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Characteristic Functions of $\mathscr{L}_1$-Spherical and $\mathscr{L}_1$-Norm Symmetric Distributions and Their Applications

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tributions and simplify that of the $\mathscr{L}_1$-norm symmetric distributions to an expression of a finite sum. These forms of c.f.'s can be used to derive the probability density functions (p.d.f.'s) of linear combinations of variables. We shall show that this gives a unified approach to the treatment of the linear function of i.i.d. random variables and their order statistics associated with double-exponential (i.e., Laplace), exponential, and uniform distributions. Some applications in reliability prediction, random weighting, and serial correlation are also shown. © 2001 Academic Press

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1. INTRODUCTION

Osiewalski and Steel (1993) introduced the class of multivariate $\mathscr{L}_p$-spherical distributions, where the symmetry is imposed through the density function. An important special class of $\mathscr{L}_p$-spherical distributions is generated by independent sampling from exponential power distribution (Box and Tiao, 1973, Chap. 3). For $p=1$ the sample comes from double-exponential distribution, for $p=2$ it corresponds to sampling from a normal, and for $p=+\infty$ it is from a uniform distribution. An n-variate random vector $\mathbf{x}$ is said to have and $\mathscr{L}_p$-spherical distribution, denoted by

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$\mathbf{x} \sim S(n, p; G)$, if $\mathbf{x} \stackrel{d}{=} R^* \cdot \mathbf{w}$, where $\mathbf{w}$ has the uniform distribution on the surface of the $\mathscr{L}_p$-sphere in R^n,

$$F_{n,p} = \left\{ \mathbf{x} = (x_1, ..., x_n)^T : -\infty < x_j < +\infty, \sum_{j=1}^{n} |x_j|^p = 1 \right\}, \tag{1.1}$$

and R^*, being independent of $\mathbf{w}$, is univariate nonnegative random variable with c.d.f. G. Based on the symmetry of a stochastic representation, Gupta and Song (1997a, b) recently studied the properties of the $\mathscr{L}$-spherical distribution. The c.f. of the $\mathscr{L}_p$-spherical distribution has not been available.

Yue and Ma (1995) developed a family of the multivariate versions of the Weibull distributions, called the multivariate $\mathscr{L}_p$-norm symmetric distributions, which are extensions of the family of multivariate $\mathscr{L}_1$-norm symmetric distributions studied by Fang and Fang (1988). An n-dimensional random vector $\mathbf{z}$ is said to have an $\mathscr{L}_p$-norm symmetric distribution, denoted by $\mathbf{z} \sim L(n, p; G)$, if $\mathbf{z} \stackrel{d}{=} R^* \cdot \mathbf{u}$, where $\mathbf{u}$ is uniformly distributed on the $\mathscr{L}_p$-norm closed simplex in R^n_+,

$$T_{n,p} = \left\{ \mathbf{x} = (x_1, ..., x_n)^T : x_j \geqslant 0, \sum_{j=1}^{n} x_j^p = 1 \right\}, \tag{1.2}$$

and R^*, being independent of $\mathbf{u}$, is univariate nonnegative random variable with c.d.f. G. When $p = 1$, we denote $F_{n,1}$ and $T_{n,1}$ by F_n and T_n respectively. The c.f. of the uniform distribution on T_n, i.e., $\mathbf{u} \sim U(T_n)$, is given by Fang *et al.* (1990, p. 116) as follows:

$$E(e^{i\mathbf{t}^T\mathbf{u}}) = \Gamma(n)\, e^{it_n} \sum_{j=0}^{\infty} \frac{i^j}{\Gamma(n+j)} \sum_{r_1 + \cdots + r_{n-1} = j} \prod_{k=1}^{n-1} (t_k - t_n)^{r_k}. \tag{1.3}$$

Note that the right side of (1.3) is a summation with infinite terms.

