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Borel-fixed ideals and reduction number ☆

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Abstract

The aim of this paper is to study the relationship between the reduction number and Borel-fixed ideals in all characteristics. Especially it is shown that $r(R/I) \leqslant r(R/I^{\text{lex}})$, where I^{lex} denotes the unique lex-segment ideal whose Hilbert function is equal to that of I. This solves a recent question by Conca.

Keywords: Reduction number; Borel-fixed ideal; Hilbert function

Introduction

Let A be a standard graded algebra over an infinite field k. An ideal $\mathfrak{q} = (z_1, \dots, z_s)$, where $z_1, \dots, z_s$ are linear forms of A, is called an *s-reduction* of A if $\mathfrak{q}_t = A_t$ for t large enough (cf. [10]). The *reduction number* of A with respect to $\mathfrak{q}$, written as $r_{\mathfrak{q}}(A)$, is the minimum number r such that $\mathfrak{q}_{r+1} = A_{r+1}$. The *s-reduction number* of A is defined as

$$r_s(A) := \min\bigl\{r_{\mathfrak{q}}(A) \mid \mathfrak{q} = (z_1, \dots, z_s) \text{ is a reduction of } A\bigr\}.$$

Let $d = \dim A$. It is well-known that a reduction $\mathfrak{q}$ of A is minimal with respect to inclusion if and only if $\mathfrak{q}$ can be generated by d elements. In this case, $k[z_1, \dots, z_d] \hookrightarrow A$ is a Noether normalization of A and the reduction number $r_{\mathfrak{q}}(A)$ is the maximum degree of the generators of A as a graded $k[z_1, \dots, z_d]$-module [16]. For short, we set $r(A) = r_d(A)$. The reduction number $r(A)$ can be used as a measure for the complexity of A. For instance,

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we can relate $r(A)$ to other important invariants of A such that the degree, the arithmetic degree and the Castelnuovo–Mumford regularity (see [13,16,17]).

Let I be an arbitrary homogeneous ideal in a polynomial ring $R = k[x_1, \dots, x_n]$. It is shown recently in [5] and [15] (see also [3]) that $r(R/I) \leqslant r(R/\operatorname{in}(I))$, where $\operatorname{in}(I)$ denotes the initial ideal of I with respect to a given term order. In particular, we have $r(R/I) = r(R/\operatorname{gin}(I))$, where $\operatorname{gin}(I)$ denotes the generic initial ideal of I with respect to the reverse lexicographic term order [14]. Since generic initial ideals are Borel-fixed (see the definition in Section 1), we may restrict the study on the reduction number to that of Borel-fixed ideals. If $\operatorname{char}(k) = 0$, Borel-fixed ideals are characterized by the so-called strong stability which gives information on their monomials [1]. Similar characterizations can be established for the positive characteristic cases [11]. But these characterizations are not good enough for certain problems. For instance, Conca [5] has raised the question whether $r(R/I) \leqslant r(R/I^{\mathrm{lex}})$, where I^{lex} denotes the unique lex-segment ideal whose Hilbert function is equal to that of I. He solved this question for $\operatorname{char}(k) = 0$ by using the strong stability, but his proof does not work for the positive characteristic cases.

The aim of this paper is to study the relationship between the s-reduction number and Borel-fixed ideals in all characteristics. By definition, Borel-fixed ideals are closed under certain specializations which is similar to the strong stability. Using this property we show that the reduction numbers of s-reductions of the quotient ring of a Borel-fixed ideal are attained by s-reductions generated by variables (Theorem 1.2). This gives a practical way to compute the s-reduction number. We will also estimate the number of monomials which can be specialized to a given monomial in the above sense (Theorem 1.7). As a consequence, we obtain a combinatorial version of the well-known Eakin–Sathaye's theorem which estimates the s-reduction number by means of the Hilbert function (Corollary 1.9 and Theorem 2.1). Furthermore, we show that the bound of Eakin–Sathaye's theorem is attained by the s-reduction number when I is a lex-segment monomial ideal (Theorem 2.4). These results help solve Conca's question for all characteristics in a more general setting, namely, that $r_s(R/I) \leqslant r_s(R/I^{\mathrm{lex}})$. Finally, since $r(R/I^{\mathrm{lex}})$ is extremal in the class of ideals with a given Hilbert function, we will estimate $r(R/I^{\mathrm{lex}})$ in terms of some standard invariants of I. We shall see that $r(R/I^{\mathrm{lex}})$ is bounded by a polynomial of $r(R/I)$ (Theorem 2.7).

