Pointwise Estimate for Linear Combinations of Bernstein–Kantorovich Operators

Shunsheng Guo, Lixia Liu, and Qiulan Qi

College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050016, People’s Republic of China

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For linear combinations of Bernstein–Kantorovich operators $K_n, f, x$, we give an equivalent theorem with $\omega_{r\lambda}^2, (f, t)$. The theorem unites the corresponding results of classical and Ditzian–Totik moduli of smoothness.

Key Words: Bernstein–Kantorovich operator; moduli of smoothness; linear combinations.

1. INTRODUCTION

Let $f$ be a function defined on $[0, 1]$. The Bernstein–Kantorovich operator is defined by

$$K_n(f, x) = (n + 1) \sum_{k=0}^{n} P_{n,k}(x) \int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}} f(t) \, dt, \quad (1.1)$$

where $P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

The linear combinations of Bernstein–Kantorovich operators (see [1, p. 116 (9.2.6)]) are given by

$$K_{n,r}(f, x) = \sum_{i=0}^{r-1} \alpha_i(n)K_n(f, x), \quad (1.2)$$

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where $n_{i}$ and $\alpha_i(n)$ satisfy
\begin{equation}
\begin{aligned}
(a) \quad n = n_0 < \cdots < n_{r-1} \leq Kn; & \quad (b) \quad \sum_{i=0}^{r-1} |\alpha_i(n)| \leq C; \\
(c) \quad K_{n,r}(1,x) = 1; & \quad (d) \quad K_{n,r}((t-x)m;x) = 0, \quad m = 1, \ldots, r-1.
\end{aligned}
\end{equation}

In [2] Ditzian introduced $\omega_{\varphi}^2(f, t)$ and gave an interesting direct result for the Bernstein operators which united the result with $\omega^2(f, t)$ and $\omega_{\varphi}^2(f, t)$. For polynomial approximation, in [3] Ditzian and Jiang used $\omega_{\varphi}^r(f, t)$ to bridge the gap between the classical estimate ($\lambda = 0$) and the recently developed estimate by Ditizan–Totik ($\lambda = 1$).

We recall that [1]
\begin{equation}
\omega_{\varphi}^r(f, t) = \sup_{0 \leq h \leq t} \sup_{x \in [0, 1]} \left| \Delta_{h\varphi}^{2r}(x)f(x) \right|, \quad \varphi^2(x) = x(1-x)
\end{equation}

is equivalent to the $K$-functional
\begin{equation}
K_{\varphi}^r(f, t^2r) = \inf_{g \in C_{1\varphi}^{2r-1}} \left\{ \|f - g\| + t^{2r}\|\varphi^{2r}\| g^{(2r)} \right\},
\end{equation}

\begin{equation}
\Delta_{\varphi}^{2r} f(x) = \sum_{k=0}^{2r} (-1)^k \binom{2r}{k} f(x + rh - kh), \quad \|\| = \|\|_{\infty}.
\end{equation}

In [5–7] we obtained equivalent theorems with $\omega_{\varphi}^r(f, t)$ for Szász-type and Bernstein-type operators. But can this kind of result be improved with $\omega_{\varphi}^{2r}(f, t)$? In this paper we obtain this kind of equivalent theorem with $\omega_{\varphi}^{2r}(f, t)$ for $K_{n,r}$, which includes the corresponding result of [6]. We point out that the Bernstein–Kantorovich operators were introduced because they apply in $L_p[0, 1], 1 \leq p \leq \infty$ (cf. [8, 9]) and not just to $C[0, 1]$. Since for the estimate by $\omega_{\varphi}^{2r}(f, t)_p(\lambda = 0)$ the converse theorem does not work, we cannot deal with $L_p[0, 1]$ by $\omega_{\varphi}^{2r}(f, t)(\lambda \in [0, 1])$. On the converse result, we just get a Steckin–Marchaud inequality in [10] for the Bernstein–Kantorovich operator. In this paper, we attempted to get a strong converse inequality; however, we failed.

Now we state our main result.

**Theorem 1.** For $f \in C[0, 1], r \in \mathbb{N}, 0 \leq \lambda \leq 1,$ and $0 < \alpha < \frac{2r}{x^r}$ we have
\begin{equation}
|K_{n,r}(f, x) - f(x)| = O\left( (n^{-\frac{1}{2}}\delta_{n}^{1-\lambda}(x))^\alpha \right) \Leftrightarrow \omega_{\varphi}^{2r}(f, t) = O(t^\alpha),
\end{equation}

where $\delta_{n}(x) = \varphi(x) + \frac{1}{\sqrt{n}}$.

