Positive Subdefinite Matrices*

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ABSTRACT

The purpose of this paper is to summarize the known results on positive subdefinite matrices and to study more deeply some of their properties. In particular we contrast these matrices, characterizing quasiconvex quadratic forms, with positive semidefinite matrices, characterizing convex quadratic forms, to stress the loss due to the generalization.

1. INTRODUCTION

Martos was the first to introduce and to characterize positive subdefinite matrices, in [6]. They were introduced in connection with the study of quadratic forms quasiconvex and pseudoconvex on the nonnegative orthant. Cottle and the author studied criteria for these matrices in the same context in [3].

The purpose of this paper is to summarize the known results on these matrices and to study more deeply some of their properties. Since quasiconvexity is a generalization of convexity, and since positive semidefinite matrices characterize convex quadratic forms, it is interesting to contrast positive subdefinite matrices with positive semidefinite matrices to see the properties that are lost through the generalization.

In Sec. 2, the basic definitions and the notation are introduced. Then the criteria for mere positive subdefiniteness, obtained by Martos in [6] and by Cottle and the author in [3], are summarized in Sec. 3. In Sec. 4 some properties of merely positive subdefinite matrices (i.e. positive subdefinite matrices that are not positive semidefinite) are analyzed: their irreducibility.

*This research was supported by National Research Council of Canada, Grant No. A 8312

LINEAR ALGEBRA AND ITS APPLICATIONS 31:233–244 (1980) 233

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the distribution of zeros in these matrices, and their singularity. Finally, sufficient conditions for the sum of two positive subdefinite matrices to be positive subdefinite are given in Sec. 5. This section presents only a partial solution to the problem of specifying necessary and sufficient conditions for this points out a major difference between these matrices and positive semidefinite matrices.

2. PRELIMINARIES

Throughout this paper $D$ will stand for a real symmetric matrix of order $n$. Associated with $D$ is the quadratic form $x^TDx$ defined for all $x \in \mathbb{R}^n$.

Recall that a semipositive vector is a nonzero nonnegative vector. Naturally $x$ is seminegative if and only if $-x$ is semipositive. The same kind of terminology applies to real matrices. For example, a semipositive matrix is a nonzero matrix with nonnegative entries.

In [6], Martos identifies a class of matrices $D$ and corresponding quadratic forms $x^TDx$ called positive subdefinite. Their defining property is

$$x^TDx < 0 \text{ implies } Dx \neq 0, \text{ and } Dx > 0 \text{ or } Dx < 0.$$  

Moreover, the quadratic form $x^TDx$ is strictly positive subdefinite if and only if

$$x^TDx < 0 \text{ implies } Dx > 0 \text{ or } Dx < 0.$$  

It is evident that positive semidefinite quadratic forms are strictly positive subdefinite (by default), and strictly positive subdefinite quadratic forms are positive subdefinite. Thus, in order to exclude the positive semidefinite quadratic forms, Martos inserts the word “merely” before “positive subdefinite.”

A criterion given below for positive subdefiniteness will refer to the number of negative eigenvalues of the matrix $D$. In the rest of the paper, whenever it is stated that a matrix $D$ has exactly one negative eigenvalue, this means that the matrix has exactly one negative eigenvalue counting multiplicities.

3. CRITERIA FOR MERE POSITIVE SUBDEFINITENESS

This section is a summary of earlier characterizations given by Martos in [6] and by Cottle and the author in [3].
Theorem 3.1 [3, Theorem 4.1]. The real symmetric matrix $D$ is merely positive subdefinite if and only if

(i) $D < 0$ and $D \neq 0$,
(ii) $D$ has exactly one negative eigenvalue.

Notice that the necessity of Theorem 3.1 is proved by Martos in [6, Theorem 1]. The following result characterizes seminegative matrices having exactly one negative eigenvalue. Relying on this result, an equivalent criterion for mere positive subdefiniteness can be stated.

Theorem 3.2 [3, Theorem 4.2]. Let $D$ be a real symmetric seminegative matrix. $D$ has exactly one negative eigenvalue if and only if $D$ has nonpositive principal minors.

Corollary 3.3. The real symmetric matrix $D$ is merely positive subdefinite if and only if

(i) $D < 0$ and $D \neq 0$,
(ii) $D$ has nonpositive principal minors.

