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Spaces of embeddings of compact polyhedra into 2-manifolds

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Abstract

Let M be a PL 2-manifold and X be a compact subpolyhedron of M and let $\mathcal{E}(X, M)$ denote the space of embeddings of X into M with the compact-open topology. In this paper we study an extension property of embeddings of X into M and show that the restriction map from the homeomorphism group of M to $\mathcal{E}(X, M)$ is a principal bundle. As an application we show that if M is a Euclidean PL 2-manifold and $\dim X \geqslant 1$ then the triple $(\mathcal{E}(X, M), \mathcal{E}^{\mathrm{LIP}}(X, M), \mathcal{E}^{\mathrm{PL}}(X, M))$ is an (s, Σ, σ)-manifold, where $\mathcal{E}_K^{\mathrm{LIP}}(X, M)$ and $\mathcal{E}_K^{\mathrm{PL}}(X, M)$ denote the subspaces of Lipschitz and PL embeddings.

Keywords: Embeddings; Homeomorphism groups; 2-manifolds; Infinite-dimensional manifolds

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1. Introduction

The investigation of the topology of the homeomorphism groups of compact 2-manifolds [8,9,11] included the use of conformal mappings in order to develop some extension properties of embeddings of a circle into an annulus and proper embeddings of an arc into a disk. In this paper we establish a similar extension property of embeddings of trees into a disk. Since every graph can be decomposed into ads (cones over finite points) and arcs connecting them, this implies an extension property of embeddings of compact polyhedra into 2-manifolds.

Suppose M is a PL 2-manifold and $K \subset X$ are compact subpolyhedra of M. Let $\mathcal{E}_K(X, M)$ denote the space of embeddings $f : X \hookrightarrow M$ with $f|_K = id$, equipped with the compact-open topology. An embedding $f : X \hookrightarrow M$ is said to be proper if $f(X \cap \partial M) \subset \partial M$ and $f(X \cap \operatorname{Int} M) \subset \operatorname{Int} M$. Let $\mathcal{E}_K(X, M)^*$ denote the subspace of proper embeddings in $\mathcal{E}_K(X, M)$, and let $\mathcal{E}_K(X, M)^*_0$ denote the connected component of the inclusion $i_X : X \subset M$ in $\mathcal{E}_K(X, M)^*$. Our result is summarized in the next statement.

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Theorem 1.1. *For every $f \in \mathcal{E}_K(X, M)^*$ and every neighborhood U of $f(X)$ in M, there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{E}_K(X, M)^*$ and a map $\varphi : \mathcal{U} \to \mathcal{H}_{K \cup (M \setminus U)}(M)_0$ such that $\varphi(g) f = g$ for each $g \in \mathcal{U}$ and $\varphi(f) = id_M$.*

Let $\mathcal{H}_X(M)$ denote the group of homeomorphisms h of M onto itself with $h|_X = id$, equipped with the compact-open topology. Let $\mathcal{H}(M)_0$ denote the identity component of $\mathcal{H}(M)$. In the study of the homotopy type of $\mathcal{H}_X(M)_0$ and $\mathcal{E}_K(X, M)_0$ the restriction map

$$\pi : \mathcal{H}_K(M)_0 \to \mathcal{E}_K(X, M)_0^*$$

plays an important role (cf. [3]). The above extension maps yield local sections of this restriction map.

Corollary 1.1. *For any open neighborhood U of X in M, the restriction map*

$$\pi : \mathcal{H}_{K \cup (M \setminus U)}(M)_0 \to \mathcal{E}_K(X, U)_0^*, \quad \pi(f) = f|_X,$$

is a principal bundle with the fiber $\mathcal{G} \equiv \mathcal{H}_{K \cup (M \setminus U)}(M)_0 \cap \mathcal{H}_X(M)$, where the subgroup $\mathcal{G}$ acts on $\mathcal{H}_{K \cup (M \setminus U)}(M)_0$ by right composition.

As an application of Extension Theorem 1.1 we can study the embedding space $\mathcal{E}_K(X, M)$ from the viewpoint of infinite dimensional topology (see §4 for basic terminologies). In [16] Sakai and Wong showed the (s, Σ, σ)-stability property of triples of spaces of embeddings of compact polyhedra and subspaces of Lipschitz and PL embeddings, and posed the question whether these triples are (s, Σ, σ)-manifolds. The 1-dimensional case is studied in [15]. In this paper we will consider the 2-dimensional case and answer the question affirmatively.

Let $\mathcal{E}_K^{\mathrm{PL}}(X, M)$ denote the subspace of PL-embeddings. When M is a Euclidean PL 2-manifold, let $\mathcal{E}_K^{\mathrm{LIP}}(X, M)$ denote the subspace of Lipschitz embeddings. The Extension Theorem enables us to reduce the ANR-property and the homotopy negligibility of embedding spaces to the ones of the homeomorphism groups. Using the characterization of (s, Σ, σ)-manifold [20] we have the following result.

Theorem 1.2. *Suppose M is a Euclidean* PL *2-manifold and $K \subset X$ are compact subpolyhedra of M. If $\dim(X \setminus K) \geqslant 1$, then the triple $(\mathcal{E}_K(X, M),\ \mathcal{E}_K^{\mathrm{LIP}}(X, M),\ \mathcal{E}_K^{\mathrm{PL}}(X, M))$ is an (s, Σ, σ)-manifold.*

Further applications of Corollary 1.1 to the study of $\mathcal{H}_X(M)$ and $\mathcal{E}_K(X, M)$ will be given in a succeeding paper. We conclude this section with some remarks. In Section 2 we study the extension property of embeddings of a tree into a disk. Section 3 contains the proofs of Theorem 1.1 and Corollary 1.1. The final Section 4 contains the proof of Theorem 1.2. Throughout the paper spaces are assumed to be separable and metrizable. A Euclidean PL n-manifold is a subpolyhedron of some Euclidean space $\mathbb{R}^m$ which is a PL-manifold with respect to the induced triangulation and is equipped with the metric induced from the standard metric of $\mathbb{R}^m$. When M is an orientable manifold, $\mathcal{H}_+(M)$ denote the subspace of orientation preserving homeomorphisms of M. Finally $i_X : X \subset Y$ denotes the inclusion map.

2. Extension property of embeddings of trees into disks

In this section we will study some extension properties of embeddings of trees into disks. The proper embedding case is a consequence of a direct application of the conformal mapping theorem on simply connected domains (cf. [11]). Thus our interest is in the case of embeddings into the interior of a disk, where we need to apply the conformal mapping theorem on a doubly connected domain one boundary circle of which is collapsed to a tree.

Throughout the section we will work on the plane $\mathbb{C}$ $(= \mathbb{R}^2)$ and use the following notations: For $z \in \mathbb{C}$ and $r > 0$, $D(z,r) = \{x \in \mathbb{C}\colon |z-x| \leqslant r\}$, $O(z,r) = \{x \in \mathbb{C}\colon |z-x| < r\}$, $C(z,r) = \{x \in \mathbb{C}\colon |z-x| = r\}$, and $D(r) = D(0,r)$, $O(r) = O(0,r)$, $C(r) = C(0,r)$. For $0 < r < s$, $A(r,s) = \{x \in \mathbb{C}\colon r \leqslant |x| \leqslant s\}$. For $A \subset \mathbb{C}$ and $\varepsilon > 0$, $O(A,\varepsilon) = \{x \in \mathbb{C}\colon |x-y| < \varepsilon$ for some $y \in A\}$ (the ε-neighborhood of A).

2.1. Proper embeddings of trees into a disk

First we recall the conformal mapping theorem on simply connected domains normalized by the three points boundary condition. Consider the family $\mathcal{J} = \{(J, w_1, w_2, w_3)\colon J$ is a simple closed curve in $\mathbb{C}$ and $w_1, w_2, w_3 \in J$ are three distinct points lying on J in counterclockwise order (with respect to the orientation induced from $\mathbb{C}$)$\}$. A sequence $\{A_n\}_{n\geqslant 1}$ of subsets of $\mathbb{C}$ is said to be uniformly locally connected if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $n \geqslant 1$ and any $x, y \in A_n$ with $|x-y| < \delta$ there exists an arc α in A_n with connecting x and y and $\operatorname{diam}\alpha < \varepsilon$.

