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# Elementary chains and compact spaces with a small diagonal

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## Abstract

It is a well known open problem if, in ZFC, each compact space with a small diagonal is metrizable. We explore properties of compact spaces with a small diagonal using elementary chains of substructures. We prove that ccc subspaces of such spaces have countable  $\pi$ -weight. We generalize a result of Gruenhage about spaces which are metrizable fibered. Finally we discover that if there is a Luzin set of reals, then every compact space with a small diagonal will have many points of countable character.

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## 0. Introduction

In [4] Hušek defined a space,  $X$ , to have an  $\omega_1$ -accessible diagonal if there is an  $\omega_1$ -sequence  $\langle \langle x_\alpha, y_\alpha \rangle : \alpha < \omega_1 \rangle$  in  $X^2$  that converges to the diagonal  $\Delta(X)$  in the sense that every neighborhood of the diagonal contains a tail of the sequence. Hušek also mentions that Van Douwen referred to spaces that do *not* have an  $\omega_1$ -accessible diagonal as *having a small diagonal*. The latter has gained more currency since Zhou's [10] and the definition has been cast in a more

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positive form:  $X$  has a small diagonal if every uncountable subset of  $X^2$  that is disjoint from the diagonal has an uncountable subset whose closure is disjoint from the diagonal. For brevity's sake we will say that a space is csD if it is a compact Hausdorff space with a small Diagonal.

There are a number of very interesting results known for csD spaces and we recommend [3] as an excellent reference. In particular, it is known that csD spaces have countable tightness [5] and that it follows from CH that csD spaces are metrizable [4,5]. One of our main results is that ccc subspaces of a csD space have countable  $\pi$ -weight. In [9] Tkachuk describes a space as *metrizable fibered* if there is a continuous map onto a metric space with the property that each point preimage (fiber) is also metrizable. Let us say that a space is *weight  $\kappa$  fibered* if the obvious generalization is satisfied: there is a map onto a space with weight at most  $\kappa$  so that each fiber also has weight at most  $\kappa$ . In [3] Gruenhage showed that a metrizable fibered csD space is metrizable. We will show that this is also true for weight  $\omega_1$  fibered spaces.

Though the main question on csD spaces is whether they are metrizable, it is at present not even known if they must have points of countable character. We uncover the surprising connection that if there is a Luzin set of reals, then each csD space does have points of countable character.

## 1. Preliminaries

We begin by citing a convenient characterization of csD spaces obtained by Gruenhage in [3]. We say that a sequence  $\langle \langle x_\alpha, y_\alpha \rangle : \alpha \in \omega_1 \rangle$  of pairs is  $\omega_1$ -separated if there is an uncountable subset  $A$  of  $\omega_1$  such that  $\{x_\alpha : \alpha \in A\}$  and  $\{y_\alpha : \alpha \in A\}$  have disjoint closures. Gruenhage showed that a compact space is csD if and only if every uncountable sequence of pairs is  $\omega_1$ -separated.

### 1.1. Elementary sequences

The key to uncovering why a non-metric compact space might not be csD is to select the right  $\omega_1$ -sequence of pairs. We will explore a method of using chains of countable elementary substructures for this purpose.

For a cardinal  $\theta$  we let  $H(\theta)$  denote the collection of all sets whose transitive closure has cardinality less than  $\theta$  (see [6, Chapter IV]). An  $\omega_1$ -sequence  $\langle M_\alpha : \alpha < \omega_1 \rangle$  of countable elementary substructures of  $H(\theta)$  that satisfies  $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$  for all  $\alpha$  and  $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$  for all limit  $\alpha$  will simply be called an *elementary sequence*.

It will be convenient to assume that the spaces considered in this paper are subspaces of  $[0, 1]^\kappa$  for some suitable cardinal  $\kappa$  (usually the weight of the space under consideration). By a basic open subset of  $[0, 1]^\kappa$  we mean a set that is a product  $\prod_{\xi \in \kappa} I_\xi$  where each  $I_\xi$  is a relatively open subinterval of  $[0, 1]$  with rational endpoints such that the set of  $\xi \in \kappa$  for which  $I_\xi \neq [0, 1]$  – its *support* – is finite. When a space  $X$  is a subspace of  $[0, 1]^\kappa$  we will use the intersections of these basic open sets with  $X$  as a base for  $X$ .

*Elementary sequences for a space.* Let  $X$  be a compact space and assume that  $X$  is a subset of  $[0, 1]^\kappa$ , where  $\kappa$  is the weight of  $X$ . An *elementary sequence for  $X$*  will be an elementary sequence  $\langle M_\alpha : \alpha < \omega_1 \rangle$  in  $H((2^\kappa)^+)$  such that  $X \in M_0$ .

For each  $\alpha \in \omega_1$  we associate two spaces with  $X$  and  $M_\alpha$ : the first is the closure of  $X \cap M_\alpha$ , which we denote  $X_\alpha$ . The second is the image of  $X$  under the projection  $\text{pr}_{M_\alpha}$  from  $[0, 1]^\kappa$  onto  $[0, 1]^{M_\alpha \cap \kappa}$ : we write  $X_{M_\alpha} = \text{pr}_{M_\alpha}[X]$ .

For each  $x \in X$  and  $\alpha \in \omega_1$  we often denote  $\text{pr}_{M_\alpha}(x)$  by  $x \upharpoonright M_\alpha$  and we write  $[x \upharpoonright M_\alpha] = \{y \in X : \text{pr}_{M_\alpha}(y) = \text{pr}_{M_\alpha}(x)\}$ .

We will apply Gruenhage's criterion to sequences of pairs associated to elementary sequences.

