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# Duality for Multiobjective *B*-vex Programming Involving *n*-Set Functions

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In the present paper we consider a class of multiobjective *B*-vex programming problems involving differentiable *B*-vex *n*-set functions and establish duality results in terms of properly efficient solutions. Further, we relate the problem to a certain saddle point of a Lagrangian and show multiobjective fractional program as a special case of the main problem. 

## 1. INTRODUCTION

Optimization theory dealing with set functions was recently developed by Morris [11], who defined the notion of local convexity, global convexity, and differentiability for set functions and established optimality conditions and Lagrangian duality that are closely parallel to similar results in nonlinear programming problems with point functions. He also discussed some computational procedures for the solution of nonlinear programs with set functions. Corley [8] extended the results presented by Morris [11] for *n*-set functions and established optimality conditions along with Lagrangian duality. Zalmai [14] mentioned several applications of nonlinear programming problems involving *n*-set functions and considered several



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practical applications for a class of nonlinear programming problems involving a single objective and differentiable $n$-set functions. Zalmai [14] also established several sufficient optimality conditions and duality results under generalized $\rho$-convexity conditions. Bector *et al.* [3] established sufficient optimality conditions and proved duality results for multiobjective programming problems with differentiable $n$-set functions. In [4], Bector *et al.* consider a class of multiobjective fractional programming problems in which the objectives are ratios of appropriately restricted differentiable $n$-set functions and introduce, along the lines of Bector [2], Wolfe's dual [13] and establish duality results in terms of properly efficient solutions. A relationship with a certain vector-valued saddle point of a Lagrangian is established. In the present paper we generalize the results presented in [4] to a multiobjective programming problem in which the functions involved are $B$-vex $n$-set functions [1, 5].

## 2. NOTATION, DEFINITIONS, AND PRELIMINARIES

Throughout the paper we assume that $(X, A, \mu)$ is a finite atomless measure space with $L_1(X, A, \mu)$ separable. We also assume that $S$ is a subset of $A^n = A \times A \times \cdots \times A$, the $n$-fold product of $\sigma$-algebra $A$ of subsets of a given set $X$. Let $d$ be the pseudometric on $A^n$ defined by

$$d((R_1, R_2, \ldots, R_n), (S_1, S_2, \ldots, S_n)) = \left[ \sum_{i=1}^{n} \mu^2(R_i \Delta S_i) \right]^{1/2},$$
$$R_i, S_i \in A, \forall\, i = 1, 2, \ldots, n,$$

where $R_i \Delta S_i$ denotes the symmetric difference for $R_i$ and $S_i$. Thus $(A^n, d)$ is a pseudosymmetric space which will serve as the domain for most of the functions used in the present paper. Thus $h \in L_1(X, A, \mu)$ and $Z \in A$ with indicator (characteristic) function $I_z \in L_\infty(X, A, \mu)$, the general integral $\int_z h\, d\mu$ will be denoted by $\langle h, I_z \rangle$.

We now give the following definitions along the lines of Zalmai [14].

DEFINITION 2.1. A set function $H: A \to R^1$ is said to be differentiable at $S^* \in A$ if there exists $DH_{S^*} \in L_1(X, A, \mu)$, called the derivative of $H$ at $S^*$, such that

$$H(S) = H(S^*) + \langle DH_{S^*}, I_{S^*} - I_{S^*} \rangle + V_H(S^*, S),$$

where

$$V_H(S^*, S) \text{ is } o[d(S^*, S)],$$

i.e.,

$$\lim_{d(S^*, S) \to 0} V_H(S^*, S)/d(S^*, S) = 0.$$

We now define the differentiation for an $n$-set function.

DEFINITION 2.2. Let $F\colon A^n \to R^1$ and $(S_1^*, S_2^*, \ldots, S*_n) \in A^n$. Then $F$ is said to have a partial derivative at $(S_1^*, S_2^*, \ldots, S*_n)$ with respect to its $i$th argument $S_i$ if the set function $H(S_i) = F(S_1^*, S_2^*, \ldots, S_{i-1}^*, S_i, S_{i+1}^*, \ldots, S_n^*,)$ has derivative $DF_{S_i^*}$ at $S_i^*$. In that case we define the $i$th partial derivative of $F$ at $(S_1^*, S_2^*, \ldots, S_n^*)$ to be

$$D_i F_{S_{1,\ldots,}^* S_n^*} = DF_{S_i^*}, \qquad i = 1, 2, \ldots, n.$$

DEFINITION 2.3. Let $F\colon A^n \to R^1$ and $(S_1^*, S_2^*, \ldots, S_n^*) \in A^n$. Then $F$ is said to be differentiable at $(S_1^*, S_2^*, \ldots, S_n^*)$ if all the partial derivatives $DF_{S_i^*}, i = 1, 2, \ldots, n$, exist and satisfy

$$F(S_1, S_2, \ldots, S_n) = F(S_1^*, S_2^*, \ldots, S_n^*) + \sum_{i=1}^{n} \left\langle D_i F_{S_{1,\ldots,}^* S_n^*}, I_{S_i} - I_{S_i^*} \right\rangle + W_F[(S_1^*, S_2^*, \ldots, S_n^*), (S_1, S_2, \ldots, S_n)],$$

where

$$W_F[(S_1^*, S_2^*, \ldots, D_n^*), (S_1, S_2, \ldots, S_n)]$$

is

$$o\{d[(S_1^*, S_2^*, \ldots, S_n^*), (S_1, S_2, \ldots, S_n)]\} \qquad \text{for all } (S_i, S_2, \ldots, S_n) \in A^n.$$

DEFINITION 2.4. Let $F\colon A^n \to R^1$ be differentiable. Then $F$ is said to be convex (strictly convex) [8, 14] if for $(R_1, R_2, \ldots, R_n)$, $(S_1, S_2, \ldots, S_n) \in A^n$

$$F(R_1, R_2, \ldots, R_n) - F(S_1, S_2, \ldots, S_n) \geqq (>) \sum_{i=1}^{n} \left\langle D_i F_{S_{1,\ldots,}S_n}, I_{R_i} - I_{S_i} \right\rangle.$$

DEFINITION 2.5. Let $F\colon A^n \to R^1$ be differentiable. Then $F$ is said to be concave (strictly concave) [8, 14] if for $(R_1, R_2, \ldots, R_n)$, $(S_1, S_2, \ldots, S_n) \in A^n$

$$F(R_1, R_2, \ldots, R_n) - F(S_1, S_2, \ldots, S_n) \leqq (<) \sum_{i=1}^{n} \left\langle D_i F_{S_{1,\ldots,}S_n}, I_{R_i} - I_{S_i} \right\rangle.$$

Next we introduce the following definition of $n$-set $B$-vex (strictly $B$-vex) function.

DEFINITION 2.6 [5]. $F: A^n \to R^1$ be differentiable and $B: A^n \times A^n \to R^1$, $B > 0$. Then $F$ is $B$-vex (strictly $B$-vex) on $A^n$, if for $R = (R_1, \ldots, R_n)$, $S = (S_1, \ldots, S_n) \in A^n$

$$B(R,S)\left[F(R_1, R_2, \ldots, R_n) - F(S_1, S_2, \ldots, S_n)\right]$$
$$\geqq (>) \sum_{i=1}^{n} \left\langle D_i F_{S_1,\ldots,S_n}, I_{R_i} - I_{S_i} \right\rangle.$$

DEFINITION 2.7 [5]. Let $F:A^n \to R^1$ be differentiable and $B:A^n \times A^n \to R^1$, $B > 0$. Then $F$ is $B$-cave (strictly $B$-cave) on $A^n$ if for $R = (R_1, R_2, \ldots, R_n)$, $S = (S_1, S_2, \ldots, S_n) \in A^n$

$$B(R,S)\left[F(R_1, R_2, \ldots, R_n) - F(S_1, S_2, \ldots, S_n)\right]$$
$$\leqq (<) \sum_{i=1}^{n} \left\langle D_i F_{S_1,\ldots,S_n}, I_{R_i} - I_{S_i} \right\rangle,$$

or equivalently, $F$ is $B$-cave (strictly $B$-cave) [5] on $A^n$ if and only if $-F$ is $B$-vex (strictly $B$-vex) on $A^n$.