Fact 2.1. *Let $z_1, z_2, z_3 \in C(1)$ be the fixed three points lying on $C(1)$ in counterclockwise order.*

(i) ([14, Corollary 2.7]) *For every $(J, w_1, w_2, w_3) \in \mathcal{J}$ there exists a unique $\varphi = \varphi(J, w_1, w_2, w_3) \in \mathcal{E}(D(1), \mathbb{C})$ such that φ maps $O(1)$ conformally onto the interior of J, $\varphi(C(1)) = J$ and $\varphi(z_i) = w_i$ $(i = 1,2,3)$.*

(ii) *If a sequence $(J_n, w_1(n), w_2(n), w_3(n))$ $(n \geqslant 1)$ converges to (J, w_1, w_2, w_3) in the following sense, then $\varphi(J_n, w_1(n), w_2(n), w_3(n))$ converges uniformly to $\varphi(J, w_1, w_2, w_3)$:*

  $(*)$ *J_n converges to J with respect to the Hausdorff metric, $\{J_n\}$ is uniformly locally connected, and $w_i(n) \to w_i$ $(i = 1,2,3)$.*

For the statement (ii) we refer to the proof of [14, Theorem 2.1, Proposition 2.3] (also see the proof of Lemma 2.3).

Lemma 2.1. *Suppose D is a disk and $C = \partial D$.*

(i) (cf. [11, Lemma 3]) *There exists a map $\Phi : \mathcal{E}(C, \mathbb{C}) \to \mathcal{E}(D, \mathbb{C})$ such that $\Phi(f)|_C = f$ $(f \in \mathcal{E}(C, \mathbb{C}))$.*

(ii) (cf. [11, Lemma 5]) *Suppose T is a tree embedded into a disk D such that $T \cap C$ coincides with the set of terminal vertices of T. Then there exists a* *map* $\Psi : \mathcal{E}_{T\cap C}(T, D)^* \to \mathcal{H}_\partial(D)$ *such that* $\Psi(f)|_T = f$ $(f \in \mathcal{E}_{T\cap C}(T, D)^*)$ *and* $\Psi(i_T) = id_D$.

Proof. We may assume that $D = D(1)$. Let $z_1, z_2, z_3 \in C(1)$ be as in Fact 2.1.

(i) Let $\mathcal{E}^{\pm} = \{f \in \mathcal{E}(C(1), \mathbb{C})\colon f$ preserves (reverses) orientation$\}$. If $f \in \mathcal{E}^+(C(1), \mathbb{C})$, then $(f(C(1)), f(z_1), f(z_2), f(z_3)) \in \mathcal{J}$ and by Fact 2.1 we obtain $\varphi(f) = \varphi(f(C(1)), f(z_1), f(z_2), f(z_3)) \in \mathcal{E}(D(1), \mathbb{C})$. If $f_n \to f$ in $\mathcal{E}^+$, then $(f(C(1)), f(z_1), f(z_2), f(z_3))$ converges to $(f(C(1)), f(z_1), f(z_2), f(z_3))$ in the sense $(*)$ of Fact 2.1(ii). Hence the map $\varphi : \mathcal{E}^+ \to \mathcal{E}(D(1), \mathbb{C})$ is continuous. Let $c : \mathcal{H}(C(1)) \to \mathcal{H}(D(1))$ be the cone extension map and let $\gamma : \mathbb{C} \to \mathbb{C}$ be the reflection $\gamma(z) = \overline{z}$. Then the extension map Φ is defined by $\Phi(f) = \varphi(f)c(\varphi(f)^{-1}f)$ for $f \in \mathcal{E}^+$ and $\Phi(f) = \gamma\Phi(\gamma f)$ for $f \in \mathcal{E}^-$.

(ii) The tree T separates the disk $D(1)$ into subdisks D_i. By (i) each disk D_i admits an extension map $\psi_i : \mathcal{E}(\partial D_i, \mathbb{C}) \to \mathcal{E}(D_i, \mathbb{C})$. Every $f \in \mathcal{E}_{T\cap C(1)}(T, D(1))^*$ can be extended to $\overline{f} \in \mathcal{E}_{C(1)}(T \cup C(1), D(1))$. The required extension map Ψ is defined by $\Psi(f)|_{D_i} = \psi_i(\overline{f}|_{\partial D_i})$. To achieve $\Psi(i_T) = id_D$, replace $\Psi(f)$ by $\Psi(f)\Psi(i_T)^{-1}$. $\square$

In the proof of Theorem 1.1 we will apply the statement (ii) to the case where T is an arc.

2.2. Embeddings of trees into the interior of a disk

Suppose T is a finite tree ($\neq 1$ pt) embedded into $O(2)$. We will use the following notation: For $a, b \in T$, let $E_T(a, b)$ denote the unique arc in T connecting a and b. Let $\{v_1, \ldots, v_n\}$ be the collection of end vertices of T. We can choose disjoint arcs $\alpha_1, \ldots, \alpha_n$ in $D(2)$ such that each α_i connects v_i with a point a_i in $C(2)$ and $\operatorname{Int}\alpha_i \subset O(2) \setminus T$. We can arrange the ordering of v_i's so that $a_1, \ldots, a_n$ lie on $C(2)$ in counterclockwise order. The labeling is unique up to the cyclic permutations. Note that T does not meet the interior of the disk surrounded by the simple closed curve $\alpha_i \cup E_T(v_i, v_{i+1}) \cup \alpha_{i+1} \cup a_i a_{i+1}$, where $v_{n+1} = v_1$ and $a_{n+1} = a_1$.

Lemma 2.2 [7, Ch. V, §1, Theorems 1.1, 1.2]. *There exists a unique real number r, $0 < r < 2$, and a unique map $h : A(r, 2) \to D(2)$ such that $h : \operatorname{Int} A(r, 2) \to O(2) \setminus T$ is a conformal map and $h(2) = 2$. Furthermore, the map h satisfies the following conditions*:

(i) *h maps $C(2)$ homeomorphically onto $C(2)$,*

(ii) *$h(C(r)) = T$ and there exists a unique collection of points $\{u_1, \ldots, u_n\}$ lying on $C(r)$ in counterclockwise order such that h maps each circular arc $u_i u_{i+1}$ homeomorphically onto the arc $E_T(v_i, v_{i+1})$, where $u_{n+1} = u_1$.*

We refer to [14, Ch. 2, Theorem 2.1] for the extension to boundary and [14, Ch. 2, §1 Prime End Theorem, §§4, 5] and [7, p. 40] for the correspondence between prime ends and boundary points. Let $\mathcal{E} = \mathcal{E}(T, O(2))$. For each $f \in \mathcal{E}$ the image $f(T)$ is a tree in $O(2)$. Hence by Lemma 2.2 there exists a unique real number r_f, $0 < r_f < 2$, and a unique map $h_f : A(r_f, 2) \to D(2)$ such that $h_f : \operatorname{Int} A(r_f, 2) \to O(2) \setminus f(T)$ is a conformal map and $h_f(2) = 2$. For $0 < r < 2$ let $\varphi_r : A(1, 2) \to A(r, 2)$ denote the radial map defined by $\varphi_r(x) = ((2-r)(|x|-1)+r)x/|x|$, and let $\mathcal{C}(A(1,2), D(2))$ denote the space of continuous maps from $A(1,2)$ to $D(2)$, with the compact-open topology. We have $h_f\varphi_{r_f} \in \mathcal{C}(A(1,2), D(2))$.

Lemma 2.3. *The map* $\Psi : \mathcal{E}(T, O(2)) \to \mathbb{R} \times \mathcal{C}(A(1,2), D(2))$, $\Psi(f) = (r_f, h_f\varphi_{r_f})$, *is continuous.*

This continuity property can be verified using the length distortion under conformal mapping [14, Proposition 2.2]. When L is a rectifiable (possibly open) curve in $\mathbb{R}^2$, we denote the length of L by $\Lambda(L)$.

Proof. Suppose $f_n \to f$ in $\mathcal{E}$. It suffices to show that the sequence $(r_n, h_n\varphi_{r_n}) \equiv (r_{f_n}, h_{f_n}\varphi_{r_{f_n}})$ has a subsequence $(r_{n_k}, h_{n_k}\varphi_{r_{n_k}})$ such that $r_{n_k} \to r_f$ and $h_{n_k}\varphi_{r_{n_k}}$ converges uniformly to $h_f\varphi_{r_f}$.

Let $R_0 > 2$ (= the radius of $D(2)$) and $\varepsilon(\rho) = 2\pi R_0/\sqrt{\log(1/\rho)}$ $(0 < \rho < 1)$.

