Degree conditions for hamiltonicity: Counting the number of missing edges

Ralph J. Faudree\textsuperscript{a}, Richard H. Schelp\textsuperscript{a}, Akira Saito\textsuperscript{b}, Ingo Schiermeyer\textsuperscript{c,1}

\textsuperscript{a}Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
\textsuperscript{b}Department of Computer Science, Nihon University, Sakurajosui 3-25-40, Setagaya-ku, Tokyo 156-8550, Japan
\textsuperscript{c}Institut für Diskrete Mathematik und Algebra, Technische Universität Bergakademie Freiberg, 09596 Freiberg, Germany

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Abstract

In 1960 Ore proved the following theorem: Let $G$ be a graph of order $n$. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices $u$ and $v$, then $G$ is hamiltonian. In this note we strengthen Ore’s theorem as follows: we determine the maximum number of pairs of nonadjacent vertices that can have degree sum less than $n$ (i.e. violate Ore’s condition) but still imply that the graph is hamiltonian. Some other sufficient conditions (i.e. Fan’s condition) for hamiltonian graphs are strengthened as well.

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1. Introduction

We use [3,4] for terminology and notation not defined here and consider finite and simple graphs only.

\textbf{Theorem 1 (Dirac [6]).} Let $G$ be a graph of order $n$ and minimum degree $\delta(G) \geq n/2$. Then $G$ is hamiltonian.

This was generalized by the following theorem of Ore.

\textbf{Theorem 2 (Ore [11]).} Let $G$ be a graph of order $n$. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices $u$ and $v$, then $G$ is hamiltonian.

At the Second International Conference on Graph Theory in Elgersburg in May 2000 the second author asked the following two questions:

\textbf{Question 1.} How many vertices can have degree less than $n/2$ (i.e. violate Dirac’s condition) but still imply the graph is hamiltonian?

E-mail addresses: rfaudree@memphis.edu (R.J. Faudree), schelpr@msci.memphis.edu (R.H. Schelp), asaito@cs.chs.nihon-u.ac.jp (A. Saito), schierme@tu-freiberg.de (I. Schiermeyer).

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**Question 2.** How many pairs of nonadjacent vertices can have degree sum less than \( n \) (i.e. violate Ore’s condition) but still imply that the graph is hamiltonian?

We now define two classes of nonhamiltonian graphs. For \( n \geq 3 \) and \( 1 \leq \delta \leq k \leq (n - 1)/2 \) let \( F(n, k) \) be the graph of order \( n \) consisting of a complete graph on \( n - k \) vertices and \( k \)-independent vertices which are completely adjacent to the same \( k \) vertices of the \( K_{n-k} \). Hence, \( F(n, k) = K_k + (K_{n-2k} \cup K_1) \) and \( F(n, k) \) has degree sequence \( d_1 = d_2 = \cdots = d_k = k \), \( d_{k+1} = \cdots = d_{n-k} = n - k - 1 \) and \( d_{n-k+1} = \cdots = d_n = n - 1 \). The graph \( H(n, k, \delta) \) is obtained from the graph \( F(n, k) \) as follows: choose one of the \( k \)-independent vertices and delete \( k - \delta \) of its edges.

2. Extended Ore’s theorem

The purpose of this note is to answer both questions if the minimum degree is additionally known, thereby giving an improved version of both Dirac’s and Ore’s theorems. The version of Dirac’s theorem is the following.

**Theorem 3.** Let \( G \) be a graph of order \( n \) and minimum degree \( \delta = \delta(G) < n/2 \). If \( |\{v \in V(G) \mid d(v) < n/2\}| \leq \delta - 1 \), then \( G \) is hamiltonian.

The condition given in Theorem 3 is best possible since the nonhamiltonian graph \( F(n, \delta) \) has \( \delta \) vertices of degree \( \delta \). The theorem is an immediate consequence of the following result of Pósa.

**Theorem 4** (Pósa, [12]). Let \( G \) be a graph of order \( n \) and degree-sequence \( d_1 \leq d_2 \leq \cdots \leq d_n \). If \( d_i > i \) for \( 1 \leq i < (n - 1)/2 \) and \( d_{(n+1)/2} > (n - 1)/2 \) for \( n \) odd, then \( G \) is hamiltonian.

We will now present a complete answer to Question 2.