For strict mere positive subdefiniteness a more efficient sufficiency test exists. Notice the parallel between this result and the criteria to verify positive definiteness.

Theorem 3.4 [3, Theorem 4.3]. If $D$ is a real symmetric seminegative matrix with negative leading principal minors, then $D$ is strictly merely positive subdefinite.

Finally, Martos gives the following criterion for strict mere positive subdefiniteness.

Theorem 3.5 [6, Theorem 2]. A merely positive subdefinite matrix is strictly merely positive subdefinite if and only if it does not contain a row (or column) of zeros.

4. PROPERTIES

This section includes a series of results establishing, for merely positive subdefinite matrices, properties like those for positive semidefinite and totally nonnegative matrices.
The first result is shown in [4] by Cottle and the author for merely positive subdefinite matrices bordered by a vector, but it can be formulated as follows.

**Theorem 4.1 [4, Lemma].** Let the real symmetric matrix $D$ of order $n$ be merely positive subdefinite. Let $d_i = [d_{i,1}, ..., d_{i,i-1}, d_{i+1,i}, ..., d_{i,n}]^T \in E^{n-1}$, and $D_i$ be the square matrix of order $n-1$ obtained by deleting the $i$th row and the $i$th column of $D$. If $d_{ii} = 0$, $d_i \neq 0$, and $D_i \neq 0$, then for any $v \in E^{n-1}$

$$v^T D_i v < 0 \implies d_i^T v \neq 0.$$

### 4.1 Irreducibility

A real symmetric matrix $D$ is **irreducible** if no permutation matrix $Q$ exists such that

$$QDQ^T = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

where $D_1$ and $D_2$ are principal submatrices of $D$.

**Theorem 4.2.** Let the real symmetric matrix $D$ be merely positive subdefinite. The matrix $D$ cannot be reduced to

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

where $D_1 \neq 0$ and $D_2 \neq 0$.

**Proof.** Clearly the principal submatrices of any positive subdefinite matrix are positive subdefinite. While the inheritance of mere positive subdefiniteness is false, we can assert that, if $D$ is merely positive subdefinite, its principal submatrices are nonpositive and positive subdefinite. If $A$ is a principal submatrix of $D$, then either $A$ is positive semidefinite, in which case $A = 0$ and $\det A = 0$, or $A$ is merely positive subdefinite, in which case $\det A < 0$. Hence, if $D$ is reducible to

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$
where \( D_1 \neq 0 \) and \( D_2 \neq 0 \), then it follows that there exist submatrices \( \overline{D}_1 \) and \( \overline{D}_2 \) of \( D_1 \) and \( D_2 \), respectively, such that \( \det \overline{D}_1 < 0 \) and \( \det \overline{D}_2 < 0 \). But

\[
\overline{D} = \begin{bmatrix}
    D_1 & 0 \\
    0 & D_2
\end{bmatrix}
\]

is a principal submatrix of \( D \), and

\[
\det \overline{D} = \det \overline{D}_1 \det \overline{D}_2 > 0.
\]

This contradiction completes the proof.

\( \square \)

**4.2. Distribution of zeros in a merely positive subdefinite matrix**

In this section, the distribution of zeros in a merely positive subdefinite matrix is analyzed.

**Theorem 4.3.** Let \( D \) be a merely positive subdefinite matrix of order \( n > 2 \). If \( d_{ii} = 0 \) and \( i \neq j \), then \( d_{ii} = 0 \) or \( d_{jj} = 0 \).

**Proof.**

\[
\begin{bmatrix}
    d_{ii} & d_{ij} \\
    d_{ji} & d_{jj}
\end{bmatrix} = \begin{bmatrix}
    d_{ii} & 0 \\
    0 & d_{jj}
\end{bmatrix}
\]

is a principal submatrix of \( D \). The determinant of this matrix is equal to \( d_{ii} d_{jj} > 0 \), since \( d_{ii} < 0 \) and \( d_{jj} < 0 \). Thus \( d_{ii} = 0 \) or \( d_{jj} = 0 \), because otherwise Corollary 3.3 implies that \( D \) is not merely positive subdefinite.