In this paper we shall employ the partial-fraction expansion, the CKS (Cambanis, Keener, and Simons) formula and the HG (Hermite–Genocchi) formula to obtain for the first time the c.f. for $\mathscr{L}_1$-spherical distributions in Theorem 1 of Section 3. Our second contribution is to simplify (1.3) as finite summations for three different situations (see Theorem 2). Analogous results are developed for the c.f.'s of $\mathbf{z} \sim L(n, 1; G)$ and $\mathbf{y} \sim U(V_n)$, where $V_n \triangleq V_n(1)$ is a special case of the open simplex in R^n_+,

$$V_n(c) = \left\{ \mathbf{x} = (x_1, ..., x_n)^T : x_j \geqslant 0, \sum_{j=1}^{n} x_j \leqslant c \right\}, \tag{1.4}$$

where c is a positive constant. In Section 4, we use the c.f. to obtain the p.d.f. of the linear combinations of variables for three kinds of cases. This leads to a unified approach to the treatment of the linear function of i.i.d.

random variables and their order statistics associated with exponential and uniform distributions. The formula (5.5.4) of David (1981, p. 103) is a direct consequence of our (4.13). The applications in reliability prediction, random weighting and serial correlation are shown in Section 5.

2. PRELIMINARIES

In order to derive the characteristic functions in the next section, we shall collect some useful formulae about the partial-fraction expansion and other multi-fold integrals on the open simplex $V_n(c)$.

2.1. *Partial-Fraction Expansion.* A lot of useful formulae can be obtained by combining the surface integral formula with the partial-fraction identity (Hazewinkel, 1990, p. 311)

$$\frac{N(x)}{(x-b_1)\cdots(x-b_n)}=\sum_{k=1}^{n}\frac{N(b_k)\,\Delta(b_1,\ldots,b_n)}{x-b_k}, \tag{2.1}$$

where $\Delta_k(b_1,\ldots,b_n)=\prod_{j\neq k,\,j=1}^{n}(b_k-b_j)^{-1}$, $N(x)$ is a polynomial of degree r, $0\leqslant r\leqslant n-1$, and $b_j\neq b_k$ for $j,\ k=1,\ldots,\ n,\ j\neq k$. Especially, taking $N(x)=-x$ in (2.1) and setting $x=0$, we have

$$0=\sum_{k=1}^{n}\Delta_k(b_1,\ldots,b_n). \tag{2.2}$$

2.2. *CKS Formula.* Let $h(\cdot)$ and $g(\cdot)$ be measurable functions on R^1_+ and R^1 respectively, and further let $g(\cdot)$ have the $(n-1)$th absolutely continuous derivatives. It can be shown by induction and through the use of (2.2) that (Cambanis *et al.* 1983, p. 225)

$$\begin{aligned}\int_{R^n_+} h\left(\sum_{j=1}^{n}x_j\right)g^{(n-1)}\left(\sum_{j=1}^{n}s_jx_j\right)d\mathbf{x}\\ =\sum_{k=1}^{n}\Delta_k(s_1,\ldots,s_n)\int_0^\infty h(u)\,g(us_k)\,du.\end{aligned} \tag{2.3}$$

From the proof of Theorem 3.1 in Cambanis *et al.* (1983, pp. 225–226), we know that (2.3) can derive the integral

$$\int_{R^n_+} h\left(\sum_{j=1}^{n}x_j\right)\prod_{j=1}^{n}\cos(s_jx_j)\,d\mathbf{x}=\sum_{k=1}^{n}\Delta_k(s_1^2,\ldots,s_n^2)\cdot B_n(s_k^2), \tag{2.4}$$

where

$$B_n(t)=\begin{cases}(-1)^{(n-1)/2}\, t^{(n-1)/2}\displaystyle\int_0^\infty \cos(u\sqrt{t})\, h(u)\, du, & n \text{ odd},\\ (-1)^{(n-2)/2}\, t^{(n-1)/2}\displaystyle\int_0^\infty \sin(u\sqrt{t})\, h(u)\, du, & n \text{ even}.\end{cases}$$

We shall call (2.4) the CKS formula in the following. An alternative version of (2.3) is given by

$$\int_{V_n(c)} h\left(\sum_{j=1}^n x_j\right) g^{(n-1)}\left(\sum_{j=1}^n s_j x_j\right) d\mathbf{x} = \sum_{k=1}^n \Delta_k(s_1, ..., s_n) \int_0^c h(u)\, g(us_k)\, du. \tag{2.5}$$

Another important formula is

$$\int_{V_n(c)} h\left(\sum_{j=1}^n x_j\right) d\mathbf{x} = \frac{1}{(n-1)!}\int_0^c h(u)\, u^{n-1}\, du, \tag{2.6}$$

which can be obtained by using (5.10) of Fang *et al.* (1990, p. 115).