We now consider the nonlinear multiobjective fractional programming problem (VP) involving differentiable $n$-set functions,

$$\text{(VP)} \qquad \text{V-min}\left[Q_1(S_1, S_2, \ldots, S_n), \ldots, Q_p(S_1, S_2, \ldots, S_n)\right]$$

subject to

$$Q_{ij}(S_1, S_2, \ldots, S_n) \leqq 0, i = 1, 2, \ldots, p, \text{ and } j = 1, 2, \ldots, m \qquad (2.2)$$

$$S = (S_1, S_2, \ldots, S_n) \in A^n, \qquad (2.3)$$

where

(i) $A^n$ is the $n$-fold product of a $\sigma$-algebra $A$ of subsets of a given set $X$,

(ii) $Q_i$ for $i = 1, 2, \ldots, p$, and $Q_{ij}$ for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, m$ are real-valued differentiable $B_i$-vex functions defined on $A^n$, and

(iii) the symbol V-min stands for vector minimization.

DEFINITION 2.8. A feasible solution $(S_1^*, S_2^*, \ldots, S_n^*) \in A^n$ for (VP) is said to be efficient for (VP) if and only if there is no other feasible solution

$(S_1, S_2, \dots, S_n) \in A$ for $A$ for (VP) such that

$$F_i(S_1, S_2, \dots, S_n) \leqq F_i(S_1^*, S_2^*, \dots, S_n^*) \qquad \text{for all } i \in (1, 2, \dots, p)$$

and

$$F_k(S_1, S_2, \dots, S_n) < F_k(S_1^*, S_2^*, \dots, S_n^*) \qquad \text{for some } k \in (1, 2, \dots, p).$$

DEFINITION 2.9. An efficient solution $(S_1^*, S_2^*, \dots, S_n^*)$ for (VP) is said to be properly efficient for (VP) if and only if there exists a scalar $M > 0$ such that for all $i \in (1, 2, \dots, p)$,

$$\begin{aligned}&\left[F_i(S_1^*, S_2^*, \dots, S_n^*) - F_i(S_1, S_2, \dots, S_n)\right]\\ &\qquad \leqq M\left[F_j(S_1, S_2, \dots, S_n) - F_j(S_1^*, S_2^*, \dots, S_n^*)\right]\end{aligned}$$

for some $j \in (1, 2, \dots, p)$ such that

$$F_j(S_1, S_2, \dots, S_n) > F_j(S_1^*, S_2^*, \dots, S_n^*)$$

whenever $(S_1, S_2, \dots, S_n) \in A^n$ is feasible for (VP) and

$$F_i(S_1, S_2, \dots, S_n) < F_i(S_1', S_2^*, \dots, S_n^*).$$

An efficient solution that is not properly efficient is said to be improperly efficient. Thus for $(S_1^*, S_2^*, \dots, S_n^*)$ to be improperly efficient for (VP) means that to every sufficiently large scalar $M > 0$, there is a feasible solution $(S_1, S_2, \dots, S_n) \in A^n$ and an $i$ such that

$$F_i(S_1, S_2, \dots, S_n) < F_i(S_1^*, S_2^*, \dots, S_n^*)$$

and

$$\begin{aligned}&\left[F_i(S_1^*, S_2^*, \dots, S_n^*) - F_i(S_1, S_2, \dots, S_n)\right]\\ &\qquad > M\left[F_j(S_1, S_2, \dots, S_n) - F_j(S_1^*, S_2^*, \dots, S_n^*)\right]\end{aligned}$$

for all $j \in (1, 2, \dots, p)$, such that

$$F_j(S_1, S_2, \dots, S_n) > F_j(S_1^*, S_2^*, \dots, S_n^*).$$

For a vector maximization problem the above definitions are modified accordingly.

In what follows, we shall need the following programming problem (PP), containing a single objective $n$-set function:

$$(\mathrm{PP}) \qquad \text{minimize } F(S_1, S_2, \ldots, S_n)$$

subject to

$$H_j(S_1, S_2, \ldots, S_n) \leqq 0, \qquad j = 1, 2, \ldots, m$$
$$(S_1, S_2, \ldots, S_n) \in A^n.$$

DEFINITION 2.10. A point $(S_1^*, S_2^*, \ldots, S_n^*) \in A^n$ is said to be a regular feasible solution for (PP) if there exists $(\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_n) \in A^n$ such that

$$H_j(S_1^*, S_2^*, \ldots, S_n^*) + \sum_{i=1}^{n} \left\langle D_i H_{jS_{1,\ldots,}^* S_n^*}, I_{\hat{S}_i} - I_{S_i^*} \right\rangle < 0, \qquad j = 1, 2, \ldots, p.$$

In what follows we shall use the following theorem for (PP) whose proof may be found in [8, 14].

THEOREM 2.1. *Let* $(S_1^*, S_2^*, \ldots, S_n^*)$ *be a regular optimal solution of* (PP). *Then there exists* $u^* = (u_1^*, u_2^*, \ldots, u_m^*) \in R_+^m$ (*nonnegative orthant of* $R^m$) *such that*

$$\left\langle D_i F_{S_{1,\ldots,}^* S_n^*} + \sum_{j=1}^{n} u_j^* D_i H_{jS_{1,\ldots,}^* S_n^*}, I_{S_i} - I_{S_i^*} \right\rangle \geqq 0, \qquad \forall S_i \in A, i = 1, 2, \ldots, n.$$
$$u_j^* H_j(S_1^*, S_2^*, \ldots, S_n^*) = 0, \qquad (j = 1, 2, \ldots, m)$$
$$H_j(S_1^*, S_2^*, \ldots, S_n^*) \leqq 0, \qquad (j = 1, 2, \ldots, m)$$
$$u^* = (u_1^*, u_2^*, \ldots, u_m^*) \geqq 0.$$

## 3. OPTIMALITY CONDITIONS

From (VP), we write the following multiple problems $(P_k^*) \equiv P_k(S_1^*, \ldots, S_n^*)$, for $k = 1, 2, \ldots, p$, each problem having a single objective $n$-set function:

$$(P_k^*) \qquad \text{minimize } Q_k(S_1, S_2, \ldots, S_n)$$

subject to

$$Q_i(S_1, S_2, \ldots, S_n) \leqq Q_i(S_1^*, S_2^*, \ldots, S_n^*), \qquad \forall i = 1, 2, \ldots, p; i \neq k \tag{3.1}$$

$$Q_{ij}(S_1, S_2, \ldots, S_n) \leqq 0, \qquad (i = 1, 2, \ldots, p; j = 1, 2, \ldots, m) \tag{3.2}$$

$$(S_1, S_2, \ldots, S_n) \in A^n \tag{3.3}$$

$$\text{for each } k = 1, 2, \ldots, p. \tag{3.4}$$

The following lemma can be proved along the lines of Chankong and Haimes [7].

LEMMA 3.1. *Let* $(S_1^*, S_2^*, \ldots, S_n^*) \in A^n$ *be*

H-1. *regular for* (VP), *and*

H-2. *regular for at least one* $(P_k^*)$, $k = 1, 2, \ldots, p$.

*Then* $(S_1^*, S_2^*, \ldots, S_n^*)$ *is an efficient solution for* (VP) *if and only if it is optimal solution of the problem* $(P_k^*)$.

LEMMA 3.2. *For* $i = 1, 2, \ldots, p$, $F_i$ *is* a $B_i$*-vex function on* $A^n$. *Then*:

(i) *For* $\alpha_i \geqq 0$, $i = 1, 2, \ldots, p$, *the function* $F = \alpha_1 F_1 + \cdots + \alpha_p F_p$ *is* $B_i$*-vex on* $A^n$.