**Definition 1.1.** An elementary  $\omega_1$ -sequence of pairs for a space  $X$  is a sequence  $\langle \langle x_\alpha, y_\alpha \rangle : \alpha \in \omega_1 \rangle$  of pairs of points from  $X$  for which there is some elementary sequence  $\langle M_\alpha : \alpha \in \omega_1 \rangle$  for  $X$  so that  $\{x_\alpha, y_\alpha\} \in M_{\alpha+1}$ ,  $x_\alpha \neq y_\alpha$  and  $x_\alpha \upharpoonright M_\alpha = y_\alpha \upharpoonright M_\alpha$  for each  $\alpha \in \omega_1$ .

The (seemingly narrow) gap between metrizable and being csD in the class of compact spaces is revealed in the next two propositions.

**Proposition 1.2.** *A compact space is metrizable if and only if it has no elementary  $\omega_1$ -sequence of pairs.*  $\square$

It is somewhat easier to prove the contrapositive form: a compact space has uncountable weight iff it has an elementary  $\omega_1$ -sequence of pairs.

**Proposition 1.3.** *A compact space  $X$  is not csD if and only if it has an elementary  $\omega_1$ -sequence of pairs that is not  $\omega_1$ -separated.*

**Proof.** We will actually prove a stronger statement: each  $\omega_1$ -sequence of pairs that is not  $\omega_1$ -separated contains an elementary  $\omega_1$ -sequence. Suppose that  $\langle \langle x_\alpha, y_\alpha \rangle : \alpha \in \omega_1 \rangle$  is an uncountable set of pairs of  $X$  that is not  $\omega_1$ -separated. Choose any elementary sequence  $\langle M_\gamma : \gamma \in \omega_1 \rangle$  for  $X$  such that  $\langle \langle x_\alpha, y_\alpha \rangle : \alpha \in \omega_1 \rangle$  is an element of  $M_0$ . For each  $\gamma \in \omega_1$ , choose  $\alpha_\gamma$  (if possible) so that  $x_{\alpha_\gamma} \upharpoonright M_\gamma = y_{\alpha_\gamma} \upharpoonright M_\gamma$ . By elementarity, if such an  $\alpha_\gamma$  exists, it may be chosen in  $M_{\gamma+1}$ . Then  $\langle \langle x_{\alpha_\gamma}, y_{\alpha_\gamma} \rangle : \gamma \in \omega_1 \rangle$  is the desired elementary  $\omega_1$ -sequence that is not  $\omega_1$ -separated.

To finish the proof we show that it is always possible to choose an  $\alpha_\gamma$ . If not then there will be a  $\gamma$  such that the set  $A$  of  $\alpha$  for which  $x_\alpha \upharpoonright M_\gamma \neq y_\alpha \upharpoonright M_\gamma$  is co-countable. For each  $\alpha \in A$ , there are basic open sets  $U_\alpha$  and  $W_\alpha$  with disjoint closures and supports in  $M_\gamma$  such that  $x_\alpha \in U_\alpha$  and  $y_\alpha \in W_\alpha$ . As  $U_\alpha$  and  $W_\alpha$  are determined by finite subsets of  $M_\gamma$  they belong to  $M_\gamma$ . As  $M_\gamma$  is countable one pair  $\langle U, W \rangle$  would be chosen uncountably often, say for  $\alpha \in B$ . The latter set would witness that our sequence is  $\omega_1$ -separated.  $\square$

## 2. Applications of elementary $\omega_1$ -sequences

There will be occasions when one countable elementary substructure will already do but in our first result a fair amount of care will go into the construction of an elementary  $\omega_1$ -sequence of pairs.

### 2.1. Cellularity and $\pi$ -weight

We need the notions of  $\pi$ -bases and local  $\pi$ -bases. A  $\pi$ -base for a space is a collection of non-empty open sets such that each non-empty open set contains one of them. A local  $\pi$ -base for a space at a point is a collection of non-empty open sets such that each neighborhood of the point contains one of them. Thus one obtains the notions of  $\pi$ -weight and  $\pi$ -character: minimum cardinalities of defining families.

A continuous surjection is said to be irreducible if no proper closed subset of the domain maps onto the range or, dually, the image of a set with non-empty interior has non-empty

interior as well. The latter formulation easily implies that  $\pi$ -weight is invariant under irreducible surjections, in both directions.

An easy application of Zorn's Lemma will show that if  $f : X \rightarrow Y$  is a continuous surjection between compact Hausdorff spaces one can find a closed subset  $Z$  of  $X$  such that  $f \upharpoonright Z : Z \rightarrow Y$  is irreducible and surjective.

The following lemma is of the nature of a reflection result in that we replace an unspecific 'uncountable' by the specific 'cardinality  $\omega_1$ '. As such it can be proven using a closing-off argument but we opt to work in the spirit of this paper and let an elementary chain do the closing off for us.

**Lemma 2.1.** *If a compact space  $X$  has a ccc subspace with uncountable  $\pi$ -weight, then it has a compact ccc subspace with  $\pi$ -weight equal to  $\omega_1$ .*

**Proof.** Let  $Y$  be a ccc subspace that does not have a countable  $\pi$ -base. Let  $K$  be the closure of  $Y$ ; then  $K$  is compact, ccc, and does not have a countable  $\pi$ -base either. Let  $\langle M_\alpha : \alpha \in \omega_1 \rangle$  be an elementary sequence for  $K$  and let  $M = \bigcup_{\alpha \in \omega_1} M_\alpha$ . For each  $\alpha \in \omega_1$  there is, by elementarity, a basic open subset  $U_\alpha$  of  $K$  that is an element of  $M_{\alpha+1}$  and does not contain any non-empty open subset of  $K$  that is a member of  $M_\alpha$ . It follows that  $K_M = \text{pr}_M[K]$  has  $\pi$ -weight  $\omega_1$  since it has weight at most  $\omega_1$  and does not have countable  $\pi$ -weight.

