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Extremal structure and Samuel compactification ☆

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ABSTRACT

Let M be a uniform space and X the Banach space of bounded and uniformly continuous functions from M into $\mathbb{R}$, provided with its supremum norm.
The aim of this paper is to discuss the connection between the geometry of X and the nature of M. In particular, we will prove that certain reconstructions of the unit ball of X by means of its extreme points admit a translation in terms of extension of uniformly continuous functions. We also analyze the impact of these properties on the Samuel compactification of M.

1. Introduction

Let M be a (not necessarily compact) uniform space and X the algebra of bounded and uniformly continuous functions from M into $\mathbb{R}$. As usual, we will consider the uniform norm on X:

$$\|x\| = \sup\{|x(t)|:\ t \in M\} \quad (x \in X).$$

The closed unit ball of X will be denoted by B_X:

$$B_X = \{x \in X:\ \|x\| \leqslant 1\}.$$

If $e \in B_X$ and the equation $e = (1-\lambda)x + \lambda y$, with $0 < \lambda < 1$ and $x, y \in B_X$, implies $e = x = y$, it is said that e is an extreme point of B_X. The symbol E_X will stand for the set of extreme points of B_X. It is not difficult to prove that

$$E_X = \{e \in X:\ |e(t)| = 1,\ \forall t \in M\}.$$

On the other hand, it is also immediate that an element $x \in X$ is invertible if, and only if, there exists $\rho > 0$ such that $x(t) > \rho$, for all $t \in M$. The set of invertible elements of X will be represented by X^{-1}.

With respect to the notions of uniformity and uniform space we will follow the criterion of [3]. In particular, only Hausdorff uniform spaces will be considered and, if $\mathcal{U}$ is a uniformity on a certain set, it must be assumed that every element of $\mathcal{U}$ is symmetric.

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If A is a subset of a certain set, the complement of A will be represented by A^c.

We have to mention, for later use, the following result by Katětov concerning the extension of uniformly continuous functions.

Theorem 1. *([4]) Let M be a uniform space, $A \subset M$ and $f : A \to [-1, 1]$ a uniformly continuous function. Then there exists $x : M \to [-1, 1]$, uniformly continuous, such that $x|_A = f$.*

Now, observe that the previously defined set E_X may be very small (if M is connected E_X contains exactly two elements).

In this paper we study conditions under which E_X is large enough to ensure the equality $B_X = \overline{\text{co}}(E_X)$, where $\overline{\text{co}}(E_X)$ denotes the closed convex hull of E_X. We analyze the influence of this geometric property over the structure of the uniform space M and the nature of an adequate compactification of M that we comment below.

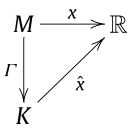
Let $\mathcal{U}$ be the original uniformity on M and $\mathcal{U}_0$ the uniformity determined by X, that is, the initial uniformity corresponding to the family of bounded and uniformly continuous functions from M into $\mathbb{R}$, with respect to $\mathcal{U}$. It is easy to see that $\mathcal{U}_0 \subset \mathcal{U}$ and that both uniformities induce the same topology. Furthermore, $(M, \mathcal{U}_0)$ is a totally bounded space. The completion of this last uniform space is called the **Samuel compactification** of M with respect to the uniformity $\mathcal{U}$ [7].

Since Hausdorff compact spaces are uniquely uniformizable, they will be considered to be provided with its corresponding uniformity whenever one refers to any notion depending on such a structure.

The Samuel compactification provides us with the following result:

Theorem 2. *Given a uniform space M there exists a Hausdorff compact space K and a mapping $\Gamma : M \to K$ such that*

i) *$\Gamma(M)$ is dense in K.*
ii) *Γ is a homeomorphism from M onto $\Gamma(M)$.*
iii) *For every bounded and uniformly continuous function $x : M \to \mathbb{R}$ there is a unique continuous function $\hat{x} : K \to \mathbb{R}$ such that $\hat{x} \circ \Gamma = x$.*



iv) *For each continuous function $f : K \to \mathbb{R}$, the function $f \circ \Gamma$ is uniformly continuous.*

Observe, in particular, that the mapping $f \mapsto f \circ \Gamma$, from $C(K)$ onto X, is an isometric isomorphism, where $C(K)$ denotes the Banach space of continuous functions from K into $\mathbb{R}$.