(ii) *Additionally, for* $i = 1, 2, \ldots, p$, *if at least one of* $F_i$ *is a strictly* $B_i$*-vex functions on* $A^n$ *with corresponding* $\alpha_i > 0$, *then* $F = \alpha_1 F_1 + \cdots + \alpha_p F_p$ *is strictly* $B_i$*-vex on* $A^n$.

THEOREM 3.1 (Necessary Conditions). *Let* $(S_1^*, S_2^*, \ldots, S_n^*) \in A^n$ *be a*

(i) *regular efficient solution for* (VP), *and*

(ii) *regular solution for* $(P_k^*)$, $k = 1, 2, \ldots, p$.

*Then there exist*

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_p^*) > 0 \quad \text{and} \quad y^{*j} = (y_{1j}^*, y_{2j}^*, \ldots, y_{pj}^*,) \geqq 0 \; (j = 1, 2, \ldots, m)$$

*such that*

$$\left\langle \sum_{i=1}^{p} \lambda_i^* D_r Q_{iS_1^*, \ldots, S_n^*} + \sum_{j=1}^{m} \sum_{i=1}^{p} y_{ij}^* D_r Q_{ijS_1^*, \ldots, S_n^*}, I_{S_r} - I_{S_r^*} \right\rangle \geqq 0, \quad \forall S_r \in A, r = 1, 2, \ldots, n, \tag{3.5}$$

$$y_{ij}^* Q_{ij}(S_1^*, S_2^*, \ldots, S_n^*) = 0 \qquad (i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, m) \quad (3.6)$$

$$Q_{ij}(S_1^*, S_2^*, \ldots, S_n^*) \leqq 0 \qquad (i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, m) \quad (3.7)$$

$$y_{ij}^* \geqq 0 \qquad (i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, m). \quad (3.8)$$

*Proof.* Since $(S_1^*, S_2^*, \ldots, S_n^*)$ is a regular efficient solution for (VP), by Lemma 3.1 it is an optimal solution for $(P_k^*)$, $k = 1, 2, \ldots, p$. Also $(S_1^*, S_2^*, \ldots, S_n^*)$ is regular for $(P_k^*)$; therefore, by Theorem 2.2, there exist

$$\tau_{ik}^* \geqq 0, \qquad i = 1, 2, \ldots, p, i \neq k, u_{ijk} \geqq 0$$

for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, m$ such that

$$\left\langle D_r Q_{kS_1^*, \ldots, S_n^*} + \sum_{\substack{i=1 \\ i \neq k}}^{p} \tau_{ik}^* D_r Q_{iS_1^*, \ldots, S_n^*} + \sum_{j=1}^{m} \sum_{i=1}^{p} u_{ijk}^* D_r Q_{ijS_1^*, \ldots, S_n^*}, I_{S_r} - I_{S_r^*} \right\rangle \geqq 0,$$

$$\forall S_r \in A, r = 1, 2, \ldots, n \quad (3.9)$$

$$u_{ijk}^* Q_{ij}(S_1^*, S_2^*, \ldots, S_n^*) = 0 \qquad (i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, m) \quad (3.10)$$

$$Q_{ij}(S_1^*, S_2^*, \ldots, S_n^*) \leqq 0 \qquad (i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, m) \quad (3.11)$$

$$u_{ij}^* \geqq 0 \qquad (i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, m) \quad (3.12)$$

$$\text{for each } k = 1, 2, \ldots, p. \quad (3.13)$$

Summing over $k$ in (3.9), (3.10), (3.12), and setting

$$\lambda_k^* = 1 + \sum_{\substack{i=1 \\ i \neq k}}^{p} \tau_{ik}^* > 0, \qquad k = 1, 2, \ldots, p,$$

$$y_{ij}^* = \sum_{k=1}^{p} u_{ijk}^* \geqq 0 \qquad (i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, m),$$

we obtain (3.5)–(3.8).

THEOREM 3.2 (Sufficient Conditions). *Assume that there exist*

$$\lambda_i^* > 0, \qquad i = 1, 2, \ldots, p, y_{ij}^* \geqq 0, i = 1, 2, \ldots, p, j = 1, 2, \ldots, m$$

*such that* (3.5)–(3.8) *hold at* $(S_1^*, S_2^*, \ldots, S_n^*)$ *for all feasible solutions of* (VP). *Then* $(S_1^*, S_2^*, \ldots, S_n^*)$ *is an efficient solution of* (VP).

*Proof.* We assume that $(S_1^*, S_2^*, \ldots, S_n^*)$ is not an efficient solution of (VP) and exhibit a contradiction. $(S_1^*, S_2^*, \ldots, S_n^*)$ not being an efficient solution of (VP) yields that there exists a feasible solution $(S_1, S_2, \dots, S_n)$ to (VP) such that

$$Q_i(S_1, S_2, \dots, S_n) \leqq Q_i(S_1^*, S_2^*, \dots, S_n^*) \qquad \text{for } i = 1, 2, \dots, p, i \neq k \tag{3.14}$$

and

$$Q_k(S_1, S_2, \dots, S_n) < Q_k(S_1^*, S_2^*, \dots, S_n^*). \tag{3.15}$$

Since the functions $Q_i$ and $Q_{ij}$ are $B_i$-vex, and

$$\lambda_i^* > 0, y_{ij}^* \geqq 0 \qquad \text{for all } 1 = 1, 2, \dots, p \text{ and } j = 1, 2, \dots, m,$$

by Lemma 3.2,

$$\lambda_i^* Q_i(S_1, S_2, \dots, S_n) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(S_1, S_2, \dots, S_n)$$

is $B_i$-vex for $i = 1, 2, \dots, p$. Hence, by Definition 2.6, there is

$$B_i\colon A^n \times A^n \to R^1, B_i > 0 \qquad i = 1, 2, \dots, p$$

such that for all $(S_1, S_2, \dots, S_n)$ we have

$$B_i(S, S^*)\left[\left(\lambda_i^* Q_i(S_1, \dots, S_n) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(S_1, \dots, S_n)\right)\right.$$

$$\geqq \left\langle D_r\left(\lambda_i^* Q_i + \sum_{j=1}^{m} y_{ij}^* Q_{ij}\right)_{S_{1,\dots,}^* S_n^*}, I_{S_r} - I_{S_r^*}\right\rangle, \tag{3.16}$$

where, for notational convenience, we have $S = (S_1, \dots, S_n)$ and $S^* = (S_1^*, \dots, S_n^*)$. Using the summation sign on both sides of (3.16) for $i = 1, 2, \dots, p$, we obtain

$$\sum_{i=1}^{p} B_i(S, S^*)\left[\left(\lambda_i^* Q(S_1, \dots, S_n) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(S_1, \dots, S_n)\right)\right.$$

$$\left. - \left(\lambda_i^* Q_i(S_1^*, \dots, S_n^*) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(S_1^*, \dots, S_n^*)\right)\right]$$

$$\geqq \left\langle \sum_{i=1}^{p} D_r\left(\lambda_i^* Q_i + \sum_{j=1}^{m} y_{ij}^* Q_{ij}\right)_{S_{1,\dots,}^* S_n^*}, I_{S_r} - I_{S_r^*}\right\rangle$$

$$= \left\langle \sum_{i=1}^{p} \left(\lambda_i^* D_r Q_{iS_{1,\dots,}^* S_n^*} + \sum_{j=1}^{m} y_{ij}^* D_r Q_{ijS_{1,\dots,}^* S_n^*}\right), I_{S_r} - I_{S_r^*}\right\rangle. \tag{3.17}$$

(3.8), (3.2) and (3.6) yield

$$\sum_{j=1}^{m} y_{ij}^{*} Q_{ij}(S_1, S_2, \ldots, S_n) \leqq \sum_{j=1}^{m} y_{ij}^{*} Q_{ij}(S_1^{*}, S_2^{*}, \ldots, S_n^{*}), \qquad i = 1, 2, \ldots, p. \tag{3.18}$$