Since  $K_M$  is a continuous image of  $K$ , it is ccc. Now choose any compact  $Z \subset K$  such that  $\text{pr}_M \upharpoonright Z : Z \rightarrow K_M$  is irreducible and onto. Since the map is irreducible,  $Z$  is ccc and has the same  $\pi$ -weight (namely  $\omega_1$ ) as  $K_M$ .  $\square$

The first main result grew out of an attempt to see if there was a more direct proof, perhaps uncovering more about the nature of csD spaces, that a csD space has countable tightness.

**Theorem 2.2.** *In a csD space every ccc subspace has countable  $\pi$ -weight.*

**Proof.** We proceed by contradiction. Since a closed subspace of a csD space will also be csD, we may apply Lemma 2.1 and assume that we have a compact space  $X$  that is csD and ccc, and has  $\pi$ -weight equal to  $\omega_1$ .

Let  $\langle M_\alpha : \alpha \in \omega_1 \rangle$  be an elementary sequence for  $X$  and let  $M = \bigcup_{\alpha \in \omega_1} M_\alpha$ . It follows that the basic open subsets of  $X$  that are members of  $M$  form a  $\pi$ -base for  $X$ . We will define an elementary  $\omega_1$ -sequence of pairs.

For each  $\alpha$  we choose our pair  $x_\alpha \neq y_\alpha \in M_{\alpha+1}$  so that  $x_\alpha \upharpoonright M_\alpha = y_\alpha \upharpoonright M_\alpha$ . We first choose a non-empty basic open set  $U_\alpha$  from  $M_{\alpha+1}$  whose closure does not contain any non-empty basic open set that is a member of  $M_\alpha$ . We may do so since  $X$  is assumed to not have a countable  $\pi$ -base. Let  $x_\alpha$  be any point in  $U_\alpha \cap M_{\alpha+1}$ . Next, we note that  $\text{pr}_{M_\alpha}[X \setminus \text{cl}U_\alpha]$  is dense in  $X_{M_\alpha}$  so that  $\text{pr}_{M_\alpha}[X \setminus U_\alpha] = X_{M_\alpha}$ . Therefore we can take a closed subset  $Z_\alpha$  of  $X$ , disjoint from  $U_\alpha$ , such that  $\text{pr}_{M_\alpha}$  maps  $Z_\alpha$  irreducibly onto  $X_{M_\alpha}$ ; by elementarity, there is such a  $Z_\alpha$  in  $M_{\alpha+1}$ .

**Claim.** *There is a point  $y_\alpha \in Z_\alpha$  such that for each neighborhood  $W$  of  $y_\alpha$ , the set  $\text{pr}_{M_\alpha}^{-1}[X_{M_\alpha} \setminus \text{pr}_{M_\alpha}[\text{cl}W]]$  is not dense in a neighborhood of  $x_\alpha$ —in dual form:  $x_\alpha$  belongs to the closure of the interior of  $\text{pr}_{M_\alpha}^{-1}[\text{pr}_{M_\alpha}[\text{cl}W]]$  for every neighborhood of  $y_\alpha$ . By elementarity, such a point  $y_\alpha$  can be chosen to be a member of  $M_{\alpha+1}$ ; it is immediate that  $y_\alpha \neq x_\alpha$  and  $x_\alpha \upharpoonright M_\alpha = y_\alpha \upharpoonright M_\alpha$ .*

*Proof:* We prove the dual form: if not then there is a cover of  $Z_\alpha$  by basic open sets  $W$  for which  $x_\alpha \notin \text{cl int pr}_{M_\alpha}^{-1}[\text{pr}_{M_\alpha}[\text{cl}W]]$ . Take a finite subcover  $\{W_1, \dots, W_n\}$ ; then  $X_{M_\alpha} = \bigcup_{i=1}^n \text{pr}_{M_\alpha}[\text{cl}W_i]$  and so  $X = \bigcup_{i=1}^n \text{pr}_{M_\alpha}^{-1}[\text{pr}_{M_\alpha}[\text{cl}W_i]]$ . Now the union of the boundaries of

these sets is nowhere dense, so that in fact  $X = \bigcup_{i=1}^n \text{cl int } \text{pr}_{M_\alpha}^{-1}[\text{pr}_{M_\alpha}[\text{cl}W_i]]$ , a contradiction as  $x_\alpha$  was assumed not to belong to that union.

Now we show that the elementary  $\omega_1$ -sequence of pairs  $\langle \langle x_\alpha, y_\alpha \rangle : \alpha \in \omega_1 \rangle$  is not  $\omega_1$ -separated. Suppose that  $U$  and  $W$  are disjoint open subsets of  $X$  that have disjoint closures, and that there is an uncountable subset  $A$  of  $\omega_1$  such that  $x_\alpha \in U$  and  $y_\alpha \in W$  for all  $\alpha \in A$ . Let  $\mathcal{U}$  and  $\mathcal{W}$  denote the families of basic open subsets of  $X$  that are contained in  $U$  and  $W$  respectively. Since  $X$  is ccc and the family of basic open sets that are members of  $M$  forms a  $\pi$ -base for  $X$  there is a  $\delta < \omega_1$  such that the unions of  $\mathcal{U}_\delta = M_\delta \cap \mathcal{U}$  and  $\mathcal{W}_\delta = M_\delta \cap \mathcal{W}$  are dense in  $U$  and  $W$  respectively. Note that each member,  $O$ , of  $\mathcal{U} \cup \mathcal{W}$  has its support in  $M_\delta$  so that it satisfies  $O = \text{pr}_{M_\alpha}^{-1}[\text{pr}_{M_\alpha}[O]]$ .

