\left\lceil b_i^{(l-1)}/\alpha_l\right\rceil$ as it can be easily verified that the minimum in (47) in case of $\sigma^n_{\alpha,b}(b)$ is achieved at any $\overline{\pi}$, where $\overline{\pi}_i \neq n$ for all $i \in K$. Therefore (54) reduces to (48) and the proof is complete. $\square$

Notice that one possible choice for $\bar{s}$ that guarantees $\bar{s} > s_i$ for all $i \in K$ is $\bar{s} = \sum_{i=1}^{k} s_i$. Theorem 13 shows that a mixed n-step MIR inequality for k constraints can be simply obtained by applying the corresponding mixed n-step MIR function $\sigma^n_{\alpha,b}$ on the coefficient vectors of the variables and the right-hand side vector. Fig. 1 shows an example of the function $\sigma^2_{\alpha,b}(d_1, d_2)$ with $\alpha = (\alpha_1, \alpha_2) = (25, 10)$ and $b = (b_1, b_2) = (39, 18)$ for $(d_1, d_2) \in [-25, 25]^2$.

As we see in Theorem 13, conditions (46) are only on the parameters α_j chosen by the user and no conditions on coefficients a_{it} in Y_m are required. An interesting question is whether it is always possible to find a positive parameter vector $\alpha \in \mathbb{R}^n$ such that it satisfies conditions (46). The answer is yes. Given the set of rows in K with the right-hand sides $b_1, \ldots, b_k$, there is an infinite number of choices for the parameter vector α that satisfy conditions (46). For $i \in K$, $j = 2, \ldots, n$, and $l \in \mathbb{N}$, define the intervals $I_i^{j,l}$ in $\mathbb{R}_+$ as follows:

$$I_i^{j,l} = \begin{cases} \left(\dfrac{b_i^{(j-1)}}{l}, \dfrac{\alpha_{j-1}}{l}\right] & \text{for } 2 \leq l < \tau_i^j, \\[2ex] \left(\dfrac{b_i^{(j-1)}}{l}, \dfrac{b_i^{(j-1)}}{l-1}\right) & \text{for } l \geq \tau_i^j \end{cases}$$

where $\tau_i^j = \left\lceil \alpha_{j-1}/(\alpha_{j-1} - b_i^{(j-1)})\right\rceil$. Then one can choose the elements of the parameter vector α in a recursive fashion as follows:

*Step*1. Pick a positive value for α_1;
*Step*2. For $j := 2, \ldots, n$ do
Pick a value for α_j such that $\alpha_j \in \cap_{i\in K} \cup_{l=2}^{+\infty} I_i^{j,l}$;

We see that in iteration j of Step 2, the set of possible values for α_j depends on the values picked for $\alpha_1, \ldots, \alpha_{j-1}$. Notice that for any i, j and l, we have $\left\lceil b_i^{(j-1)}/\alpha_j\right\rceil = l$ if $\alpha_j \in I_i^{j,l}$. Based on the definitions of τ_i^j and the intervals $I_i^{j,l}$, it can be easily verified that each α_j picked from the set in Step 2 satisfies the conditions $\alpha_j\left\lceil b_i^{(j-1)}/\alpha_j\right\rceil \leq \alpha_{j-1}$ for $i \in K$. Moreover, observe that for each $j \in \{2, \ldots, n\}$, the set $\cap_{i\in K} \cup_{l=2}^{+\infty} I_i^{j,l}$ contains the interval $\left(0, \min\{b_i^{(j-1)}/(\tau_i^j - 1), i \in K\}\right)$ except for the discrete values $b_i^{(j-1)}/l, l \in \mathbb{N}, l \geq \tau_i^j$. Therefore there are always infinitely many choices for each α_j. We note that the intervals presented in [24] for the 2-step MIR inequality are the special case of $I_i^{j,l}$ for $n = 2$, $k = 1$, and $\alpha_1 = 1$.

6. Mixed *n*-step MIR inequalities for special structures

The capacitated lot-sizing problem [3,19,25] and the capacitated facility location problem [3,20,21] have been studied for years. In this section, we introduce useful generalizations of these two problems, which we refer to as the multi-module lot-sizing problem (MML) and the multi-module facility location problem (MMF), respectively, and show that the mixed n-step MIR inequalities can be used to generate valid inequalities for them. The mixed n-step MIR inequalities for MML generalize the (k, l, S, I) inequalities for the constant-capacity lot-sizing problem (CCL) [1,19] and the mixed n-step MIR inequalities for MMF generalize the mixed MIR inequalities for the constant-capacity facility location problem (CCF) [1,20,21].

6.1. Multi-module lot-sizing (MML)

We first define the multi-module lot-sizing (MML) problem. Let $T := \{1, \dots, m\}$ be the set of time periods and $\{\alpha_1, \dots, \alpha_n\}$ be the set of capacities of n available capacity modules. In each period the total capacity can be the summation of some integer multiples of $\alpha_1, \dots, \alpha_n$. In MML the goal is to find a production plan that minimizes the sum of production, inventory, and module setup costs over all periods while meeting the demands (without backlogging) and satisfying capacity constraints. Let x_t be the production, s_t be the inventory at the end of period t, and z_t^j be the number of modules of capacity $\alpha_j, j = 1, \dots, n$, used in period t. Then MML is $\min\{\sum_{t\in T} p_t x_t + \sum_{t\in T} h_t s_t + \sum_{t\in T} \sum_{j=1}^n f_t^j z_t^j : (x, s, z) \in X^{\mathrm{MML}}\}$, where

$$X^{\mathrm{MML}} = \Bigg\{ (x, s, z) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{Z}_+^{m\times n} : s_{t-1} + x_t = d_t + s_t,\ t \in T \tag{55}$$

$$x_t \le \sum_{j=1}^n \alpha_j z_t^j,\ t \in T \Bigg\}, \tag{56}$$

and d_t, p_t, h_t, and f_t^j are the demand, production cost per unit, inventory cost per unit, and the setup cost per module of capacity $\alpha_j, j = 1, \dots, n$, in period t, respectively, and $s_0 = 0$.