(3.14), (3.15), and (3.18) along with $\lambda_i^{*} > 0$, $i = 1, 2, \ldots, p$, and $B_i$: $A^n \times A^n \to R^1$, $B_1 > 0$, $i = 1, 2, \ldots, p$, yield

$$\sum_{i=1}^{p} B_i(S, S^{*}) \left[ \left( \lambda_i^{*} Q_i(S_1, \ldots, S_n) + \sum_{j=1}^{m} y_{ij}^{*} Q_{ij}(S_1, \ldots, S_n) \right) - \left( \lambda_i^{*} Q_i(S_1^{*}, \ldots, S_n^{*}) + \sum_{j=1}^{m} y_{ij}^{*} Q_{ij}(S_1^{*}, \ldots, S_n^{*}) \right) \right] < 0. \tag{3.19}$$

(3.17) and (3.19) yield

$$\left\langle \sum_{i=1}^{p} \left( \lambda_i^{*} D_r Q_{iS_1^{*}, \ldots, S_n^{*}} + \sum_{j=1}^{m} y_{ij}^{*} D_r Q_{ijS_1^{*}, \ldots, S_n^{*}} \right), I_{S_r} - I_{S_r^{*}} \right\rangle < 0,$$

which contradicts (3.5). Hence the result.

*Remark* 3.1. We observe that Theorem 3.2 can be strengthened if the assumption (ii) (for $i = 1, 2, \ldots, p$ each $Q_i$ and each $Q_{ij}$ is a real-valued differentiable $B_i$-vex functions on $A^n$ for $j = 1, 2, \ldots, m$) in (VP) is replaced by the following:

(i) For all $i = 1, 2, \ldots, p$ each

$$\lambda_i^{*} Q_i(S_1, S_2, \ldots, S_n) + \sum_{j=1}^{m} y_{ij}^{*} Q_{ij}(S_1, S_2, \ldots, S_n)$$

is a differentiable $B_i$-vex function on $A^n$, or

(ii)

$$\sum_{i=1}^{p} \lambda_1^{*} \left( Q_i(S_1, S_2, \ldots, S_n) + \sum_{j=1}^{m} y_{ij}^{*} Q_{ij}(S_1, S_2, \ldots, S_n) \right)$$

is a differentiable $B_i$-vex function on $A^n$.

## 4. DUAL PROBLEM AND DUALITY THEOREMS

The above necessary and sufficient theorems (Theorems 3.1 and 3.2) provide us the motivation for introducing the following vector maximization problem as the Wolfe-type [13] dual problem (VD) for the primal vector minimization problem (VP).

(VD)

$$\text{maximize} \left( Q_1(T_1, \ldots, T_n) + \sum_{j=1}^{m} y_{1j} Q_{1j}(T_1, \ldots, T_n), \ldots, Q_p(T_1, \ldots, T_n) + \sum_{j=1}^{m} y_{pj} Q_{pj}(T_1, \ldots, T_n) \right) \tag{4.1}$$

subject to

$$\left\langle D_r \sum_{i=1}^{p} \lambda_i \left( Q_i(T_1, \ldots, T_n) + \sum_{j=1}^{m} y_{ij} Q_{ij}(T_1, \ldots, T_n) \right)_{T_1^*, \ldots, T_n^*}, I_{S_r} - I_{T_r^*} \right\rangle \geqq 0,$$

$$\text{for all } S_r \in A,\ r = 1, 2, \ldots, n, \tag{4.2}$$

$$\lambda_i > 0,\ y_{ij} \geqq 0\ (i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, m),\ (T_1, T_2, \ldots, T_n) \in A^n. \tag{4.3}$$

Henceforth, we shall use, for notational convenience, the following notations for (VP) and (VD):

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) > 0, \qquad T = (T_1, T_2, \ldots, T_n) \in A^n,$$
$$S = (S_1, S_2, \ldots, S_n) \in A^n.$$

Also,

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & \cdots & y_{1m} \\ y_{21} & y_{22} & \cdots & \cdots & y_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{p1} & y_{p2} & \cdots & \cdots & y_{pm} \end{bmatrix} \in R^{p \times m}$$

is the matrix of Lagrange multipliers for the constraints of (VP),

$$y^i = (y_{i1}, y_{i2}, \ldots, y_{im}), \qquad i = 1, 2, \ldots, m,$$

$$F(S) = \left[Q_1(S_1, S_2, \ldots, S_n), \ldots, Q_p(S_1, S_2, \ldots, S_n)\right] \tag{4.4}$$

$$Q_i(S) = Q_i(S_1, S_2, \ldots, S_n), \qquad i = 1, 2, \ldots, p \tag{4.5}$$

$$Q_{ij}(S) = Q_{ij}(S_1, S_2, \ldots, S_n), \qquad i = 1, 2, \ldots, p; j = 1, 2, \ldots, m \tag{4.6}$$

$$L_i(T, y^i) = Q_i(T_1, T_2, \ldots, T_n) + \sum_{j=1}^{m} y_{ij} Q_{ij}(T_1, T_2, \ldots, T_n), \qquad i = 1, 2, \ldots, p \tag{4.7}$$

$$L(T, Y) = \left[L_1(T, y^1), L_2(T, y^2), \ldots, L_p(T, y^p)\right]. \tag{4.8}$$

Thus, we have the vector minimization problem as the primal problem (VP),

(VP) V-minimize $F(S)$

subject to

$$Q_{ij}(S) \leqq 0 \qquad (i = 1, 2, \ldots, p; j = 1, 2, \ldots, m) \tag{4.9}$$

$$S \in A^n$$

and the following vector maximization problem as the vector dual program (VD):

(VD) V-maximize $L(T, Y)$

subject to

$$\left\langle D_r \sum_{i=1}^{p} \lambda_i L_i(T, y^i)_{T^*_{1,\ldots,}T^*_n}, I_{T_r} - I_{T^*_r} \right\rangle \geqq 0, \qquad \text{for all } T \in A^n \tag{4.10}$$

$$\lambda > 0, Y \geqq 0, Y \in R^{p \times m}, \lambda \in R^p, y^i \in R^m, T \in A^n. \tag{4.11}$$

THEOREM 4.1 (Weak Duality). *Let $S \in A^n$ be feasible to* (VP) *and* $(\lambda, T, Y)$ *be feasible to* (VD). *Then* $F(S) \nleq L(T, Y)$.

*Proof.* If possible let $F(S) \leq L(T, Y)$. This implies

$$Q_i(S) \leqq L_i(T, y^i), \qquad i = 1, 2, \ldots, p, i \neq k$$

$$Q_k(S) < L_k(T, y^i).$$

This along with (4.9) and (4.12) yields

$$Q_i(S) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S) \leqq L_i(T, y^i), \qquad i = 1, 2, \ldots, p, i \neq k$$

$$Q_k(S) + \sum_{j=1}^{m} y_{kj} Q_{kj}(S) < L_k(T, y^i).$$

Since $\lambda_i > 0$ and $B_i(S, T) > 0$ for all $i = 1, 2, \ldots, p$, from the above we have

$$\lambda_i B_i(S, T)\left[\left(Q_i(S) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S)\right) - \left(Q_i(T) + \sum_{j=1}^{m} y_{ij} Q_{ij}(T)\right)\right] \leqq 0,$$
$$i = 1, 2, \ldots, p, i \neq k$$

$$\lambda_k B_k(S, T)\left[\left(Q_k(S) + \sum_{j=1}^{m} y_{kj} Q_{kj}(S)\right) - \left(Q_k(T) + \sum_{j=1}^{m} y_{kj} Q_{kj}(T)\right)\right] < 0,$$

respectively. This leads to

$$\sum_{i=1}^{p} \lambda_i B_i(S, T)\left[\left(Q_i(S) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S)\right) - \left(Q_i(T) + \sum_{j=1}^{m} y_{ij} Q_{ij}(T)\right)\right] < 0. \tag{4.12}$$

(4.12) along with the $B_i$-vexity assumption on functions $Q_i$ and $Q_{ij}$ for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, m$ yields

$$\left\langle D_r \sum_{i=1}^{p} \lambda_i L_i(T, y^i)_{T_1^*, \ldots, T_n^*}, I_{S_r} - I_{T_r^*} \right\rangle < 0, \qquad \text{for } S \in A^n$$

being a primal feasible solution. This contradicts (4.10). Hence $F(S) \nleq L(T, Y)$.