Let  $\alpha \in A$  be larger than  $\delta$ . By construction each member of  $\mathcal{U}_\delta$  is disjoint from  $\text{cl}W$  and because its support is in  $M_\alpha$  it is also disjoint from  $\text{pr}_{M_\alpha}^{-1}[\text{pr}_{M_\alpha}[\text{cl}W]]$ . It follows that  $\bigcup \mathcal{U}_\delta$  is contained in  $\text{pr}_{M_\alpha}^{-1}[X_{M_\alpha} \setminus \text{pr}_{M_\alpha}[\text{cl}W]]$ . Since  $\bigcup \mathcal{U}_\alpha$  is dense in  $U$ , this contradicts the conditions in the claim.  $\square$

**Remark 2.3.** From [Theorem 2.2](#) it follows that no closed subset of a csD space can map onto the Tychonoff cube  $[0, 1]^{\omega_1}$ , and therefore, by Šapírovskii’s famous result from [7] that csD spaces have (many) points of countable  $\pi$ -character: every closed subset must have a point of countable  $\pi$ -character (relative to the closed set).

The theorem does not (yet) imply that csD spaces have countable tightness: for this one would need to show that the hereditary  $\pi$ -character is countable, see [8].

### 2.2. Weight $\omega_1$ fibered

We will see in [Theorem 2.9](#) that a similar application of elementary sequences will imply that a csD space will have a property stronger than countable tightness. This approach was inspired by the Juhász–Szentmiklóssy proof from [5] where it is shown that if a compact space does not have countable tightness, then it will contain a converging (free)  $\omega_1$ -sequence (also making essential use of Šapírovskii’s result). A csD space cannot contain a (co-countably) converging  $\omega_1$ -sequence. We will need a strengthening of this result. A point  $x$  is commonly called *condensation point* of a set  $A$  if every neighborhood of  $x$  contains uncountably many points of  $A$ .

**Proposition 2.4.** *Let  $A$  be an uncountable subset of a csD space  $X$  whose set of condensation points is metrizable, then there is a co-countable subset  $B$  of  $A$  with a metrizable closure.*

**Proof.** Let  $\langle M_\alpha : \alpha \in \omega_1 \rangle$  be an elementary sequence for  $X$  such that  $A \in M_0$  and put  $M = \bigcup_{\alpha < \omega_1} M_\alpha$ . Let  $K$  denote the (closed) set of condensation points of  $A$ ; it is also a member of  $M_0$ . Since  $K$  is compact metrizable and a member of  $M_0$ , it follows that  $\text{pr}_{M_0} \upharpoonright K$  is one-to-one.

We prove by contradiction that  $A \setminus K$  is countable. If  $A \setminus K$  is uncountable then we may choose for each  $\alpha$  a pair of points  $\langle x_\alpha, y_\alpha \rangle$  such that  $x_\alpha \in A \setminus K$ ,  $y_\alpha \in K$  and  $\text{pr}_{M_\alpha}(x_\alpha) = \text{pr}_{M_\alpha}(y_\alpha)$ .

Let  $J$  be any uncountable subset of  $\omega_1$  and let  $y \in K$  be a condensation point of  $\{x_\alpha : \alpha \in J\}$ . We show that  $y$  also belongs to the closure of  $\{y_\alpha : \alpha \in J\}$ .

Let  $U$  be a basic open neighborhood of  $y$  that is in  $M$  and take  $\beta$  such that the support of  $U$  is contained in  $M_\beta$ . There are uncountably many  $\alpha$  in  $J \setminus \beta$  for which  $x_\alpha \in U$  and for these  $\alpha$  we have  $y_\alpha \upharpoonright \beta = x_\alpha \upharpoonright \beta$  and hence also  $y_\alpha \in U$ . It follows that  $y \upharpoonright M$  is in the closure of  $\{y_\alpha \upharpoonright M : \alpha \in J\}$  and hence, because  $\text{pr}_M$  is one-to-one on  $K$  that  $y$  is in the closure of  $\{y_\alpha : \alpha \in J\}$ .  $\square$

We are now ready to prove our result about weight  $\omega_1$  fibered spaces. Remember from the introduction that  $X$  is weight  $\omega_1$  fibered if there is a continuous map  $f : X \rightarrow Y$  such that  $Y$  and every fiber  $f^{-1}(y)$  have weight at most  $\omega_1$ .