When $\alpha_1 = \alpha_2 = \cdots = \alpha_n = C$, the capacity constraints (56) simplify to $x_t \le C \sum_{j=1}^n z_t^j,\ t \in T$. Now for each $t \in T$, one can replace all variables $z_t^j, j = 1, \dots, n$ in MML, with a single variable y_t, where $y_t = z_t^{j_t}$ and $j_t \in \{1, \dots, n\}$ is the index for which $f_t^{j_t} = \min\{f_t^j, j = \{1, \dots, n\}\}$. As a result of this, the MML reduces to CCL. Pochet and Wolsey [19] presented the so-called (k, l, S, I) inequalities for X^{CCL}, the set of feasible solutions to CCL. In [1] it is shown that these inequalities are simply the mixed 1-step MIR inequalities generated from base inequalities formed by aggregating and relaxing the flow balance constraints (55). These results are also true for the special case of CCL in which $y_t \in \{0, 1\}, t \in T$.

Here we show that the mixed n-step MIR can be used to get valid inequalities for X^{MML}. These inequalities generalize the (k, l, S, I) inequalities for X^{CCL} to the case of multiple capacity modules. First, we construct the *base inequalities* for which the mixed n-step MIR inequalities will be written. We follow the notation of [19] as much as possible. For any $k, l \in T$, where $k < l$, let $S \subseteq \{k, \dots, l\}$. For $i \in S$, let $S_i = S \cap \{k, \dots, i\}$ and $b_i = \sum_{t=k}^{n_i-1} d_t$, where

$$n_i = \begin{cases} \min\{t : t \in S \setminus S_i\}, & \text{if } S \setminus S_i \neq \emptyset \\ l + 1, & \text{if } S \setminus S_i = \emptyset. \end{cases}$$

Adding up equalities (55) from period k to period $n_i - 1$, we get

$$s_{k-1} + \sum_{t=k}^{n_i-1} x_t = b_i + s_{n_i-1}. \tag{57}$$

Note that $S_i \subseteq \{k, \dots, n_i - 1\}$ by definition. If we relax $x_t, t \in S_i$, in (57) to its upper bound based on (56) and drop $s_{n_i-1} (\ge 0)$, we get the following valid inequality:

$$s_{k-1} + \sum_{t\in\{k,\dots,n_i-1\}\setminus S_i} x_t + \sum_{t\in S_i} \sum_{j=1}^n \alpha_j z_t^j \ge b_i. \tag{58}$$

Setting $v_i := s_{k-1} + \sum_{t\in\{k,\dots,n_i-1\}\setminus S_i} x_t$ and $y_j^i := \sum_{t\in S_i} z_t^j, j = 1, \dots, n$, inequality (58) becomes

$$\sum_{j=1}^n \alpha_j y_j^i + v_i \ge b_i, \tag{59}$$

which is of the same form as the defining inequalities of $Q^{m,n}$ (notice that $v_i \in \mathbb{R}_+, y_j^i \in \mathbb{Z}_+, j = 1, \dots, n$). Let $I \subseteq S$. We get an inequality like (59) for each $i \in I$. Without loss of generality and for simplicity of notation assume that the parameter vector for mixed n-step MIR is $\alpha = (\alpha_1, \dots, \alpha_n)$ and also $I = \{1, \dots, |I|\}$ such that $b_{i-1}^{(n)} \le b_i^{(n)}, i \in I$. Now if $\alpha_j \left\lceil b_i^{(j-1)}/\alpha_j \right\rceil \le \alpha_{j-1}, j = 2, \dots, n, i \in I$, then by letting $\overline{v} = s_{k-1} + \sum_{t\in\{k,\dots,n_{|I|}-1\}\setminus S} x_t$ (note that $\overline{v} \ge v_i$ for all $i \in I$), based on Corollary 3, the mixed n-step MIR inequalities

$$s_{k-1} + \sum_{t\in\{k,\dots,n_{|I|}-1\}\setminus S} x_t \geq \sum_{i=1}^{|I|} \left(b_i^{(n)} - b_{i-1}^{(n)}\right) \phi^i(y^i), \tag{60}$$

$$s_{k-1} + \sum_{t\in\{k,\dots,n_{|I|}-1\}\setminus S} x_t \geq \sum_{i=1}^{|I|} \left(b_i^{(n)} - b_{i-1}^{(n)}\right) \phi^i(y^i) + \left(\alpha_n - b_{|I|}^{(n)}\right)\left(\phi_n^1(y^1) - 1\right) \tag{61}$$

are valid for X^{MML}, where $y_j^i = \sum_{t\in S_i} z_t^j$. We refer to inequalities (60) and (61) as the *type I and type II multi-module* (k, l, S, I) *inequalities*. The (k, l, S, I) inequalities for X^{CCL} presented in [1,19] are the special case of (60) for $n = 1$ (the constant capacity case).