For natural numbers \( n \) and \( \delta \), define \( g(n, \delta) \) by

\[
g(n, \delta) = \begin{cases} 
\infty & \text{if } n \leq 2\delta, \\
\frac{1}{8}(n^2 - 1) & \text{if } 2\delta + 1 \leq n \leq 6\delta - 3 \text{ and } n \text{ is odd}, \\
\frac{1}{8}(n^2 + 2n - 8) & \text{if } 2\delta + 2 \leq n \leq 6\delta - 4 \text{ and } n \text{ is even}, \\
\delta n - \frac{2}{2}\delta^2 - \frac{1}{2}\delta & \text{if } n \geq 6\delta - 2.
\end{cases}
\]

For a graph \( G \) of order \( n \), define \( N_2(G) \) and \( n_2(G) \) by

\[
N_2(G) = \{xy \in E(\overline{G}) : d(x) + d(y) < n\} \quad \text{and} \quad n_2(G) = |N_2(G)|.
\]

**Theorem 5** (Extended Ore’s theorem). Let \( G \) be a graph of order \( n \) with minimum degree \( \delta \). If \( n_2(G) < g(n, \delta) \), then \( G \) is hamiltonian.

Theorem 5 is sharp for \( \delta \geq 2 \). If \( n \geq 6\delta - 2 \), let \( G = F(n, \delta) \). Then \( |V(G)| = n, \delta(G) = \delta \) and \( n_2(G) = \delta(n - 2\delta) + (\frac{\delta}{2}) = g(n, \delta) \). However, \( G \) is not hamiltonian. If \( 2\delta + 1 \leq n \leq 6\delta - 3 \) and \( n \) is odd, let \( G = H(n, (n - 1)/2, \delta) \). Note that \( F(n, (n - 1)/2) = K_{(n-1)/2} + ((n + 1)/2)K_1. \) Then \( |V(G)| = n \) and \( \delta(G) = \delta \). Although there exist pairs of nonadjacent vertices \( x, v \) with \( d(x) = \delta(G) \) and \( v \in V(K_{(n-1)/2}) \), they do not belong to \( N_2(G) \) since \( d(v) \geq n - 2 \). Thus, even for \( \delta \geq 1 \),

\[
n_2(G) = \left( \frac{n+1}{2} \right) = \frac{1}{8}(n^2 - 1) = g(n, \delta).
\]

Furthermore, \( G \) is not hamiltonian. If \( 2\delta + 2 \leq n \leq 6\delta - 4 \) and \( n \) is even, let \( G = H(n, (n - 2)/2, \delta) \). Then \( |V(G)| = n \) and \( \delta(G) = \delta \). Since \( N_2(G) = \{uv : u, v \in V(K_2 \cup (n - 2)/2K_1), uv \not\in K_2\} \), \( n_2(G) = \left( \frac{\delta^2}{2} \right) - 1 = \frac{1}{8}(n^2 + 2n - 8) = g(n, \delta) \). However, \( G \) is not hamiltonian.
Theorem 6 (Erdős [7]). Let $G$ be a graph of order $n \geq 3$ with minimum degree $\delta$. If either $\delta \geq \frac{1}{2} n$ or $|E(G)| > f(n, \delta)$ holds, then $G$ is hamiltonian.

Proof of Theorem 5. If $\delta \geq \frac{1}{2} n$, then $G$ is hamiltonian by Dirac’s theorem. Thus, we may assume $\delta < \frac{1}{2} n$, or $n \geq 2\delta + 1$.

Note that in this range, $f(n, \delta) + g(n, \delta) = \binom{n}{2}$. Let $G' = C_n(G)$ be the Bondy–Chvátal closure of $G$ [2]. Then $E(G') = N_2(G') \subset N_2(G)$ and hence $|E(G')| \leq n_2(G) < g(n, \delta)$, which implies $|E(G')| > f(n, \delta)$. By Theorem 6 $G'$ is hamiltonian, and hence $G$ is hamiltonian. $\square$

A more careful detailed argument can be used to show that the graph $G$ of Theorem 5 satisfies $C_n(G) = K_n$, so as a corollary $G$ is hamiltonian. This can be also shown using the following two facts. Sachs [13] has shown that every graph satisfying the Erdős condition also satisfies the Pósa condition. Next it has been shown (cf. [9]) that every graph satisfying the Pósa condition also satisfies $C_n(G) = K_n$.

Theorem 7. Let $G$ be a graph of order $n$. If $n_2(G) < g(n, \delta)$, then $C_n(G) = K_n$.

The following sufficient condition for hamiltonian graphs is due to Chvátal and Erdős.

Theorem 8 (Chvátal and Erdős, [5]). Let $G$ be a simple graph of order $n \geq 3$. If $\alpha(G) \leq \kappa(G)$, then $G$ is hamiltonian.