\( \square \)

Notice the parallel between this result and the following property for positive semidefinite matrices shown in [2, Theorem 3]: if the real square matrix \( A \) of order 2 is positive semidefinite, then \( a_{11} = 0 \) implies that \( a_{12} + a_{21} = 0 \). Notice also that Theorem 4.3 is a special case of Theorem 4.2.

The following result shown by Cottle and the author is used to further characterize the distribution of zeros in a merely positive subdefinite matrix.

**Theorem 4.4 [3, Corollary 5.1].** If any row in a nonzero principal submatrix of a merely positive subdefinite matrix equals zero, then the corresponding row of the entire matrix equals zero.
**Corollary 4.5.** Let $D$ be a merely positive subdefinite matrix of order $n > 2$. If $d_{ii} = 0$ and $i \neq j$, then either $d_{ii} = d_{jj} = 0$ or $d_{ii} \neq 0$ and all the entries of the $i$th (or $j$th) row and column are equal to zero.

This corollary gives a condition for a row (column) of a merely positive subdefinite matrix to be equal to zero. The study is pursued further to characterize submatrices that have all their entries equal to zero. To ease the notation, the matrix $D$ is partitioned as follows:

$$D = \begin{bmatrix} D_1 & D_3 & \tilde{d}_n \\ D_3^T & D_2 & \tilde{d}_n \\ \tilde{d}_n & \tilde{d}_n & d_{nn} \end{bmatrix}.$$

Notice that the results shown using the last row can be generalized to any row.

**Theorem 4.6.** Let $D$ be a merely positive subdefinite matrix of order $n$, and $D_1$ and $D_2$ be submatrices of order $n_1$ and $n_2$, respectively ($n_1 + n_2 = n - 1$). If $d_{nn} = 0$, $\tilde{d}_n = 0$, $\tilde{d}_n < 0$, and $n_1 > 0$, then $D_2 = 0$.

**Proof.** Under the hypotheses of the theorem, the matrix $D$ reduces to

$$D = \begin{bmatrix} D_1 & D_3 & \tilde{d}_n \\ D_3^T & D_2 & 0 \\ \tilde{d}_n & 0 & 0 \end{bmatrix}.$$

For each diagonal element $d_{n_i + i, n_i + i}$ of $D_2$, $i = 1, 2, \ldots, n_2$, the principal submatrix of $D$

$$\tilde{D}_i = \begin{bmatrix} d_{11} & d_{1n_i + i} & d_{1n} \\ d_{1n_i + i} & d_{n_i + i, n_i + i} & 0 \\ d_{1n} & 0 & 0 \end{bmatrix}$$

has a determinant equal to

$$\det \tilde{D}_i = -d_{1n_i}^2 \cdot d_{n_i + i, n_i + i}.$$
But $\det \overline{D}_i < 0$, and $d_{n_1+i,n_1+i} < 0$, $d_{in} < 0$. Hence $d_{n_1+i,n_1+i} = 0$.

For nondiagonal entries, $i \neq j$, $1 < i, j < n_2$, the principal submatrix of $D$

$$
\overline{D}_q = \begin{bmatrix}
 d_{11} & d_{1n_1+i} & d_{1n_1+i} & d_{1n} \\
 d_{1n_1+i} & 0 & d_{n_1+i,n_1+i} & 0 \\
 d_{1n_1+i} & d_{n_1+i,n_1+i} & 0 & 0 \\
 d_{1n} & 0 & 0 & 0
\end{bmatrix}
$$

has a determinant equal to

$$
\det \overline{D}_q = d_{1n}^2 d_{n_1+i,n_1+i}^2.
$$

Thus $\det \overline{D}_q < 0$, $d_{n_1+i,n_1+i} < 0$, $d_{1n} < 0$ imply that $d_{n_1+i,n_1+i} = 0$.

Hence $D_2 = 0$.

Corollary 4.5 and Theorem 4.6 indicate that, whenever a diagonal entry $d_i$ of $D$ is equal to zero, either the $i$th row and the $i$th column are equal to zero or the submatrix composed of the rows and columns of $D$ corresponding to the zero elements of the $i$th row has all its entries equal to zero.

4.3 Singularity

In [5, pp. 98–100], Gantmacher studies the inheritance of singularity for totally nonnegative square matrices. Recall that a totally nonnegative square matrix has all its minors nonnegative. It is shown that if $A$ is a singular principal submatrix of a totally nonnegative square matrix $A$, any submatrix $B$ of $A$ having $A$ as a principal submatrix is singular. A similar result holds for merely positive subdefinite matrices.