Remark 2. A special case of MML is when in each period t only modules of a specific capacity C_t are available but the capacity of modules in different periods are not necessarily the same. This is the well-known capacitated lot-sizing problem (CL) [3,25]. The set of feasible solutions in this case is

$$X^{CL} = \left\{(x, s, z) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{Z}_+^m : s_{t-1} + x_t = d_t + s_t, t \in T; x_t \leq C_t z_t, t \in T\right\}.$$

We note that in many studies the special case of binary z_t variables is considered [3,25]. The mixed n-step MIR inequalities (60) and (61) can be easily specialized to X^{CL}. Assume that $\{\alpha_1, \dots, \alpha_n\}$ is the set of distinct capacity values, i.e. for any $t \in T$, $C_t = \alpha_j$ for some $j \in \{1, \dots, n\}$. So without loss of generality we assume that the parameter vector is $\alpha = (\alpha_1, \dots, \alpha_n)$. Then the only difference in the above derivation is that (58) becomes $s_{k-1} + \sum_{t\in\{k,\dots,n_i-1\}\setminus S_i} x_t + \sum_{t\in S_i} C_t z_t \geq b_i$, and therefore in (60) and (61), we must set $y_j^i = \sum_{t\in S_i : C_t=\alpha_j} z_t$ for $i \in I, j = 1, \dots, n$. □

Considering an $i \in I$, recall that $b_i = \sum_{t=k}^{n_i-1} d_t$, i.e. b_i is the total demand in periods k to $n_i - 1$. The n-step MIR conditions on b_i and the module capacities $\alpha_1, \dots, \alpha_n$, i.e.

$$\alpha_j \left\lceil b_i^{(j-1)}/\alpha_j \right\rceil \leq \alpha_{j-1}, \quad j = 2, \dots, n, \tag{62}$$

which are required for validity of (60) and (61) have an interesting interpretation. First note that for $j = 2, \dots, n$, we have $b_i^{(j)} > 0$ and $\alpha_j > 0$, and therefore $\left\lceil b_i^{(j-1)}/\alpha_j \right\rceil \geq 1$. This along with (62) means the module capacities must be in a non-increasing order, i.e. $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Now given a $j \in \{2, \dots, n\}$, consider a strategy for opening a total capacity of b_i that only uses modules $\alpha_1, \dots, \alpha_j$ and opens units of the largest module possible without the total opened capacity exceeding b_i. When this is not possible anymore, one additional unit of α_j is opened to make the total opened capacity greater than or equal to b_i. Clearly under this strategy the total capacity opened, denoted by O_j, will be $O_j = \sum_{l=1}^{j-1} \alpha_l \left\lfloor b_i^{(l-1)}/\alpha_l \right\rfloor + \alpha_j \left\lceil b_i^{(j-1)}/\alpha_j \right\rceil$. Now it is easy to see that conditions (62) are equivalent to $O_1 \geq O_2 \geq \cdots \geq O_n$. As a result, the n-step MIR conditions (62) mean that the module capacities $\alpha_1, \dots, \alpha_n$ should be such that if we consider more of them in the strategy above (i.e. we increase j), the total opened capacity for covering the demand b_i using this strategy will decrease or remain the same.

Example 2. Consider the MML with two capacity modules $\alpha = (\alpha_1, \alpha_2) = (9, 4)$ and 6 time periods with demands $(d_1, d_2, d_3, d_4, d_5, d_6) = (4, 10, 17, 6, 1, 11)$. Now let $k = 2$, $l = 6$ and choose $S = \{3, 5, 6\}$ and $I = \{3, 5\}$. Therefore $S_3 = \{3\}, S_5 = \{3, 5\}, n_3 = 5, n_5 = 6, b_3 = 33, b_5 = 34$. The base inequalities (59) corresponding to time periods $i = 3$ and $i = 5$ are

$$9y_1^3 + 4y_2^3 + \upsilon_3 \geq 33,$$
$$9y_1^5 + 4y_2^5 + \upsilon_5 \geq 34,$$

where $\upsilon_3 = \upsilon_5 = s_1 + x_2 + x_4, y_1^3 = z_3^1, y_2^3 = z_3^2, y_1^5 = z_3^1 + z_5^1$ and $y_2^5 = z_3^2 + z_5^2$. Note that we have $b_3^{(1)} = 6, b_5^{(1)} = 7$, $b_3^{(2)} = 2, b_5^{(2)} = 3$, and $\left\lceil b_3^{(1)}/\alpha_2 \right\rceil = \left\lceil b_5^{(1)}/\alpha_2 \right\rceil = 2$. We see that the conditions $\alpha_2 \left\lceil b_i^{(1)}/\alpha_2 \right\rceil \leq \alpha_1, i = 3, 5$, are satisfied. Therefore, the type I and type II mixed 2-step MIR inequalities obtained from mixing the two base inequalities are (note that $b_3^{(2)} < b_5^{(2)}$):

$$\overline{\upsilon} \geq 2(8 - y_2^3 - 2y_1^3) + (8 - y_2^5 - 2y_1^5),$$
$$\overline{\upsilon} \geq 2(8 - y_2^3 - 2y_1^3) + (8 - y_2^5 - 2y_1^5) + (8 - y_2^3 - 2y_1^3 - 1),$$

respectively, where $\overline{\upsilon} = s_1 + x_2 + x_4$. Written in terms of the original variables, these inequalities are

$$s_1 + x_2 + x_4 \geq 2(8 - z_3^2 - 2z_3^1) + (8 - z_3^2 - z_5^2 - 2z_3^1 - 2z_5^1),$$
$$s_1 + x_2 + x_4 \geq 2(8 - z_3^2 - 2z_3^1) + (8 - z_3^2 - z_5^2 - 2z_3^1 - 2z_5^1) + (8 - z_3^2 - 2z_3^1 - 1). \quad \square$$