**Theorem 2.5.** *Each csD space that is weight  $\omega_1$  fibered is metrizable.*

**Proof.** Let  $X$  be a compact space that is weight  $\omega_1$  fibered but not metrizable. We will prove that  $X$  is not csD. Let  $\langle M_\alpha : \alpha \in \omega_1 \rangle$  be an elementary sequence for  $X$ . As usual we are assuming that  $X$  is a subspace of  $[0, 1]^\kappa$  for some cardinal  $\kappa \in M_0$ . Let  $M$  denote the union  $\bigcup_{\alpha \in \omega_1} M_\alpha$  and we recall that  $[x \upharpoonright M]$  denotes the set  $\{y \in X : y \upharpoonright M = x \upharpoonright M\}$ . Since  $X$  is weight  $\omega_1$  fibered, this will be witnessed by the elementary substructure  $M_0$ ; thus it is routine to verify that, for each  $x \in X$ , the set  $[x \upharpoonright M]$  has weight at most  $\omega_1$ . By Hušek's result, we may assume that each set  $[x \upharpoonright M]$  is metrizable. On the other hand, and also by Hušek's theorem, we may also assume that the weight of  $X$  is greater than  $\omega_1$ , so we may fix an  $x \in X$  such that  $[x \upharpoonright M] \neq \{x\}$ . We complete the proof by establishing two lemmas of independent interest. If  $[x \upharpoonright M]$  is not a  $G_\delta$ -set, then Lemma 2.6 will complete the proof. On the other hand if  $[x \upharpoonright M]$  is a  $G_\delta$ -set, then there will be some  $\delta < \omega_1$  such that  $[x \upharpoonright M] = [x \upharpoonright M_\delta]$  is metrizable and not a singleton. In this case Lemma 2.7 will complete the proof.  $\square$

**Lemma 2.6.** *If  $X$  is a space for which there is an elementary chain  $\langle M_\alpha : \alpha \in \omega_1 \rangle$  for  $X$  and a point  $x$  in  $X$  such that  $[x \upharpoonright M]$  is metrizable but not a  $G_\delta$ -set, then  $X$  is not csD.*

**Proof.** Since we are assuming that  $[x \upharpoonright M]$  is not a  $G_\delta$ -set there is no  $\delta \in \omega_1$  such that  $[x \upharpoonright M] = [x \upharpoonright M_\delta]$ . For each  $\alpha \in \omega_1$  choose  $x_\alpha \in [x \upharpoonright M_\alpha] \setminus [x \upharpoonright M]$ . For each  $\alpha$ , there is a  $\beta_\alpha$  such that  $x_\alpha \notin [x \upharpoonright M_{\beta_\alpha}]$ , hence the set  $A = \{x_\alpha : \alpha \in \omega_1\}$  is uncountable. The set of condensation points of  $A$  is contained in  $[x \upharpoonright M]$ ; and so, by Proposition 2.4,  $X$  is not csD.  $\square$

**Lemma 2.7.** *If for some  $x \in X$ , there is an elementary chain  $\langle M_\alpha : \alpha \in \omega_1 \rangle$  such that  $[x \upharpoonright M]$  is metrizable but not a singleton, then  $X$  is not csD.*

**Proof.** Let  $\langle M_\alpha : \alpha \in \omega_1 \rangle$  be an elementary chain and suppose that  $[x \upharpoonright M]$  is metrizable but not a singleton. By Lemma 2.6, we may assume that  $[x \upharpoonright M]$  is a  $G_\delta$ -set. That is, we may assume that there is a  $\delta \in \omega_1$  such that  $[x \upharpoonright M] = [x \upharpoonright M_\delta]$ . So  $[x \upharpoonright M_\delta]$  is metrizable and equal to  $[x \upharpoonright M_\alpha]$  for all  $\alpha \geq \delta$ . We apply elementarity to this statement to choose an elementary  $\omega_1$ -sequence that will not be  $\omega_1$ -separated. For  $\alpha \geq \delta$ , we make the observation that  $M_{\alpha+1}$  is a model of the statement

$$(\exists x \in X)([x \upharpoonright M_\alpha] = [x \upharpoonright M_\delta] \text{ is metrizable and not equal to } \{x\}).$$

By elementarity, we can take such a point  $x_\alpha \in X \cap M_{\alpha+1}$  and take  $y_\alpha \in [x_\alpha \upharpoonright M_\alpha] \cap M_{\alpha+1}$  witnessing that  $[x_\alpha \upharpoonright M_\alpha] \neq \{x_\alpha\}$ . Since  $[x_\alpha \upharpoonright M_\alpha]$  is a compact metrizable set which is a member of  $M_{\alpha+1}$ , the basic open sets in  $M_{\alpha+1}$  will contain a base for it.

The remainder of the proof follows that of Gruenhage for the metrizable fibered case (see [3]), because the subspace  $\bigcup\{[x_\alpha \upharpoonright M_\delta] : \alpha \in \omega_1 \setminus \delta\}$  is metrizable fibered over  $X_\delta$ . Indeed, it follows from the construction that  $x_\alpha \upharpoonright M_\delta \neq x_\beta \upharpoonright M_\delta$  for  $\delta < \alpha < \beta$ . Let  $J$  be any uncountable subset of  $\omega_1$ . Working in the space  $X_\delta$ , there is an  $\alpha \in J$  such that  $\text{pr}_{M_\delta}(x_\alpha)$  is a condensation point of the set  $\{\text{pr}_{M_\delta}(x_\beta) : \beta \in J\}$ . Since  $\text{pr}_{M_\delta}$  is a closed map and  $[x_\alpha \upharpoonright M_\delta]$  is compact, there is a point  $z \in [x_\alpha \upharpoonright M_\delta]$  that is a condensation point of  $\{x_\beta : \beta \in J \setminus \alpha\}$ . Although  $z$  itself may not be a member of  $M_{\alpha+1}$ , the basic open sets from  $M_{\alpha+1}$  contain a local base at  $z$ . In addition, for each basic open neighborhood  $U$  of  $z$  from  $M_{\alpha+1}$  and for each  $\beta \in J \setminus \alpha + 1$  we have  $x_\beta \in U$

if and only if  $y_\beta \in U$ . It therefore follows that  $z$  is also a condensation point of  $\{y_\beta : \beta \in J\}$ . This shows that  $\langle \langle x_\beta, y_\beta \rangle : \beta \in \omega_1 \rangle$  is not  $\omega_1$ -separated; and completes the proof that  $X$  is not csD.  $\square$

### 2.3. Luzin sets

A set of reals is a Luzin set if every uncountable subset is dense in some interval. Luzin sets exist if the Continuum Hypothesis holds but Martin's Axiom plus the negation of Continuum Hypothesis implies they do not exist. The existence of Luzin sets has some influence on the structure of csD spaces.