**Remark 1.** For $\lambda = 1$, this is the result of [1, (9.3.3)].

Throughout this paper, $C$ denotes a constant independent of $n$ and $x$, but it is not necessarily the same in different cases.
2. LEMMAS AND DIRECT THEOREM

To prove our main theorem, we need the following lemmas:

**Lemma 2.1.** Let \( f \in C[0, 1] \), \( f^{(2r-1)} \in A.C_{\infty} \), and \( r\lambda - m > 0 \). We have
\[
|\varphi^{2r\lambda - 2m}(x)f^{(2r-m)}(x)| \leq C(\|f\| + \|\varphi^{2r\lambda} f^{(2r)}\|). \quad (2.1)
\]

**Proof.** We observe that (see [1, p. 136])
\[
|f^{(2r-m)}(\frac{1}{2})| \leq C(\|f\|_{\frac{1}{2}, \frac{1}{1}} + \|f^{(2r)}\|_{\frac{1}{1}, \frac{1}{1}}) \leq C(\|f\| + \|\varphi^{2r\lambda} f^{(2r)}\|). \quad (2.2)
\]

For \( r\lambda - m > 0 \), when \( x \) is near 0 \( (x \leq \frac{1}{2}) \),
\[
|f^{(2r-m)}(x) - f^{(2r-m)}(\frac{1}{2})| \leq \int_{x}^{\frac{1}{2}} |f^{(2r-m+1)}(u)| du
\]
\[
\leq \|x^{r\lambda - m+1}f^{(2r-m+1)}(x)\|_{[0, \frac{1}{2}]} \int_{x}^{\frac{1}{2}} u^{-r\lambda + m-1} du
\]
\[
\leq C\|x^{r\lambda - m+1}f^{(2r-m+1)}(x)\|_{[0, \frac{1}{2}]} x^{-r\lambda + m},
\]
which implies by (2.2)
\[
|x^{r\lambda - m}f^{(2r-m)}(x)| \leq C(\|f\| + \|\varphi^{2r\lambda} f^{(2r)}\| + \|x^{r\lambda - m+1}f^{(2r-m+1)}(x)\|). \quad (2.3)
\]

From these, by induction one has
\[
|\varphi^{2r\lambda - 2m}(x)f^{(2r-m)}(x)| \leq C(\|f\| + \|\varphi^{2r\lambda} f^{(2r)}\|).
\]

This is (2.1) for \( x \in [0, \frac{1}{2}] \). We can treat the case \( x \in (\frac{1}{2}, 1] \) similarly.

**Lemma 2.2.** For \( f^{(2r)} \in C[0, 1] \), let \( R_{\Delta}(f, t, x) \) is \( f^{(2r)} \times (u)du \), \( r \in N \), and \( 0 \leq \lambda \leq 1 \). Then for \( x \in E_{n} = [\frac{1}{n}, 1 - \frac{1}{n}] \), one has
\[
|K_{n,r}(R_{\Delta}(f, t, x), x)| \leq C(n^{-\frac{1}{r}}\varphi^{1-\lambda}(x))^{2r}\|\varphi^{2r\lambda} f^{(2r)}\|. \quad (2.4)
\]

**Proof.** From [1, (9.5.3)] \( K_{n,r}((t - x)^{2r}, x) \leq Cn^{-r}\varphi^{2r}(x) \), \( x \in E_{n} \), and
\[
|t - u|^{2r-1}\varphi^{2r\lambda}(u) \leq |t - x|^{2r-1}\varphi^{2r\lambda}(x),
\]
for \( u \) is between \( x \) and \( t \) (see [1, p. 141]). So we have
\[
|K_{n,r}\left(\int_{x}^{t}(t - u)^{2r-1}f^{(2r)}(u)du, x\right)| \leq \|\varphi^{2r\lambda} f^{(2r)}\|K_{n,r}\left(\int_{x}^{t} |t - u|^{2r-1}\varphi^{2r\lambda}(u)du, x\right)
\]
\[
\leq \varphi^{-2r\lambda}(x)\|\varphi^{2r\lambda} f^{(2r)}\|K_{n,r}((t - x)^{2r}, x)
\]
\[
\leq C(n^{-\frac{1}{r}}\varphi^{1-\lambda}(x))^{2r}\|\varphi^{2r\lambda} f^{(2r)}\|. 
\]
THEOREM 2. For \( f(x) \in C[0, 1], r \in N, \) and \( 0 \leq \lambda \leq 1, J = \max\{i|\lambda - 2r + i \leq 0, i \leq 2r - 1\}. \) Then

\[
|K_{n,r}(f, x) - f(x)| \leq C \left( \sum_{i=r}^J \omega^r_i(f, (n^{-r}\varphi^{2(i-r)}(x))^{1/i}) + \omega_n^r(f, n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)) + \frac{\varphi^{2r(1-\lambda)}(x)}{n^r} \|f\| \right). \tag{2.5}
\]

Proof.