(i) Passing to a subsequence we may assume $r_n \to r_0$ for some r_0, $0 \leqslant r_0 \leqslant 2$. First we will show that $0 < r_0 < 2$. (a) Suppose $r_0 = 2$. Take ρ, $0 < \rho < 1$, such that $\varepsilon(\rho) < d(f(T), C(2))$. Choose $n \geqslant 1$ such that $\varepsilon(\rho) < d(f_n(T), C(2))$ and $|r_n - r_0| < \rho$. We can apply [14, Proposition 2.2] for any point $c \in C(2)$ (with $R = 2$) to find ρ_0, $\rho < \rho_0 < \sqrt{\rho}$, such that $\Lambda(h_n(L)) < \varepsilon(\rho)$, where L is one of the two arc components of $C(c, \rho_0) \cap A(r_n, 2)$ which connects $C(r_n)$ and $C(2)$. This implies $d(f_n(T), C(2)) < \varepsilon(\rho)$, a contradiction. (b) Suppose $r_0 = 0$. Take ρ, $0 < \rho < 1$, such that $\varepsilon(\rho) < \operatorname{diam} f(T)$. Choose $n \geqslant 1$ such that $\varepsilon(\rho) < \operatorname{diam} f_n(T)$ and $r_n < \rho$. By [14, Proposition 2.2] there exists ρ_0, $\rho < \rho_0 < \sqrt{\rho}$ such that $\Lambda(h_n(C(\rho_0))) < \varepsilon(\rho)$. Since $f_n(T)$ is contained in the interior of the circle $h_n(C(\rho_0))$, we have $\operatorname{diam} f_n(T) < \varepsilon(\rho)$, a contradiction.

(ii) Next we will show that the sequence $h_n : A(r_n, 2) \to D(2)$ $(n \geqslant 1)$ is equicontinuous, that is, for every $\varepsilon > 0$ there exists a $\rho > 0$ such that $|h_n(z) - h_n(w)| < \varepsilon$ for any $n \geqslant 1$ and $z, w \in A(r_n, 2)$ with $|z - w| < \rho$. Let $\varepsilon > 0$ be given. We may assume that $\varepsilon < d(C(2), f_n(T))$ for each $n \geqslant 1$. Since the sequence $C(2)$, $f_n(T)$ $(n \geqslant 1)$ is uniformly locally connected, there exists a δ, $0 < \delta < \varepsilon/2$, such that if $z, w \in f_n(T)$ (respectively $C(2)$) and $|z - w| < \delta$, then there exists an arc A in $f_n(T)$ (respectively $C(2)$) connecting z and w and with $\operatorname{diam} A < \varepsilon/2$. Choose ρ, $0 < \rho < 1$, such that $\varepsilon(\rho) < \delta$ and $2\sqrt{\rho} < 2 - \max_{n \geqslant 0} r_n$. Suppose $z, w \in A(r_n, 2)$ and $|z - w| < \rho$. By [14, Proposition 2.2] (with $c = z$) we have ρ_0, $\rho < \rho_0 < \sqrt{\rho}$, such that $\Lambda(h_n(L)) < \varepsilon(\rho)$, where $L = C(z, \rho_0) \cap A(r_n, 2)$. Since $z, w \in D \equiv D(z, \rho_0) \cap A(r_n, 2)$, it suffices to show that $\operatorname{diam} h_n(D) < \varepsilon$. By the choice of ρ, $D(z, \rho_0)$ meets at most one of $C(2)$ and $C(r_n)$. If $D(z, \rho_0) \subset A(r_n, 2)$ or $D(z, \rho_0) \supset D(0, r_n)$, then $L = C(z, \rho_0)$ and $h_n(D)$ is a disk bounded by $h_n(L)$, so $\operatorname{diam} h_n(D) < \varepsilon(\rho)$. Otherwise, L is an arc connecting two points P, Q with either (a) $P, Q \in C(2)$ or (b) $P, Q \in C(r_n)$. In both cases $|h_n(P) - h_n(Q)| \leqslant \Lambda(h_n(L)) < \delta$, hence by the choice of δ, we have an arc A in $C(2)$ (respectively $f_n(T)$) connecting $h_n(P)$ and $h_n(Q)$ and $\operatorname{diam} A < \varepsilon/2$. In the case (a) $h_n(L)$ separates $D(2)$ into the subdisk $h_n(D)$ and another subdisk. Since $h_n(D) \cap f_n(T) = \emptyset$ and $d(C(2), f_n(T)) > \varepsilon$, the Jordan curve $h_n(L) \cup A$ bounds the disk $h_n(D)$, so $\operatorname{diam} h_n(D) < \varepsilon$. In the case (b) the Jordan curve

$h_n(L) \cup A$ bounds a disk E in $D(2)$ with $\operatorname{diam} E < \varepsilon$. Since $h_n(A(r_n, 2) \setminus (D \cup C(r_n)))$ is contained in the exterior of E and $h_n(\operatorname{Int} D) \cap \partial E = \emptyset$, it follows that $h_n(\operatorname{Int} D) = \operatorname{Int} E \setminus f_n(T)$ so $h_n(D) = E$.

(iii) Since the sup-metric $d(\varphi_{r_n}, \varphi_{r_0}) = |r_n - r_0| \to 0$ $(n \to \infty)$, the sequence $h_n\varphi_{r_n}$ $(n \geqslant 1)$ is also equicontinuous. By the Ascoli-Arzelà theorem, passing to a subsequence, we may assume that $h_n\varphi_{r_n}$ converges to a map $h_0' : A(1,2) \to D(2)$. Set $h_0 = h_0'\varphi_{r_0}^{-1}$. Then $h_0(A(r_0, 2)) = D(2)$, $h_0(C(2)) = C(2)$, $h_0(C(r_0)) = f(T)$ and $h_0(2) = 2$. Since the sequence of univalent analytic maps $h_n : \operatorname{Int} A(r_n, 2) \to \mathbb{C}$ converges weakly uniformly to the map $h_0 : \operatorname{Int} A(r_0, 2) \to \mathbb{C}$ (i.e., for each compact subset K of $\operatorname{Int} A(r_0, 2)$, $h_n|_K$ (n large) converges uniformly to $h_0|_K$) and h_0 is not constant, $h_0 : \operatorname{Int} A(r_0, 2) \to \mathbb{C}$ is also a univalent analytic map [19, Ch. 3, Theorem 3.3]. It follows that $h_0(\operatorname{Int} A(r_0, 2)) = O(2) \setminus f(T)$ and $h_0 : \operatorname{Int} A(r_0, 2) \to O(2) \setminus f(T)$ is a conformal map, so $(r_0, h_0) = (r_f, h_f)$ by the uniqueness in Lemma 2.2. This completes the proof. □

Let $i : T \hookrightarrow O(2)$ denote the inclusion and set

$$\mathcal{E}_+ \equiv \mathcal{E}_+\big(T, O(2)\big) = \big\{ f \in \mathcal{E}\text{: there exists an } h \in \mathcal{H}_+(D(2)) \text{ with } hi = f \big\},$$

which is an open neighborhood of i in $\mathcal{E}$.

Proposition 2.1.

(i) *There exists a canonical map $\Phi = \Phi_T : \mathcal{E}_+ \to \mathcal{H}_+(D(2))$ such that $\Phi(f)i = f$ $(f \in \mathcal{E}_+)$ and $\Phi(i) = id$.*

(ii) *There exists a neighborhood $\mathcal{U}$ of i in $\mathcal{E}$ and a map $\varphi : \mathcal{U} \to \mathcal{H}_\partial(D(2))$ such that $\varphi(f)i = f$ $(f \in \mathcal{U})$ and $\varphi(i) = id_D$.*

Proof. (i) Let $f \in \mathcal{E}_+$. Comparing two maps $h_f\varphi_{r_f}$, $fh_i\varphi_{r_i} : C(1) \to f(T)$, we obtain a unique map $\Theta_0(f) \in \mathcal{H}_+(C(1))$ such that $h_f\varphi_{r_f}\Theta_0(f) = fh_i\varphi_{r_i}$. Extend $\Theta_0(f)$ radially to $\Theta(f) \in \mathcal{H}_+(A(1,2))$ by $\Theta(f)(rz) = r\Theta_0(f)(z)$ $(z \in C(1),\ 1 \leqslant r \leqslant 2)$. The required map $\Phi(f)$ is defined as the unique map $\Phi(f) \in \mathcal{H}_+(D(2))$ with $h_f\varphi_{r_f}\Theta(f) = \Phi(f)h_i\varphi_{r_i}$. In claim below we will show that the map Θ_0 is continuous. This implies the continuity of the map Φ.

(ii) Since $\Phi(i) = id$, if we take a sufficiently small neighborhood $\mathcal{U}$ of i, then $\Phi(f)|_{C(2)}$ is close to $id_{C(2)}$ for $f \in \mathcal{U}$, and we can use a collar of $C(2)$ in $D(2)$ and a local contraction of a neighborhood of $id_{C(2)}$ in $\mathcal{H}(C(2))$ to modify the map $\Phi|_{\mathcal{U}}$ to obtain the desired map φ. □

Claim. *The map $\Theta_0 : \mathcal{E}_+ \to \mathcal{H}_+(C(1))$ is continuous.*

Proof. Under the notations of Lemma 2.2, let $g_f = h_f\varphi_{r_f}$ and $x_j(f) = \varphi_{r_f}^{-1}(u_j)$. For the inclusion $i : T \subset D(2)$, we abbreviate as $g = g_i$ and $x_j = x_j(i)$. Let $L_j = x_jx_{j+1}$ (the circular arc in $C(1)$). Also let $\tilde{f} = \Theta_0(f)$. Note that g_f is continuous in f (Lemma 2.3),

$g_f \tilde{f} = gf$, $\tilde{f}(x_j) = x_j(f) = g_f^{-1}(f(v_j))$ and that g_f maps $\tilde{f}(L_j)$ homeomorphically onto $f(E_T(v_j, v_{j+1}))$.