**Theorem 2.8.** *If there is a Luzin set, then every csD space contains points that are  $G_\delta$ -sets.*

**Proof.** Let  $X$  be a csD space. If  $X$  has any isolated points then there is nothing to prove, so assume that it does not. Let  $M$  be a countable elementary substructure of  $H(\theta)$  that contains  $X$ . Then  $\text{pr}_M[X]$  is a compact metrizable space with no isolated points. If there is any point  $x \in X$  such that  $[x \upharpoonright M] = \{x\}$  then we are done as well. Let  $Y$  be a dense subset of  $\text{pr}_M[X]$  that is homeomorphic to the space of irrational numbers, and let  $L \subset Y$  be a dense Luzin set of cardinality  $\omega_1$ . For each  $z \in L$  choose distinct  $x_z, y_z \in X$  so that  $\text{pr}_M(x_z) = \text{pr}_M(y_z) = z$ . We show that  $\langle \langle x_z, y_z \rangle : z \in L \rangle$  is not  $\omega_1$ -separated. Let  $A$  be an uncountable subset of  $L$ ; we show that the closures of  $\{x_z : z \in A\}$  and  $\{y_z : z \in A\}$  intersect.

Since  $L$  is Luzin, there is a basic open set  $W$  in  $M$  such that  $A$  contains a dense subset of  $\text{pr}_M[W]$ . Since  $X$  has a dense set of points of countable  $\pi$ -character we may choose  $x \in W \cap M$  so that it has a countable local  $\pi$ -base  $\mathcal{B}$  consisting of basic open sets from  $M$  that are contained in  $W$ . However, since each member of  $\mathcal{B}$  belongs to  $M$  and  $A \cap W$  is dense in  $W$  each member of  $\mathcal{B}$  contains  $x_z$  and  $y_z$  for some  $z$ ; this implies that  $x$  is in the closure of both  $\{x_z : z \in A\}$  and  $\{y_z : z \in A\}$ .  $\square$

### 2.4. Local $\pi$ -bases and nets

A family  $\mathcal{F}$  of nonempty closed sets is a local  $\pi$ -net at a point  $x$  if every neighborhood of  $x$  contains a member of  $\mathcal{F}$ . If  $\mathcal{F}$  is a countable family of  $G_\delta$ -sets and is a local  $\pi$ -net at  $x$ , then  $x$  has a countable local  $\pi$ -base, provided the ambient space is compact.

**Theorem 2.9.** *Let  $K$  be a compact  $G_{\omega_1}$ -set in a csD space  $X$ . Then each countable local  $\pi$ -base in  $K$  expands to a countable local  $\pi$ -net in  $X$  consisting of  $G_\delta$ -sets.*

**Proof.** For ease of exposition we will assume that  $X$  is zero-dimensional; the modifications for the general case are tedious but straightforward. Let  $x \in K$  and let  $\{b_n : n \in \omega\}$  be a family of relatively clopen subset of  $K$  such that each neighborhood of  $x$  contains one. For each  $n$  let  $\mathcal{K}_n = \{K_\alpha^n : \alpha \in \omega_1\}$  be a filter base of clopen sets such that  $b_n = \bigcap \mathcal{K}_n$ . Fix an ultrafilter  $\mathcal{U}$  on  $\omega$  so that for each neighborhood  $U$  of  $x$ , the set  $\{n : b_n \subset U\}$  is a member of  $\mathcal{U}$ . Fix an elementary  $\omega_1$ -sequence  $\langle M_\alpha : \alpha \in \omega_1 \rangle$  for  $X$  so that  $x, K, \{\mathcal{K}_n : n \in \omega\}$  and  $\mathcal{U}$  are in  $M_0$ . For each  $\alpha \in \omega_1$ , assume that the family  $\{Z(\alpha, n) : n \in \omega\}$  is not a local  $\pi$ -net at  $x$  where  $Z(\alpha, n) = \bigcap \{K_\beta^n : \beta \in M_\alpha\}$ . Let  $U_\alpha \in M_{\alpha+1}$  be a clopen set containing  $x$  so that  $Z(\alpha, n) \setminus U_\alpha \neq \emptyset$  for each  $n \in \omega$ . Choose  $y_\alpha \in M_{\alpha+1}$  so that  $y_\alpha$  is in the  $\mathcal{U}$ -limit of the sequence  $\langle Z(\alpha, n) \setminus U_\alpha : n \in \omega \rangle$ . It follows that  $\text{pr}_{M_\alpha}(y_\alpha) = \text{pr}_{M_\alpha}(x)$ , and so  $\langle \langle x, y_\alpha \rangle : \alpha \in \omega_1 \rangle$  is an elementary  $\omega_1$ -sequence and we prove it is not  $\omega_1$ -separated by showing that  $\langle y_\alpha : \alpha < \omega_1 \rangle$  converges co-countably to  $x$ .