Throughout this paper, if $Q \subset R$ is an ideal which generates a reduction of R/I, then we will denote its reduction number by $r_Q(R/I)$.

1. Borel-fixed ideals

Let I be a monomial ideal of the polynomial ring $R = k[x_1, \dots, x_n]$. Let $\mathcal{B}$ denote the Borel subgroup of $\mathrm{GL}(n, k)$ which consists of the upper triangular invertible matrices. Then I is called a *Borel-fixed* ideal if for all $g \in \mathcal{B}$, $g(I) = I$. We say that a monomial x^B is a *Borel specialization* of a monomial x^A if x^B can be obtained from x^A by replacing every variable x_i of x^A by a variable x_{j_i} with $j_i \leqslant i$. The name comes from the simple fact that any Borel-fixed monomial ideal is closed under Borel specialization.

Lemma 1.1. *Let I be a Borel-fixed monomial ideal. If I contains x^A then I contains any Borel specialization of x^A.*

Proof. Let x^B be a monomial obtained from x^A by replacing each variable x_i by a variable x_{j_i} with $j_i \leqslant i$, $i = 1, \dots, n$. Let g be the element of the Borel group $\mathcal{B}$ defined by the linear transformation

$$g(x_i) = \begin{cases} x_i & \text{if } j_i = i, \\ x_i + x_{j_i} & \text{if } j_i \neq i. \end{cases}$$

Then x^B is a monomial of $g(x^A)$. Since $g(I) = I$, this implies $x^B \in I$. □

Let $d = \dim R/I$. If I is a Borel-fixed ideal, every associated prime ideal of I has the form $(x_1, \dots, x_i)$ for $i \geqslant n - d$ (see, e.g., [8, Corollary 15.25]). From this it follows that s variables of R generate an s-reduction of R/I if and only if they are of the form $x_{i_1}, \dots, x_{i_{s-d}}, x_{n-d+1}, \dots, x_n$ with $1 \leqslant i_1 < \cdots < i_{s-d} \leqslant n - d$. It is clear that $r_{(x_{i_1},\dots,x_{i_{s-d}},x_{n-d+1},\dots,x_n)}(R/I)$ is the least integer r such that all monomials of degree $r + 1$ in the remaining variables are contained in I. The following result shows that the computation of the reduction numbers of all s-reductions of R/I can be reduced to the above class of s-reductions.

Theorem 1.2. *Let I be a Borel-fixed ideal and $s \geqslant d = \dim R/I$. Then*

(i) *For every s-reduction $\mathfrak{q}$ of R/I, there exist variables $x_{i_1}, \dots, x_{i_{s-d}}$ with $1 \leqslant i_1 < \cdots < i_{s-d} \leqslant n - d$ such that*

$$r_{\mathfrak{q}}(R/I) = r_{(x_{i_1},\dots,x_{i_{s-d}},x_{n-d+1},\dots,x_n)}(R/I).$$

(ii) $r_s(R/I) = r_{(x_{n-s+1},\dots,x_n)}(R/I)$.

Proof. Let $y_1, \dots, y_s$ be linear forms of R which generates $\mathfrak{q}$ in R/I. Without restriction we may assume that

$$y_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{it_i}x_{t_i} \quad (i = 1, \dots, s)$$

with $a_{it_i} \neq 0$ for different indices $t_1, \dots, t_s$. Let g be the element of the Borel group $\mathcal{B}$ defined by the linear transformation

$$g(x_j) = \begin{cases} x_j & \text{if } j \notin \{t_1, \dots, t_s\}, \\ y_i & \text{if } j = t_i,\ 1 \leqslant i \leqslant s. \end{cases}$$