(1) First we will show the following statement:

(∗) Suppose $f \in \mathcal{E}_+$, U is any open neighborhood of $x_j(f)$ in $\mathbb{C}$ and A_j is a small compact neighborhood of x_j in $C(1)$ such that $g_f(\tilde{f}(A_j)) \cap g_f(A(1,2) \setminus U) = \emptyset$ (hence $\tilde{f}(A_j) \subset U$). If f' is sufficiently close to f, then $\tilde{f}'(A_j) \subset U$. In particular, $x_j(f) \in C(1)$ is continuous in f.

In fact, there exists an $\varepsilon > 0$ such that $O(fg(A_j), \varepsilon) \cap O(g_f(A(1,2) \setminus U), \varepsilon) = \emptyset$. If f' is sufficiently close to f then the sup-metric $d(f', f) < \varepsilon$ and $d(g_{f'}, g_f) < \varepsilon$. Hence, $f'g(A_j) = g_{f'}\tilde{f}'(A_j)$ does not meet $g_{f'}(A(1,2) \setminus U)$, so $g_{f'}\tilde{f}'(A_j) \subset U$.

(2) To show that $\tilde{f}$ is continuous in f, let $f \in \mathcal{E}_+$ and $\varepsilon > 0$ be given. It suffices to show that for each $j = 1, \ldots, n$ there exists a small neighborhood $\mathcal{U}$ of f in $\mathcal{E}_+$ such that $\tilde{f}$ and $\tilde{f}'$ are ε-close on L_j for every $f' \in \mathcal{U}$.

Set $U_j = O(x_j(f), \varepsilon/2)$ and $U_{j+1} = O(x_{j+1}(f), \varepsilon/2)$, and let A_j and A_{j+1} be small circular arc neighborhoods of x_j and x_{j+1} in $C(1)$ as in (1) with respect to U_j and U_{j+1}, respectively. Set $K_j = cl(L_j \setminus (A_j \cup A_{j+1}))$ and choose small circular arc neighborhoods C_j and C_{j+1} of $x_j(f)$ and $x_{j+1}(f)$ in $C(1)$ such that $g_f\tilde{f}(K_j)$ meets neither $g_f(C_j)$ nor $g_f(C_{j+1})$. Choose $\delta_1 > 0$ such that $O(g_f\tilde{f}(K_j), \delta_1)$ meets neither $O(g_f(C_j), \delta_1)$ nor $O(g_f(C_{j+1}), \delta_1)$. By the compactness argument there exists δ, $0 < \delta < \delta_1$, such that for any $x \in L_j$, $g_f(\tilde{f}(L_j) \setminus O(\tilde{f}(x), \varepsilon)) \cap O(g_f\tilde{f}(x), 2\delta) = \emptyset$.

By (1) there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{E}_+$ such that if $f' \in \mathcal{U}$, then $\tilde{f}'(A_j) \subset U_j$, $\tilde{f}'(A_{j+1}) \subset U_{j+1}$, $\tilde{f}'(x_j) \in C_j$, $\tilde{f}'(x_{j+1}) \in C_{j+1}$ and $d(f, f') < \delta$, $d(g_{f'}, g_f) < \delta$. Since $\tilde{f}'$ is orientation preserving, $\tilde{f}'(x_j) \in C_j$ and $\tilde{f}'(x_{j+1}) \in C_{j+1}$, it follows that $\tilde{f}'(L_j) \subset \tilde{f}(L_j) \cup C_j \cup C_{j+1}$. If $x \in A_j$, then $\tilde{f}'(x), \tilde{f}(x) \in U_j$ so that $d(\tilde{f}'(x), \tilde{f}(x)) < \varepsilon$. For each $x \in A_{j+1}$ we have the same conclusion. Suppose $x \in K_j$. Since $g_{f'}\tilde{f}'(x) = f'g(x)$ is δ-close to $fg(x) = g_f\tilde{f}(x) \in g_f\tilde{f}(K_j)$ and $g_{f'}(C_j) \subset O(g_f(C_j), \delta)$, we have $\tilde{f}'(x) \notin C_j$. Similarly $\tilde{f}'(x) \notin C_{j+1}$, and so $\tilde{f}'(x) \in \tilde{f}(L_j)$. Since $g_f\tilde{f}(x) = fg(x)$ is δ-close to $f'g(x) = g_{f'}\tilde{f}'(x)$ and the latter is also δ-close to $g_f\tilde{f}'(x)$, we have $g_f\tilde{f}'(x) \in O(g_f\tilde{f}(x), 2\delta)$. Hence by the choice of δ, $\tilde{f}'(x) \in O(\tilde{f}(x), \varepsilon)$. This completes the proof. □

Finally we will see a symmetry property of the map Φ_T in Proposition 2.1(i). For $z \in C(1)$ let $\theta_z : \mathbb{C} \to \mathbb{C}$ denote the rotation $\theta_z(w) = z \cdot w$ and let $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection, $\gamma(x, y) = (x, -y)$.

Lemma 2.4.

(i) $\Phi_T(\theta_z f) = \theta_z \Phi_T(f)$ $(f \in \mathcal{E}_+, z \in C(1))$.

(ii) $\Phi_{\gamma(T)}(\gamma f \gamma) = \gamma \Phi_T(f)\gamma$ $(f \in \mathcal{E}_+)$. *In particular, if T is a segment in the x-axis, then* $\Phi_T(\gamma f) = \gamma \Phi_T(f)\gamma$ $(f \in \mathcal{E})$.

Proof. (i) Let $f \in \mathcal{E}_+$, $z \in C(1)$ and let $w_0 \in C(2)$ be the unique point such that $\theta_z h_f \theta_z^{-1}(w_0) = 2$. Under Lemma 2.2, $(r_f, \theta_z h_f \theta_z^{-1} \theta_w)$ corresponds to $\theta_z f$, where $w = w_0/2$. Thus $\Theta(\theta_z f) = \theta_w^{-1} \theta_z \Theta(f)$ and the conclusion follows from

$$\begin{aligned}\Phi(\theta_z f)h_i\varphi_{r_i} &= \big(\theta_z h_f \theta_z^{-1}\theta_w\big)\varphi_{r_f}\Theta(\theta_z f)\\ &= \big(\theta_z h_f \theta_z^{-1}\theta_w\big)\varphi_{r_f}\theta_w^{-1}\theta_z\Theta(f)\\ &= \theta_z h_f\varphi_{r_f}\Theta(f) = \theta_z\Phi(f)h_i\varphi_{r_i}.\end{aligned}$$

(ii) Since $(r_i, \gamma h_i \gamma)$ corresponds to $\gamma(T)$ and $(r_f, \gamma h_f \gamma)$ corresponds to $\gamma f(T)$, it follows that $\Theta_{\gamma(T)}(\gamma f \gamma) = \gamma\Theta_T(f)\gamma$. The conclusion follows from

$$\begin{aligned}\big(\gamma\Phi(f)\gamma\big)\big(\gamma h_i\gamma\varphi_{r_i}\big) &= \gamma\big(\Phi(f)h_i\varphi_{r_i}\big)\gamma\\ &= \gamma\big(h_f\varphi_{r_f}\Theta(f)\big)\gamma\\ &= \big(\gamma h_f\gamma\varphi_{r_f}\big)\big(\gamma\Theta(f)\gamma\big). \qquad \square\end{aligned}$$

3. Extension property of embeddings of compact polyhedra into 2-manifolds

In this section we prove Theorem 1.1 and Corollary 1.1. First we consider the case where M is compact.

Lemma 3.1. *Suppose M is a compact* PL 2-*manifold and $K \subset X$ are compact subpolyhedra of M. Then there exists an open neighborhood $\mathcal{U}$ of i_X in $\mathcal{E}_K(X, M)^*$ and a map $\varphi : \mathcal{U} \to \mathcal{H}_K(M)$ such that $\varphi(f)|_X = f$ $(f \in \mathcal{U})$ and $\varphi(i_X) = id_M$.*

Proof. We may assume that $K = \emptyset$, since if φ satisfies the above condition in the case where $K = \emptyset$ then we have $\varphi(\mathcal{U} \cap \mathcal{E}_K(X, M)^*) \subset \mathcal{H}_K(M)$ for any $K \subset X$.