Case 1. \( x \in E_n = [\frac{1}{n}, 1 - \frac{1}{n}] \). According to the definition of \( K_{\varphi^r}(f, t') \) and \( \omega^{\varphi^r}_i(f, t) \sim K_{\varphi^r}(f, t') \), we can choose \( g = g_{n, x, \lambda} \) for the fixed \( x \) and \( \lambda \) such that

\[
\|f - g\| + (n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^{2r}\|\varphi^{2\lambda}g^{2r}\| \leq C\omega^r_n(f, n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)). \tag{2.6}
\]

As

\[
|K_{n,r}(f, x) - f(x)| \leq C\|f - g\| + |K_{n,r}(g, x) - g(x)|,
\]

now we estimate the second term. Writing \( R_{n,i}(x) = K_{n,r}((t - x)^i, x) \), \( \tilde{\Delta}_h f(x) = f(x + h) - f(x), \) \( \tilde{\Delta}_h f(x) = \tilde{\Delta}_h (\tilde{\Delta}_h^{-1} f(x)) \),

\[
\omega^i(f, t) = \sup_{0<h \leq t} \sup_{x \in [0, +\infty]} |\tilde{\Delta}_h f(x)|.
\]

From [1, p. 26], we know \( \omega^i(f, t) \sim \omega^i(f, t) \).

We define [4]

\[
T_{n,i}(f, x) = -\frac{1}{i!} \sgn R_{n,i}(x) \tilde{\Delta}_i^{(j)} \left( \frac{\varphi^r_n(x)\tilde{\Delta}_h f(x)}{\varphi^r_n(x)}, r \leq i \leq J. \right.
\]

By a simple calculation, we can get [4]

\[
T_{n,i}(t - x)^i, x = \begin{cases} -R_{n,i}(x), & i = j; \\ C_{ij} |R_{n,i}(x)|^i \sgn R_{n,i}(x), & i < j \leq 2r - 1; \\ 0, & i > j; \end{cases} \tag{2.7}
\]

\[
K_{n,r}((t - x)^i, x) = 0 \quad (1 \leq i \leq r - 1), \tag{2.8}
\]

where \( C_{i,j} \) are constants that depend on \( i, j \) but not on \( n \) and \( x \).

Let \( T_{n,i,j}(g, x) = -\left( C_{i,j} / j! \right) \sgn R_{n,i}(x) \tilde{\Delta}_h^{(j)} \left( g(x) \right. \) \( (i < j_1 \leq J) \).

Generally, if

\[
T_{n,i,j_1,...,j_k}(t - x)^h, x = C_{i,j_1,...,j_k} |R_{n,i}(x)|^{\frac{h}{j_k!}} \times \sgn R_{n,i}(x) \quad (i < j_1 < \cdots < j_k \leq J),
\]

then we define the operator

\[
T_{n,i,j_1,...,j_k}(g, x) = -\frac{C_{i,j_1,...,j_k}}{j_k!} \sgn R_{n,i}(x) \tilde{\Delta}_h^{(j_k)} \left( g(x) \right.
\]
These operators have the properties [4]
\[ |T_{n,i,j_1,\ldots,j_k}(g, x)| \leq C\omega^h(g, |R_{n,i}(x)|^\frac{1}{h}), \]
where \( C_{i,j_1,\ldots,j_k} \) are constants that depend on \( i, j_1, \ldots, j_k \) but not on \( n \) and \( x \). Now let
\[ A_n(g, x) = K_{n,r}(g, x) + \sum_{i=0}^{J} T_{n,i}(g, x) + \sum_{r \leq i < j_1 < \cdots < j_k \leq J} T_{n,i,j_1,\ldots,j_k}(g, x), \]
where the second sum is taken on all finite sequences \( j_1, \ldots, j_k \) which satisfy \( r \leq i < j_1 < \cdots < j_k \leq J \). By computation we have
\[ \|A_n\| \leq M + \sum_{i=0}^{J} \frac{2^i}{i!} + \sum_{r \leq i < j_1 < \cdots < j_k \leq J} |C_{i,j_1,\ldots,j_k}| \frac{2^h}{j_k!} \leq C. \quad (2.9) \]
Using the Taylor expansion,
\[ g(t) = g(x) + (t-x)g'(x) + \cdots + \frac{1}{J!} g^{(J)}(x)(t-x)^J + \frac{1}{(J+1)!} g^{(J+1)}(x)(t-x)^{J+1} + \cdots \]
\[ + \frac{1}{(2r-1)!} g^{(2r-1)}(x)(t-x)^{2r-1} + R_{2r}(g, t, x), \]
where \( R_{2r}(g, t, x) = \frac{1}{(2r-1)!} \int_{x}^{t} (t-u)^{2r-1} g^{(2r)}(u) du \). By [1, (9.5.3)], we know that for \( x \in E_n \),
\[ K_{n,r}((t-x)^{2r-i}, x) \leq Cn^{-r} \varphi^{2r-2i}(x), \quad i \leq r. \]
So for \( x \in E_n, \varphi^{-2}(x) \leq n, \) and \( J+1 \leq j \leq 2r-1, \) we have
\[ |A_n((t-x)^j, x)| \leq C|R_{n,i}(x)|^j \leq Cn^{-r} \varphi^{2(j-r)}(x). \quad (2.10) \]
From the definition of \( A_n(g, x) \), we can write
\[ A_n(g, x) - g(x) = \sum_{j=J+1}^{2r-1} \frac{1}{j!} A_n((t-x)^j, x)g^{(j)}(x) + A_n(R_{2r}(g, . , x), x) \]
\[ \quad := I_1 + I_2. \quad (2.11) \]
First we estimate $I_1$. From (2.10) and Lemma 2.1, when $r\lambda - 2r + j > 0$, by Lemma 2.1, we can get