Then $g((x_{t_1}, \dots, x_{t_s})) = g((y_1, \dots, y_s))$. Since $g(I) = I$, this implies that $x_{t_1}, \dots, x_{t_s}$ generate an s-reduction of R/I with

$$r_{\mathfrak{q}}(R/I) = r_{(x_{t_1},\dots,x_{t_s})}(R/I).$$

As observed before, $x_{t_1}, \ldots, x_{t_s}$ must be of the form $x_{i_1}, \ldots, x_{i_{s-d}}, x_{n-d+1}, \ldots, x_n$ with $1 \leqslant i_1 < \cdots < i_{s-d} \leqslant n-d$. This proves (i).

To prove (ii) choose $\mathfrak{q}$ such that $r_s(R/I) = r_{\mathfrak{q}}(R/I)$. By (i) there exist variables $x_{t_1}, \ldots, x_{t_s}$ such that $r_{\mathfrak{q}}(R/I) = r_{(x_{t_1},\ldots,x_{t_s})}(R/I)$. Note that $r_{(x_{t_1},\ldots,x_{t_s})}(R/I)$ is the least integer r such that all monomials of degree $r+1$ in the remaining variables are contained in I and that all monomials of degree $r+1$ in $x_1, \ldots, x_{n-s}$ are their Borel specializations. By Lemma 1.1, the latter monomials are contained in I, too. This implies

$$r_{(x_{t_1},\ldots,x_{t_s})}(R/I) \geqslant r_{(x_{n-s+1},\ldots,x_n)}(R/I) \geqslant r_s(R/I).$$

So we conclude that $r_s(R/I) = r_{(x_{n-s+1},\ldots,x_n)}(R/I)$. $\square$

The case $s = d$ of Theorem 1.2 was already proved by Bresinsky and Hoa [3, Theorem 11]. They showed that all minimal reductions of R/I have the same reduction number. But their arguments can not be extended to the general case. By Theorem 1.2(i), there are at most $\binom{n-d}{s-d}$ different reduction numbers for the s-reductions. This number $\binom{n-d}{s-d}$ can be attained if $\operatorname{char}(k) > 0$. This displays a different behaviour than in the case $s = d$.

Example 1.3. Assume that $\operatorname{char}(k) = p$. Let $d \leqslant s < n$ and $1 < a_1 < \cdots < a_{n-d}$ be integers. Then

$$I = \left(x_1^{p^{a_1}}, \ldots, x_{n-s}^{p^{a_{n-s}}}\right) \subseteq R = k[x_1, \ldots, x_n]$$

is a Borel-fixed ideal. For the s-reduction $Q = (x_{i_1}, \ldots, x_{i_{s-d}}, x_{n-d+1}, \ldots, x_n)$ of R/I with $1 \leqslant i_1 < \cdots < i_{s-d} \leqslant n-d$ we have

$$r_Q(R/I) = p^{a_{j_1}} + \cdots + p^{a_{j_{n-s}}} - n + s,$$

where $\{j_1, \ldots, j_{n-s}\} = \{1, \ldots, n-d\} \setminus \{i_1, \ldots, i_{s-d}\}$. Hence the s-reductions of R/I have exactly $\binom{n-d}{s-d}$ different reduction numbers. Moreover, we have

$$r_s(R/I) = p^{a_1} + \cdots + p^{a_{n-s}} - n + s.$$

If $\operatorname{char}(k) = 0$, Borel-fixed ideals are characterized by a closure property stronger than that of Borel specialization. Recall that a monomial ideal I is called *strongly stable* if whenever $x^A \in I$ and x^A is divided by x_i, then $x^A x_j / x_i \in I$ for all $j \leqslant i$. Any strongly stable monomial ideal is Borel-fixed. The converse holds if $\operatorname{char}(k) = 0$ [1, Proposition 2.7]. In this case we can easily compute the reduction number of R/I by the following result.