COROLLARY 4.1. *Let $S^*$ be a feasible solution of* (VP) *and let* $(\lambda^*, T^*, Y^*)$ *be a feasible solution of* (VD) *with* $F(S^*) = L(T^*, Y^*)$. *Then* $S^*$ *and* $(\lambda^*, S^*, Y^*)$ *are efficient solutions of* (VP) *and* (VD), *respectively.*

COROLLARY 4.2. *Let $S$ be a feasible solution of* (VP) *and* $(\lambda, T, Y)$ *be a feasible solution of* (VD). *Then there exists* $\lambda^* > 0$, $\lambda^* \in R^p$ *such that*

$$\sum_{i=1}^{p} \lambda_i^* Q_i(S) \geqq \sum_{i=1}^{p} \lambda_i^* L_i(T, y^i).$$

THEOREM 4.2 (Strong Duality). *Suppose $S^* \in A^n$ satisfies* H-1 *and* H-2 *of Lemma* 3.1 *and is a properly efficient solution of* (VP). *Then there exist $\lambda^* \in R^p, Y^* \in R^{p\times m}, \lambda^* > 0, Y^* \geqq 0$ such that $(\lambda^*, S^*, Y^*)$ is a properly efficient solution of* (VD) *and the objective value of* (VP) *at $S^*$ is equal to the objective value of* (VD) *at $(\lambda^*, S^*, Y^*)$.*

*Proof.* Since $S^*$ is a properly efficient solution of (VP) and satisfies H-1 and H-2 of Lemma 3.1, by Theorem 3.1, there exist

$$\lambda^* \in R^p \text{ and } U^* \in R^{p\times m},\ \lambda^* > 0, u^* \geqq 0$$

such that

$$\left\langle \sum_{i=1}^{p} \lambda_i^* D_r Q_{iS_1^*,\dots,{}^*S_n^*} + \sum_{j=1}^{m}\sum_{i=1}^{p} u_{ij}^* D_r Q_{ijS_1^*,\dots,S_n^*},\, I_{S_r} - I_{S_r}^* \right\rangle \geqq 0,$$

$$\forall S_r \in A, r = 1,2,\dots,n.$$

$$u_{ij}^* Q_{ij}(S_1^*, S_2^*, \dots, S_n^*) = 0 \qquad (i = 1,2,\dots,p;\ j = 1,2,\dots,m)$$

$$Q_{ij}(S_1^*, S_2^*, \dots, S_n^*) \leqq 0 \qquad (i = 1,2,\dots,p;\ j = 1,2,\dots,m)$$

$$\lambda_i^* > 0, u_{ij}^* \geqq 0 \qquad (i = 1,2,\dots,p;\ j = 1,2,\dots,m).$$

Since $\lambda_i^* > 0$, setting

$$u_{ij}^* = \lambda_i^* y_{ij}^* \qquad (i = 1,2,\dots,p;\ j = 1,2,\dots,m),$$

we obtain

$$\left\langle \sum_{i=1}^{p} \lambda_i^* D_r Q_{iS_1^*,\dots,S_n^*} + \sum_{j=1}^{m}\sum_{i=1}^{p} \lambda_i^* y_{ij}^* D_r Q_{ijS_1^*,\dots,S_n^*},\, I_{S_r} - I_{S_r^*} \right\rangle \geqq 0,$$

$$\forall S_r \in A, r = 1,2,\dots,n, \quad (4.13)$$

$$\lambda_i^* y_{ij}^* Q_{ij}(S_1^*, S_2^*, \dots, S_n^*) = 0 \qquad (i = 1,2,\dots,p;\ j = 1,2,\dots,m) \quad (4.14)$$

$$Q_{ij}(S_1^*, S_2^*, \dots, S_n^*) \leqq 0 \qquad (i = 1,2,\dots,p;\ j = 1,2,\dots,m) \quad (4.15)$$

$$\lambda_i^* > 0,\ \lambda_i^* y_{ij}^* \geqq 0 \qquad (i = 1,2,\dots,p;\ j = 1,2,\dots,m). \quad (4.16)$$

(4.7), (4.13), and (4.16) yield

$$\left\langle D_r \sum_{i=1}^{p} \lambda_i^* L_i(S, y^i)_{S_1^*,\dots,S_n^*},\, I_{S_r} - I_{S_r^*} \right\rangle \geqq 0$$

and

$$\lambda_i^* > 0, y_{ij}^* \geqq 0 \qquad (i = 1, 2, \dots, p;\ j = 1, 2, \dots, m).$$

This yields that $(\lambda^*, S^*, Y^*)$ is a feasible solution to (VD). From (4.14) we get

$$\sum_{j=1}^{m} y_{ij}^* Q_{ij}(S^*) = 0 \qquad \text{for all } i = 1, 2, \dots, p.$$

Therefore,

$$Q_i(S^*) = Q_i(S^*) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(S^*) \qquad \text{for all } i = 1, 2, \dots, p;$$

that is,

$$F(S^*) = L(S^*, Y^*).$$

This implies that the objective value of (VP) at $S^*$ is equal to the objective value of (VD) at $(\lambda^*, S^*, Y^*)$. Using Corollary 4.1 we have that $(\lambda^*, S^*, Y^*)$ is an efficient solution of (VD). We next show that $(\lambda^*, S^*, Y^*)$ is a properly efficient solution of (VD). For doing so we assume that $(\lambda^*, S^*, Y^*)$ is not a properly efficient solution of (VD) and exhibit a contradiction. If $(\lambda^*, S^*, Y^*)$ is not a properly efficient solution of (VD), then it is only improperly efficient. Therefore, to every sufficiently large scalar $M > 0$, there is a solution $(\tilde{\lambda}, \tilde{S}, \tilde{Y})$ feasible to (VD) such that for some $i$, $L_i(\tilde{S}, \tilde{y}^i) > L_i(S^*, y^{*i})$ and

$$L_i\left(\tilde{S}, \tilde{y}^i\right) - L_i(S^*, y^{*i}) > M\left[L_k(S^*, y^{*k}) - L_k\left(\tilde{S}, \tilde{y}^k\right)\right]$$

for all $M > 0$ and all $k = 1, 2, \dots, p$ such that $L_k(S^*, y^{*k}) > L_k(\tilde{S}, \tilde{y}^k)$. Equivalently, $L_i(\tilde{S}, \tilde{y}^i)$ is infinitely better than $L_i(S^*, y^{*i})$ for some $i$, whereas $L_k(S^*, y^{*k})$ is at most finitely better than $L_k(\tilde{S}, \tilde{y}^k)$ for any $k = 1, 2, \dots, p$. Hence for any $\lambda > 0$, $\lambda \in R^p$,

$$\sum_{i=1}^{p} \lambda_i L_i\left(\tilde{S}, \tilde{y}_i\right) > \sum_{i=1}^{p} \lambda_i L_i(S^*, y^{*i}), \tag{4.17}$$

but

$$\sum_{j=1}^{m} y_{ij}^* Q_{ij}(S^*) = 0 \qquad \text{for all } i = 1, 2, \dots, p;$$

therefore, by (4.17) we have for any $\lambda > 0$, $\lambda \in R^p$,

$$\sum_{i=1}^{p} \lambda_i L_i\big(\tilde{S}, \tilde{y}^i\big) > \sum_{i=1}^{p} \lambda_i Q_i(S^*), \tag{4.18}$$

but (4.18) contradicts Corollary 4.2, and therefore, $(\lambda^*, S^*, Y^*)$ is a properly efficient solution of (VD).