6.2. Multi-module facility location (MMF)

We first define the multi-module facility location (MMF) problem. Let $P := \{1, \ldots, n_P\}$ be a set of potential facilities, $Q := \{1, \ldots, n_Q\}$ be a set of clients, and $\{\alpha_1, \ldots, \alpha_n\}$ be the set of capacities for n capacity modules. In MMF the goal is to decide the capacity of facilities and assign the demand of clients to facilities such that the summation of capacity setup costs and distribution costs is minimized while the demands and the capacity constraints are satisfied. The capacity of each facility is the summation of some integer multiples of $\alpha_1, \ldots, \alpha_n$. Let x_{pq} be the portion of demand of client q satisfied by facility p, and u_p^j be the number of capacity modules installed in facility p. Then MMF is $\min\{\sum_{p\in P}\sum_{q\in Q} c_{pq}x_{pq} + \sum_{p\in P} f_p^j u_p^j : (x, u) \in X^{\mathrm{MMF}}\}$, where

$$X^{\mathrm{MMF}} = \left\{ (x, u) \in \mathbb{R}_+^{n_P n_Q} \times \mathbb{Z}_+^{n_P n} : \sum_{p\in P} x_{pq} = d_q, q \in Q \right. \tag{63}$$

$$\left. \sum_{q\in Q} x_{pq} \le \sum_{j=1}^{n} \alpha_j u_p^j, p \in P \right\}, \tag{64}$$

and d_q, c_{pq}, and f_p^j are the demand of client q, the distribution cost per unit between facility p and client q, and the setup cost per module of capacity α_j, $j = 1, \ldots, n$, in facility p, respectively.

Let $I := \{1, 2, \ldots, n_I\}$, and for $i \in I$, choose $S_i \subseteq P$ and $K_i \subseteq Q$. Let $b_i := \sum_{q\in K_i} d_q$ be the total demand of clients in K_i. Adding the demand constraints (63) for $q \in K_i$, we get

$$\sum_{p\in P} w_p^i = b_i \tag{65}$$

where $w_p^i = \sum_{q\in K_i} x_{pq}$ is the total demand of clients in K_i satisfied by facility p. Now by (64), we have $w_p^i \le \sum_{j=1}^{n} \alpha_j u_p^j$. Therefore for $p \in S_i$, we relax w_p^i in (65) to its upper bound to get

$$\sum_{p\in P\setminus S_i} w_p^i + \sum_{p\in S_i}\sum_{j=1}^{n} \alpha_j u_p^j \ge b_i, \quad i \in I. \tag{66}$$

When there is only one module size, i.e. $\alpha_j = C, j = 1, \ldots, n$, the capacity constraints (64) simplify to $\sum_{q\in Q} x_{pq} \le C\sum_{j=1}^{n} u_p^j$, $p \in P$. Now for each $p \in P$, one can replace all variables $u_p^j, j = 1, \ldots, n$ in MMF, with a single variable y_p, where $y_p = z_p^{j_p}$ and $j_p \in \{1, \ldots, n\}$ is the index for which $f_p^{j_p} = \min\{f_p^j, j = \{1, \ldots, n\}\}$. As a result of this, the MMF reduces to CCF. We denote the feasible set of CCF by X^{CCF}. In this case, inequalities (66) reduce to $\sum_{p\in P\setminus S_i} w_p^i + \sum_{p\in S_i} Cy_p \ge b_i, i \in I$. These were used as base inequalities by Günlük and Pochet [1] to generate mixed 1-step MIR inequalities for X^{CCF}. According to [1], in the case where $y_p, p \in P$, are restricted to be binary, if $K_i \subset K_{i+1}$ and $S_i \subset S_{i+1}$ for all i, these mixed 1-step MIR inequalities are the same inequalities introduced by Aardal et al. [20,21] for X^{CCF} and define facets or high-dimensional faces for its convex hull as shown in [20,21].

Here we show that the mixed n-step MIR inequalities can be used to get valid inequalities for X^{MMF}. These inequalities generalize the inequalities presented in [1] for X^{CCF} to the case of multiple capacities. Defining $v_i := \sum_{p\in P\setminus S_i} w_p^i$ and $y_j^i := \sum_{p\in S_i} u_p^j$, for $i \in I$, inequality (66) becomes

$$v_i + \sum_{j=1}^{n} \alpha_j y_j^i \ge b_i, \quad i \in I. \tag{67}$$

Notice that $v_i \in \mathbb{R}_+, y_j^i \in \mathbb{Z}_+, i \in I, j = 1, \ldots, n$. Without loss of generality assume that the parameter vector for mixed n-step MIR is $\alpha = (\alpha_1, \ldots, \alpha_n)$ and also the indices in I are such that $b_{i-1}^{(n)} \le b_i^{(n)}, i \in I$. Now if $\alpha_j \left\lceil b_i^{(j-1)}/\alpha_j \right\rceil \le \alpha_{j-1}, j = 2, \ldots, n, i \in I$, by letting $\overline{v} = \sum_{(p,q)\in T} x_{pq}$, where $T = \{(p, q) : p \in P \setminus S_i, q \in K_i \text{ for some } i \in I\}$ (note that $\overline{v} \ge v_i$ for all $i \in I$), based on Corollary 3, the mixed n-step MIR inequalities

$$\sum_{(p,q)\in T} x_{pq} \ge \sum_{i=1}^{n_I} \left(b_i^{(n)} - b_{i-1}^{(n)}\right) \phi^i(y^i), \tag{68}$$

$$\sum_{(p,q)\in T} x_{pq} \ge \sum_{i=1}^{n_I} \left(b_i^{(n)} - b_{i-1}^{(n)}\right) \phi^i(y^i) + \left(\alpha_n - b_{n_I}^{(n)}\right)\left(\phi_n^1(y^1) - 1\right) \tag{69}$$

are valid for X^{MMF}, where $y_j^i = \sum_{p\in S_i} u_p^j$. The inequalities for X^{CCF} presented in [1] are the special case of (68) for $n = 1$ (the constant capacity case).