Corollary 1.4. *Let I be a strongly stable monomial ideal. For any $s \geqslant \dim R/I$ we have*

$$r_s(R/I) = \min\left\{t \mid x_{n-s}^{t+1} \in I\right\}.$$

Proof. By Theorem 1.2(ii) we have to prove that

$$r_{(x_{n-s+1},\dots,x_n)}(R/I) = \min\{t \mid x_{n-s}^{t+1} \in I\}.$$

Hence, it is sufficient to show that if $x_{n-s}^{t+1} \in I$ then all monomials of degree $t+1$ in $x_1, \dots, x_{n-s}$ are contained in I. But this follows from the strong stability of I. $\square$

Example 1.3 shows that Lemma 1.4 does not hold if I is not strongly stable.

If $\operatorname{char}(k) = 0$, the number of possible reduction numbers for the s-reductions of R/I is much smaller than in the case $\operatorname{char}(k) > 0$. In fact, for any s-reduction $Q = (x_{i_1}, \dots, x_{i_{s-d}}, x_{n-d+1}, \dots, x_n)$ with $1 \leqslant i_1 < \dots < i_{s-d} \leqslant n-d$, we can show similarly as above that

$$r_Q(R/I) = \min\{t \mid x_{j_{n-s}}^{t+1} \in I\},$$

where j_{n-s} is the largest index outside the set $\{i_1, \dots, i_{s-d}, n-d+1, \dots, n\}$. Since there at most $s-d+1$ such indices, Theorem 1.2(i) shows that there are at most $s-d+1$ different reduction numbers for the s-reductions.

Example 1.5. Let I be the ideal generated by all monomials bigger or equal a monomial in the list $x_1^{a_1}, \dots, x_{n-d}^{a_{n-d}}$ with respect to the graded lexicographic order, where $1 < a_1 < \dots < a_{n-d}$. It is easy to see that this ideal is strongly stable and the s-reductions of R/I have exactly $s-d+1$ different reduction numbers.

The set of all monomials which can be Borel-specialized to x^A will be denoted by $P(x^A)$. If we can estimate the cardinality $|P(x^A)|$ of $P(x^A)$, we can decide when $x^A \in I$, depending on the behavior of the Hilbert function of I.

Lemma 1.6. *Let I be a Borel-fixed ideal. Assume that $\dim_k(R/I)_t < |P(x^A)|$ for $t = \deg x^A$. Then $x^A \in I$.*

Proof. If $x^A \notin I$, then $P(x^A) \cap I = \emptyset$ by Lemma 1.1. Since $P(x^A)$ consists of monomials of degree t, this implies $\dim_k(R/I)_t \geqslant |P(x^A)|$, a contradiction. $\square$

Theorem 1.7. *Suppose $x^A = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_s}^{\alpha_{i_s}}$ with $\alpha_{i_1}, \dots, \alpha_{i_s} > 0$, $1 \leqslant i_1 < \dots < i_s \leqslant n$. Put $i_{s+1} = n+1$. Then*

$$\left|P\left(x^A\right)\right| \geqslant \sum_{t=1}^{s} \binom{\alpha_{i_1} + \dots + \alpha_{i_t} + i_{t+1} - i_t - 1}{i_{t+1} - i_t - 1} - s + 1.$$

Proof. The cases $n = 0$ and $\deg x^A = 0$ are trivial because $x^A = 1$. Assume that $n \geqslant 1$ and $\deg x^A > 0$.

If $i_s = n$, we let $x^B = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_{s-1}}^{\alpha_{i_{s-1}}}$ and consider x^B as a monomial in the polynomial ring $S = k[x_1, \dots, x_{n-1}]$. Any monomial of $P(x^A)$ is the product of a monomial of

$P(x^B) \cap S$ with $x_n^{\alpha_n}$. The converse also holds. Hence $|P(x^A)| = |P(x^B) \cap S|$. Using induction on n we may assume that

$$\big|P\big(x^B\big) \cap S\big| \geqslant \sum_{t=1}^{s-1} \binom{\alpha_{i_1} + \cdots + \alpha_{i_t} + i_{t+1} - i_t - 1}{i_{t+1} - i_t - 1} - (s-1) + 1.$$

Since $i_{s+1} = n + 1 = i_s + 1$, we have

$$\binom{\alpha_{i_1} + \cdots + \alpha_{i_s} + i_{s+1} - i_s - 1}{i_{s+1} - i_s - 1} = 1.$$