Bondy [1] has shown that every graph satisfying Ore’s condition also satisfies the Chvátal–Erdős condition.

The following examples show that a graph satisfying the extended Ore condition need not satisfy the Chvátal–Erdős condition. Let $H = K_2 + (K_p \cup K_q)$ with $p = q$ or $q + 1$. Now delete edges in $K_p$ and/or $K_q$ such that

1. a Hamilton cycle is not affected
2. $n_2(G) < g(n, \delta)$,

where $G$ is the graph obtained by deleting edges from $H$. Then $\kappa(G) = 2$, but $\alpha(G) \geq 3$.

Example. $p = q$ or $q + 1$.

Delete one edge in $K_p$. Then for

$n$ even: $n_2(G) = 1 + 2(n/2 - 1) = n - 1 < \frac{1}{8} (n^2 + 2n - 8)$ for $n \geq 7$.
$n$ odd: $n_2(G) = 1 + 2((n - 1)/2 - 1) = n - 2 < \frac{1}{8} (n^2 - 1)$ for $n \geq 6$.

Question. Are there other sufficient conditions for hamiltonian graphs, which are implied by Ore’s condition but not by Ore’s extended condition?

3. Extended Fan’s theorem

Theorem 9 (Fan [8]). Let $G$ be a 2-connected graph of order $n$. If $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices $u$ and $v$ with $d(u, v) = 2$, then $G$ is hamiltonian.
For natural numbers $n$ and $\delta$, define $g(n, \delta)$ by

$$
g(n, \delta) = \begin{cases} 
1 & \text{if } \delta = 2, \\
2 & \text{if } 3 \leq \delta < \frac{n+1}{3}, \\
\binom{\delta}{2} & \text{if } \frac{n+2}{3} \leq \delta < \frac{n-1}{2}, \\
\binom{\delta+1}{2} & \text{if } \delta = \frac{n-1}{2}, \\
\infty & \text{if } \delta \geq \frac{n}{2}.
\end{cases}
$$

For a graph $G$ of order $n$, define $N_2(G)$ and $n_2(G)$ by

$$
N_2(G) = \left\{ xy \in E(\overline{G}) : d(x, y) = 2 \text{ and } \max\{d(x), d(y)\} < \frac{n}{2} \right\}
$$

and

$$
n_2(G) = |N_2(G)|.
$$

**Theorem 10 (Extended Fan’s theorem).** Let $G$ be a 2-connected graph of order $n$ with minimum degree $\delta$. If $n_2(G) < g(n, \delta)$, then $G$ is hamiltonian.

Theorem 10 is sharp for $n \geq 6$, $\delta = 2, n \geq 18$, $3 \leq \delta \leq (n-6)/4$ and $n \geq 5$, $(n+2)/3 \leq \delta < n/2$. If $\delta = 2$, let $G = F(n, 2)$. Then $|V(G)| = n$, $\delta(G) = 2$ and $n_2(G) = 1$ for $n \geq 6$. However, $G$ is not hamiltonian.

If $3 \leq \delta \leq (n-6)/4$, let $G$ consist of three disjoint graphs $G_1 = K_{n-2\delta-2}$, $G_2 = K_{\delta+1}$ and $G_3 = K_{\delta-1}$. Now add the four edges $u_1v_1, u_2v_2, u_1w_1, u_2w_2$ for six vertices $u_1, u_2 \in V(G_1), v_1, v_2 \in V(G_2), w_1, w_2 \in V(G_3)$. Then $|V(G)| = n$, $\delta(G) = \delta$ and $n_2(G) = 2$. However, $G$ is not hamiltonian.

If $2 \leq \delta < n/2$, let $G = F(n, \delta)$. Then $\delta(G) = \delta$ and $n_2(G) = \binom{\delta}{2}$, if $\delta < (n-1)/2$, and $n_2(G) = \binom{\delta+1}{2}$, if $\delta = (n-1)/2$. However, $G$ is not hamiltonian.

**Proof of Theorem 10.** If $\delta = 2$ we have $n_2(G) = 0$ and $G$ is hamiltonian by Fan’s theorem.

Next let $3 \leq \delta \leq (n+1)/3$. We may assume that $n_2(G) = 1$. Let $A = \{v \in V(G) : d(v) \geq n/2\}$ and $B = V(G) - A$.

First observe that either exactly one component of $B$ is missing one edge or all components of $B$ are complete.

We now distinguish two cases.