A pair (K, Γ) satisfying the first three conditions of the above theorem is called a **uniform compactification** of M. The Stone–Čech compactification is a remarkable example.

If (K, Γ) is a uniform compactification of M, condition iii) holds, in fact, for uniformly continuous functions valued in any Hausdorff compact space.

In order to highlight the role of condition iv) in Theorem 2 we state the following result which can be easily proved.

Proposition 3. *Let M be a uniform space and (K, Γ) a uniform compactification of M. The following statements are equivalent*:

i) *For each continuous function $f : K \to \mathbb{R}$, the function $f \circ \Gamma$ is uniformly continuous.*
ii) *Γ is uniformly continuous.*
iii) *For any uniform compactification (Q, Ψ) of M there exists a surjective and continuous mapping $F : Q \to K$ such that $F \circ \Psi = \Gamma$.*

From this proposition it can be deduced that if (K_1, Γ_1) and (K_2, Γ_2) are uniform compactifications of M which satisfy condition iv) in Theorem 2, then the spaces K_1 and K_2 are homeomorphic.

Therefore, up to a homeomorphism, the Samuel compactification of M, usually denoted by sM, is the unique uniform compactification of M satisfying the four conditions in Theorem 2. In [8], an interesting study of the Samuel compactification of metric spaces can be found. The terminology in this reference (the minimum uniform compactification) is clear in the light of Proposition 3.

2. Extreme points and extension of uniformly continuous functions

The extremal structure of continuous function spaces has been studied over a long time period. For any compact Hausdorff space K, W.G. Bade proved that the unit ball of $C(K)$ is the closed convex hull of its extreme points if, and only if, K is zero-dimensional (see [1] and [2]). Later, N.T. Peck [6, Theorem 4] established that, if K is a zero-dimensional compact metric space, then $B_{C(K)}$ is the sequentially convex hull of its extreme points (a formulation of this property can be found in the statement viii) of Theorem 4). A similar result was obtained by D. Oates without the hypothesis of metrizability of K [5].

In this paper, the aforementioned results will be extended to the context of uniformly continuous function spaces.

According to previously settled notation, let M be a not necessarily compact uniform space and X the Banach space of bounded and uniformly continuous functions from M into $\mathbb{R}$. In this section we will show that the abundance of extreme points in B_X is, in a certain sense, equivalent to the abundance of invertible elements in X. At the same time, such conditions determine an attractive kind of disconnectedness of M and a property of extension of uniformly continuous functions. Moreover, we will study the impact of these properties over the Samuel compactification of M. In essence, our goal is the identification of those uniform spaces that have a zero-dimensional Samuel compactification.

We will make use of the following concept:

Given two subsets A and B of M, it is said that A is **uniformly contained** in B if there exists $U \in \mathcal{U}$ (the original uniformity of M) such that $U(A) \subset B$. In such a case, we will write $A \subset_u B$.

Obviously, the relation of uniform inclusion is more restrictive (in general) than the relation of inclusion. Notice, in this connection, that the condition $A \subset_u A$ implies, in particular, that A is open-and-closed.

Theorem 4. *Let M and X be as in the introduction. The following statements are equivalent*:

i) $B_X = \overline{\text{co}}(E_X)$.
ii) *For each $x \in B_X$ there exists $y \in \text{co}(E_X)$ such that $\|x - y\| < 1$.*
iii) *For any two subsets A and B of M such that $A \subset_u B$ there exists G, uniformly contained in itself, such that $A \subset G \subset B$.*
iv) *The set X^{-1} of invertible elements of X is dense in X.*
v) *For every $x \in X$, there exists $y \in X^{-1}$ such that $\|x - y\| < 1$.*
vi) *Given $D \subset M$ and a uniformly continuous function $f : D \to \{-1, 1\}$ there exists a uniformly continuous function $e : M \to \{-1, 1\}$ such that $e|_D = f$.*
vii) $B_X = \lambda E_X + (1 - \lambda) B_X$, *for all* $\lambda \in]0, \frac{1}{2}[$.
viii) *For every $x \in B_X$ there exist a sequence $\{e_n\}$ in E_X and a sequence $\{\lambda_n\}$ in $[0, 1]$, with $\sum_{n=1}^{\infty} \lambda_n = 1$, such that $x = \sum_{n=1}^{\infty} \lambda_n e_n$.*

Proof. The second condition is an immediate consequence of the first one.