Remark 3. A special case of MMF is when each facility p can have only modules of a specific capacity C_p but the capacity of modules in different facilities are not necessarily the same. This is the well-known capacitated facility location (CF) problem [3,20,21]. The set of feasible solutions in this case is

$$X^{CF} = \left\{ (x, u) \in \mathbb{R}_+^{n_P n_Q} \times \mathbb{Z}_+^{n_P} : \sum_{p \in P} x_{pq} = d_q, q \in Q; \sum_{q \in Q} x_{pq} \le C_p u_p, p \in P \right\}.$$

We note that in many studies the special case of binary u_p variables is considered [3,20,21]. The mixed n-step MIR inequalities (68) and (69) can be easily specialized to X^{CF} very similar to the way (60) and (61) were specialized to X^{CL} in Remark 2 with $y_j^i = \sum_{p \in S_i : C_p = \alpha_j} u_p$ for $i \in I, j = 1, \dots, n$. □

Considering an $i \in I$, the n-step MIR conditions on the demand b_i and the module capacities $\alpha_1, \dots, \alpha_n$, i.e. $\alpha_j \left\lceil b_i^{(j-1)} / \alpha_j \right\rceil \le \alpha_{j-1}, j = 2, \dots, n$, which are required for validity of (68) and (69) have an interpretation similar to the one described in Section 6.1.

7. Computational results

In this section, we present our preliminary computational results on using the mixed n-step MIR inequalities for general MIP in solving small MIPLIB instances as well as using the mixed n-step MIR inequalities (60) in solving multi-module lot-sizing (MML) instances.

7.1. MIPLIB instances

In the first part of our computational study, we investigate the differential impact of adding 2-row mixed 1-step MIR and 2-row mixed 2-step MIR cuts over MIR (i.e. 1-step MIR) cuts on small MIPLIB instances. It is known that the separation problem for MIR cuts is strongly NP-complete [26], so naturally, one does not expect the existence of an efficient exact separation algorithm for the MIR cuts. The complexity and existence of an efficient exact separation for the n-step MIR cuts for $n \ge 2$, and the mixed n-step MIR cuts for $n \ge 1$, are open problems. These problems have not been addressed even for the 2-step MIR [27] and the mixed 1-step MIR [1], which were introduced before n-step MIR [16] (we note that Dash and Günlük [14] formulated the separation problem for the mixed 1-step MIR cuts as mixed integer programs). Given the more complicated structure of n-step MIR and mixed n-step MIR cuts, the exact separation problems for these cuts and determining their complexity do not seem to be easy.

As a result, in our study we used a heuristic separation algorithm based on the ideas of the heuristic proposed by Marchand and Wolsey [28] for 1-step MIR cuts. To our knowledge, this separation heuristic (or its variants) is the only existing heuristic which works well for application of general purpose MIR-based cuts on instances such as those in MIPLIB, which are generally quite sparse and have bounds on a large number of integer variables. The aggregation and bound substitution elements of this heuristic provide suitable base inequalities to apply n-step MIR functions. The details of our separation heuristic are as follows.

We used the aggregation and bound substitution heuristics of [28] to generate the base inequalities for which the cuts are developed. Given an instance and the optimal solution of its LP relaxation, we converted the constraints of the problem to equality constraints by adding necessary slack variables and used the aggregation heuristic of [28] to aggregate the constraints of the problem according to the procedure presented in [28] (the MAXAGGR parameter of [28] was set to 6). We then applied criterion (a) of the bound substitution heuristic in [28] (which uses the optimal LP relaxation solution) to generate base constraints of the form of the defining constraints of Y_m.

For each instance we performed three experiments. In each experiment, the cuts were generated only at the root node and from the base constraints developed as explained above. In the first experiment, denoted by 1MIR, we added only 1-step MIR cuts to the problem. For each base constraint, we generated the 1-step MIR cuts (see Section 2) by setting the parameter α_1 equal to each one of the positive coefficients of integer variables in the base constraint and added those cuts that were violated by the optimal LP relaxation solution to the problem. Rounds of 1-step MIR cuts were added until no more violated cuts could be generated. After each round, the LP relaxation was re-optimized and its new solution was used in generating the cuts in the next round.

In the second experiment, denoted by 1MIR1MIX, we added one round of mixed 1-step MIR cuts *in addition to* the 1-step MIR cuts that were added in experiment 1MIR. More specifically, after adding the cuts of 1MIR, we re-optimized the LP relaxation and used the new LP relaxation solution in separation with mixed 1-step MIR cuts. We only considered 2-row mixing ($k = 2$). All pairs of the base constraints were considered for mixing. For each pair, we generated a set of mixed 1-step MIR cuts according to Theorem 13 (we used $\bar{s} = s_1 + s_2$) by setting the value of the parameter α_1 equal to each one of the positive coefficients of integer variables in the two base constraints. Out of all the cuts generated by these choices of α_1, we added to the problem those that were violated by the optimal LP relaxation solution.

The third experiment, denoted by (1MIR2MIX), is similar to 1MIR1MIX, however we added one round of mixed 2-step MIR cuts (Section 5) instead of mixed 1-step MIR cuts. The details are the same as 1MIR1MIX. The only difference is in choosing

Table 1
Results of computational experiments on small MIPLIB instances.