**Theorem 4.7.** Let $D$ be a merely positive subdefinite matrix of order $n > 2$. If some principal submatrix of order $n - 1$ is singular, then $D$ is singular.

**Proof.** With no loss of generality, assume that $D$ is partitioned as

$$
D = \begin{bmatrix}
 \overline{D} & d_n \\
 d_n^T & d_{nn}
\end{bmatrix},
$$

and assume that $\overline{D}$ is singular (i.e. $\det \overline{D} = 0$).
If \( \overline{D} \) is positive semidefinite, then \( \overline{D} = 0 \) by the inheritance of positive subdefiniteness (see proof of Theorem 4.2). Hence \( \det D = 0 \), and \( D \) is singular.

Suppose that \( \overline{D} \) is merely positive subdefinite. Assume, for contradiction, that \( D \) is nonsingular. Let \( a \in E^{n-1} \) be such that \( a \neq 0 \) and \( \overline{D}a = 0 \). Notice that such an \( a \) exists because \( \overline{D} \) is singular. Consider the vector \( [a^T, 0]^T \in E^n \):

\[
\begin{bmatrix}
\overline{D} & d_n \\
d_n^T & d_{nn}
\end{bmatrix}
\begin{bmatrix}
a \\
0
\end{bmatrix} =
\begin{bmatrix}
\overline{D}a \\
d_n^Ta
\end{bmatrix} =
\begin{bmatrix}
0 \\
d_n^Ta
\end{bmatrix}.
\]

Since \( D \) is nonsingular, it follows that \( d_n^Ta \neq 0 \).

\( D \) merely positive subdefinite implies the existence of a vector \( b \in E^{n-1} \) such that \( b^TDb < 0 \). Consider the vector \( [(b + \alpha a)^T, 0]^T \in E^n \). It follows that

\[
\begin{bmatrix}
(b + \alpha a)^T, 0
\end{bmatrix}
\begin{bmatrix}
\overline{D} & d_n \\
d_n^T & d_{nn}
\end{bmatrix}
\begin{bmatrix}
(b + \alpha a) \\
0
\end{bmatrix} = b^T\overline{D}b + 2ab^T\overline{D}a + \alpha^2a^T\overline{D}a
\]

\[
= b^T\overline{D}b < 0,
\]

\[
\begin{bmatrix}
\overline{D} & d_n \\
d_n^T & d_{nn}
\end{bmatrix}
\begin{bmatrix}
b + \alpha a \\
0
\end{bmatrix} =
\begin{bmatrix}
\overline{D}b + a\overline{D}a \\
d_n^Tb + \alpha d_n^Ta
\end{bmatrix} =
\begin{bmatrix}
\overline{D}b \\
d_n^Tb + \alpha d_n^Ta
\end{bmatrix}.
\]

If \( d_n^Ta \neq 0 \), by a suitable choice of \( \alpha \), the scalar

\[
d_n^Tb + \alpha d_n^Ta
\]

can be made to have arbitrary sign and, in particular, it can be made to be opposite to that of \( \overline{D}b \). This contradicts the definition of positive subdefiniteness for \( D \).

**Corollary 4.8.** Let \( D \) be a merely positive subdefinite matrix of order \( n > 2 \). If \( \det D < 0 \), then all the principal minors of order \( k > 2 \) are negative.

The converse of Corollary 4.8 is not true. Indeed, the matrix

\[
\mathcal{D} =
\begin{bmatrix}
0 & -1 & -1 \\
-1 & 0 & -1 \\
-1 & -1 & -2
\end{bmatrix}
\]
is merely positive subdefinite with all its principal minors of order 2 negative. But \( \det D = 0 \).

5. SUMMATION OF TWO POSITIVE SUBDEFINITE MATRICES

Since criteria for mere positive subdefiniteness are given in terms of eigenvalues or in terms of principal minors, it is difficult to study the properties of the sum of two such matrices. An interesting, but very hard, problem would be to determine necessary and sufficient conditions for the positive subdefiniteness of the sum of two positive subdefinite matrices. Unfortunately, only a partial solution to this problem is known to the author.