Let  $W$  be any neighborhood of  $x$  and let  $U = \{n \in \omega : b_n \subset W\}$ . For each  $n \in U$  there is, by compactness some  $\alpha_n < \omega_1$  so that  $K_{\alpha_n}^n \subset W$ . Let  $\alpha \in \omega_1$  be larger than all  $\alpha_n$ . It follows that  $Z(\delta, n) \subset W$  for all  $n \in U$  and all  $\delta \geq \alpha$ . Since  $y_\delta$  is in the closure of  $\bigcup\{Z(\delta, n) : n \in U\}$ , we find that  $\{y_\delta : \delta > \alpha\} \subset W$ .  $\square$

**Corollary 2.10.** *If  $A$  is an uncountable subset of a csD space, then  $A$  has a condensation point which has a countable local  $\pi$ -base in  $\text{cl}A$ .*  $\square$

## 2.5. OCA and sequential compactness

In the paper [1] it is shown that the Proper Forcing Axiom (PFA) implies that all csD spaces are metrizable. We can use the results of this paper to give a shorter proof for sequentially compact csD spaces, and one that uses only a consequence of Todorćević's open coloring axiom (OCA). PFA implies that compact spaces with countable tightness are sequential, but we do not know if OCA (or ZFC!) implies that csD spaces are sequentially compact.

First we prove a strengthening of Gruenhage's result that shows how badly non-metrizably fibered a csD non-metrizable space would have to be.

**Lemma 2.11.** *If  $\{M_\alpha : \alpha \in \omega_1\}$  is an elementary chain for a non-metrizable csD space  $X$ , then there is a  $\delta \in \omega_1$  such that the set of non-metrizable sets in  $\{[x \upharpoonright M_\delta] \cap X_0 : x \in X\}$  contains a perfect set.*

**Proof.** It is implicit in [4] that a non-metrizable csD space (of countable tightness) will contain a separable non-metrizable subspace. By elementarity  $M_0$  will contain such a separable set, and so  $X_0 = \text{cl}(X \cap M_0)$  will itself not be metrizable. Fix any elementary  $\omega_1$ -sequence of pairs  $\langle \langle x_\alpha, y_\alpha \rangle : \alpha \in \omega_1 \rangle$  for the sequence  $\langle M_\alpha : \alpha \in \omega_1 \rangle$ , but chosen so that  $\{x_\alpha, y_\alpha\} \subset X_0$  for all  $\alpha$ . Let  $A$  be an uncountable subset of  $\omega_1$  witnessing that the sequence is  $\omega_1$ -separated. Find a  $\delta \in \omega_1$  so that for each basic open set  $U$  from  $M_\delta$  the implication “if  $U \cap \{x_\alpha : \alpha \in A\}$  is uncountable, then  $U \cap \{x_\alpha : \alpha \in A \setminus M_\beta\}$  is infinite for all  $\beta < \delta$ ” holds. Let  $K \subset X_{M_\delta}$  be the projection of  $\text{cl}\{x_\alpha : \alpha \in A \setminus M_\delta\}$  by the map  $\text{pr}_{M_\delta}$ . It follows from the choice of  $\delta$  that  $K$  is a perfect set. In addition each point in  $K$  is a limit point of  $\text{pr}_{M_\delta}[\{x_\alpha : \alpha \in A \setminus M_\gamma\}]$  for all  $\gamma \in \omega_1$ .

We show that for each  $z$  in  $K$  the set  $[z] = \text{pr}_{M_\delta}^{-1}(z)$  is not metrizable. We assume we have a  $z$  in  $K$  such that  $[z]$  is metrizable and derive a contradiction. For each  $x \in [z]$  the set  $[x \upharpoonright M]$  is metrizable and a  $G_\delta$ -set because it is a subset of  $[z]$ . By Lemma 2.7 it follows that  $[x \upharpoonright M] = \{x\}$ . Therefore, the mapping  $\text{pr}_M$  restricted to  $[z]$  is a homeomorphism. Since  $\text{pr}_M[[z]]$  is a compact metrizable subset of  $[0, 1]^{K \cap M}$  there is some  $\alpha_z < \omega_1$  such that the basic open sets in  $M_{\alpha_z}$  contain a base for  $[z]$ . Now choose a sequence  $\langle \beta_n : n \in \omega \rangle$  in  $A \setminus \alpha_z$  so that  $\langle \text{pr}_{M_\delta}(x_{\beta_n}) : n \in \omega \rangle$  converges to  $z$ . There is a point  $x \in [z]$  that is a cluster point of the sequence  $\langle x_{\beta_n} : n \in \omega \rangle$ . By thinning out we can assume that the latter sequence converges to  $x$ . Since the sequence  $\langle \text{pr}_{M_{\alpha_z}}(x_{\beta_n}) : n \in \omega \rangle$  converges to  $\text{pr}_{M_{\alpha_z}}(x)$  in  $X_{\alpha_z}$ , the sequence  $\langle \text{pr}_{M_{\alpha_z}}(y_{\beta_n}) : n \in \omega \rangle$  converges to  $\text{pr}_{M_{\alpha_z}}(x)$  as well. Since  $\langle y_{\beta_n} : n \in \omega \rangle$  accumulates at some point in  $[z]$  and  $\text{pr}_{M_{\alpha_z}}$  is one-to-one on  $[z]$ , it follows that  $x$  is a limit of  $\langle y_{\beta_n} : n \in \omega \rangle$ .  $\square$

We will make use of the following application of OCA by Todorćević. Let  $\mathcal{X}$  be a family of disjoint pairs of subsets of a countable set  $S$ . Say that the family  $\mathcal{X}$  is *countably separated* if there is a countable family  $\mathcal{Y}$  of subsets of  $S$  such that for each pair  $\langle a, b \rangle \in \mathcal{X}$ , there is a  $Y \in \mathcal{Y}$  such that  $a \setminus Y$  and  $b \cap Y$  are both finite — in case  $\mathcal{Y}$  has just one element we say that  $\mathcal{X}$  is *separated*. The following result is taken from [2, p. 145] and is attributed to Todorćević.