$$
|I_1| \leq C|n^{-r}\varphi^{2(r-j-\alpha)}(x)g(j)(x)|
$$

$$
\leq Cn^{-r}\varphi^{2r(1-\lambda)}(x)|\varphi^{2(j-2r+j)}(x)g(j)(x)|
$$

$$
\leq Cn^{-r}\varphi^{2r(1-\lambda)}(x)(\|g\| + \|\varphi^{2rJ(2r)}\|).
$$

(2.12)

Now we estimate $I_2$,

$$
|T_{n,i_1,i_2,...,i_k}(R_r(g,t,x))| \leq C\left|\sum_{m=0}^{j_k} \left(\frac{j_k}{m}\right)^{x+(j_k-m)} \int_{R_{n,i}(x)} \left((x+(j_k-m))\right)
$$

$$
\times |R_{n,i}(x)|^{1/2} - u)^{2r-1} g^{2r}(u) du
$$

$$
\leq C\|\varphi^{2rJ(2r)}\|\varphi^{-2rJ}|R_{n,i}(x)|^{2r/2}
$$

$$
\leq Cn^{-r}\varphi^{2r(1-\lambda)}(x)\|\varphi^{2rJ(2r)}\|.
$$

By this and Lemma 2.2 we obtain

$$
|I_2| \leq Cn^{-r}\varphi^{2r(1-\lambda)}(x)\|\varphi^{2rJ(2r)}\|.
$$

(2.13)

From (2.11)–(2.13) and the definition of $A_n(g,x)$, we get

$$
|K_{n,r}(g,x) - g(x)| \leq |A_n(g,x) - g(x)|
$$

$$
+ \left|\sum_{i=1}^{j} T_{n,i}(g,x) + \sum_{1<i_1<...<i_k} T_{n,i_1,i_2,...,i_k}(g,x)\right|
$$

$$
\leq C\left(\sum_{i=1}^{j} \omega^i(g, (n^{-r}\varphi^{2(r-j-\alpha)}(x)))^i\right)
$$

$$
+ n^{-r}\varphi^{2r(1-\lambda)}(x)(\|g\| + \|\varphi^{2rJ(2r)}\|)
$$

$$
\leq C\left(\|f - g\| + \sum_{i=1}^{j} \omega^i(f, (n^{-r}\varphi^{2(r-j-\alpha)}(x)))^i\right)
$$

$$
+ n^{-r}\varphi^{2r(1-\lambda)}(x)(\|f\| + \|\varphi^{2rJ(2r)}\|).
$$

(2.6)

By (2.6), we can deduce (2.5).

Case 2. $x \in E^c_n = [0, \frac{1}{n}] \cap (1 - \frac{1}{n}, 1]$. For fixed $x$, we can choose $g \equiv g_{n,x}$ satisfying $\|f - g\| + n^{-r}\|g(\cdot)\|^2 \leq C\omega(f, \frac{1}{n})$. From [1, (9.5.12) and (9.5.13)],

$$
B_{n,r}((t-x)^{2r-2-j}, x) = \sum_{m=j}^{r-j-1} \varphi^{2r-2-j-2m}(x)q_m(x)\sum_{s=0}^{r-1} C_n^{-r+j-m}
$$

$$
= n^{-2r} \sum_{m=j}^{r-j-1} \varphi^{2r-2-j-2m}(x)n^{2r}q_m(x)d_{r-j+m}(n).
$$
Using Theorem 2 has been proved.