(1) The case when $\partial M = \emptyset$: We fix a triangulation of X and let S_k $(k = 0, 1, 2)$ denote the set of k-simplices of this triangulation and $X^{(1)}$ denote the 1-skeleton of X. For each $\sigma \in S_1$ with ends v, w we choose two disjoint subarcs σ_v, σ_w of σ with $v \in \sigma_v$, $w \in \sigma_w$ and a subarc e_σ of $\operatorname{Int}\sigma$ with $\operatorname{Int} e_\sigma \supset cl(\sigma \setminus (\sigma_v \cup \sigma_w))$. For each $v \in S_0$ set $T_v = \{v\} \cup (\bigcup_{v\in\sigma\in S_1} \sigma_v)$, which is an ad or a single point. We choose two disjoint families of closed disks $\{D_v\}_{v\in S_0}$ and $\{E_\sigma\}_{\sigma\in S_1}$ in M such that (i) $T_v \subset \operatorname{Int} D_v$ $(v \in S_0)$ and (ii) $X^{(1)} \cap E_\sigma = e_\sigma$ and $\operatorname{Int} e_\sigma \subset \operatorname{Int} E_\sigma$ (i.e., e_σ is a proper arc of E_σ).

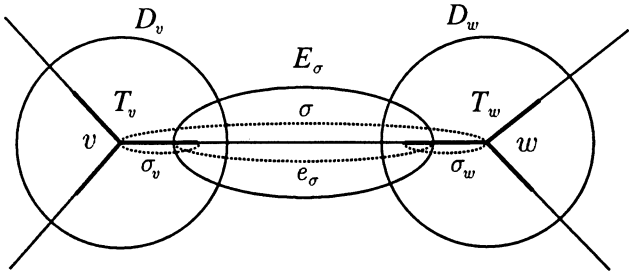

Fig. 1(a).

By Proposition 2.1(ii) for each $v \in S_0$ there exists a neighborhood $\mathcal{V}_v$ of i_{T_v} in $\mathcal{E}(T_v, \mathrm{Int}\, D_v)$ and an extension map $\alpha_v : \mathcal{V}_v \to \mathcal{H}_\partial(D_v)$. In turn, by Lemma 2.1(ii) for each $\sigma \in S_1$ there exists a neighborhood $\mathcal{W}_\sigma$ of i_{e_σ} in $\mathcal{E}_{\partial e_\sigma}(e_\sigma, E_\sigma)^*$ and an extension map $\beta_\sigma : \mathcal{W}_\sigma \to \mathcal{H}_\partial(E_\sigma)$. If $\mathcal{U}$ is a sufficiently small neighborhood of i_X in $\mathcal{E}(X, M)$, then for any $f \in \mathcal{U}$ we have $f|_{T_v} \in \mathcal{V}_v$ for every $v \in S_0$ and we can define a map $\lambda : \mathcal{U} \to \mathcal{H}(M)$ by

$$\lambda(f) = \begin{cases} \alpha_v(f|_{T_v}) & \text{on } D_v, \\ id & \text{on } M \setminus \bigcup_v D_v. \end{cases}$$

Since $\lambda(i_X) = id_M$ and $\lambda(f)^{-1} f|_{T_v} = i_{T_v}$ $(v \in S_0)$, if $\mathcal{U}$ is small enough, then $\lambda(f)^{-1} f$ is sufficiently close to i_X so that $\lambda(f)^{-1} f|_{e_\sigma} \in \mathcal{W}_\sigma$. Hence we can define a map $\mu : \mathcal{U} \to \mathcal{H}(M)$ by

$$\mu(f) = \begin{cases} \beta_\sigma(\lambda(f)^{-1} f|_{e_\sigma}) & \text{on } E_\sigma, \\ id & \text{on } M \setminus \bigcup_\sigma E_\sigma. \end{cases}$$

Then $\mu(i_X) = id_M$ and $\hat{f} \equiv \mu(f)^{-1}\lambda(f)^{-1} f$ is equal to the identity map on $X^{(1)}$ for each $f \in \mathcal{U}$. Since $\hat{f}(\sigma) = \sigma$ $(\sigma \in S_2)$, we can define a map $\nu : \mathcal{U} \to \mathcal{H}(M)$ by $\nu(f)|_X = \hat{f}$ and $\nu(f)|_{M \setminus X} = id$. Since $\nu(i_X) = id_M$ and $\nu(f)^{-1}\mu(f)^{-1}\lambda(f)^{-1} f = i_X$, the map $\varphi : \mathcal{U} \to \mathcal{H}(M)$, $\varphi(f) = \lambda(f)\mu(f)\nu(f)$ $(f \in \mathcal{U})$ satisfies the desired conditions.

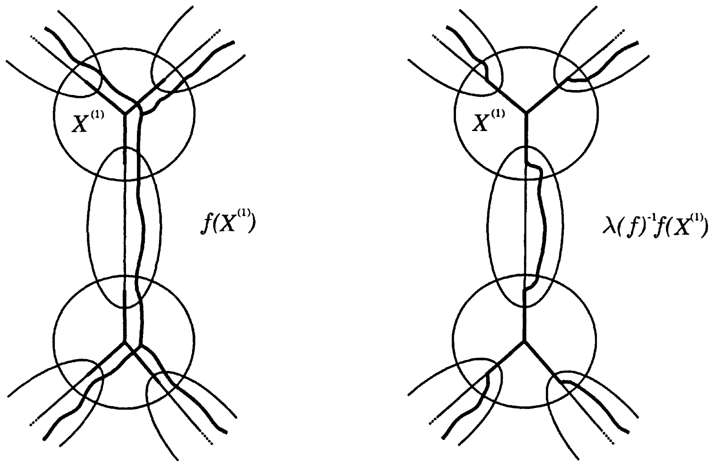

Fig. 1(b). Fig. 1(c).

(2) The case when $\partial M \neq \emptyset$: We can use the double $N = M \cup_{\partial M} M$. Since X is a subpolyhedron of M, $Y = X \cap \partial M$ is also a subpolyhedron of ∂M.

(i) By (1) (where $K \neq \emptyset$) we have a neighborhood $\mathcal{V}_0$ of $i_{X \cup \partial M}$ in $\mathcal{E}_{\partial M}(X \cup \partial M, N)$ and an extension map $\psi_0 : \mathcal{V}_0 \to \mathcal{H}_{\partial M}(N)$. We can extend every $f \in \mathcal{E}_Y(X, M)^*$ to an $f_0 \in \mathcal{E}_{\partial M}(X \cup \partial M, N)$ by the identity on ∂M. If $\mathcal{V}$ is a small neighborhood of i_X in $\mathcal{E}_Y(X, M)^*$, then for every $f \in \mathcal{V}$ we have $f_0 \in \mathcal{V}_0$, so $\psi(f_0)$ is defined and $\psi_0(f_0)(M) = M$. Thus we have an extension map $\psi : \mathcal{V} \to \mathcal{H}_{\partial M}(M)$, $\psi(f) = \psi_0(f_0)|_M$.

(ii) Since $\mathcal{H}(\partial M)$ is locally contractible, using a collar of ∂M in M, we have a neighborhood $\mathcal{W}$ of $id_{\partial M}$ in $\mathcal{H}(\partial M)$ and a map $F : \mathcal{W} \to \mathcal{H}(M)$ such that $F(g)|_{\partial M} = g$ $(g \in \mathcal{W})$ and $F(id_{\partial M}) = id_M$. We can easily verify a 1-dimensional version of Lemma 3.1 and find a neighborhood $\mathcal{W}_0$ of i_Y in $\mathcal{E}(Y, \partial M)$ and an extension map $\lambda_0 : \mathcal{W}_0 \to \mathcal{H}(\partial M)$. We may assume that $\lambda_0(\mathcal{W}_0) \subset \mathcal{W}$. Hence if $\mathcal{U}$ is a small neighborhood of i_X in $\mathcal{E}(X, M)^*$, then we have a map $\lambda : \mathcal{U} \to \mathcal{H}(M)$, $\lambda(f) = F(\lambda_0(f|_Y))$. Then $\lambda(f)|_Y = f|_Y$ $(f \in \mathcal{U})$ and $\lambda(id_X) = id_M$. If $\mathcal{U}$ is small, then we have $\lambda(f)^{-1} f \in \mathcal{V}$ and the required extension map $\varphi : \mathcal{U} \to \mathcal{H}(M)$ is defined by $\varphi(f) = \lambda(f)\psi(\lambda(f)^{-1} f)$. □

Lemma 3.2. *If M is a compact* PL 2*-manifold and X is a compact subpolyhedron of M, then $\mathcal{H}_X(M)$ is an ANR.*

