NOTE

Generalized Split Graphs and Ramsey Numbers

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A graph $G$ is called a $(p, q)$-split graph if its vertex set can be partitioned into $A, B$ so that the order of the largest independent set in $A$ is at most $p$ and the order of the largest complete subgraph in $B$ is at most $q$. Applying a well-known theorem of Erdős and Rado for $A$-systems, it is shown that for fixed $p, q$, $(p, q)$-split graphs can be characterized by excluding a finite set of forbidden subgraphs, called $(p, q)$-split critical graphs. The order of the largest $(p, q)$-split critical graph, $f(p, q)$, relates to classical Ramsey numbers $R(s, t)$ through the inequalities

$$2\left(\phi(F(R(p + 2, q + 2))) + 1\right) \geq f(p, q) \geq R(p + 2, q + 2) - 1$$

where $F(t)$ is the smallest number of $t$-element sets ensuring a $t + 1$-element $A$-system. Apart from $f(1, 1) = 5$, all values of $f(p, q)$ are unknown.

Split graphs have been introduced by Földes and Hammer in [FH] as graphs whose vertices can be partitioned into a complete graph and an independent set. It was proved in [FH] that split graphs can be characterized by the exclusion of three induced subgraphs: $C_4, 2K_2$, and $C_5$. (The same result is obtained independently in a slightly more general form in [GL] in the context of 2-track interval systems.) A natural generalization of split graphs have been considered in [EG]; here we shall use a special case of that definition. First some more or less standard terminology is summarized.

We consider finite undirected simple graphs $G = (V, E)$, where $V, E$ are the vertex set and edge set of $G$, respectively. The numbers $|V|, |E|$ are called the order and the size of the graph $G$. For $A \subseteq V$, $G[A]$ denotes the subgraph of $G$ induced by $A$, if $G$ is clear from the context, $[A]$ will be used. As usual, $\alpha(G)$ denotes the order of the largest independent set of $G$.

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and \( o(G) \) denotes the order of the largest complete subgraph of \( G \). The set \( \{1, 2, \ldots, k\} \) is abbreviated as \( [k] \).

A graph \( G = (V, E) \) is called a \((p, q)\)-split graph if \( V \) can be partitioned into sets \( A, B \) so that \( x(G[A]) \leq p \) and \( o(G[B]) \leq q \). The partition \( [A, B] \) will be called a \((p, q)\)-split partition of \( G \).

A graph \( G \) is called \((p, q)\)-split critical if \( G \) is not a \((p, q)\)-split graph but all proper induced subgraphs of \( G \) are \((p, q)\)-split graphs. In a \((p, q)\)-split critical graph \( G \) each vertex \( v \) defines a \((p, q)\)-split partition \([A_v, B_v]\) on \( V(G) \setminus v \) and there exists an \( o\)-witness, \( S_v \), which is an independent set of \( p + 1 \) vertices containing \( v \) and disjoint from \( B_v \). Similarly, there exists an \( o\)-witness, \( K_v \), a complete subgraph of \( G \) with \( q + 1 \) vertices which contains \( v \) and is disjoint from \( A_v \).

Note that a graph \( G \) is \((p, q)\)-split if and only if its complement \( G \) is \((q, p)\)-split. A similar remark is valid for split critical graphs. In particular, \((p, p)\)-split graphs and \((p, p)\)-split critical graphs form self-complementary families of graphs.

In [EG] the minimum order of split critical graphs studied (under a more general definition). This paper is focused on the maximum order of a \((p, q)\)-split critical graph. It is not clear a priori that the maximum is finite, the main result of the paper is the proof of the finiteness (Theorem 1). This allows us to define the function \( f(p, q) \) as the maximum number of vertices in a \((p, q)\)-split critical graph. It is worth mentioning that Theorem 1 is an existence theorem, giving an upper bound which is probably very far from the actual value of \( f(p, q) \) which might be difficult to determine in general.

A more restricted family of graphs, the perfect \((p, q)\)-split critical graphs is the same as the class of graphs studied by Kézdy et al. in [KSW]. They proved (Theorem 2.5 in [KSW]) that there are finitely many perfect \((p, q)\)-split critical graphs. Theorem 1 is a more general result but the bound of Kézdy et al. for the maximum number of vertices of a perfect \((p, q)\)-split critical graph is better than the general bound coming from Theorem 1.