**Proposition 2.12 (OCA).** *If a family  $\mathcal{X}$  of disjoint pairs of subsets of a countable set  $S$  is not countably separated, then there is an uncountable subcollection  $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\}$  of  $\mathcal{X}$  with the property that whenever  $\alpha \neq \beta$  in  $\omega_1$  the set  $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$  is not empty. In particular, for all uncountable  $A \subset \omega_1$  the collection  $\{(a_\alpha, b_\alpha) : \alpha \in A\}$  is not separated.*

**Theorem 2.13 (OCA).** *If  $X$  is a sequentially compact csD space then  $X$  is metrizable.*

**Proof.** Assume that  $X$  is not metrizable, and apply Lemma 2.11. Let  $S$  denote the countable set  $M_0 \cap X$ . We work in the non-metrizable subspace  $X_0$ . Let  $f$  denote the projection map from  $X_0$  onto  $X_{M_0}$ , which is onto by elementarity. Let  $Z$  be a perfect set of points of  $X_0$  with the property that  $f^{-1}(z) = [z]$  is not metrizable for each  $z \in Z$ . Let  $\{\mathcal{Y}_z : z \in Z\}$  be a listing of all countable sequences of subsets of  $S$ . For each  $z \in Z$  we will show that there are disjoint subsets  $a_z$  and  $b_z$  of  $S$  that are not separated by  $\mathcal{Y}_z$  and converge to distinct points of  $[z]$ .

Let  $\{Y_n : n \in \omega\}$  be an enumeration of  $\mathcal{Y}_z$ . For each  $n \in \omega$ , let  $Y_n^0 = S \setminus Y_n$  and  $Y_n^1 = Y_n$ . For each function  $h \in 2^\omega$ , let  $[z]_h$  denote the closed set  $\bigcap_{n \in \omega} \text{cl} \bigcap_{i < n} Y_i^{h(i)}$ . Since  $[z]$  is not metrizable, there must be some  $h \in 2^\omega$  such that  $[z]_h$  is not a singleton. Choose open sets  $U$  and  $W$  of  $X_0$ , with disjoint closures, that both intersect  $[z]_h$ . Additionally, fix a descending neighborhood base  $\{U_n : n \in \omega\}$  for the  $G_\delta$ -set  $[z]$ . Each of the families  $\{U \cap U_n \cap \bigcap_{i < n} Y_i^{h(i)} : n \in \omega\}$  and  $\{W \cap U_n \cap \bigcap_{i < n} Y_i^{h(i)} : n \in \omega\}$  are descending sequences of infinite subsets of  $S$ . There are infinite sets  $a_z$  and  $b_z$  such that for each  $n$ ,  $a_z$  is almost contained in  $U \cap U_n \cap \bigcap_{i < n} Y_i^{h(i)}$  and  $b_z$  is almost contained in  $W \cap U_n \cap \bigcap_{i < n} Y_i^{h(i)}$ . Since we are assuming that  $X$  is sequentially compact, we may assume that  $a_z$  converges to a point  $x_z \in [z] \cap \text{cl}U$  and  $b_z$  converges to a point  $y_z \in [z] \cap \text{cl}W$ .

Now we apply Proposition 2.12 to the family  $\mathcal{X} = \{(a_z, b_z) : z \in Z\}$ . It is evident by the construction that this family is not countably separated. Therefore there is an uncountable subset  $Y$  of  $Z$  such that for all uncountable  $A \subset Y$  the families  $\{a_z : z \in A\}$  and  $\{b_z : z \in A\}$  cannot be separated.

It now follows easily that the  $\omega_1$ -sequence of pairs  $\langle (x_z, y_z) : z \in Y \rangle$  is not  $\omega_1$ -separated. To see this assume that  $U$  is an open set containing  $\{x_z : z \in A\}$  for some uncountable  $A$ . Then  $a_z \setminus U$  will be finite for each  $z \in A$ . By considering all possible uncountable subsets of  $A$  it follows that for all but countably many  $z$  in  $A$  the intersection  $b_z \cap U$  is infinite and hence  $y_z \in \text{cl}U$  for all these  $z$ .  $\square$

### 3. Questions

Needless to say, the main open problem is to determine if every csD space is metrizable. However, here are some other questions which certainly seem difficult and interesting.

**Question 3.1.** Assume that  $X$  is an infinite csD space.

- (1) Is  $X$  sequentially compact? Does it even contain a non-trivial converging sequence? Does  $X$  contain a point of countable character?
- (2) If  $X$  is first countable, is it metrizable?
- (3) Does  $X$  contain a copy of the Cantor set?
- (4) Can the cardinality of  $X$  be greater than  $2^{\aleph_0}$ ?
- (5) Is each countably compact subset closed?
- (6) Does Martin's Axiom imply that  $X$  is metrizable?
- (7) If  $X$  is hereditarily separable, is it metrizable? Gruenhage [3] has shown that hereditarily Lindelöf csD spaces are metrizable.

Another question is what happens if we modify the csD property by analogy with separated versus countably separated families of pairs.

**Question 3.2.** Define  $X$  to be  $\sigma$ -sD to mean that for each collection  $\{\langle x_\alpha, y_\alpha \rangle : \alpha \in \omega_1\}$  of pairs from  $X$ , there is a countable cover  $\{A_n : n \in \omega\}$  of  $\omega_1$ , such that for each  $n$  the sets  $\{x_\alpha : \alpha \in A_n\}$  and  $\{y_\alpha : \alpha \in A_n\}$  have disjoint closures. If  $X$  is  $\sigma$ -sD, is it metrizable?

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