Proof. Let $\pi : \mathcal{H}(M) \to \mathcal{E}(X, M)^*$, $\pi(h) = h|_X$, denote the restriction map. By Lemma 3.1 (with $K = \emptyset$) there exists an open neighborhood $\mathcal{U}$ of i_X in $\mathcal{E}(X, M)^*$ and a map $\varphi : \mathcal{U} \to \mathcal{H}(M)$ such that $\varphi(f)|_X = f$. Then $\Phi : \mathcal{U} \times \mathcal{H}_X(M) \cong \pi^{-1}(\mathcal{U})$, $\Phi(f, h) = \varphi(f)h$, is a homeomorphism with the inverse $\Phi^{-1}(k) = (k|_X, \varphi(k|_X)^{-1}k)$. Since $\mathcal{H}(M)$ is an ANR [11] and $\pi^{-1}(\mathcal{U})$ is open in $\mathcal{H}(M)$, $\mathcal{H}_X(M)$ is also an ANR. □

Proof of Theorem 1.1. Theorem 1.1 can be reduced to Lemma 3.1 by the following observations:

(i) Since there exists an $h \in \mathcal{H}_{K \cup (M \setminus U)}(M)_0$ such that hf is a PL embedding (cf. [3, Appendix]) we may assume that f is a PL-embedding. Replacing X by $f(X)$, we may assume that $f = i_X : X \subset M$.

(ii) Taking a compact PL-submanifold neighborhood N of X in U and replacing (M, X, K) by $(N, X \cup \mathrm{Fr}_M N, K \cup \mathrm{Fr}_M N)$, we may assume that M is compact and $U = M$.

(iii) If M is compact then $\mathcal{H}_K(M)_0$ is open in $\mathcal{H}_K(M)$ by Lemma 3.2. Hence we can take a smaller $\mathcal{U}$ to attain $\varphi(\mathcal{U}) \subset \mathcal{H}_K(M)_0$. □

Proof of Corollary 1.1. Let $f \in \mathcal{E}_K(X, U)^*_0$ and let $\mathcal{U}_f$, φ_f be as in Theorem 1.1. If $\mathcal{U}_f \cap \mathrm{Im}\,\pi \neq \emptyset$ then $\mathcal{U}_f \subset \mathrm{Im}\,\pi$. In fact, if $h \in \mathcal{H}_{K \cup (M \setminus U)}(M)_0$ and $\pi(h) = h|_X \in \mathcal{U}_f$, then for any $g \in \mathcal{U}_f$ we have $g = \pi(\varphi_f(g)\varphi_f(h|_X)^{-1}h)$. Hence $\mathrm{Im}\,\pi$ is clopen in $\mathcal{E}_K(X, U)^*_0$, so $\mathrm{Im}\,\pi = \mathcal{E}_K(X, U)^*_0$ and $\mathcal{U}_f \subset \mathcal{E}_K(X, U)^*_0$. Choose an $h_f \in \mathcal{H}_{K \cup (M \setminus U)}(M)_0$ with $h_f|_X = f$ and define a local trivialization $\Phi : \mathcal{U}_f \times \mathcal{G} \cong \pi^{-1}(\mathcal{U}_f)$ by $\Phi(g, h) = \varphi_f(g)h_f h$. □

By a similar argument we can also show the following statements.

Corollary 3.1. *Suppose $K \subset Y \subset X$ are compact subpolyhedra of a* PL 2*-manifold M.*

(i) *For any open neighborhood U of X in M the restriction map $\pi : \mathcal{H}_{K \cup (M \setminus U)}(M) \to$ $\mathrm{Im}\,\pi \subset \mathcal{E}_K(X, U)^*$ is a principal bundle with the fiber $\mathcal{H}_{X \cup (M \setminus U)}(M)$ and* $\mathrm{Im}\,\pi$ *is clopen in $\mathcal{E}_K(X, U)^*$.*

(ii) *The restriction map $p : \mathcal{E}_K(X, M)^* \to$* $\mathrm{Im}\, p \subset \mathcal{E}_K(Y, M)^*$ *is locally trivial and* $\mathrm{Im}\, p$ *is clopen in $\mathcal{E}_K(Y, M)^*$.*

4. The spaces of embeddings into 2-manifolds

In this final section we will prove Theorem 1.2.

4.1. Basic facts on infinite-dimensional manifolds

First we recall some basic facts on infinite-dimensional manifolds. As for the model spaces we follow the standard convention:

$$s = (-\infty, \infty)^{\infty} \quad (\cong \ell_2),$$

$$\Sigma = \left\{(x_n) \in s\colon \sup_n |x_n| < \infty\right\},$$

$$\sigma = \left\{(x_n) \in s\colon x_n = 0 \text{ (almost all } n)\right\}.$$

A triple (X, X_1, X_2) means a triple of a space X and subspaces $X_1 \supset X_2$. A triple (X, X_1, X_2) is said to be a (s, Σ, σ)-manifold if each point of X admits an open neighborhood U in X and an open set V in s such that $(U, U \cap X_1, U \cap X_2) \cong (V, V \cap \Sigma, V \cap \sigma)$ (a homeomorphism of triples). In [20] we have obtained a characterization of (s, Σ, σ)-manifolds in terms of some class conditions, a stability condition and the homotopy negligible complement condition. A space is σ-(fd-)compact if it is a countable union of (finite-dimensional) compact subsets. A triple (X, X_1, X_2) is said to be (s, Σ, σ)-stable if $(X \times s, X_1 \times \Sigma, X_2 \times \sigma) \cong (X, X_1, X_2)$. We say that a subset Y of X has the homotopy negligible (h.n.) complement in X if there exists a homotopy $\varphi_t : X \to X$ $(0 \leqslant t \leqslant 1)$ such that $\varphi_0 = id_X$ and $\varphi_t(X) \subset Y$ $(0 < t \leqslant 1)$. The homotopy φ_t is called an absorbing homotopy of X into Y.

Fact 4.1.

(i) *Y has the h.n. complement in X iff each point $x \in X$ has an open neighborhood U and a homotopy $\varphi : U \times [0, 1] \to X$ such that $\varphi_0 = i_U\colon U \subset X$ and $\varphi_t(U) \subset Y$ $(0 < t \leqslant 1)$.*

(ii) *If Y has the h.n. complement in X, then X is an ANR iff Y is an ANR by* [10].

(iii) ([17]) *Suppose X is an ANR. Then Y has the h.n. complement in X iff for any open set U of X the inclusion $U \cap Y \subset U$ is a weak homotopy equivalence. Hence if both $Y \subset X$ and $Z \subset Y$ have the h.n. complement, then so does $Z \subset X$.*

In (i) $U \cap Y$ has the h.n. complement in U and local absorbing homotopies can be uniformized to a global one [13].

We will apply the following characterization of (s, Σ, σ)-manifolds [20].

Proposition 4.1. *A triple (X, X_1, X_2) is an (s, Σ, σ)-manifold iff*

(i) *X is a separable completely metrizable ANR, X_1 is σ-compact and X_2 is σ-fd-compact,*

(ii) *X_2 has the h.n. complement in X,*

(iii) *(X, X_1, X_2) is (s, Σ, σ)-stable.*

We refer to [18] for related topics in infinite-dimensional topology.

4.2. The spaces of embeddings into 2-manifolds

First we summarize the stability property and the class property of embedding spaces. Suppose (X,d) and (Y,ρ) are metric spaces. An embedding $f:X\to Y$ is said to be L-Lipschitz ($L\geqslant 1$) if $\frac{1}{L}d(x,y)\leqslant\rho(f(x),f(y))\leqslant Ld(x,y)$ for any $x,y\in X$.

Lemma 4.1 [16, Theorems 1.2]. *Suppose M is a Euclidean* PL 2*-manifold and $K\subset X$ are compact subpolyhedra of M. If* $\dim(X\setminus K)\geqslant 1$, *then the triples $(\mathcal{E}_K(X,M),\mathcal{E}_K^{\mathrm{LIP}}(X,M),\mathcal{E}_K^{\mathrm{PL}}(X,M))$ and $(\mathcal{E}_K(X,M)^*,\mathcal{E}_K^{\mathrm{LIP}}(X,M)^*,\mathcal{E}_K^{\mathrm{PL}}(X,M)^*)$ are (s,Σ,σ)-stable.*

Lemma 4.2.