So we get

$$\big|P\big(x^A\big)\big| = \big|P\big(x^B\big) \cap S\big| \geqslant \sum_{t=1}^{s} \binom{\alpha_{i_1} + \cdots + \alpha_{i_t} + i_{t+1} - i_t - 1}{i_{t+1} - i_t - 1} - s + 1.$$

If $i_s < n$, we divide $P(A)$ into two disjunct parts P_1 and P_2. The first part P_1 consists of monomials divided by x_{i_1}, and the second part P_2 consists of monomials not divided by x_{i_1}. Set $x^C = x_{i_1}^{\alpha_1 - 1} x_{i_2}^{\alpha_2} \cdots x_{i_s}^{\alpha_s}$. Every monomial of P_1 is the product of x_{i_1} with a monomial of $P(x^C)$. The converse also holds. Hence $|P_1| = |P(x^C)|$. Using induction on $\deg(x^A)$ we may assume that

$$\begin{aligned} \big|P\big(x^C\big)\big| &\geqslant \sum_{t=1}^{s} \binom{\alpha_{i_1} + \cdots + \alpha_{i_t} + i_{t+1} - i_t - 2}{i_{t+1} - i_t - 1} - s + 1 \\ &\geqslant \binom{\alpha_{i_1} + \cdots + \alpha_{i_s} + i_{s+1} - i_s - 2}{i_{s+1} - i_s - 1}. \end{aligned}$$

Note that the sum should starts from $t = 2$ to s if $a_{i_1} = 1$. In this case, the above formula holds because $\binom{\alpha_{i_1} - 2}{\alpha_{i_1} + i_2 - i_1 - 1} = \binom{i_2 - i_1 - 1}{0} = 1$. To estimate $|P_2|$ let $x^D = x_{i_1+1}^{\alpha_{i_1}} \cdots x_{i_s+1}^{\alpha_{i_s}}$. It is obvious that every monomial of $P(x^D)$ does not contain x_{i_1} and can be Borel-specialized to x^A. Therefore, $P(x^D)$ is contained in P_2. Using induction on i_s we may assume that

$$\begin{aligned} \big|P\big(x^D\big)\big| \geqslant &\sum_{t=1}^{s-1} \binom{\alpha_{i_1} + \cdots + \alpha_{i_t} + i_{t+1} - i_t - 1}{i_{t+1} - i_t - 1} \\ &+ \binom{\alpha_{i_1} + \cdots + \alpha_{i_s} + i_{s+1} - i_s - 2}{i_{s+1} - i_s - 2} - s + 1. \end{aligned}$$

Summing up we obtain

$$\begin{aligned}|P| &= |P_1| + |P_2| \geqslant \left|P\left(x^C\right)\right| + \left|P\left(x^D\right)\right| \\ &\geqslant \binom{\alpha_{i_1} + \cdots + \alpha_{i_s} + i_{s+1} - i_s - 2}{i_{s+1} - i_s - 1} + \sum_{t=1}^{s-1} \binom{\alpha_{i_1} + \cdots + \alpha_{i_t} + i_{t+1} - i_t - 1}{i_{t+1} - i_t - 1} \\ &\quad + \binom{\alpha_{i_1} + \cdots + \alpha_{i_s} + i_{s+1} - i_s - 2}{i_{s+1} - i_s - 2} - s + 1 \\ &= \sum_{t=1}^{s} \binom{\alpha_{i_1} + \cdots + \alpha_{i_t} + i_{t+1} - i_t - 1}{i_{t+1} - i_t - 1} - s + 1. \qquad \square\end{aligned}$$

The bound of Theorem 1.7 is far from being the best possible as one can realize from the proof. However, it is sharp in many cases.

Example 1.8. If $R = k[x_1, x_2, x_3]$ we have $P(x_1x_3) = \{x_1x_3, x_2x_3\}$. Hence

$$\left|P(x_1x_3)\right| = 2 = \binom{3-1+1-1}{1} + \binom{4-3+1-1}{1} - 2 + 1.$$

An interesting application of Theorem 1.7 is the following bound for the reduction number.