*Remark* 4.1. We observe that Theorems 4.1 and 4.2 can be strengthened if assumption (ii) (for $i = 1, 2, \dots, p$ each $Q_i$ and each $Q_{ij}$ is a real-valued differentiable $B_i$-vex function on $A^n$ for $j = 1, 2, \dots, m$) in (VP) is replaced by the following:

(i) For all $i = 1, 2, \dots, p$ each $L_i(T, y^i)$ is a differentiable $B_i$-vex function on all feasible solutions of (VP) and (VD), or

(ii) $\lambda^T L(T, Y)$ is a differentiable $B$-vex function on all feasible solutions of (VP) and (VD).

We now prove the following Mangasarian-type converse duality theorem [10] for (VP) and (VD).

THEOREM 4.3 (Strict Converse Theorem). *Suppose $S^*$ satisfies* H-1 *and* H-2 *of Lemma* 3.1, *and is a properly efficient solution of* (VP). *Let* $(\lambda^*, T^*, Y^*)$ *be a properly efficient solution of* (VD). *If at least one of* $Q_i, i = 1, 2, \dots, p$, *is strictly* $B_i$*-vex and/or at least one of* $Q_{ij}$, *with the corresponding* $y_{ij} > 0, i = 1, 2, \dots, p;\ j = 1, 2, \dots, m$, *is strictly* $B_i$*-vex, then* $S^* = T^*$; *that is,* $T^*$ *is a properly efficient solution of* (VP) *and* $F(S^*) = L(T^*, Y^*)$.

*Proof.* We assume that $S^* \neq T^*$ and exhibit a contradiction. By Theorem 4.2, there exist

$$\lambda_i^* > 0,\ y_{ij}^* \geqq 0,\ i = 1, 2, \dots, p;\ j = 1, 2, \dots, m,$$

such that (4.13)–(4.16) hold and $(\lambda^*, S^*, Y^*)$ is a properly efficient solution for (VD), and

$$\begin{aligned} F(S^*) &= L(S^*, Y^*) \\ &= L(T^*, Y^*). \end{aligned} \tag{4.19}$$

This yields

$$L(S^*, Y^*) - L(T^*, Y^*) = 0. \tag{4.20}$$

From (4.16) and (4.20) we have

$$\left(Q_i(S^*) + \sum_{j=1}^{m} y_{ij}^* \, y_{ij}^* Q_{ij}(S^*)\right) - \left(Q_i(T^*) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(T^*)\right) = 0, \quad i = 1, 2, \ldots, p. \tag{4.21}$$

Since $\lambda_i^* > 0$, $i = 1, 2, \ldots, p$, and for all feasible $(S, \lambda, T, Y)$, $B_i(S, T^*) > 0$, $i = 1, 2, \ldots, p$, (4.21) yields

$$\sum_{i=1}^{p} \lambda_i^* \, B_i(S^*, T^*)\left[\left(Q_i(S^*) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(S^*)\right) - \left(Q_i(T^*) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(T^*)\right)\right] = 0. \tag{4.22}$$

Now we are given the hypothesis that at least one of $Q_i$, $i = 1, 2, \ldots, p$, is strictly $B_i$-vex and/or at least one of $Q_{ij}$, with the corresponding $y_{ij} > 0$, $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, m$, is strictly $B_i$-vex. To be specific we assume that $Q_k$ is strictly $B_k$-vex. We can handle the strict $B$-vexity of other functions exactly along the same lines. This, by Lemma 3.2 (ii), implies that the function $Q_i + \sum_{j=1}^{m} y_{ij}^* Q_{ij}$ is strictly $B_i$-vex for $i = k$. Therefore, for $S^* \neq T^*$ we have

$$\sum_{i=1}^{p} \lambda_i^* B_i(S^*, T^*)\left[\left(Q_i(S^*) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(S^*)\right) - \left(Q_i(T^*) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(T^*)\right)\right] > \left\langle D_r \sum_{i=1}^{p} \lambda_i L_i(T, y^i)_{T_1^*, \ldots, T_n^*}, I_{S_r^*} - I_{T_r^*} \right\rangle. \tag{4.23}$$

Using (4.10) in (4.23) we have

$$\sum_{i=1}^{p} \lambda_i^* B_i(S^*, T^*)\left[\left(Q_i(S^*) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(S^*)\right) - \left(Q_i(T^*) + \sum_{j=1}^{m} y_{ij}^* Q_{ij}(T^*)\right)\right] > 0. \tag{4.24}$$

(4.24) contradicts (4.22). Hence $S^* = T^*$. Also from (4.19), $F(S^*) = L(S^*, Y^*)$.

*Remark* 4.2. We observe that Theorem 4.3 can be strengthened if assumption (ii) (for $i = 1, 2, \ldots, p$ each $Q_i$ and each $Q_{ij}$ is a real valued differentiable $B_i$-vex functions on $A^n$ for $j = 1, 2, \ldots, m$) in (VP) is replaced by the following.

(i) For all $i = 1, 2, \ldots, p$ each $L_i(T, y^i)$ is a differentiable $B_i$-vex function on all feasible solutions of (VP) and (VD), or

(ii) $\lambda^T L(T, Y)$ is a differentiable $B$-vex function on all feasible solutions of (VP) and (VD).

## 5. VECTOR-VALUED LAGRANGIAN AND VECTOR SADDLE POINT

In this section we consider the vector-valued Lagrangian $L: A^n \times R_+^{p \times m} \to R^p$ given by

$$L(S, Y) = \left[ L_1(S, y^1), L_2(S, y^2), \ldots, L_p(S, y^p) \right],$$

where $L_i(S, y^i)$, as already defined, is

$$L_i(S, y^i) = Q_i(S) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S), \qquad i \in I = \{1, 2, \ldots, p\}$$

and $R_+^{p \times m}$ contains nonnegative elements.

Along the lines of Rodder [12], we define a vector saddle point (or generalized saddle point) of $L$ and study its relationship to (VP).

DEFINITION 5.1. A point $(S^*, Y^*) \in R_+^{p \times m}$ is said to be a vector saddle point of the vector-valued Lagrangian $L$ if

$$L(S^*, Y) \ngeq L(S^*, Y^*) \qquad \text{for all } Y \in R_+^{p \times m}$$

and

$$L(S^*, Y^*) \ngeq L(S, Y^*) \qquad \text{for all } S \in A^n.$$

We now prove the following two theorems along the lines of Mangasarian [10].