The immediate corollary of Theorem 1 is that \((p, q)\)-split graphs can be characterized by the exclusion of finitely many induced subgraphs (Corollary 1). An explicit description of these graphs (like \( C_4, 2K_2, C_5 \) for \((1, 1)\)-split critical graphs) is not expected since (as Proposition 2 will show) all Ramsey graphs are among them.

The easiest example of a \((p, q)\)-split critical graph is the graph \((p + 1)K_{q+1} \), i.e. \( p + 1 \) vertex-disjoint copies of the complete graph \( K_{q+1} \). This example is of minimum order as the next proposition shows.

**Proposition 1.** Graphs of order at most \( pq + p + q \) are \((p, q)\)-split graphs.
Proof. Let $G$ be a graph of order at most $pq + p + q$. Select the maximum number of vertex disjoint complete subgraphs of $G$, each with $q + 1$ vertices. The vertices covered by these complete graphs span a subgraph of $G$ with no independent set of size $p + 1$ and the uncovered vertices span a subgraph of $G$ with no complete subgraph of size $q + 1$. Thus $G$ is a $(p, q)$-split graph.

To get much larger examples of $(p, q)$-split critical graphs, let $R(s, t)$ be the classical Ramsey number, the smallest integer $N$ for which every graph $G$ of order $N$ satisfies either $\alpha(G) \geq s$ or $\omega(G) \geq t$. $(s, t)$-Ramsey graphs are the graphs $G$ of order $R(s, t) - 1$ for which $\alpha(G) < s$ and $\omega(G) < t$. Ramsey numbers (and Ramsey graphs) are known only for small values of $s$ and $t$. It is known that the $(3, 3)$-, $(3, 4)$-, and $(4, 4)$-Ramsey graphs are unique (the first is the pentagon, the others can be found e.g. in [GRS]).

Proposition 2. $(p + 2, q + 2)$-Ramsey graphs are $(p, q)$-split critical.

Proof. Let $G$ be a $(p + 2, q + 2)$-Ramsey graph. Assume that $G$ is $(p, q)$-split with split partition $A, B$. Then one can add a new vertex to $G$ adjacent to all vertices of $B$ (and to no vertices of $A$). The resulting graph $G^*$ has $R(p + 2, q + 2)$ vertices and $\alpha(G^*) \leq p + 1$, $\omega(G^*) \leq q + 1$. This contradicts the definition of the Ramsey number $R(p + 2, q + 2)$. Thus $G$ is not a $(p, q)$-split graph. On the other hand, for each $v \in V(G)$ the sets $A_v$, $B_v$ can be defined as the set of vertices non-adjacent, respectively adjacent to $v$. Since $G$ is a Ramsey graph, $\alpha(G[A_v]) \leq p$, $\omega(G[B_v]) \leq q$ follows immediately. Therefore $G$ is $(p, q)$-split critical.

It is tempting to conjecture that the Ramsey graphs are the largest split critical graphs. For $(1, 1)$-split critical graphs this follows from the split graph characterization theorem cited above. However, for $p = 1, q = 2$, the $(3, 4)$-Ramsey graph has eight vertices ($R(3, 4) = 9$) but the graph obtained from the regular 9-gon by adding three pairwise non-intersecting shortest diagonals is an example of a $(1, 2)$-split critical graph. In fact, it is not clear whether this is a largest $(1, 2)$-split critical graph (even the claim that there is a largest one seems to be nontrivial). The following graph $G_{18}$ on 18 vertices also beats by one the famous $(4, 4)$-Ramsey graph. Let $M$ denote the six-vertex graph obtained by joining a new vertex to two non-consecutive vertices of a five cycle. The graph $M$ has two vertices with the same set of neighbors, call them special vertices. Then $G_{18}$ is defined by arranging 18 vertices into a $3 \times 6$ matrix in which each column forms a triangle and each row is isomorphic to a copy of $M$ arranged so that the six special vertices of the three copies occupy distinct columns.

Proposition 3. The graph $G_{18}$ is a $(2, 2)$-split critical graph.
Proof. Assume that \([A, B] \) is a split partition of \(G_{18}\). Since \(\omega([B]) \leq 2\), \(B\) has at most two vertices from each column of \(G_{18}\). Therefore one can select a six element subset \(C \subseteq A\) with all vertices of \(C\) from different columns. Now \(\tau([C]) \leq 2\) because \(C \subseteq A\) and \(\omega([C]) \leq 2\) because only the columns form triangles in \(G_{18}\). This contradicts to \(R(3, 3) = 6\).