Corollary 1.9. *Let I be a Borel-fixed monomial ideal. Assume that*

$$\dim_k(R/I)_t < \binom{s+t}{t}$$

for some integers $s, t \geqslant 1$. Then $x_{n-s+1}, \ldots, x_n$ generate a reduction of R/I with

$$r_{(x_{n-s+1},\ldots,x_n)}(R/I) \leqslant t - 1.$$

Proof. We have to show that the ideal $(I, x_{n-s+1}, \ldots, x_n)$ contains every monomial x^A of degree t in $x_1, \ldots, x_{n-s}$. If we write $x^A = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_s}^{\alpha_{i_s}}$ with $1 \leqslant i_1 < \cdots < i_s \leqslant n - s$ and $\alpha_{i_1} + \cdots + \alpha_{i_s} = t$, then Theorem 1.7 gives

$$\left|P\left(x^A\right)\right| \geqslant \binom{n - i_s + t}{t} \geqslant \binom{s+t}{t} > \left|P\left(x^A\right)\right|.$$

By Lemma 1.6, this implies $x^A \in I$. $\square$

2. Eakin–Sathaye's theorem

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over an *infinite* field k of arbitrary characteristic. In this section we will deal with the reduction number of R/I for an arbitrary homogeneous ideal I. Let us first recall the following theorem of Eakin and Sathaye.

Theorem 2.1 [7, Theorem 1]. *Let I be an arbitrary homogeneous ideal in R. Assume that*

$$\dim_k(R/I)_t < \binom{s+t}{t}$$

for some integers $s, t \geqslant 1$. Choose s generic linear forms $y_1, \dots, y_s$, that is in a non-empty open subset of the parameter space of s linear forms of R. Then $y_1, \dots, y_s$ generate a reduction of R/I with

$$r_{(y_1,\dots,y_s)}(R/I) \leqslant t-1.$$

Eakin–Sathaye's theorem provides an efficient way to estimate the reduction number (see, e.g., [17, Corollary 3.4 and Theorem 4.2]). We shall see that Corollary 1.9 (though formulated for Borel-fixed ideals and a fixed reduction) is equivalent to Eakin–Sathaye's theorem. For that we need the following observations.

First, the reduction number of a reduction generated by generic elements is the smallest one among reductions generated by the same number of generators.

Lemma 2.2. *For every integer $s \geqslant \dim R/I$ choose s generic linear forms $y_1, \dots, y_s$ in R. Then $y_1, \dots, y_s$ generate a reduction of R/I with*

$$r_{(y_1,\dots,y_s)}(R/I) = r_s(R/I).$$

Proof. The statement was already proved for the case $s = \dim R$ in [14, Lemma 4.2]. The proof for arbitrary $s \geqslant \dim R$ is similar, hence we omit it. □

Secondly, the smallest reduction number does not change when passing to any generic initial ideal.

Theorem 2.3. *Let $\operatorname{gin}(I)$ denote the generic initial ideal of I with respect to the reverse lexicographic term order. For every integer $s \geqslant \dim R/I$ we have*

$$r_s(S/I) = r_s\big(S/\operatorname{gin}(I)\big).$$

Proof. The statement was already proved for the case $s = \dim R$ in [14, Theorem 4.3]. The case of arbitrary $s \geqslant \dim R/I$ can be proved in the same manner (though not trivial). □

Now we are able to show that Eakin–Sathaye's theorem can be deduced from Corollary 1.9. Since the proof relies only on properties of Gröbner basis and Borel-fixed ideals, it can be viewed as a combinatorial proof.

