Note

The Johnson–Schechtman space has the 6-bounded approximation property ✩

Indrek Zolk

Faculty of Mathematics and Computer Science, Tartu University, J. Liivi 2, 50409 Tartu, Estonia

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A B S T R A C T

In 2001 G. Godefroy proved that a subspace $X_{JS}$ of $c_0$ constructed by W.B. Johnson and G. Schechtman in 1996 has the $\lambda$-bounded approximation property with $\lambda \leq 8$. This paper slightly improves Godefroy’s proof establishing that $\lambda \leq 6$.

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1. Let $X$ be a Banach space (over $K = \mathbb{R}$ or $\mathbb{C}$). If for every compact set $K$ and every $\epsilon > 0$ there is a bounded linear finite rank operator $T$ on $X$ such that $\|Tx - x\| \leq \epsilon$ for every $x \in K$, then $X$ is said to have the approximation property. If the norms of such operators are uniformly bounded by $\lambda$, then $X$ is said to have the $\lambda$-bounded approximation property. The 1-bounded approximation property is called the metric approximation property.

A sequence of bounded linear finite rank operators $(P_m)$ on $X$ such that $P_mP_n = P_{\min(m,n)}$, $m, n \in \mathbb{N}$, and $\lim_m P_m x = x$ for every $x \in X$, is called a finite dimensional decomposition of $X$. The number $\sup_m \|P_m\|$ is called the decomposition constant of $(P_m)$. A Banach space having a finite dimensional decomposition with the decomposition constant $\lambda$ also enjoys the $\lambda$-bounded approximation property (for more about these properties see, e.g., [1]).

It is a well-known result of Grothendieck [4, Chapter I, “Proposition” 37] that if there exists a Banach space which fails the approximation property, then there also exists a subspace of $c_0$ that fails the approximation property (see, e.g., [9, p. 37]). Hence, relying on Enflo’s theorem [2], let $Y = \bigcup_n Y_n$ be a subspace of $c_0$ failing the approximation property, where $(Y_n)$ is an increasing sequence of finite dimensional subspaces of $Y$. We denote by $c(Y_n)$ the Banach space of norm-convergent sequences $(y_n) \subset Y$, where $y_n \in Y_n$, $n \in \mathbb{N}$, with respect to the supremum norm.

The Johnson–Schechtman space $X_{JS}$ (constructed by W.B. Johnson and G. Schechtman in 1996 and published in [6]) is an isomorphic copy of $c(Y_n)$ in $c_0$. The key of the construction is [7, p. 51, observation of J. Lindenstrauss]: if a Banach space $X$ has a subspace so that both the subspace and the quotient space with respect to it embed into $c_0$, then so does $X$ itself.

G. Godefroy has proven in [3, Theorem VI.3] that $X_{JS}$ has a finite dimensional decomposition with the decomposition constant not exceeding 8. He wrote in [3, Ch. VII, §VI] that no effort had been made in the proof to tighten the constant and it is unlikely that 8 were the critical value. The main aim of this paper is to tighten the constant to 6.

2. The following—the main result of this paper—is a slight improvement of [3, Theorem VI.3].

Theorem 1. The Johnson–Schechtman space $X_{JS}$ has a finite dimensional decomposition with the decomposition constant not greater than 6, but $X_{JS}$ fails the metric approximation property.

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E-mail address: indrek.zolk@ut.ee.
The proof in [3] goes in two parts: first the construction of $X_{JS}$ and the finite dimensional decomposition, and second, showing that $X_{JS}$ fails the metric approximation property. We need to go through only the first part. For the second part, we refer the reader to [3, p. 21].

**Proof.** Let $Y = \bigcup Y_n$ be a subspace of $c_0$ failing the approximation property, $\dim Y_n < \infty$, $n \in \mathbb{N}$, and $Y_1 \subset Y_2 \subset \cdots$. Define a quotient map $L: c(Y_n) \rightarrow Y$ by $L(y_n) = \lim_n y_n$, thus ker $L = c_0(Y_n)$ (the subspace of $c(Y_n)$ consisting of norm-decaying sequences). Define an isometric embedding $T: c_0(Y_n) \rightarrow c_0$ (for instance, having $(y_n) \in c_0(Y_n)$, where $y_n = (\xi_k^n)_{k \in \mathbb{N}}$, one can define $T(y_n) = (\xi_1^n, \xi_2^n, \xi_3^n, \xi_4^n, \xi_5^n, \xi_6^n, \xi_7^n, \ldots)$.

Extend $T$ to an operator $\tilde{T}: c(Y_n) \rightarrow c_0$ using Sobczyk’s theorem [12]: we have $\|\tilde{T}\|_{c(Y_n)} = \|T\| + \|\tilde{T}\|_{\leq 2\|T\|}$. Let the coordinate functions on $c_0$ be $e_k^n, k \in \mathbb{N}$; we need the expression of $\tilde{T}(y_n) = (x_k^n - t_k^n(y_n))_k$, where $x_k^n$ are Hahn–Banach extensions of functionals $y_k^n = T^*(e_k^n) \in c_0(Y_n)^*$. Also, $x_k^n, t_k^n \in \|T\|B_{c_0(Y_n)}$ and $t_k^n$ is null on $c_0(Y_n)$ for all $k$ (see, e.g., [3, proof of Theorem II.1]). It can be easily verified that $V : c(Y_n) \rightarrow c_0 \oplus_{\infty} c_0 \cong c_0, V(y_n) = (\tilde{T}(y_n), L(y_n))$, is an isomorphism into $c_0$ with $\|V\| \leq 2$. The next step in [3, proof of Theorem VI.3] now yields $\|V^{-1}|_{\text{ran} V} \leq 4$; we shall present an argument that gives $\|V^{-1}|_{\text{ran} V} \leq 3$.