Let \(v\) be a vertex of \(G_{18}\) and \(w\) be the special vertex in the column of \(v\) (\(v = w\) is possible). Then \(A_v\) is defined by removing \(w\) from the row of \(w\) and \(B_v = V(G_{18})\setminus (A_v \cup \{v\})\). Since \(A_v\) is a five-cycle and \(B_v\) has two vertices from each column, \((A_v, B_v)\) is a \((2, 2)\)-split partition of \(V(G_{18})\). In terms of \(f(p, q)\), the preceding remarks show that \(f(1, 1) = 5\), \(f(1, 2) \geq 9\), \(f(2, 2) \geq 18\). It would be interesting to determine \(f(2, 2)\); the antisymmetry of \(G_{18}\) suggests that there are much larger examples of \((2, 2)\)-split critical graphs.

In the proof of Theorem 1 we shall refer to the diagonal case of a well-known theorem of Erdős and Rado ([ER]) on \(\Delta\)-systems (later several other names, like star and sunflower were introduced for \(\Delta\)-systems).

A hypergraph (set system) with edges \(e_1, e_2, \ldots, e_t\) is called a \(2\)-system, if any two distinct edges intersect in the same set \(K\), i.e. \(e_i \cap e_j = K\) for all \(1 \leq i < j \leq t\). The set \(K\) is called the kernel and the sets \(e_i \setminus K\) are the rays (or petals) of the \(\Delta\)-system. The rank of a hypergraph is the cardinality of its largest edge. The well-known theorem of Erdős and Rado is generally stated for simple \(\Delta\)-uniform hypergraphs but it is immediate to check that it remains true if only rank \(\Delta\) is assumed and multiple edges are also allowed. With this consideration the diagonal case of the Erdős–Rado theorem ([ER]) can be stated as follows.

**Theorem A.** A (not necessarily simple) hypergraph of rank \(\Delta\) with more than \(F(r) = r! (r^r)\) edges contains a \(\Delta\)-system with \(\Delta + 1\) edges.

**Theorem 1.** For any fixed pair of positive integers \(p, q\) there are finitely many \((p, q)\)-split critical graphs.

Proof. Assume that \(G\) is a \((p, q)\)-split critical graph. Let \(A\) denote a subset of \(V = V(G)\) such that \(\tau(G[A]) \leq p\) and \(|A|\) is largest with this property. Set \(V = [n]\), \(A = [m]\) and, for convenience, \(A = V\setminus A\). We are going to show that \(m\) is bounded by a function \(g(p, q)\). This will imply the theorem because the same argument can be applied to \(G\) to show that \(|B| \leq g(q, p)\) for every \(B \subseteq V\) such that \(\omega(G[B]) \leq q\). Then, using a \((p, q)\)-split partition \([A_v, B_v]\) of \(V(G)\setminus v\) (with arbitrary \(v \in V\)) we obtain

\[|V(G)| \leq g(p, q) + g(q, p) + 1.\]
For each \( i \in A \), consider a \((p, q)\)-split partition \([A_i, B_i]\) of \( V \setminus i \) such that \(|A_i \setminus A|\) is as small as possible. Set

\[
X_i = A \setminus A_i, \quad Y_i = A \setminus A_i.
\]

Since \( Y_i \subseteq A \), \( \alpha([Y_i]) \leq p \). On the other hand, \( Y_i \setminus i \subseteq B_i \) therefore \( \omega([Y_i \setminus i]) \leq q \). Thus, by Ramsey's theorem, \(|Y_i| < R(p + 1, q + 2)\). Also, from the choice of \( A \), \(|X_i| \leq |Y_i|\). Using \( p + 2 \) to get a more symmetric formula,

\[
|X_i| < |Y_i| < R(p + 2, q + 2) = r.
\]

The sets \( X_i \) and \( Y_i \) form hypergraphs with \( m \) edges and the rank of both hypergraphs is at most \( r \). Set

\[
g(p, q) = f(f(r))
\]

where \( F \) is the Erdos-Rado function from Theorem A. The proof will be finished by proving the following claim.