Combinatorial proof of Theorem 2.1. By Lemma 2.2, we have to show that $r_s(R/I) \leqslant t-1$. Let $\operatorname{gin}(I)$ denote the generic initial ideal of I with respect to the reverse lexicographic term order. From the theory of Gröbner bases we know that $\operatorname{gin}(I)$ is a Borel-fixed monomial ideal with $\dim_k(R/\operatorname{gin}(I))_t = \dim_k(R/I)_t$ (see, e.g., [8, Theorem 15.3]). By Corollary 1.9, the assumption $\dim_k(R/I)_t < \binom{s+t}{t}$ implies

$$r_s\bigl(R/\operatorname{gin}(I)\bigr) \leqslant r_{(x_{n-s+1},\dots,x_n)}\bigl(R/\operatorname{gin}(I)\bigr) \leqslant t-1.$$

Now, we only need to apply Theorem 2.3 to get back to $r_s(R/I)$. $\square$

On the other hand, Corollary 1.9 can be deduced from Eakin–Sathaye's theorem because according to Theorem 1.2(ii) and Lemma 2.2 we have

$$r_{(x_{n-s+1},\dots,x_n)}(R/I) = r_s(R/I) = r_{(y_1,\dots,y_s)}(R/I)$$

for any Borel-fixed ideal I.

We shall see that the bound of Eakin–Sathaye's theorem is attained exactly by lex-segment ideals. Recall that a *lex-segment* ideal is a monomial ideal I such that if $x^A \in I$ then $x^B \in I$ for any monomial $x^B \geqslant x^A$ with respect to the lexicographic term order. It is easy to see that lex-segment ideals are strongly stable.

Theorem 2.4. *Let I be a lex-segment ideal. Then*

$$r_s(R/I) = \min\left\{t \mid \dim_k(R/I)_t < \binom{s+t}{t}\right\} - 1.$$

Proof. By Theorem 2.1 and Lemma 2.2 we have $r_s(R/I) \leqslant r-1$, where

$$r := \min\left\{t \mid \dim_k(R/I)_t < \binom{s+t}{t}\right\}.$$

It remains to show that $r_s(R/I) \geqslant r-1$. Assume to the contrary that $r_s(R/I) < r-1$. By Theorem 1.2(ii) we have $r_{(x_{n-s+1},\dots,x_n)}(R/I) = r_s(R/I) < r-1$. Using Corollary 1.4 we can deduce that $x_{n-s}^{r-1} \in I$. By the definition of a lex-segment ideal, this implies that every monomial of degree $r-1$ which involves one of the variables $x_1,\dots,x_{n-s-1}$ is contained in I. Equivalently, the monomials of degree $r-1$ not contained in I involve only the $s+1$ variables $x_{n-s},\dots,x_n$. Since $x_{n-s}^{r-1} \in I$, this implies

$$\dim_k(R/I)_{r-1} < \binom{s+r-1}{r-1}.$$

This contradicts to the definition of r. $\square$

Given a homogeneous ideal I in R, we denote by I^{lex} the unique lex-segment ideal whose Hilbert function is equal to that of I. It is well-known that the Betti numbers of R/I^{lex} are extremal in the class of ideals with a given Hilbert function [2,9,11]. If $\operatorname{char}(k) = 0$, Conca showed that the reduction number $r(R/I^{\mathrm{lex}})$ is extremal in this sense [5, Proposition 10]. He raised the question whether this result holds for all characteristics. The following result will settle Conca's question in the affirmative.

Corollary 2.5. *Let I be an arbitrary homogeneous ideal in R and $s \geqslant \dim R/I$. Then*

$$r_s(R/I) \leqslant r_s\big(R/I^{\mathrm{lex}}\big).$$

Proof. According to Theorem 2.4 we have

$$r_s\big(R/I^{\mathrm{lex}}\big) = \min\left\{t \mid \dim_k(R/I)_t < \binom{s+t}{t}\right\} - 1.$$

By Theorem 2.1, this implies $r_s(R/I) \leqslant r_s(R/I^{\mathrm{lex}})$. $\square$

By Corollary 2.5, $r(R/I^{\mathrm{lex}})$ is extremal in the class of ideals with a given Hilbert function. So it is of interest to estimate $r(R/I^{\mathrm{lex}})$ in terms of other invariants of I.