Assume that $\|V^{-1}|_{\text{ran} V} \geq 3$. As there exists a sequence $(y_n) \in c(Y_n)$ such that $\|y_n\| > 3$, $\lim_n \|y_n\| < 1$ and $\|\tilde{T}(y_n)\| < 1$. Let $N \in \mathbb{N}$ be an index such that $\sup_{n \geq N} \|y_n\| < 1$. Split $(y_n)$ into two parts: $(y_0^n) = (y_1, \ldots, y_{N-1}, 0, 0, \ldots)$ and $(y_N^n) = (y_N - y_0^n)$. Of course $\|y_0^n\| > 3, (y_N^n) \in c_0(Y_n)$ and $\|y_N^n\| < 1$.

Due to the inequality $\sup_n \|y_1^n(y_N^n)\| = \|T(y_N^n)\| = \|y_N^n\| > 3$, we find an index $m \in \mathbb{N}$ for which $\|y_m^n(y_N^n)\| > 3$. As $\sup_n \|x_m^n(y_N^n)\| = \|\tilde{T}(y_N^n)\| < 1$, we also have the inequality $\|y_m^n(y_N^n)\| < 3$. Bearing in mind that $t_m^n(y_N^n) = t_m^n(y_N^n)$, we have

\[
|x_m^n(y_N^n)| + 1 \leq |x_m^n(y_0^n) - t_m^n(y_N^n)| + |x_m^n(y_0^n)| + 1 + 2 + \|t_m^n\| \leq \|y_m^n(y_0^n)\| = |x_m^n(y_0^n)| < 1,
\]

a contradiction. Therefore $\|V^{-1}|_{\text{ran} V} \leq 3$.

Denote $|X_5| = \text{ran} V$ and $P_m = V Q_m V^{-1}$, where $Q_m(y_n) = (y_1, \ldots, y_{m-1}, y_m, y_{m+1}, \ldots), m \in \mathbb{N}$. It is straightforward to verify that $(P_m)$ is a finite dimensional decomposition of $X_{JS}$ and $\sup_m \|P_m\| \leq 6$. □

**Remark 2.** By [10, Corollary 2.5 and Remark 2.4], $X_{JS}$ fails the metric $A$-approximation property for any operator ideal $A$ which is contained in the union of weakly compact, strictly singular, and completely continuous operators. By [11, Corollary 3.8], there exist a separable reflexive Banach space $Z$ and a compact linear operator $T : X_{JS} \rightarrow Z$ such that for every net $(T_a)$ of finite rank operators from $X_{JS}$ to $Z$ converging strongly to $T$, there holds $\sup_a \|T_a\| > \|T\|$; in particular, $X_{JS}$ fails the weak metric approximation property (see [8]).

The following corollary is immediate.

**Corollary 3.** The Johnson–Schechtman space $X_{JS}$ has the $\lambda$-bounded approximation property with $\lambda \leq 6$.

Note that the proof of Theorem 1 is useful for any subspace of $c_0$ as a starting point, yielding a finite dimensional decomposition with the decomposition constant not greater than $6$ on the constructed space. Also note that every Banach space constructed in this manner has the commuting bounded approximation property; hence this construction cannot provide any information on a well-known open problem whether every Banach space with the bounded approximation property has the commuting bounded approximation property.

3. One says that a Banach space $X$ is $M$-embedded if the canonical projection $\pi_X$ from $X^{***}$ onto $X^*$ satisfies the inequality

\[
\|X^{***} - \pi_X X^{***}\| + \|\pi_X X^{***}\| \leq \|X^{***}\|, \quad X^{***} \in X^{***}.
\]

$M$-embeddedness inherits to subspaces and quotient spaces (see, e.g., [5, p. 111]). A well-known example of an $M$-embedded Banach space is $c_0$.

The Banach–Mazur distance between Banach spaces $X$ and $Y$ is defined as $d_{BM}(X, Y) = \inf\{\|T\|\|T^{-1}\| : T : X \rightarrow Y$ is an isomorphism\}. (If $X$ and $Y$ are not isomorphic, one defines $d_{BM}(X, Y) = \infty$.)
Theorem 4. (See [3, Corollary VI.2].) Let $X$ be a separable $M$-embedded space. If there exists a Banach space $Y$ with the metric approximation property such that $d_{BM}(X, Y) < 2$, then $X$ has the metric approximation property.

Merging the last result (note that it also applies to subspaces and quotient spaces of $c_0$) with Theorem 1, we have:

Corollary 5. For every Banach space $Y$ with the metric approximation property, there holds $d_{BM}(X_{JS}, Y) \geq 2$. On the other hand, there exists a Banach space $Y$ with the metric approximation property for which $d_{BM}(X_{JS}, Y) \leq 6$.

The question which is the greatest value of $\lambda$ that would guarantee the metric approximation property to pass over from a Banach space $Y$ to any separable $M$-embedded space $X$ with $d_{BM}(X, Y) < \lambda$, is yet open.

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