	Instance	flugpl	gt2	lseu	mas74	mas76	mod008	p0033	rgn
DEFAULT	*zlp*	1,167,190	13,460.2	834.68	10,482.8	38,893.9	290.93	2520.57	48.8
	zmip	1,201,500.0	21,166.0	1120.0	11,801.2	40,005.1	307.0	3089.0	82.2
	Time	0.0	0.0	0.1	278.6	50.2	0.2	0.0	0.1
	Nodes	94	1	101	2,672,210	403,345	577	6	523
1MIR	*Cuts*	0	22	47	81	116	98	28	31
	zcut	1,167,190.0	20,592	918.9	10,575.6	39,024.0	298.9	2598.1	57.6
	Time	0.0	0.0	0.1	305.6	19.5	0.1	0.0	0.3
	Nodes	94	1	123	2,764,416	178,778	45	1	1050
	Gapclosed	0.00	92.55	29.50	7.04	11.71	49.32	13.64	26.39
1MIR1MIX	*Cuts*	0	29	23	35	40	19	23	0
	zcut	1,167,190.0	20,592.9	942.9	10,580.0	39,036.0	299.4	2628.2	57.6
	Time	0.0	0.0	0.1	325.0	32.5	0.1	0.0	0.3
	Nodes	94	1	140	2,954,935	281,679	22	3	1050
	Gapclosed	0.00	92.56	37.93	7.37	12.79	52.49	18.93	26.39
1MIR2MIX	*Cuts*	0	472	75	347	138	1432	11	0
	zcut	1,167,190.0	20,726.5	998.9	10,583.1	39,056.2	300.3	2636.3	57.6
	Time	0.0	0.1	0.1	316.4	78.8	0.2	0.0	0.3
	Nodes	94	1	132	2,734,233	296,988	39	1	1050
	Gapclosed	0.00	94.30	57.57	7.61	14.61	58.58	20.36	26.39

parameters α_1 and α_2. For each pair of the base constraints, we constructed a list consisting of all positive coefficients of integer variables in the two base inequalities and then considered all pairs of parameters from this list that satisfy the 2-step MIR condition, i.e. conditions (46) for $n = k = 2$. Out of all the cuts generated by these choices of α_1 and α_2, we added to the problem those that were violated by the LP relaxation solution.

We note that in the experiments above, our method of choosing values for the parameters α_1 and α_2 (choosing from the coefficients of base constraints) was motivated by the facet-defining conditions for the n-step MIR inequalities presented in [17].

We limited our experiments to small instances in MIPLIB libraries. More specifically, we selected all instances from MIPLIB 3.0, 2003, and 2010 which have less than 40 rows and less than 1000 columns. Out of these instances, we ignored one infeasible instance (`p2m2p1m1p0n100` from MIPLIB 2010) as well as the following instances: `enigma` from MIPLIB 3.0 because it has an integrality gap of zero as well as `markshare1` and `markshare2` from MIPLIB 2003, and `markshare_5_0` from MIPLIB 2010, because their solution time using CPLEX 11.0 [29] even with no cuts was prohibitively long. This left us with 8 instances which are from MIPLIB 3.0 and 2003.

In all three experiments, we solved the LP relaxation after adding the cuts and found its optimal solution. We then dropped the cuts that were inactive at this optimal solution and solved the MIP with active cuts. We also solved the LP relaxation and MIP without adding any of our own cuts (denoted by DEFAULT). We used CPLEX 11.0 with its default options in all our experiments. The program was coded in Microsoft Visual C++ and run on a PC with Intel Quad Core 2.4 GHz processor with 4 MB of RAM. The results are presented in Table 1. The *cuts* row shows the number of 1-step MIR cuts in 1MIR, number of mixed 1-step MIR cuts (in addition to 1-step MIR cuts) in 1MIR1MIX, and number of mixed 2-step MIR (in addition to 1-step MIR cuts) in 1MIR2MIX. The *nodes* and *time* rows show the number of branch-and-bound nodes and time (in seconds) to solve the MIP to optimality. The *gapclosed* row shows the percentage of the integrality gap closed by the cuts in each experiment, i.e. $gapclosed = 100(zcut - zlp)/(zmip - zlp)$, where *zlp*, *zcut*, and *zmip* are the optimal objective values of the LP relaxation with no cuts, LP relaxation with the cuts, and MIP, respectively.

Comparing the percentage of integrality gap that is closed among the three experiments, we see that in all instances except `flugpl` and `rgn`, for which our separation did not result in any mixed 1-step or 2-step MIR cuts, adding mixed 1-step MIR cuts over 1-step MIR cuts has improved the closed gap. The maximum improvement is $37.93\% - 29.50\% = 8.43\%$ (for `lseu`). More interestingly, in these instances adding mixed 2-step MIR cuts over 1-step MIR cuts has improved the closed gap more than adding mixed 1-step MIR cuts over 1-step MIR cuts. For 1MIR2MIX, the maximum improvement over 1MIR is $57.57\% - 29.50\% = 28.07\%$ (for `lseu`). These results are quite promising in light of the fact that MIPLIB instances are notorious with respect to gap improvement beyond what is achieved by 1-step MIR [30].