\[ B_{n,r}((t-x)^{2r-2j+1}, x) = \sum_{m=j-1}^{r-j-1} \varphi^{2r-2j-2m}(x)q_m(x)dr_{j+m+1}(n), \]

where \( B_{n,r} \) is the combination of Bernstein operators, \( q_m \) is a fixed bounded polynomial, and \( \sum_{j=0}^{r-1} C_i n^{\alpha} = d_\rho(n) = O(n^{-\rho}) \). When \( j = 0 \), we have

\[ B_{n,r}((t-x)^{2r}, x) = n^{-2r} \sum_{l=0}^{r-1} (n\varphi^2(x))^{r-l} q_l(x)dr_{r+l}(n)n^{r+l}, \]

\[ B_{n,r}((t-x)^{2r+1}, x) = n^{-2r} \varphi^2(x) \sum_{l=0}^{r} (n\varphi^2(x))^{r-l} q_l(x)dr_{r+l}(n)n^{r+l}. \]

Using \( n\varphi^2(x) \leq C \) for \( x \in \mathbb{E}_n^\circ \) and the relation of \( B_n \) and \( K_n \) [1, (9.5.14)],

\[ K_n((t-x)^{2r}, x) = \frac{d}{dx} \left[ B_{n+1} \left( \frac{(t-x)^{2r+1}}{2r+1}, x \right) \right] + B_{n+1}((t-x)^{2r}, x), \]

we know (see [1, Chap. 9.2, the definition of \( K_{n,r}(f, x) \)])

\[ |K_{n,r}((t-x)^{2r}, x)| = \sum_{s=0}^{r-1} C_s K_{n,s-1}((t-x)^{2r}, x) \]

\[ = \sum_{s=0}^{r-1} C_s B_{n,s}((t-x)^{2r}, x) + \sum_{s=0}^{r-1} C_s \frac{d}{dx} B_{n,s}((t-x)^{2r+1}, x) \]

\[ = B_{n,r}((t-x)^{2r}, x) + \frac{d}{dx}(B_{n,r}((t-x)^{2r+1}, x)). \]

So, \( |K_{n,r}((t-x)^{2r}, x)| \leq Cn^{-2r} \); by the Hölder inequality we have \( |K_{n,r}((t-x)^{i}, x)| \leq Cn^{-\gamma}. \) From \( K_{n,r}((t-x)^i, x) = 0, 0 < i \leq r-1 \), we have

\[ |K_{n,r}(f, x) - f(x)| \leq C\|f - g\| + |K_{n,r}(g, x) - g(x)| \]

\[ \leq C\|f - g\| + \left| K_{n,r} \left( \int_0^1 (t-u)^{r-1} g^{(r)}(u) \, du, x \right) \right| \]

\[ \leq C \left( \|f - g\| + n^{-r}\|g^{(r)}\| \right) \leq C\omega^r \left( f, (n^{-r})^\frac{1}{2} \right). \]

Theorem 2 has been proved.

3. PROOF OF THEOREM 1

To prove Theorem 1, we need the following lemmas:

**Lemma 3.1.** For \( 0 < \alpha < \frac{2r}{2^r-1}, 0 \leq \lambda \leq 1 \), if \( \omega^{2r}_\psi(f, t) = O(t^\alpha) \), then

\[ \omega'(f, t) = O(t^{\alpha(1-\frac{1}{2})}), \quad r \leq i \leq 2r. \]
From [1, (3.1.5)], we get
\[
\omega^{2r}(f, t) = \omega^{2r}(f, (t^{1 - \frac{1}{r'}})^{\frac{1}{r'}}) \\
\leq C\omega^{2r}(f, t^{1 - \frac{1}{r'}}) \leq C^r(1 - \frac{1}{r'}). \tag{3.1}
\]
From [1, (4.3.1)] and \(0 < \alpha(1 - \frac{1}{r}) < r\), we have
\[
\omega^{2r-1}(f, t) \leq C^r \bigg( \int_t^\infty \frac{u^{\alpha(1 - \frac{1}{r})}}{u^{2r}} \, du + \|f\| \bigg) \leq C^r(1 - \frac{1}{r'}). \tag{3.2}
\]
By induction, we can obtain Lemma 3.1.

By [6] or [11, (2.6)] with the H"older inequality, we can easily get

**Lemma 3.2.** For \(0 < t < \frac{1}{12}, \frac{\mu}{2} < x < 1 - \frac{\mu}{2}, 0 \leq \beta \leq 2r\), we have
\[
\int \cdots \int_x^\infty \varphi^{-\beta}(x + u_1 + \cdots + u_{2r}) \, du_1 \cdots du_{2r} \leq C^r \varphi^{-\beta}(x). \tag{3.3}
\]

**Lemma 3.3.** For \(r < N, 0 \leq \lambda \leq 1\), and \(0 < \alpha < 2r\), we have
\[
\begin{align*}
|\varphi^{2\lambda}(x)K_n^{(2r)}(f, x)| &\leq Cn^\beta \delta_n^{2\lambda-2r}(x)\|f\|, \quad (3.1) \\
|\varphi^{2\lambda}(x)K_n^{(2r)}(f, x)| &\leq C\|\varphi^{2\lambda}f^{(2r)}\|. \quad (3.2)
\end{align*}
\]