Claim. \( m \leq g(p, q) \). If the claim is not true then the definition of \( g \) implies that for some \( I \subseteq [m], \ |I| = r + 1 \), the hypergraphs \( \{X_i : i \in I\} \) and \( \{Y_i : i \in I\} \) are both \( A \)-systems with \( r + 1 \) edges. By rearranging indices, we may assume that \( I = [r + 1] \). Let \( X, Y \) denote the kernels of these \( A \)-systems and let \( X^*, Y^* \) denote the petals, i.e. \( X^* = X \setminus X, Y^* = Y \setminus Y \) for \( i \in [r + 1] \).

We prove first that there is a non-empty petal \( X^*_i \) for some \( i \in [r + 1] \). If this were not true then \( X = X_i \) for all \( i \in [r + 1] \). Since \( |Y| \leq r \), there exists \( j \in [r + 1] \) such that \( j \not\in Y \), therefore \( j \in Y^*_i \). Now for any \( \alpha \)-witness \( S_j \), \( S_j \cap Y = \emptyset \) therefore \( S_j \) can have non-empty intersection with at most \( p + 1 \) sets \( Y_i \). Since \( r + 1 > p + 1 \) (in fact, \( r + 1 > p + 1 \)), there is a \( k \in [r + 1] \) such that \( S_j \cap Y_k = \emptyset \). Since \( X_j = X_k (= X) \), this implies that

\[
S_j \subseteq ((A \setminus X_k) \cup X_j) = ((A \setminus Y_k) \cup X_k) = A_k,
\]

contradicting to the assumption \( \alpha([A_k]) \leq p \).

Based on the previous paragraph, we may assume that \( X^*_i \neq \emptyset \). Consider the partition \( A^*_i, B^*_i \) of \( V \setminus \{1\} \) where \( A^*_i \) is defined as

\[
A^*_i = (A_1 \setminus X^*_i) \cup (B_1 \cap Y^*_i)
\]

and \( B^*_i = V \setminus (A^*_i \cup \{1\}) \). We are going to show that \([A^*_i, B^*_i]\) is a \((p, q)\)-split partition of \( V \setminus \{1\} \), i.e. \( \alpha([A^*_i]) \leq p \) and \( \omega([B^*_i]) \leq q \). Indeed, if \( A^*_i \)
has an independent set $S$ with $|S| = p + 1$, from $S \cap Y = \emptyset$, $S$ can have non-empty intersection with at most $p + 1$ $Y_j$ sets. Therefore, for some $k \in [r + 1]$, $S \cap Y_k = \emptyset$. But then $S \subseteq (A \setminus Y_k) \cup X \subseteq A_k$, contradicting $\alpha([A_k]) \leq p$. Assume that $B^*_k = (Y \setminus \{1\}) \cup (A \setminus X)$ has a complete subgraph $K$ with $|K| = q + 1$. Select $k \in [r + 1]$ such that $X_k \cap K$ and $\{k\} \cap K$ are both empty sets. Such a choice is possible, since $r + 1 > q + 1$, and in fact $r + 1 > q + 1$. Now the vertices of $K$ are covered by $(Y_k \setminus \{k\}) \cup (A \setminus X_k) \subseteq B_k$ which contradicts $\omega([B_k]) \leq q$. Therefore $[A^*_k, B^*_k]$ is a $(p, q)$-split partition of $V \setminus \{1\}$. This contradicts to the choice of $A_1$, because

$$|A^*_k \setminus A| = |A_1 \setminus (A \cup X^*_k)| = |X_1 \setminus X^*_k| < |X_1| = |A_1 \setminus A|$$

This final contradiction finishes the proof of the claim and the proof of Theorem 1.

**Corollary 1.** For fixed $p, q$, $(p, q)$-split graphs can be characterized by the exclusion of finitely many forbidden subgraphs.

Combining Proposition 2 and the actual upper bound of Theorem 1 leads to the following estimates on $f(p, q)$.

**Corollary 2.** $R(p + 2, q + 2) - 1 \leq f(p, q) \leq 2F(1)(R(p + 2, q + 2)) + 1$.

**Remarks.** Theorem 1 and its proof remain true for hypergraphs of fixed rank. Therefore Corollary 1 is also true for hypergraphs but Proposition 2 (and the lower bound of Corollary 2) collapses. The upper bound of $f(p, q)$ can certainly be improved. For example, Imre Bárány noted [B] that with a slight modification of the proof of Theorem 1 the iteration of $F$ can be avoided by doubling the inner function $R(p + 2, q + 2)$.

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**REFERENCES**