It is well known that the sum of two positive semidefinite matrices has the same property. The same result is not true in general for positive subdefinite matrices. Indeed, consider \( D_1 \neq 0 \) and \( D_2 \neq 0 \), two merely positive subdefinite matrices of order \( n_1 \) and \( n_2 \), respectively. The matrices

\[
\begin{bmatrix}
  D_1 & 0 \\
  0 & 0 
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  0 & 0 \\
  0 & D_2 
\end{bmatrix}
\]

are merely positive subdefinite matrices of order \( n_1 + n_2 \), but their sum

\[
\begin{bmatrix}
  D_1 & 0 \\
  0 & D_2 
\end{bmatrix}
\]

is not merely positive subdefinite according to Theorem 4.2.

Two different sets of sufficient conditions for the sum of two positive subdefinite matrices to have the same property are derived using two different approaches.

Notice that in general, if \( A \) and \( B \) are merely positive subdefinite matrices of order \( n \), then \( D = A + B \) is a seminegative matrix irreducible to

\[
\begin{bmatrix}
  D_1 & 0 \\
  0 & D_2 
\end{bmatrix}
\]

where \( D_1 \neq 0 \) and \( D_2 \neq 0 \). This follows from Theorem 4.2 applied to \( A \) and \( B \) and because \( A < 0 \) and \( B < 0 \). Hence, from Perron-Frobenius theory [9, Theorem 2.1], \( D \) has at least one negative eigenvalue. The difficulty is to determine necessary and sufficient conditions on \( A \) and \( B \) for \( D \) to have a unique negative eigenvalue.
The first set of sufficient conditions relies on the following well-known result shown in Ralston’s book [8].

**Theorem 5.1** [8, Theorem 10.14]. Let \( \lambda \) and \( \mathbf{x} \) be an eigenvalue and a corresponding eigenvector of a square matrix \( D \). Let \( \mathbf{y} \) be any vector such that \( \mathbf{x}^T\mathbf{y} = 1 \). Then the matrix \( D - \lambda \mathbf{xy}^T \) has the same eigenvalues as \( D \) except that \( \lambda \) is replaced by 0.

**Theorem 5.2.** Let \( A \) and \( B \) be two real symmetric matrices of order \( n \) having a unique negative eigenvalue. Let \( \alpha \) and \( a \) be this eigenvalue and the corresponding normalized eigenvector of \( A \), and let \( \beta \) and \( b \) be this eigenvalue and the corresponding normalized eigenvector of \( B \). If \( \alpha = b \), then the matrix \( A + B \) has a unique negative eigenvalue \( \alpha + \beta \).

*Proof.* Since \( Aa = \alpha a \) and \( Ba = \beta a \), it follows that \( (A + B)a = (\alpha + \beta)a \), and consequently \( \alpha + \beta < 0 \) is an eigenvalue of \( A + B \).

Furthermore, Theorem 5.1 implies that \( A - \alpha aa^T \) and \( B - \beta ba^T \) are positive semidefinite matrices. Thus \( (A + B) - (\alpha + \beta)aa^T = (A - \alpha aa^T) + (B - \beta ba^T) \) is positive semidefinite, because it is equal to the sum of two positive semidefinite matrices. Hence \( \alpha + \beta \) is the unique negative eigenvalue of \( A + B \).

A similar argument is used to show the following result.

**Theorem 5.3.** Let \( A \) and \( B \) be two real symmetric matrices of order \( n \). Suppose that \( A \) is positive semidefinite and that \( B \) has exactly one negative eigenvalue \( \beta \) with the corresponding normalized eigenvector \( a \). If \( a \) is an eigenvector of \( A \) corresponding to the eigenvalue \( \alpha \), then

(i) \( \alpha < |\beta| \) implies that \( A + B \) has exactly one negative eigenvalue \( \alpha + \beta \),

(ii) \( \alpha > |\beta| \) implies that \( A + B \) is positive semidefinite.

**Corollary 5.4.**

(i) Let \( A \) and \( B \) be two merely positive subdefinite matrices of order \( n \). If \( A \) and \( B \) have the same normalized eigenvector corresponding to their respective negative eigenvalue, then \( A + B \) is merely positive subdefinite.