7.2. Multi-module lot-sizing instances

In the second part of our computational study, we studied the performance of the mixed 2-step MIR cuts (60) in solving randomly generated MML instances with two capacity modules. Here we also used a heuristic separation algorithm. Our separation is designed based on the method presented in Section 6.1 to generate inequality (60). Using the notation of Section 6.1, given an instance and the optimal solution of its LP relaxation, denoted by $(\bar{x}, \bar{s}, \bar{z})$, our heuristic is as follows. We considered all possible choices $k, l \in \{1, \dots, T\}$ such that $k < l$. For each choice of k and l, we generated all the cuts obtained by three heuristic choices for the set S, i.e. $S = \{k, \dots, l\}$, $S = \{t \in \{k, \dots, l\} : \bar{z}_t^1 > 0 \textit{ or } \bar{z}_t^2 > 0\}$, and

Table 2
Results of computational experiments on MML instances.

Instance		DEFAULT				2MIX				
(α_1, α_2)	(f_t^1, f_t^2)	*zlp*	*zmip*	*Time*	*Nodes*	*Cuts*	*zcut*	*Time*	*Nodes*	*Gapclosed*
(180, 80)	(1000, 600)	559,248	567,703	0.3	517	729	566,565	0.4	73	86.54
		646,576	654,258	0.2	506	509	653,332	0.2	17	87.95
		615,880	623,663	0.1	261	443	622,775	0.1	1	88.59
		612,767	620,872	0.0	58	589	620,185	0.2	2	91.52
		571,612	580,115	0.2	470	607	579,458	0.1	1	92.27
	(5000, 2600)	761,700	785,624	109.7	508,198	572	782,166	5.0	2534	85.55
		812,633	835,040	53.1	228,982	741	831,892	7.7	1942	85.95
		831,488	852,734	61.2	240,425	567	849,985	4.8	2603	87.06
		812,841	832,604	30.3	145,749	520	830,666	0.9	399	90.19
		761,053	782,019	39.8	164,846	570	780,009	1.2	564	90.41
(270, 130)	(1000, 600)	730,889	741,886	0.0	43	488	740,768	0.2	22	89.83
		590,107	598,604	0.0	29	664	597,766	0.3	9	90.14
		616,219	627,391	0.3	412	578	626,296	0.2	1	90.20
		619,897	630,661	0.0	18	721	629,622	0.3	22	90.35
		541,672	550,644	0.0	157	458	549,868	0.1	1	91.35
	(5000, 2600)	604,703	629,971	19.2	86,812	742	626,920	4.9	3288	87.93
		749,124	774,130	2.0	6,809	517	771,468	0.9	453	89.35
		703,081	726,339	0.5	1,161	652	724,118	0.6	123	90.45
		660,877	684,319	0.6	1,439	651	682,235	0.6	183	91.11
		669,220	691,974	0.6	973	612	690,164	0.5	43	92.05

$S = \{t \in \{k, \ldots, l\}: \bar{z}_t^1 \notin \mathbb{Z} \text{ or } \bar{z}_t^2 \notin \mathbb{Z}\}$. Similar to the previous section, we only considered 2-row mixing (i.e. $|I| = 2$). Therefore our choices for I included all possible two-element subsets of S. For each I, we generated inequality (60) if $\alpha_2 \left\lceil b_i^{(1)}/\alpha_2 \right\rceil \leq \alpha_1$ for $i \in I$ and added it as a cut if it was violated by the optimal LP relaxation solution. As before, all the cuts were added to the root node.

We created random MML instances with two capacity modules ($n = 2$) for this experiment. All our instances had 60 time periods, i.e. $T = \{1, \ldots, 60\}$. The holding cost in all periods was 10, i.e. $h_t = 10, t \in T$. Demand d_t and production cost p_t in each period were integers drawn from *uniform*[10, 190] and *uniform*[81, 119], respectively. In [31] it was observed that the difficulty of capacitated lot-sizing (CL) instances is a function of tightness of the capacities with respect to the demand and the ratio of the setup cost to holding cost. Therefore, we used two sets of capacity modules: $\alpha = (\alpha_1, \alpha_2) = (180, 80)$ and $\alpha = (\alpha_1, \alpha_2) = (270, 130)$, the former resulting in harder instances than the latter. We also used two sets of setup costs for these modules: $(f_t^1, f_t^2) = (1000, 600)$, $t \in T$, and $(f_t^1, f_t^2) = (5000, 2600)$, $t \in T$, the former resulting in easier instances than the latter. We generated 5 instances for each combination of α and (f_t^1, f_t^2), i.e. a total of 20 instances. We note that some of the instance generation and separation ideas we used here are inspired by the ideas used in [31] for CL problems.

For each instance, we solved the LP relaxation and MIP without adding any of our own cuts (denoted by DEFAULT). We also solved the LP relaxation after adding the cuts, found its optimal solution, dropped the cuts that were inactive at this optimal solution, and solved the MIP with active cuts (denoted by 2MIX). The software and hardware platforms we used was the same as those used for MIPLIB instances. The results are presented in Table 2. The definitions of column labels are the same as the definitions of row labels for Table 1 described in Section 7.1.

Table 2 shows that the mixed 2-step MIR cuts are very effective in solving the MML problems. The percentage of integrality gap closed by these cuts is between 85.55% and 92.27% (the average is 89.44%). We also observe that adding the cuts has reduced the number of nodes in almost all instances by several orders of magnitude, especially in harder instances (which have larger number of nodes and solution times). In harder instances, the solution time has also substantially reduced.