THEOREM 5.1. *Suppose $S^* \in A^n$ satisfies* H-1 *and* H-2 *of Lemma* 3.1 *and is a properly efficient solution of* (VP). *Then there exist $\lambda_i^* > 0$, $i \in I$, and $Y^* \in R_+^{p \times m}$ such that $(S^*, Y^*)$ is a vector saddle point of the vector-valued Lagrangian $L(T, Y)$.*

*Proof.* Since $S^* \in A^n$ satisfies H-1 and H-2 of Lemma 3.1 and is a properly efficient solution of (VP), there exist $\lambda_i^* > 0$, $i \in I$, and $Y^* \in R_+^{p\times m}$ such that

$$\left\langle D_r \sum_{i=1}^{p} \lambda_i L_i(S, Y^{*i})_{S_1^*,\ldots,S_n^*}, I_{S_r} - I_{S_r^*} \right\rangle \geqq 0,$$

$$\text{for all (VP)-feasible } S \in A^n, \quad (5.1)$$

$$y_{ij}^* Q_{ij}(S_1^*, S_2^*, \ldots, S_n^*) = 0 \qquad (i = 1,2,\ldots,p;\ j = 1,2,\ldots,m) \quad (5.2)$$

$$Q_{ij}(S_1^*, S_2^*, \ldots, S_n^*) \leqq 0 \qquad (i = 1,2,\ldots,p;\ j = 1,2,\ldots,m) \quad (5.3)$$

$$\lambda_i^* > 0,\ y_{ij}^* \geqq 0 \qquad (i = 1,2,\ldots,p;\ j = 1,2,\ldots,m). \quad (5.4)$$

We first prove that

$$L(S^*, Y^*) \ngeq L(S, Y^*) \qquad \text{for all } S \in A^n.$$

If possible let $L(S^*, Y^*) \geq L(\bar{S}, Y^*)$ for some (VP)-feasible solution $\bar{S} \in A^n$. This implies

$$L_i(S^*, y^{*i}) \geqq L_i(\bar{S}, y^{*i}) \qquad \text{for all } i \in I, i \neq k$$

and

$$L_k(S^*, y^{*k}) > L_k(\bar{S}, y^{*k}).$$

This yields

$$\sum_{i=1}^{p} \lambda_i^* \left( L_i(\bar{S}, y^{*i}) - L_i(S^*, y^{*i}) \right) < 0 \qquad \text{for some (VP)-feasible } \bar{S}. \quad (5.5)$$

Using the above hypothesis along with Theorem 2.2, (5.5) in conjunction with (5.2)–(5.4) gives

$$\left\langle D_r \sum_{i=1}^{p} \lambda_i L_i(S, y^{*i})_{S_1^*,\ldots,S_n^*}, I_{\bar{S}_r} - I_{S_r^*} \right\rangle < 0, \qquad \text{for some (VP)-feasible } \bar{S}, \quad (5.6)$$

but (5.6) contradicts (5.1). Thus,

$$L(S^*, Y^*) \ngeq L(S, Y^*) \qquad \text{for all (VP)-feasible } S \in A^n.$$

To prove the other part of the vector saddle point inequality we have

$$L_i\left(S^*, y^{*i}\right) - L_i\left(S^*, y^i\right) = -\sum_{j=1}^{m} y_{ij} Q_{ij}(S^*) \geqq 0, \qquad i \in I.$$

Thus,

$$L(S^*, Y) \ngeq L(S^*, Y^*) \qquad \text{for all } Y \in R_+^{p \times m}.$$

We shall now prove Theorem 5.2 under a somewhat restricted vector minimization (RVP) model of the vector minimization problem (VP). We consider the (RVP) model

$$\text{(RVP)} \qquad V\text{-min}\left[Q_1(S), Q_2(S), \ldots, Q_p(S)\right]$$

subject to

$$H_j(S) \leqq 0, \qquad j = 1, 2, \ldots, m,$$

$$S \in A^n,$$

where

(i) $A^n$ is the $n$-fold product of a $\sigma$-algebra $A$ of subsets of a given set $X$,

(ii) $Q_i$ for $i = 1, 2, \ldots, p$, and $H_j$ for $j = 1, 2, \ldots, m$, are real-valued differentiable $B_i$-vex functions defined on $A^n$, and

(iii) the symbol $V$-min stands for vector minimization.

We write the constraint set of (RVP) as

$$\Gamma = \left\{S \in A^n \colon\ H_j(S) \leqq 0 \qquad j = 1, 2, \ldots, m\right\}.$$

For $i = 1, 2, \ldots, p$, we now introduce a differentiable mapping $T_i\colon\ R^1 \to R^1$ for which $(T_i)^{-1}$ exists and $T_i > 0$ and $T(0) = 0$ such that $T^{-1}(0) = 0$ and write

$$Q_{ij}(S) = T_i\left(H_j(S)\right)$$

and modify the set $\Gamma$ to obtain an equivalent constraint set $\Gamma_\mu$ as follows:

$$\Gamma_\mu = \left\{S \in A^n \colon\ Q_{ij}(S) \equiv T_i\left(H_j(S)\right) \leqq 0, \right.$$
$$\left. i = 1, 2, \ldots, p, j = 1, 2, \ldots, m\right\}.$$

Constraint sets $\Gamma$ and $\Gamma_\mu$ are equivalent in the sense that $S \in A^n$ is in $\Gamma$ if and only if it is in $\Gamma_\mu$. We then have the modified vector minimization

(MVP) of (RVP)

$$\text{(MVP)} \qquad V\text{-}\min_{S \in \Gamma} \left[ Q_1(S), Q_2(S), \ldots, Q_p(S) \right],$$

where

(i)

$$\Gamma_\mu = \Big\{ S \in A^n \colon\ Q_{ij}(S) \equiv T_i\big(H_j(S)\big) \leqq 0,$$
$$i = 1, 2, \ldots, p, j = 1, 2, \ldots, m \Big\},$$

(ii) $A^n$ is the $n$-fold product of a $\sigma$-algebra $A$ of subsets of a given set $X$,

(iii) $Q_i$ for $i = 1, 2, \ldots, p$, and $Q_{ij}$ for $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, m$, are real-valued differentiable $B_i$-vex functions defined on $A^n$.

We see that (MVP) is of the same type as (VP). Therefore, a dual problem for (MVP) will be of the same type as (VD). Also, we can easily define a saddle point for (MVP) in a manner similar to Definition 5.1. Furthermore, it is easy to see that (RVP) and (MVP) are equivalent in the following sense:

(i) $S \in A^n$ is in $\Gamma$ if and only if it is in $\Gamma_\mu$.

(ii) $S \in \Gamma$ is an efficient (properly efficient) solution of (RVP) if and only if it is an efficient (properly efficient) solution of (MVP).

THEOREM 5.2. *Let* $(S^*, Y^*)$ *be a vector saddle point of the vector-valued Lagrangian* $L(S, Y)$ *for* (MVP); *then* $S^*$ *is* (RVP)-*feasible,*

$$\sum_{j=1}^{m} y_{ij}^* H_j(S^*) = 0 \qquad \text{for at least one } i = 1, 2, \ldots, p.$$

*Furthermore* $S^*$ *is an efficient solution of* (RVP).

*Proof.* $(S^*, Y^*)$ is a vector saddle point of the vector-valued Lagrangian $L(S, Y)$ for (MVP) yields

$$L(S^*, Y) \ngeq L(S^*, Y^*) \qquad \text{for all } Y \in R_+^{p \times m}$$

and

$$L(S^*, Y^*) \ngeq L(S, Y^*) \qquad \text{for all } S \in A^n.$$

We first consider $L(S^*, Y) \ngeq L(S^*, Y^*)$ for all $Y \in R_+^{p \times m}$. This implies that for any $Y \in R_+^{p \times m}$,

$$L_i\big(S^*, y^i\big) - L_i\big(S^*, y^{*i}\big) \leqq 0 \qquad \text{for at least one } i$$

$$\sum_{j=1}^{m} (y_{ij} - y_{ij}^*) Q_{ij}(S^*) \leqq 0 \qquad \text{for at least one } i. \tag{5.7}$$

Assume that (5.7) holds for $i = k$ (say), where $1 \leqq k \leqq p$. Then, in (5.7), setting

$$y^k = y^{*k} + e^j, e^j = (0, \dots, 0, 1, 0, \dots, 0) \in R^m,$$

with 1 being at the $j$th place, we get

$$Q_{kj}(S^*) \leq 0 \qquad \text{for } j = 1, 2, \dots, m.$$

Thus we have

$$T_k\big(H_j(S^*)\big) \leqq 0 \qquad \text{for } j = 1, 2, \dots, m.$$

This along with the assumption that $T_i\colon\ R^1 \to R^1$, $(T_i)^{-1}$ exists, $T_i > 0$ and $T_i^{-1}(0) = 0$, for $i = 1, 2, \dots, p$, gives

$$H_j(S^*)) \leqq 0 \qquad \text{for } j = 1, 2, \dots, m.$$

Thus, $S^*$ is (RVP)-feasible. Now

$$y_{ij}^* \geqq 0, i \in I, \quad \text{and} \quad Q_{ij}(S^*) \leqq 0, j = 1, 2, \dots, m;$$

therefore,

$$\sum_{j=1}^{m} y_{ij}^* Q_{ij}(S^*) \leqq 0, \qquad i \in I.$$

Setting $Y = 0$ in $L(S^*, Y) \ngeqq L(S^*, Y^*)$, we obtain,

$$-\sum_{j=1}^{m} y_{ij}^* Q_{ij}(S^*) \leqq 0, \qquad \text{for at least one } i \in I.$$