**Case 1:** $\omega(G[B]) \geq 2$. If $G[B_i] \cong K_p - e$ for some component $B_i$ of $G[B]$, then $G[B_i]$ is hamilton-connected for all $p \geq 2$, since $d(x) + d(y) \geq 2(p - 2) \geq p + 1$ for all pairs $x, y \in V(B_i)$. If $p = 3$ or 4, then we can always find two vertices $x, y \in V(B_i)$ such that $x, y$ are hamilton-connected in $G[B_i]$ and $x, y$ are connected by two independent edges with $G[A]$. Moreover, $N_A(B_i) \cap N_A(B_j) = \emptyset$ for every pair of components $B_i, B_j$ of $B$.

If all components of $G[B]$ are complete, then there are two components $B_i, B_j$ and two vertices $x \in V(B_i), y \in V(B_j)$ such that $x, y$ have a common neighbour in $A$.

In both subcases it is possible (as in Veldman’s proof [14]) to construct a Hamilton cycle.

**Case 2:** $\omega(G[B]) = 1$. Then $G[B] \cong K_p - e$. Thus, $\Delta(G[B]) = p - 1 < n/2$ implying $|A| > n/2 - 1$. Now we can follow the first subcase of Case 1 above.

Finally, let $(n+2)/3 \leq \delta < n/2$. By a theorem of Nash-Williams [10] we know that if $G$ is 2-connected and $\delta(G) \geq (n+2)/3$, then every longest cycle is a dominating cycle. Let $C$ be a longest (nonhamiltonian) dominating cycle, and let $u \in V(G) - V(C)$ be a vertex with $d(u) = k$. According to an orientation of the cycle $C$, let $v_1, v_2, \ldots, v_k$ be the successors of the neighbours of $u$ on $C$. Then $I = \{u, v_1, v_2, \ldots, v_k\}$ is an independent set satisfying $|I| \geq \delta + 1 \geq (n+5)/3$. It is well-known that $d(x) + d(y) \leq n - 1$ for all pairs of vertices $x, y \in I$. Hence at most one vertex of $I$ has degree $\geq n/2$. Since $|V(G) - I| \leq (2n - 5)/3$, we have $N(x) \cap N(y) \neq \emptyset$ for all $x, y \in I$. Therefore, $n_2(G) \geq \binom{|I|-1}{2} \geq \binom{\delta}{2}$, if $\delta < (n-1)/2$, a contradiction. If $\delta = (n-1)/2$, then $n_2(G) \geq \binom{\delta+1}{2}$, a contradiction as well. \[\square\]
Concluding remarks. For \((n - 6)/4 < \delta \leq (n - 4)/4\) and \((n - 4)/4 < \delta \leq (n - 3)/3\) there are examples with \(n_2(G) = 2n - 10\) and \(n_2(G) = 2n + 2\delta - 8\), respectively.

If \(2 \leq \delta \leq (n - 3)/3\), let \(G\) consist of three disjoint graphs \(G_1 = K_{n-2\delta-2}\), \(G_2 = K_{\delta+1}\) and \(G_3 = K_{\delta+1}\). Now add the four edges \(u_1v_1, u_2v_2, u_1w_1, u_2w_2\) for six vertices \(u_1, u_2 \in V(G_2), v_1, v_2 \in V(G_1), w_1, w_2 \in V(G_3)\). Then \(|V(G)| = n\) and \(2 \leq \delta \leq (n - 3)/3\). If \((n - 6)/4 < \delta \leq (n - 4)/4\), then \(n_2(G) = 2n - 10\), and if \((n - 4)/4 < \delta \leq (n - 3)/3\), then \(n_2(G) = 2n + 2\delta - 8\). However, \(G\) is not hamiltonian.

These examples indicate, that in the definition of \(g(n, \delta)\), the value 2 for \(3 \leq \delta \leq (n + 1)/3\) might be replaced by a function which is linear in \(\delta\).

We have been able to verify this by showing the following result. However, since the proof is technical and long, we only give a sketch of proof.

**Theorem 11.** Let \(G\) be a 2-connected graph of order \(n\) with minimum degree \(\delta > n/4\). If \(n_2(G) < 4\delta - 8\) and \(n\) is sufficiently large, then \(G\) is hamiltonian.

**Proof (Sketch).** Suppose there is a nonhamiltonian graph \(G\). Then \(G\) has a longest cycle of circumference \(c(G) \geq 2\delta\). We deduce that the graph \(H = G[V(G) - V(C)]\) is a single component, which is hamilton-connected. Next we show that \(|NC(H)| = 2\) and that \(G\) has maximum degree \(\Delta(G) < n/2\). Finally, we obtain \(n_2(G) \geq 4\delta - 8\), a contradiction. □

**References**