Assume that the second statement holds and let A and B be subsets of M such that $A \subset_u B$. It is then immediate that the function $f : A \cup B^c \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 1 & \text{if } t \in A, \\ -1 & \text{if } t \in B^c, \end{cases}$$

is uniformly continuous. By Theorem 1 there exists a uniformly continuous function $x : M \to [-1, 1]$ such that $x|_{A \cup B^c} = f$. Hence, $x \in B_X$ and, by the hypothesis, there exists $y \in \text{co}(E_X)$ such that $\|x - y\| < 1$. Taking into account that the set $y(M)$ is finite, we may define the following real numbers:

$$\alpha = \min\bigl([0, 1] \cap y(M)\bigr) \quad \text{and} \quad \beta = \max\bigl([-1, 0[\cap y(M)\bigr).$$

Moreover, let $\varepsilon = \alpha - \beta$. The uniform continuity of y gives a $U \in \mathcal{U}$ such that

$$(t, t') \in U \quad \Rightarrow \quad \bigl|y(t) - y(t')\bigr| < \varepsilon.$$

Consider $G = \{t \in M\colon\ y(t) \geqslant \alpha\}$. Then, $G^c = \{t \in M\colon\ y(t) \leqslant \beta\}$ and, for $t \in G$ and $t' \in G^c$,

$$\bigl|y(t) - y(t')\bigr| = y(t) - y(t') \geqslant \alpha - \beta = \varepsilon.$$

In consequence, $(t, t') \notin U$ and so $U(G) \subset G$.

On the other hand, given $t \in A$, it is clear that $x(t) = 1$ and thus $y(t) > 0$. From that, it follows that $t \in G$. Similarly, if $t \in B^c$, then $x(t) = -1$ and therefore $y(t) < 0$. This implies that $t \in G^c$. In this way, we conclude that $A \subset G \subset B$ and hence ii) $\Rightarrow$ iii).

Next, we will prove that iii) $\Rightarrow$ iv). For this purpose, let $x \in X$, $\varepsilon \in]0, 1[$ and consider

$$A = \left\{t \in M\colon\ \bigl|x(t)\bigr| \geqslant \frac{\varepsilon}{2}\right\} \quad \text{and} \quad B = \left\{t \in M\colon\ \bigl|x(t)\bigr| \geqslant \frac{\varepsilon}{4}\right\}.$$

Since x is uniformly continuous there is $U \in \mathcal{U}$ such that

$$(t, t') \in U \quad \Rightarrow \quad \bigl|x(t) - x(t')\bigr| < \frac{\varepsilon}{4}.$$

Given $t' \in U(A)$ let $t \in A$ be such that $(t, t') \in U$. Then,

$$\bigl|x(t')\bigr| = \bigl|x(t) - \bigl(x(t) - x(t')\bigr)\bigr| \geqslant \bigl|x(t)\bigr| - \bigl|x(t) - x(t')\bigr| \geqslant \frac{\varepsilon}{2} - \frac{\varepsilon}{4} = \frac{\varepsilon}{4},$$

and then $t' \in B$. Therefore $U(A) \subset B$ and thus $A \subset_u B$. By iii), there is $G \subset M$ such that $G \subset_u G$ and $A \subset G \subset B$. Let $y : M \to \mathbb{R}$ be the function defined by

$$y(t) = \begin{cases} x(t) & \text{if } t \in G, \\ \frac{\varepsilon}{2} & \text{if } t \in G^c. \end{cases}$$

The condition $G \subset_u G$ ensures that y is uniformly continuous. Moreover, it is bounded and $|y(t)| \geqslant \frac{\varepsilon}{4}$, for every $t \in M$. This shows that y is an invertible element of the algebra X. On the other hand, given $t \in G$, $|x(t) - y(t)| = 0$ and, if $t \in G^c$, $|x(t) - y(t)| < \varepsilon$. Therefore, $\|x - y\| < \varepsilon$.