This along with

$$\sum_{j=1}^{m} y_{ij}^* Q_{ij}(S^*) \leqq 0, \qquad i \in I,$$

gives

$$\sum_{j=1}^{m} y_{ij}^{*} Q_{ij}(S^*) = 0 \qquad \text{for at least one } i \in I.$$

Using the assumption that $T_i$: $R^1 \to R^1$, $(T_i)^{-1}$ exists, $T_i > 0$ and $T_i^{-1}(0) = 0$ for $i = 1, 2, \ldots, p$ we have

$$\sum_{j=1}^{m} y_{ij}^{*} H_j(S^*) = 0, \qquad \text{for at least one } i \in I.$$

For any (RVP)-feasible $S$ (hence for any (MVP)-feasible $S$) and for all $i \in I$, we have

$$\begin{aligned} Q_i(S) - Q_i(S^*) &= L_i\big(S, y^{*i}\big) - L_i\big(S^*, y^{*i}\big) - \sum_{j=1}^{m} y_{ij}^{*} Q_{ij}(S) \\ &\geqq L_i\big(S, y^{*i}\big) - L_i\big(S^*, y^{*i}\big), \end{aligned} \tag{5.8}$$

But $L(S^*, Y^*) \ngeq L(S, Y^*)$ for (RVP)-feasible $S$. Hence (5.8) implies that there do not exist any (VP)-feasible $S$ such that $F(S) \leq F(S^*)$. Hence $S^*$ is efficient to (RVP).

## 6. SPECIAL CASES

In the present section we consider a vector-valued fractional programming problem (VFP) involving $n$-set functions and relate it to a special case of (VP).

$$\text{(VFP)} \quad V\text{-minimize}\left[\frac{F_1(S_1, S_2, \ldots, S_n)}{G_1(S_1, S_2, \ldots, S_n)}, \ldots, \frac{F_p(S_1, S_2, \ldots, S_n)}{G_p(S_1, S_2, \ldots, S_n)}\right]$$

subject to

$$H_j(S_1, S_2, \ldots, S_n) \leqq 0, \qquad j = 1, 2, \ldots, m$$

$$(S_1, S_2, \ldots, S_n) \in A^n,$$

where

(i) $A^n$ is the $n$-fold product of a $\sigma$-algebra $A$ of subsets of a given set $X$,

(ii) $F_i, G_i$ for $i = 1, 2, \ldots, p$ and $H_j$ for $j = 1, 2, \ldots, m$, are real-valued differentiable functions defined on $A^n$,

(iii) for $i = 1, 2, \ldots, p$, $F_i$ is a convex and nonnegative function, $G_i$ is a concave and positive function, and whenever a $G_i$ is both convex and concave the corresponding $F_i$, $i = 1, 2, \ldots, p$, is not necessarily restricted to be nonnegative,

(iv) for $j = 1, 2, \ldots, m$, $H_j$ is a convex function.

From (VFP) we now obtain, along the lines of Bector [2], Bector *et al.* [4], and Chandra *et al.* [6], an equivalent problem (EVFP). It may be verified that the constraint set and the set of efficient (properly efficient) points of (VFP) are equivalent, respectively, to the constraint set and the set of efficient (properly efficient) points of (EVFP):

(EVFP)

$$V\text{-minimize}\left[\frac{F_1(S_1, S_2, \ldots, S_n)}{G_1(S_1, S_2, \ldots, S_n)}, \ldots, \frac{F_p(S_1, S_2, \ldots, S_n)}{G_p(S_1, S_2, \ldots, S_n)}\right] \tag{6.1}$$

subject to

$$\frac{H_j(S_1, S_2, \ldots, S_n)}{G_i(S_1, SS_2, \ldots, S_n)} \leqq 0 \qquad (i = 1, 2, \ldots, p; j = 1, 2, \ldots, m) \tag{6.2}$$

$$(S_1, S_2, \ldots, S_n) \in A^n. \tag{6.3}$$

Writing

$$Q_i(S_1, S_2, \ldots, S_n) \quad \text{for} \quad \frac{F_i(S_1, S_2, \ldots, S_n)}{G_i(S_1, S_2, \ldots, S_n)}, \qquad i = 1, 2, \ldots, p$$

in (6.1), and

$$Q_{ij}(S_1, S_2, \ldots, S_n) \quad \text{for} \quad \frac{H_j(S_1, S_2, \ldots, S_n)}{G_i(S_1, S_2, \ldots, S_n)},$$
$$i = 1, 2, \ldots, p; j = 1, 2, \ldots, m$$

in (6.2), we see that (EVFP) reduces to (VP).

We now state the following Lemma 6.1, which can be proved easily.

LEMMA 6.1. *Let $F, G$: $A^n \to R^1$ be differentiable functions and let $Q = F/G$.*

(*i*) *$G$ is concave and strictly positive.*

(*ii*) *If $F$ is convex and nonnegative ($F$ need not be nonnegative if $G$ is both convex and concave), then $Q$ is a $B$-vex function on $A^n$ with $B(R, S) = G(R)/G(S) > 0$ for all $R$ and $S$ in $A^n$.*

*Remark* 6.1. In Lemma 6.1, if at least one of the functions $F$ and $G$ is strictly convex/strictly concave, $Q = F/G$ is a strictly $B$-vex function on $A^n$.

We now introduce the dual program (DFP) to (VFP),

$$\text{(DFP)} \qquad V\text{-maximize} \quad L(T, Y)$$

subject to

$$\left\langle D_r \sum_{i=1}^{p} \lambda_i L_i(T, Y^i)_{T_1^*, \dots, T_n^*}, I_{T_r} - I_{T_r^*} \right\rangle \geqq 0, \qquad \text{for all } T \in A^n$$

$$\lambda > 0, y \geq 0, Y \in R^{p \times m}, \lambda \in R^p, y^i \in R^m, T \in A^n,$$

where we assume that

(i)

$$L_i(T, y^i) = \left[ F_i(T) + \sum_{j=1}^{m} H_j(T) \right] / G_i(T), \qquad i = 1, 2, \dots, p,$$

and

(ii) if a function $G_i$ is not both convex and concave on $A^n$, then the corresponding function

$$F_i(T) + \sum_{j=1}^{m} y_{ij} H_j(T) \geqq 0 \quad \text{for all } y_{ij} \geqq 0,$$

$$i = 1, 2, \dots, p, j = 1, 2, \dots, m, T \in A^n.$$

In view of assumptions (i) and (ii) of (DFP); Lemma 6.1, Remarks 3.1, 4.1, 4.2, and 6.1; and the results of (RVP) and (MVP), the duality results relating (VFP) and (DFP) are easily established.

## 7. CONCLUSION

In this paper we present necessary and sufficient optimality conditions and a Wolfe-type [13] dual for a vector-valued primal problem in which each of the objective functions and the constraint functions is an appropriately restricted $B$-vex $n$-set functions. Weak, strong, and strictly converse duality theorems are proved. The concept of a vector saddle point for a vector-valued Lagrangian for $n$-set functions is introduced and its relationship to the vector-valued primal problem is studied. Results for a certain multiobjective fractional programming problem are shown to follow as a special case. By introducing the concept of $(\rho, B)$-vex functions, the results presented in this paper can be easily generated further.

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