**Proof.** From [1, (3.1.5)], \(\|\varphi^{2r}(x)K_n^{(2r)}(f, x)\| \leq Cn^\beta\|f\|\), for \(x \in E_n\), we have
\[
|\varphi^{2\lambda}(x)K_n^{(2r)}(f, x)| \leq \varphi^{2\lambda(\lambda-1)}(x)|\varphi^{2r}(x)K_n^{(2r)}(f, x)| \\
\leq Cn^\beta \varphi^{2\lambda(\lambda-1)}(x)\|f\| \leq Cn^\beta \delta_n^{2\lambda(\lambda-1)}(x)\|f\|.
\]

For \(x \in E_n^c\), from the procedure of the proof of [1, (9.4.1)] and noticing \(\|\varphi^{2\lambda}\|_{L_r(E_n)} \sim n^{-\lambda}\), we can easily get
\[
|\varphi^{2\lambda}(x)K_n^{(2r)}(f, x)| \leq Cn^{2r}n^{-\lambda}\|f\| \leq Cn^\beta \delta_n^{2\lambda(\lambda-1)}(x)\|f\|.
\]
So we have (3.1).

Now, we prove (3.2). When \(\lambda \neq 0\), by the Jensen inequality, we have (cf. [1, (9.4.4)])
\[
|K_n^{(2r)}(f, x)| \leq \frac{n!}{(n-2r)!} \sum_{k=0}^{n-2r} |\Delta^{2r}a_k(n+1)|P_{n-2r, k}(x) \\
\leq \frac{n!}{(n-2r)!} \left( \sum_{k=0}^{n-2r} P_{n-2r, k}(x)(|\Delta^{2r}a_k(n+1)|)^{\frac{1}{2}} \right)^{\lambda}. 
\]
where $a_k(n) = n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u) \, du$, $\Delta a_k = a_{k+1} - a_k$, and $\Delta^m a_k = \Delta^{m-1} a_k$.

Therefore (cf. [1, p. 153])

$$|\phi^{2r}(x)R_n^{(2r)}(f, x)| \leq \frac{n!}{(n-2r)!} \left( \sum_{k=0}^{n-2r} \phi^{2r}(x) P_{n-2r,k}(x) (|\Delta^{2r} a_k(n+1)|)^{\frac{1}{2}} \right)^{\lambda}$$

$$= \left( \frac{n!}{(n-2r)!} \right)^{1-\lambda}$$

$$\times \left( \sum_{k=0}^{n-2r} P_{n,k+r}(x)(k + r) \cdots (k + 1)(n - r - k) \cdots \right.$$

$$\times (n - 2r - k + 1)(|\Delta^{2r} a_k(n+1)|)^{\frac{1}{2}} \left. \right)^{\lambda}$$

$$\leq C \left( \frac{n!}{(n-2r)!} \right)^{1-\lambda}$$

$$\left( n' P_{n,r}(x)|\Delta^{2r} a_0(n+1)|^{\frac{1}{2}} + n' P_{n,n-r}(x)$$

$$\times |\Delta^{2r} a_{n-2r}(n+1)|^{\frac{1}{2}} + n^{2r-1} \sum_{k=1}^{n-2r-1} P_{n,k+r}(x)$$

$$\times \left( \frac{k}{n} \left( 1 - \frac{k}{n} \right) \right)^{r} |\Delta^{2r} a_k(n+1)|^{\frac{1}{2}} \right)^{\lambda}$$

$$: = C \left( \frac{n!}{(n-2r)!} \right)^{1-\lambda} (I_1 + I_2 + I_3)^{\lambda}. \quad (3.3)$$

For $k = 1, 2, \ldots, n-2r-1$ (cf. [1, p. 154])

$$|\Delta^{2r} a_k(n+1)|^{\frac{1}{2}} \leq C \left( n^{-2r+1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f^{(2r)}(u)| \, du \right)^{\frac{1}{2}}$$

$$\leq C \left( n^{-2r+1} \|\phi^{2r}\| \int_{\frac{k}{n}}^{\frac{k+1}{n}} (u(1-u))^{-\lambda} \, du \right)^{\frac{1}{2}}$$

$$\leq C \left( n^{-2r+1} \|\phi^{2r}\| \frac{k}{n} (1 - \frac{k}{n})^{-\lambda} \right)^{\frac{1}{2}}$$

$$= C n^{r} \|\phi^{2r}\|^{\frac{1}{2}} \left( \frac{k}{n} \left( 1 - \frac{k}{n} \right) \right)^{-\frac{1}{2}}.$$