(ii) Let \( A \) and \( B \) be two real symmetric matrices of order \( n \), positive semidefinite and merely positive subdefinite, respectively. Suppose that the normalized eigenvector \( a \) corresponding to the negative eigenvalue \( \beta \) of \( B \) is an eigenvector corresponding to an eigenvalue \( \alpha \) of \( A \). If \( \alpha < |\beta| \) and \( A + B < 0 \), then \( A + B \) is merely positive subdefinite, and if \( \alpha > |\beta| \), then \( A + B \) is positive semidefinite.
Unfortunately these conditions are not necessary in general. Indeed, consider the following merely positive subdefinite matrices:

\[
A = \begin{bmatrix}
-1 & -1 \\
-1 & 0
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
-1 & -1 \\
-1 & -1
\end{bmatrix}.
\]

The eigenvalues of \(A\) are \((-1 + \sqrt{5})/2\) and \((-1 - \sqrt{5})/2\), and the eigenvalues of \(B\) are 0 and \(-2\). Hence \(\alpha = (-1 + \sqrt{5})/2\) and \(\beta = -2\), and

\[
\sqrt{\frac{\sqrt{5}}{2} + \frac{5}{2}} \quad a = \begin{bmatrix}
\frac{1 + \sqrt{5}}{2} \\
\frac{1}{1}
\end{bmatrix} \quad \text{and} \quad \sqrt{2} \quad b = \begin{bmatrix}
1 \\
1
\end{bmatrix}.
\]

Thus the conditions in Corollary 5.4 (i) are not satisfied, but the matrix

\[
A + B = \begin{bmatrix}
-2 & -2 \\
-2 & -1
\end{bmatrix}
\]

is merely positive subdefinite.

The second set of sufficient conditions relies on results in Wilkinson’s book [10], on the relations between the eigenvalues of a matrix \(D = A + B\) and the eigenvalues of \(A\) and \(B\).

Denote by \(\alpha_1 > \alpha_2 > \cdots > \alpha_n\) the eigenvalues of \(A\), by \(\beta_1 > \beta_2 > \cdots > \beta_n\) the eigenvalues of \(B\), and by \(\delta_1 > \delta_2 > \cdots > \delta_n\) the eigenvalues of \(D\). Since \(A\) and \(B\) are merely positive subdefinite,

\[
\alpha_1 > \alpha_2 > \cdots > \alpha_{n-1} > 0 > \alpha_n,
\]

\[
\beta_1 > \beta_2 > \cdots > \beta_{n-1} > 0 > \beta_n.
\]

In [10, p. 102] it is shown that

\[
\delta_{n-1} > \alpha_{n-1} + \beta_n,
\]

\[
\delta_{n-1} > \alpha_n + \beta_{n-1}.
\]

Thus \(\delta_{n-1} > \max\{\alpha_{n-1} + \beta_n, \beta_{n-1} + \alpha_n\}\). Hence, since \(\delta_n < 0\) by Perron-Frobenius theory [9], it follows that if

\[
\max\{\alpha_{n-1} + \beta_n, \beta_{n-1} + \alpha_n\} > 0,
\]

then \(D\) is merely positive subdefinite.
The preceding example illustrates that this condition is not necessary. Indeed $\alpha_1 = (-1 + \sqrt{5})/2$, $\alpha_2 = (-1 - \sqrt{5})/2$, $\beta_1 = 0$, and $\beta_2 = -2$ imply that

$$\max \{ \alpha_2 + \beta_1, \alpha_1 + \beta_2 \} = \max \left\{ \frac{-1 - \sqrt{5}}{2}, \frac{-1 + \sqrt{5}}{2} - 2 \right\} < 0.$$ 

If $A$ is positive semidefinite and $B$ is merely positive subdefinite, then

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0,$$

$$\beta_1 \geq \beta_2 \geq \cdots \beta_{n-1} > 0 > \beta_n.$$

In [10, p. 102], it is shown that

$$\delta_n \geq \alpha_n + \beta_n,$$

and thus, if $\alpha_n + \beta_n > 0$, $D = A + B$ is positive semidefinite. On the other hand, if $\alpha_n + \beta_n < 0$, if $D = A + B < 0$, and if $\max \{ \alpha_{n-1} + \beta_n, \alpha_n + \beta_{n-1} \} > 0$, then $D$ is merely positive subdefinite.

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Received 25 July 1979; revised 29 August 1979