Obviously iv) $\Rightarrow$ v) and we will see now that v) $\Rightarrow$ vi). To this end, let $D \subset M$ and $f : D \to \{-1, 1\}$ a uniformly continuous function. Then there exists $x : M \to [-1, 1]$, uniformly continuous, such that $x|_D = f$. By the hypothesis, there is $y \in X^{-1}$ with $\|x - y\| < 1$. The function $e : M \to \{-1, 1\}$ defined by $e(t) = \frac{y(t)}{|y(t)|}$, for each $t \in M$, is a uniformly continuous extension of f. In order to verify this last statement, fix a point $t \in D$. If $f(t) = 1$, the inequality $|x(t) - y(t)| < 1$ guarantees that $y(t) > 0$. Thereby, $e(t) = 1 = f(t)$. Similarly, if $f(t) = -1$, then $y(t) < 0$ and, in consequence, $e(t) = -1 = f(t)$.

With the objective of proving that vi) $\Rightarrow$ vii), consider $\lambda \in]0, \frac{1}{2}[$ and $x \in B_X$. Then, define $D = \{t \in M\colon |x(t)| \geqslant 1 - 2\lambda\}$ and let $f : D \to \{-1, 1\}$ given by $f(t) = \frac{x(t)}{|x(t)|}$, for every $t \in D$. By the hypothesis, there is a uniformly continuous function $e : M \to \{-1, 1\}$ such that $e|_D = f$. Accordingly, the function $y = \frac{x - \lambda e}{1 - \lambda}$ belongs to X and we will show that $\|y\| \leqslant 1$. For this purpose, fix $t \in M$. If $t \in D$,

$$\left|y(t)\right| = \frac{|x(t) - \lambda e(t)|}{1 - \lambda} = \frac{||x(t)| - \lambda|}{1 - \lambda} \leqslant 1.$$

On the other hand, if $t \in D^C$, $|y(t)| = \frac{|x(t) - \lambda e(t)|}{1 - \lambda} \leqslant \frac{|x(t)| + \lambda}{1 - \lambda} \leqslant \frac{1 - 2\lambda + \lambda}{1 - \lambda} = 1$. To conclude this implication, it suffices to observe that $x = \lambda e + (1 - \lambda)y$.

With the aim of proving that vii) $\Rightarrow$ viii), let $x \in B_X$ and $\lambda \in]0, \frac{1}{2}[$. In accordance with the assumption, there are $e_1 \in E_X$ and $y_1 \in B_X$ such that $x = \lambda e_1 + (1 - \lambda)y_1$. For the same reason, $y_1 = \lambda e_2 + (1 - \lambda)y_2$ for some $e_2 \in E_X$ and $y_2 \in B_X$. By proceeding in this way, we find a sequence $\{e_n\}$ of extreme points of B_X and a sequence $\{y_n\}$ in B_X such that $y_n = \lambda e_{n+1} + (1 - \lambda)y_{n+1}$, for every $n \in \mathbb{N}$. Consequently, for any natural number n,

$$x = \lambda e_1 + (1 - \lambda)\lambda e_2 + \cdots + (1 - \lambda)^n \lambda e_{n+1} + (1 - \lambda)^{n+1} y_{n+1}.$$

From here it can be deduced immediately that $x = \sum_{n=1}^{\infty} (1 - \lambda)^{n-1} \lambda e_n$. Observe now that $\sum_{n=1}^{\infty} (1 - \lambda)^{n-1} \lambda = 1$.

Finally, it is clear that viii) $\Rightarrow$ i). $\square$

From now on, a uniform space M will be called **uniformly disconnected** if M satisfies condition iii) of the above theorem.