8. Concluding remarks

We showed that mixing can be generalized to n-step MIR resulting in the mixed n-step MIR inequalities for a generalization of the mixing set called the n-mixing set. The parameters $\alpha_1, \ldots, \alpha_n$ must satisfy the same conditions required for the validity of n-step MIR inequalities. As a special case these conditions are automatically satisfied if the parameters $\alpha_1, \ldots, \alpha_n$ are divisible. Moreover, the type I and type II mixed n-step MIR inequalities are strong in the sense that they define facets for the n-mixing set. We also showed that mixed n-step MIR can be used to generate cuts based on multiple constraints for general MIPs as well as multi-module lot-sizing and facility location problems. The mixed n-step MIR encompasses, as the special case corresponding to $n = 1$, the inequalities that were previously generated based on mixing of MIR inequalities for the mixing set [1] as well as lot-sizing and facility location problems with a constant capacity [19–21]. Our preliminary computational results on applying mixed n-step MIR inequalities in solving multi-module lot-sizing instances and small MIPLIB instances justify their effectiveness.

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Appendix

Proof of Lemma 5. If (a) is true, by substituting the point P in constraint i of $Q^{m,n}$, we get $\sum_{j=1}^{t-1} \alpha_j \left\lfloor \beta_i^{(j-1)}/\alpha_j \right\rfloor + \alpha_t \left\lceil \beta_i^{(t-1)}/\alpha_t \right\rceil + \hat{v} \geq \beta_i$, or $\alpha_t \left\lceil \beta_i^{(t-1)}/\alpha_t \right\rceil + \hat{v} \geq \beta_i^{(t-1)}$, which is trivial since $\hat{v} \geq 0$. If (b) is true, by substituting the point P in constraint i of $Q^{m,n}$, we get $\sum_{j=1}^{t} \alpha_j \left\lfloor \beta_i^{(j-1)}/\alpha_j \right\rfloor + \hat{v} \geq \beta_i$, or $\hat{v} \geq \beta_i^{(t)}$, which is true based on (b). □

Proof of Lemma 6. For $i \in M$ and $t = 1, \ldots, n$, we have

$$\begin{aligned}
\phi^i(p^{i,t}) &= \prod_{l=1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^{t-1} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rfloor - \prod_{l=t+1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rceil \\
&= \prod_{l=2}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil - \left\lfloor \frac{\beta_i}{\alpha_1} \right\rfloor \right) - \sum_{j=2}^{t-1} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rfloor - \prod_{l=t}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil \\
&= \prod_{l=2}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=2}^{t-1} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rfloor - \prod_{l=t}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil \\
&= \prod_{l=3}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{\beta_i^{(1)}}{\alpha_2} \right\rceil - \left\lfloor \frac{\beta_i^{(1)}}{\alpha_2} \right\rfloor \right) - \sum_{j=3}^{t-1} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rfloor - \prod_{l=t}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil \\
&= \cdots = \prod_{l=t}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil - \prod_{l=t}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil = 0.
\end{aligned}$$

Notice that for $i \in K$ we have $q^{i,n} = p^{i,n} + e_n$, where $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$. Based on (10), it is easy to see that $\phi^i(q^{i,n}) = \phi^i(p^{i,n}) + 1 = 1$. □

Proof of Lemma 9. If (a) is true, by substituting the point P in constraint 1 of $Q^{m,n}$, we get

$$\sum_{j=1}^{t-1} \alpha_j \left\lfloor \beta_1^{(j-1)}/\alpha_j \right\rfloor - \alpha_{t-1} + \alpha_t \left(2 \left\lfloor \beta_1^{(t-1)}/\alpha_t \right\rfloor + 1 \right) + \sum_{j=t+1}^{n} \alpha_j \left\lfloor \beta_1^{(j-1)}/\alpha_j \right\rfloor + \hat{v} \geq \beta_1.$$

This simplifies to $\hat{v} \geq \beta_1^{(n)} + \alpha_{t-1} - \alpha_t \left\lceil \beta_1^{(t-1)}/\alpha_t \right\rceil$, which is true by (a). If (b) is true, by substituting the point P in constraint 1 of $Q^{m,n}$, we get $\sum_{j=1}^{n} \alpha_j \left\lfloor \beta_1^{(j-1)}/\alpha_j \right\rfloor - \alpha_n + \hat{v} \geq \beta_1$, or $\hat{v} \geq \alpha_n + \beta_1^{(n)}$, which is true by (b). □

Proof of Lemma 10. The function $\phi^1(y^1)$ can be written as

$$\begin{aligned}
\phi^1(y^1) &= \prod_{l=1}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil y_j^1 \\
&= \prod_{l=2}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil + \prod_{l=2}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{\beta_1}{\alpha_1} \right\rfloor - \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil y_j^1 \\
&= \prod_{l=2}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil + \prod_{l=2}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lfloor \frac{\beta_1}{\alpha_1} \right\rfloor - y_1^1 \right) - \sum_{j=2}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil y_j^1 \\
&= \prod_{l=3}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil + \sum_{j=1}^{2} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor - y_j^1 \right) - \sum_{j=3}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil y_j^1 \\
&= \cdots = 1 + \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor - y_j^1 \right).
\end{aligned} \tag{A.1}$$

Based on (A.1), for $t = 2, \ldots, n$ we have

$$\phi^1(r^t) = 1 + \prod_{l=t}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil + \prod_{l=t+1}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lfloor \frac{\beta_1^{(t-1)}}{\alpha_t} \right\rfloor - 2 \left\lfloor \frac{\beta_1^{(t-1)}}{\alpha_t} \right\rfloor - 1 \right)$$

$$= 1 + \prod_{l=t}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil - \prod_{l=t}^{n} \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil = 1,$$

and

$$\phi^1(s) = 1 + \left(\left\lfloor \frac{\beta_1^{(n-1)}}{\alpha_n} \right\rfloor - \left\lfloor \frac{\beta_1^{(n-1)}}{\alpha_n} \right\rfloor + 1 \right) = 2. \quad \square$$

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