So,

$$I_3 \leq C n^{2r(1-\frac{1}{2})} \|\phi^{2r}\|^{\frac{1}{2}}. \quad (3.4)$$
Therefore we get
\[ n^{-r+1} \int_0^{\frac{n}{r+1}} u^r |f^{(2r)}(u)| \, du \]
\[ \leq \left( n^{-r+1} \| \varphi^{2r \lambda} f^{(2r)} \| \int_0^{\frac{n}{r+1}} u^{r(1-\lambda)}(1 - u)^{\lambda} \, du \right)^\frac{1}{r} \]
\[ \leq C(n^{-2r+2\lambda} \| \varphi^{2r \lambda} f^{(2r)} \|) \frac{1}{r}. \]

Therefore we get
\[ I_1 \leq Cn^{-\frac{1}{2}(2r-\lambda)+r} \| \varphi^{2r \lambda} f^{(2r)} \| \frac{1}{r}. \] (3.5)

Similarly, we have
\[ I_2 \leq Cn^{-\frac{1}{2}(2r-\lambda)+r} \| \varphi^{2r \lambda} f^{(2r)} \| \frac{1}{r}. \] (3.6)

Combining (3.3), (3.4), (3.5), and (3.6), we get for \( \lambda \neq 0 \),
\[ |\varphi^{2r \lambda}(x) K_{n}^{(2r)}(f, x)| \leq C \| \varphi^{2r \lambda} f^{(2r)} \|. \]

For \( \lambda = 0 \),
\[ |K_{n}^{(2r)}(f, x)| \leq \frac{n!}{(n-2r)!} \sum_{k=0}^{n-2r} |\Delta^{2r} a_k(n+1) P_{n-2r, k}(x)\]
\[ \leq n^{2r} (P_{n-2r, 0}(x)) |\Delta^{2r} a_0(n+1)| + P_{n-2r, n-2r}(x) |\Delta^{2r} a_{n-2r}(n+1)| \]
\[ + \sum_{k=1}^{n-2r-1} |\Delta^{2r} a_k(n+1)| P_{n-2r, k}(x). \]

From the procedure of the proof in the case where \( \lambda \neq 0 \), we have
\[ |\Delta^{2r} a_k(n+1)| \leq C n^{-2r} \| f^{(2r)} \|(k = 1, 2, \ldots, n-2r-1), |\Delta^{2r} a_0(n+1)| \leq C n^{-2r} \| f^{(2r)} \|, |\Delta^{2r} a_{n-2r}(n+1)| \leq C n^{-2r} \| f^{(2r)} \| \]

Thus we have (3.2). This finishes the proof of Lemma 3.3.

The proof of Theorem 1:

\( \Leftarrow \): By Theorem 2,
\[ |K_{n, r}(f, x) - f(x)| \leq C \left( \sum_{i=r}^{j} \omega^i(f, n^{-r} \varphi^{2(i-r)}(x)) \right)^\frac{1}{r} + \omega^2(f, n^{-1/2} \varphi^{1-\lambda}(x)) \]
\[ + n^{-r} \varphi^{2r(1-\lambda)}(x) \| f \|. \] (3.7)

For the first term on the right side of (3.7), from Lemma 3.1, we cannot obtain \( O((n^{-1/2} \varphi^{-1}(x))^a) \). For example, if \( i = r, \omega^i(f, n^{-r})^\frac{1}{r} = O((n^{-r})^{a/2}) \neq O((n^{-1/2} \varphi^{-1}(x))^a) \). In fact, if \( (n^{-r})^{a/2} \leq C(n^{-1/2} \varphi^{-1}(x))^a \), then \( \left( \frac{1}{\sqrt{n}(x)} \right)^{a(1-\lambda)} \leq C \). Let \( x \to 0 \); this is impossible. So, we cannot
get the result as in the following:
\[
\omega^2_{\varphi}(f, t) = O(t^a) \Rightarrow |K_{n,r}(f, x) - f(x)| = O\left(n^{-1/2} \varphi^{1-\lambda}(x)\right).
\]

But from (3.7) and Lemma 3.1, for \( \alpha < \frac{2r}{\lambda - 2} \), we have
\[
|K_{n,r}(f, x) - f(x)| \leq C\left(\sum_{i=0}^{r} \omega^i(f, n^{-r} \delta_n^{2(i-r)}(x))^{\frac{1}{r}} + \omega^2_{\varphi}(f, n^{-1/2} \delta_n^{1-\lambda}(x)) + n^{-r} \delta_n^{2r(1-\lambda)}(x) \|f\|\right)
\]
\[
\leq C\left(\sum_{i=0}^{\frac{2r}{\lambda-2}} (n^{-r} \delta_n^{2(i-r)}(x))^{\frac{\alpha(1-\lambda/2)}{\alpha}} + \left(n^{-1/2} \delta_n^{1-\lambda}(x)\right)^{\alpha} \right).
\]

(3.8)

For \( r \leq i \leq J, r\lambda - 2r + i \leq 0 \); noticing \( \delta_n^{-1}(x) \leq \sqrt{n} \), then
\[
(n^{-r} \delta_n^{2(i-r)}(x))^{\frac{\alpha(1-\lambda/2)}{\alpha}} \leq \left(n^{-1/2} \delta_n^{1-\lambda}(x)\right)^{\alpha}.
\]

Hence, the relation of “\( \Leftarrow \)” in (1.6) holds.