Let M be a uniform space and $K = sM$, the Samuel compactification of M. We have already observed that spaces X and $C(K)$ are isometrically isomorphic. Therefore, $B_X = \overline{\text{co}}(E_X)$ if, and only if, $B_{C(K)} = \overline{\text{co}}(E_{C(K)})$. On the other hand, this last equality holds true if, and only if, K is zero-dimensional. Thus, the equivalence between statements i) and iii) of Theorem 4 has the following consequence:

Corollary 5. *A uniform space M is uniformly disconnected if, and only if, the Samuel compactification of M is zero-dimensional.*

As it can be easily checked, if M is uniformly disconnected then M has no non-trivial connected sets (M is totally disconnected). In fact, if the uniformity of M is metrizable, we have the following result:

Proposition 6. *Every uniformly disconnected metric space is strongly zero-dimensional.*

Proof. Let M be a uniformly disconnected metric space and consider a pair A, B of subsets of M such that A is closed, B is open and $A \subset B$. It suffices to prove the existence of an open-and-closed set $G \subset M$ such that $A \subset G \subset B$ (see [3, Theorem 6.2.4]).

To avoid the trivial case we will assume $A \neq \emptyset$. First, let us show the existence of an open-and-closed covering $\{G_n\colon n \in \mathbb{N}\}$ of M, consisting of pairwise disjoint sets, such that, for each $n \in \mathbb{N}$, $G_n \cap A = \emptyset$ or $G_n \subset B$:

For every natural number n, define $E_n = \{t \in M\colon d(t, B^c) \geqslant \frac{1}{n}\}$ and $F_n = \{t \in M\colon d(t, A) \geqslant \frac{1}{n}\}$. Since $E_n \subset_u B$, there exists $P_n \subset M$, with $P_n \subset_u P_n$ (that is, $d(P_n, P_n^c) > 0$), such that $E_n \subset P_n \subset B$. Likewise, $F_n \subset_u A^c$ and thus there exists $Q_n \subset M$, with $Q_n \subset_u Q_n$, such that $F_n \subset Q_n \subset A^c$. In particular, $Q_n \cap A = \emptyset$.

In order to prove that $\{P_n\colon n \in \mathbb{N}\} \cup \{Q_n\colon n \in \mathbb{N}\}$ is a covering of M, fix a point $t \in M$. If $t \in A$ it is clear that $t \notin B^c$ and therefore $d(t, B^c) > 0$. Consequently, there is $n \in \mathbb{N}$ such that $t \in E_n \subset P_n$. On the other hand, if $t \notin A$, then $d(t, A) > 0$ and there is $n \in \mathbb{N}$ such that $t \in F_n \subset Q_n$.

Given $n \in \mathbb{N}$, define $R_{2n} = P_n$ and $R_{2n-1} = Q_n$. The family $\{R_n\colon n \in \mathbb{N}\}$ is an open-and-closed covering of M. Finally, the family $\{G_n\colon n \in \mathbb{N}\}$ given by

$$G_1 = R_1, \qquad G_{n+1} = R_{n+1} \setminus \bigcup_{j=1}^{n} R_j, \quad \text{for every } n \in \mathbb{N},$$

is a covering of M consisting of pairwise disjoint open-and-closed sets. It is also clear that, for every $n \in \mathbb{N}$, $G_{2n-1} \cap A = \emptyset$ and $G_{2n} \subset B$. Therefore $\{G_n\colon n \in \mathbb{N}\}$ satisfies the required properties.

To finish the proof, define $I = \{n \in \mathbb{N}\colon G_n \cap A \neq \emptyset\}$ and $G = \bigcup_{n\in I} G_n$. Obviously $G_n \subset B$, for every $n \in I$ and hence $G \subset B$. Moreover, given $n \in \mathbb{N}\setminus I$, $G_n \cap A = \emptyset$. Thus,

$$A = \left(\bigcup_{n\in\mathbb{N}} G_n\right) \cap A = \bigcup_{n\in\mathbb{N}} (G_n \cap A) = \bigcup_{n\in I} (G_n \cap A) \subset G.$$

Since $G^c = \bigcup_{n\in\mathbb{N}\setminus I} G_n$ it is also clear that G is an open-and-closed set. □

The reverse of the above result is not true in general. For example, the set of rational numbers with its usual metric is strongly zero-dimensional but it is not uniformly disconnected. In order to check this last assertion observe that every uniformly continuous function from $\mathbb{Q}$ into $\{-1, 1\}$ can be continuously extended to $\mathbb{R}$ and, hence, it is constant. Therefore, $\mathbb{Q}$ does not satisfy statement vi) of Theorem 4 and, hence, it does not satisfy statement iii) either, that is, $\mathbb{Q}$ is not uniformly disconnected.

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