(1) *Suppose X is a compact metric space, K is a closed subset of X and Y is a locally compact, separable metric space. Then* (i) *$\mathcal{E}_K(X,Y)$ is separable, completely metrizable, and* (ii) *$\mathcal{E}_K^{\mathrm{LIP}}(X,Y)$ is σ-compact.*

(2) ([5]) *If X is a compact polyhedron, K is a subpolyhedron of X, and Y is a locally compact polyhedron, then $\mathcal{E}_K^{\mathrm{PL}}(X,Y)$ is σ-fd-compact.*

Proof. (1) (i) $\mathcal{C}(X,Y)$ is completely metrizable by the sup-metric, and $\mathcal{E}(X,Y)$ is G_δ in $\mathcal{C}(X,Y)$.

(ii) For $L\geqslant 1$ let $\mathcal{E}^{\mathrm{LIP}(L)}(X,Y)$ denote the subspace of L-Lipschitz embeddings. If we write $Y=\bigcup_{n=1}^\infty Y_n$ (Y_n is compact and $Y_n\subset\operatorname{Int}Y_{n+1}, n\geqslant 1$), then $\mathcal{E}^{\mathrm{LIP}}(X,Y)=\bigcup_{n=1}^\infty\mathcal{E}^{\mathrm{LIP}(n)}(X,Y_n)$. Since $\mathcal{E}^{\mathrm{LIP}(n)}(X,Y_n)$ is equicontinuous and closed in $\mathcal{C}(X,Y_n)$, it is compact by Arzela–Ascoli Theorem [2, Ch. XII, Theorem 6.4]. Hence $\mathcal{E}^{\mathrm{LIP}}(X,Y)$ is σ-compact. □

For the proper PL-embedding case we need some basic facts:

Fact 4.2.

(1) *Suppose A is a* PL *disk* (*or a* PL *arc*) *and $a\in\operatorname{Int}A$. Then there exists a map $\varphi:\operatorname{Int}A\to\mathcal{H}_{\partial A}^{\mathrm{PL}}(A)$ such that $\varphi_x(a)=x$ ($x\in\operatorname{Int}A$) and $\varphi_a=id_A$.*

(2) *Suppose N is a* PL 1*-manifold with $\partial N=\emptyset$, Y is a compact subpolyhedron of N, U is an open neighborhood of Y in N. Then there exists an open neighborhood $\mathcal{U}$ of i_Y in $\mathcal{E}^{\mathrm{PL}}(Y,N)$ and a map $\varphi:\mathcal{U}\to\mathcal{H}_{N\setminus U}^{\mathrm{PL}}(N)$ such that $\varphi(f)|_Y=f$ and $\varphi(i_Y)=id_N$.*

(3) *Suppose M is a* PL 2*-manifold, N is a compact* 1*-submanifold of ∂M and U is an open neighborhood of N in M. Then there exists an open neighborhood $\mathcal{U}$ of $id_{\partial M}$ in $\mathcal{H}_{\partial M\setminus N}^{\mathrm{PL}}(\partial M)$ and a map $\varphi:\mathcal{U}\to\mathcal{H}_{M\setminus U}^{\mathrm{PL}}(M)$ such that $\varphi(f)|_{\partial M}=f$ and $\varphi(id_{\partial M})=id_M$.*

(4) *Suppose M is a* PL 2*-manifold, Y is a compact subpolyhedron of ∂M and U is an open neighborhood of Y in M. Then there exists an open neighborhood $\mathcal{V}$ of i_Y in $\mathcal{E}^{\mathrm{PL}}(Y,\partial M)$ and a map $\varphi:\mathcal{V}\to\mathcal{H}_{M\setminus U}^{\mathrm{PL}}(M)$ such that $\varphi(g)|_Y=g$ and $\varphi(i_Y)=id_M$.*

Comment.

(3) Using a PL-collar of ∂M in M, the assertion follows from the following remarks:

(3-i) If A is a PL arc (or a PL open arc), then there exists a map $\varphi : \mathcal{H}_+^{\mathrm{PL}}(A) \to \mathcal{H}^{\mathrm{PL}}(A \times [0,1])$ such that $\varphi(f)$ is an isotopy from f to id_A (i.e., $\varphi(f)(x,t) = (*, t)$, $\varphi(f)(x,0) = f(x)$ and $\varphi(f)(x,1) = (x,1)$) for each $f \in \mathcal{H}_+^{\mathrm{PL}}(A)$ and $\varphi(id_A) = id_{A \times [0,1]}$.

(3-ii) Suppose S is a PL circle. Then there exists an open neighborhood $\mathcal{U}$ of id_S in $\mathcal{H}^{\mathrm{PL}}(S)$ and a map $\varphi : \mathcal{U} \to \mathcal{H}^{\mathrm{PL}}(S \times [0,1])$ such that $\varphi(f)$ is an isotopy from f to id_S for each $f \in \mathcal{U}$ and $\varphi(id_S) = id_{S \times [0,1]}$.

In (3-i) we may assume that $A = [0,1]$ (or $A = \mathbb{R}$). Then $\varphi(f)$ is defined as the linear isotopy $\varphi(f)(x,t) = ((1-t)f(x) + tx, t)$.

(4) This follows from (2) and (3).

Lemma 4.3. *If M is a* PL 2*-manifold and $K \subset X$ are compact subpolyhedra of M, then* (i) *$\mathcal{E}_K(X, M)^*$ is completely metrizable and* (ii) *$\mathcal{E}_K^{\mathrm{PL}}(X, M)^*$ is σ-fd-compact.*

Proof. (i) $\mathcal{E}_K(X, M)^*$ is G_δ in $\mathcal{E}_K(X, M)$.

(ii) We may assume that $K = \emptyset$. It suffices to show that each $f \in \mathcal{E}^{\mathrm{PL}}(X, M)^*$ has a σ-fd-compact neighborhood. Since $\mathcal{E}_K^{\mathrm{PL}}(X, M)^* \cong \mathcal{E}_K^{\mathrm{PL}}(f(X), M)^*$, we may assume that $f = i_X$. Choose a sequence of small collars C_n of ∂M in M pinched at $Y = X \cap \partial M$ such that C_n becomes thinner and thinner and also the angle between $\mathrm{Fr}_M C_n$ and ∂M at $\mathrm{Fr}_{\partial M} Y$ becomes smaller and smaller as $n \to \infty$. Let $M_n = cl(M \setminus C_n)$. Then $\mathcal{E}_Y^{\mathrm{PL}}(X, M)^* = \bigcup_n \mathcal{E}_Y^{\mathrm{PL}}(X, M_n)$ and $\mathcal{E}^{\mathrm{PL}}(Y, \partial M)$ are σ-fd-compact by [5].

By Fact 4.2(4) there exists an open neighborhood $\mathcal{V}$ of i_Y in $\mathcal{E}^{\mathrm{PL}}(Y, \partial M)$ and a map $\varphi : \mathcal{V} \to \mathcal{H}^{\mathrm{PL}}(M)$ such that $\varphi(g)|_Y = g$ and $\varphi(i_Y) = id_M$. Let $\psi : \mathcal{E}^{\mathrm{PL}}(X, M)^* \to \mathcal{E}^{\mathrm{PL}}(Y, \partial M)$ be the restriction map, $\psi(f) = f|_Y$ and let $\mathcal{U} = \psi^{-1}(\mathcal{V})$. Then $\Phi : \mathcal{V} \times \mathcal{E}_Y^{\mathrm{PL}}(X, M)^* \to \mathcal{U}$, $\Phi(g, h) = \varphi(g)h$, is a homeomorphism with the inverse $\Phi^{-1}(f) = (f|_Y, \varphi(f|_Y)^{-1} f)$. Hence $\mathcal{U}$ is also σ-fd-compact. This implies the conclusion. $\square$

Next we verify the ANR-condition and the h.n. complement condition.

Fact 4.3 [4,6]. *Suppose M is a compact* PL 2*-manifold and X is a compact subpolyhedron of M. Then $\mathcal{H}_X^{\mathrm{PL}}(M)$ has the h.n. complement in $\mathcal{H}_X(M)$.*

Comment. By [4, p. 10] (a comment on a relative version) $\mathcal{H}_X^{\mathrm{PL}}(M)$ is (uniformly) locally contractible. Since $\mathcal{H}_X(M)$ is an ANR, by [6] $\mathcal{H}_X^{\mathrm{PL}}(M)$ has the h.n. complement in $\mathcal{H}_X(M)$. Note that in dimension 2, the local contractibility of $\mathcal{H}_X^{\mathrm{PL}}(M)$ at id_M simply reduces to the case where $X = \emptyset$ by the following splitting argument:

(1) We may assume that X has no isolated points in $\mathrm{Int}\, M$. If X has the isolated points x_i $(i = 1, \ldots, n)$ in $\mathrm{Int}\, M$, then we can choose mutually disjoint PL disk neighborhood D_i of x_i in $\mathrm{Int}\, M \setminus X_0$, where $X_0 = X \setminus \{x_1, \ldots, x_n\}$. By Fact 4.2(1) there exists a map $\varphi : \prod_{i=1}^n \mathrm{Int}\, D_i \to \mathcal{H}_{X_0}^{\mathrm{PL}}(M)$ such that $\varphi(y_1, \ldots, y_n)(x_i) = y_i$ and $\varphi(x_1, \ldots, x_n) = id_M$.