Remark 2. In Theorem 1 (1.6), when \( \alpha > \frac{2r}{\lambda - 2} \), the “\( \Leftarrow \)” is not true. We observe that \( f(t) = t^r, r \geq 2 \). Obviously, \( \omega^2_{\varphi}(f, t) = 0 \). Let \( x = n^{-1} \); we have
\[
K_{n,r}(f, x) - f(x) = K_{n,r}((t-x)^r, x) \sim \frac{1}{n^r}.
\]

If the relation “\( \Leftarrow \)” is right, then \( K_{n,r}(f, x) - f(x) = O(n^{-1/2} \delta_n^{1-\lambda}(x)) \sim n^{-\alpha(1-\lambda/2)} \), but we know, when \( \alpha > \frac{2r}{\lambda - 2}, \alpha(1-\lambda/2) > r \), so “\( \Leftarrow \)” is not true.

“\( \Rightarrow \)” Let \( \gamma_{n,\lambda}(x) = n^{-\frac{1}{2}} \delta_n^{1-\lambda}(x) \). If \( K_{n,r}(f, x) - f(x) = O(\gamma_{n,\lambda}^a(x))\), for every \( n : n > 2r \), we have
\[
\left|\Delta_{\varphi(x)}^2 f(x)\right| \leq C_{\gamma_{n,\lambda}}^a(x) + \sum_{i=0}^{r-1} |C_i(n)| \int \cdots \int_{\frac{\varphi(x)}{2}} \frac{\varphi(x)}{2} \frac{\varphi(x)}{2}
\]
\[
\times \left|K_n^{(2r)}(f, x + \sum_{j=1}^{2r} u_j)\right| du_1 \cdots du_{2r}
\]
\[
\leq C_{\gamma_{n,\lambda}}^a(x) + \sum_{i=0}^{r-1} |C_i(n)| \int \cdots \int_{\frac{\varphi(x)}{2}} \frac{\varphi(x)}{2} \frac{\varphi(x)}{2}.
\]
\[ K_n^{(2r)}(f - g, x + \sum_{j=1}^{2r} u_j) \, du_1 \cdots du_{2r} \]
\[ + \sum_{j=0}^{r-1} C(n) \int \cdots \int_{-\frac{\gamma_n}{2r}}^{\frac{\gamma_n}{2r}} K_n^{(2r)}(g, x + \sum_{j=1}^{2r} u_j) \, du_1 \cdots du_{2r} \]
\[ := C_{n,\lambda}(x) + J_1 + J_2. \quad (3.9) \]

Combining Lemma 3.2 and Lemma 3.3, we have
\[ J_1 \leq Ct^{2r} \gamma_{n,\lambda}(x) \| f - g \|. \quad (3.10) \]
\[ J_2 \leq Ct^{2r} \| \varphi^{2r} \| (2r). \quad (3.11) \]

Using (3.9), (3.10), and (3.11), choosing appropriate \( g \), we can obtain
\[ \left| \Delta_{2r}^{(2r)} f(x) \right| \leq C \left( \gamma_{n,\lambda}(x) + t^{2r} \gamma_{n,\lambda}^{-2r}(x) \omega^{2r}_{\varphi}(f, \gamma_{n,\lambda}(x)) \right). \]

For every fixed \( h : 0 < h < \frac{1}{16r} \) and every \( x : x \geq rt \), we can choose \( n \) such that \( \gamma_{n,\lambda}(x) \leq h < 2\gamma_{n,\lambda}(x) \). Then
\[ \left| \Delta_{2r}^{(2r)} f(x) \right| \leq C \left( h^\alpha + \left( \frac{t}{h} \right)^{2r} \omega^{2r}_\varphi(f, h) \right). \]
So,
\[ \omega^{2r}_\varphi(f, t) \leq C \left( h^\alpha + \left( \frac{t}{h} \right)^{2r} \omega^{2r}_\varphi(f, h) \right), \]
which yields the assertion of Theorem 1 by the Berens–Lorentz lemma.

**Remark 3.** For Szász–Kantorovich and Baskakov–Kantorovich operators, one can get similar results with the same method.

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**REFERENCES**