Lemma 2.6. *Let I be an arbitrary homogeneous ideal in R and $d = \dim R/I \geqslant 1$. Let Q be an ideal generated by d linear forms of R which forms a reduction in R/I. Put $e = \ell(R/Q + I)$. Then*

$$r\big(R/I^{\mathrm{lex}}\big) \leqslant d(e-2) + 1.$$

Proof. By [12, Theorem 2.2] we know that

$$\dim_k(R/I)_t \leqslant (e-1)\binom{t+d-2}{d-1} + \binom{t+d-1}{d-1}.$$

For $t = d(e-2) + 2$ we have

$$(e-1)\binom{de-d}{d-1} + \binom{de-d+1}{d-1} < \binom{de-d+2}{d}.$$

Hence the conclusion follows from Theorem 2.4. $\square$

We would like to point out that a bound for $r(R/I)$ in terms of e should be smaller. In fact, we always have

$$r(R/I) \leqslant r_Q(R/I) \leqslant \ell(R/Q + I) - 1 = e - 1.$$

If R/I is a Cohen–Macaulay ring, e is equal to the degree (multiplicity) of I. If R/I is not a Cohen–Macaulay ring, we may replace e by the extended (cohomological) degree of I introduced in [6].

Theorem 2.7. *Let I be an arbitrary homogeneous ideal in R and $d = \dim R/I \geqslant 1$. Let $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s$ be the degrees of the minimal homogeneous generators of I. Then*

(i) $r(R/I^{\mathrm{lex}}) \leqslant d\big[\binom{r(R/I)+n-d}{n-d} - 2\big] + 1$,
(ii) $r(R/I^{\mathrm{lex}}) \leqslant d(a_1 \cdots a_{n-d} - 2) + 1$.

Proof. Without loss of generality we may assume that $Q = (x_{n-d+1}, \dots, x_n)$ forms a minimal reduction of R/I with $r_Q(R/I) = r(R/I)$. Since $R_t = (Q + I)_t$ for $t \geqslant r(R/I) + 1$, we have

$$\begin{aligned}\ell(R/Q+I) &\leqslant \sum_{t=0}^{r(R/I)} \dim_k(R/Q+I)_t \\ &\leqslant \sum_{t=0}^{r(R/I)} \dim_k(R/Q)_t = \binom{r(R/I)+n-d}{n-d}.\end{aligned}$$

Hence (i) follows from Lemma 2.6. To prove (ii) we put $R' = k[x_1, \dots, x_{n-d}]$ and $I' = (I + Q) \cap R'$. Then I' is generated by forms of degrees $a_1' \leqslant a_1$, $a_2' \leqslant a_2, \dots$ and $\ell(R/Q + I) = \ell(R'/I')$. By [4] we can choose a regular sequence $f_1, \dots, f_{n-d}$ in I' such that $\deg(f_i) = a_i'$, $i = 1, \dots, n-d$. It is well-known that $\ell(R'/(f_1, \dots, f_{n-d})) = a_1 \cdots a_{n-d}$. Hence

$$\ell(R/Q+I) \leqslant a_1' \cdots a_{n-d}' \leqslant a_1 \cdots a_{n-d}.$$

Thus, (ii) follows from Lemma 2.6. □

Finally we give some examples which show that the bounds of Theorem 2.7 are sharp.

Example 2.8. Let $I = (x_1, \dots, x_{n-d})^2$. It is easy to see that $r(R/I) = 1$ and

$$\dim_k(R/I)_t = \binom{d+t-1}{d-1} + (n-d)\binom{d+t-2}{d-1}$$

for all $t \geqslant 1$. By Theorem 2.4 we have

$$\begin{aligned} r\big(R/I^{\mathrm{lex}}\big) &= \min\left\{t;\ \binom{d+t-1}{d-1} + (n-d)\binom{d+t-2}{d-1} < \binom{d+t}{d}\right\} - 1 \\ &= d(n-d-1) + 1.\end{aligned}$$

This is exactly the bound (i) of Theorem 2.7.

Example 2.9. Consider the one-dimensional ideal $I = (x_1^a) \subset R = k[x_1, x_2]$, $a \geqslant 1$. We have $\dim_k(R/I)_t = a$ for all $t \geqslant a - 1$. Hence Theorem 2.4 gives

$$r\big(R/I^{\mathrm{lex}}\big) = \min\{t \mid a < t + 1\} - 1 = a - 1.$$

This shows that the bound (ii) of Theorem 2.7 is sharp.

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