Then $\mathcal{U}=\{f\in\mathcal{H}^{\mathrm{PL}}_{X_0}(M)\colon\ f(x_i)\in\operatorname{Int}D_i\ \ (i=1,\dots,n)\}$ is an open neighborhood of id_M in $\mathcal{H}^{\mathrm{PL}}_{X_0}(M)$ and

$$\Phi:\Bigl(\prod\operatorname{Int}D_i\Bigr)\times\mathcal{H}^{\mathrm{PL}}_X(M)\to\mathcal{U},\quad \Phi(y_1,\dots,y_n,g)=\varphi(y_1,\dots,y_n)g,$$

is a homeomorphism with the inverse

$$\Phi^{-1}(f)=\bigl(f(x_1),\dots,f(x_n),\varphi(f(x_1),\dots,f(x_n))^{-1}f\bigr).$$

Hence if $\mathcal{H}^{\mathrm{PL}}_{X_0}(M)$ is locally contractible, then $\mathcal{H}^{\mathrm{PL}}_X(M)$ is also locally contractible.

(2) Cutting M along $\mathrm{Fr}_M X$ we may assume that $X\subset\partial M$.

(3) By Fact 4.2(4) there exists an open neighborhood $\mathcal{V}$ of i_X in $\mathcal{E}^{\mathrm{PL}}(X,\partial M)$ and a map $\varphi:\mathcal{V}\to\mathcal{H}^{\mathrm{PL}}(M)$ such that $\varphi(g)|_X=g$ and $\varphi(i_X)=id_M$. Let $\psi:\mathcal{H}^{\mathrm{PL}}(M)\to\mathcal{E}^{\mathrm{PL}}(X,\partial M)$ be the restriction map, $\psi(f)=f|_X$ and let $\mathcal{U}=\psi^{-1}(\mathcal{V})$. Then $\mathcal{U}$ is an open neighborhood of id_M in $\mathcal{H}^{\mathrm{PL}}(M)$ and $\Phi:\mathcal{V}\times\mathcal{H}^{\mathrm{PL}}_X(M)\to\mathcal{U}$, $\Phi(g,h)=\varphi(g)h$, is a homeomorphism with the inverse $\Phi^{-1}(f)=(f|_X,\varphi(f|_X)^{-1}f)$. Since $\mathcal{H}^{\mathrm{PL}}(M)$ is locally contractible [4], $\mathcal{H}^{\mathrm{PL}}_X(M)$ is also locally contractible.

Suppose M is a PL 2-manifold and $K\subset X$ are compact subpolyhedra of M.

Lemma 4.4.

(1) (i) $\mathcal{E}_K(X,M)^*$ *is an ANR and* (ii) $\mathcal{E}^{\mathrm{PL}}_K(X,M)^*$ *has the h.n. complement in* $\mathcal{E}_K(X,M)^*$.

(2) (i) $\mathcal{E}_K(X,M)$ *is an ANR and* (ii) $\mathcal{E}^{\mathrm{PL}}_K(X,M)$ *has the h.n. complement in* $\mathcal{E}_K(X,M)$.

Proof. (1) (i) For every $f\in\mathcal{E}_K(X,M)^*$, take a compact PL 2-submanifold neighborhood N of $f(X)$ in M and consider the map $\pi:\mathcal{H}_{K\cup(M\setminus\mathrm{Int}_M N)}(M)\to\mathcal{E}_K(X,M)^*$, $\pi(h)=hf$. By Theorem 1.1 there exists an open neighborhood $\mathcal{U}$ of f in $\mathcal{E}_K(X,M)^*$ and a map $\varphi:\mathcal{U}\to\mathcal{H}_{K\cup(M\setminus\mathrm{Int}_M N)}(M)$ such that $\pi\varphi(g)=g$ $(g\in\mathcal{U})$. Since $\mathcal{H}_{K\cup(M\setminus\mathrm{Int}_M N)}(M)\cong\mathcal{H}_{K\cup\mathrm{Fr}_M N}(N)$ is an ANR by Lemma 3.2, so is $\mathcal{U}$. Hence $\mathcal{E}_K(X,M)^*$ is an ANR.

(ii) By Fact 4.1(i) it suffices to show that every $f\in\mathcal{E}_K(X,M)^*$ admits a neighborhood $\mathcal{U}$ and a homotopy $F_t:\mathcal{U}\to\mathcal{E}_K(X,M)^*$ such that $F_0=i_{\mathcal{U}}$ and $F_t(\mathcal{U})\subset\mathcal{E}^{\mathrm{PL}}_K(X,M)^*$ $(0<t\leqslant 1)$. Take a compact PL 2-submanifold N of M with $f(X)\subset U\equiv\mathrm{Int}_M N$. Let $\varphi:\mathcal{U}\to\mathcal{H}_{K\cup(M\setminus U)}(M)$ be given by Theorem 1.1. Since $(\mathcal{H}_{K\cup(M\setminus U)}(M),\mathcal{H}^{\mathrm{PL}}_{K\cup(M\setminus U)}(M))\cong(\mathcal{H}_{K\cup(N\setminus U)}(N),\mathcal{H}^{\mathrm{PL}}_{K\cup(N\setminus U)}(N))$, by Fact 4.2 we have an absorbing homotopy χ_t of $\mathcal{H}_{K\cup(M\setminus U)}(M)$ into $\mathcal{H}^{\mathrm{PL}}_{K\cup(M\setminus U)}(M)$. There exists a $h\in\mathcal{H}_{K\cup(M\setminus U)}(M)$ such that $hf\in\mathcal{E}^{\mathrm{PL}}_K(X,M)^*$. Define F_t by $F_t(g)=\chi_t(\varphi(g)h^{-1})hf$ $(g\in\mathcal{U})$.

(2) There exists an $f\in\mathcal{E}^{\mathrm{PL}}_K(X,M)$ with $f(X\setminus K)\subset\operatorname{Int}M$. It induces a homeomorphism $(\mathcal{E}_K(f(X),M),\ \mathcal{E}^{\mathrm{PL}}_K(f(X),M))\cong(\mathcal{E}_K(X,M),\mathcal{E}^{\mathrm{PL}}_K(X,M)):g\mapsto gf$. Hence we may assume that $X\setminus K\subset\operatorname{Int}M$. Pushing towards $\operatorname{Int}M$ using a collar of ∂M pinched on $\partial M\cap K$, it follows that $\mathcal{E}_K(X,M)^*$ has the h.n. complement in $\mathcal{E}_K(X,M)$. Thus (i) follows from (1)(i) and Fact 4.1(ii), and (ii) follows from (1)(ii), Fact 4.1(iii) and $\mathcal{E}^{\mathrm{PL}}_K(X,M)^*\subset\mathcal{E}^{\mathrm{PL}}_K(X,M)$. □

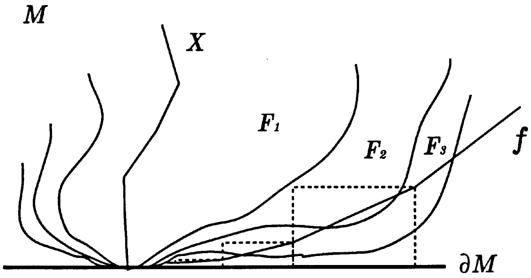

Fig. 2.

Theorem 1.2 follows from Proposition 4.1 and the above lemmas. For the proper embeddings we have a pair version.

Proposition 4.2. *If* $\dim(X \setminus K) \geqslant 1$*, then* $(\mathcal{E}_K(X,M)^*, \mathcal{E}_K^{\mathrm{PL}}(X,M)^*)$ *is an* (s,σ)*-manifold.*

Remark 4.1. In general, $\mathcal{E}_K^{\mathrm{LIP}}(X,M)^*$ is *not* σ-compact. For example, suppose X is a proper arc in M and $K = \partial X$. If $\mathcal{E}_K^{\mathrm{LIP}}(X,M)^* = \bigcup_{i \geqslant 1} \mathcal{F}_i$, $\mathcal{F}_i$ is compact, then $F_i = \{f(x) \mid f \in \mathcal{F}_i,\ x \in X\}$ is a compact subset of M with $F_i \cap \partial M = K$. By a simple diagonal argument we can define an $f \in \mathcal{E}_K^{\mathrm{LIP}}(X,M)^*$ such that $f(X) \not\subset F_i$ for each $i \geqslant 1$. Fig. 2 indicates how to define such an f near an end point of X.

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