2.3. *HG Formula.* The classical Hermite–Genocchi (HG) formula (Karlin *et al.*, 1986, p. 71) can be stated as

$$\int_{T_n} g^{(n-1)}\left(\sum_{j=1}^n s_j x_j\right) d\mathbf{x} = \sqrt{n}\sum_{k=1}^n g(s_k)\, \Delta_k(s_1, ..., s_n), \tag{2.7}$$

where $d\mathbf{x}$ denotes the volume element of T_n and $g(\cdot)$ has the same meaning as in (2.3).

The following three lemmas will be used in the sequel and their proofs are omitted.

LEMMA 1. *Assume that* $t \in R^1_+$, $b \in R^1$, *and* $b \neq 0$. *Let* $\xi_n = \xi_n(t; b) \triangleq \int_0^t e^{bx} \cdot nx^{n-1}\, dx$, *then*

$$\xi_n(t; b) = n!(-b)^{-n} - n!\, t^n e^{bt} \sum_{k=1}^n (-bt)^{-k}/(n-k)!. \tag{2.8}$$

LEMMA 2. *Assume that* $t \in R^1_+$, $a, b \in R^1$, *and* $b \neq 0$. *Let* $\eta_n = \eta_n(t; a, b) \triangleq \int_0^t \cos(a + x/b) \cdot x^{n-1}\, dx$ *and* $\zeta_n = \zeta_n(t; a, b) \triangleq \int_0^t \sin(a + x/b) \cdot x^{n-1}\, dx$. *Denote the largest integer not exceeding* x *by* $[x]$. *Then*

$$\eta_n = \sin(a+t/b) \sum_{k=0}^{[(n+1)/2]-2} \frac{(-1)^k (n-1)!\, b^{2k+1} t^{n-2k-1}}{(n-2k-1)!}$$
$$+ \cos(a+t/b) \sum_{k=0}^{[(n+1)/2]-2} \frac{(-1)^k (n-1)!\, b^{2k+2} t^{n-2k-2}}{(n-2k-2)!}$$
$$+ (-1)^{[(n+1)/2]-1} (n-1)!\, b^n \cdot C_n(t; a, b), \tag{2.9}$$

$$\zeta_n = -\cos(a+t/b) \sum_{k=0}^{[(n+1)/2]-2} \frac{(-1)^k (n-1)!\, b^{2k+1} t^{n-2k-1}}{(n-2k-1)!}$$
$$+ \sin(a+t/b) \sum_{k=0}^{[(n+1)/2]-2} \frac{(-1)^k (n-1)!\, b^{2k+2} t^{n-2k-2}}{(n-2k-2)!}$$
$$+ (-1)^{[(n+1)/2]-1} (n-1)!\, b^n \cdot D_n(t; a, b), \tag{2.10}$$

where

$$C_n(t; a, b) = \begin{cases} \sin(a+t/b) - \sin(a), & n \text{ odd}, \\ (t/b)\sin(a+t/b) + \cos(a+t/b) - \cos(a), & n \text{ even}. \end{cases}$$

$$D_n(t; a, b) = \begin{cases} \cos(a) - \cos(a+t/b), & n \text{ odd}, \\ -(t/b)\cos(a+t/b) + \sin(a+t/b) - \sin(a), & n \text{ even}. \end{cases}$$

LEMMA 3. *Assume that* $t \in R^1_+$, $b \in R^1$, *and define*

$$\delta_n(t) \triangleq \int_{V_{n-1}(t)} \cos\left(t - \sum_{j=1}^{n-1} u_j\right) \prod_{j=1}^{n-1} \cos(u_j)\, du_j.$$

Then we have

$$\delta_n(t) = \frac{(t/2)^{n-1}}{(n-1)!} \sum_{k=0}^{n-1} k \binom{n-1}{k}^2 t^{-k}$$
$$\times \sum_{j=0}^{n-1-k} \binom{n-1-k}{j} (-t)^{-j}\, \eta_{k+j}(t; t, -0.5), \tag{2.11}$$

where the function η_{k+j} *is given by* (2.9).

3. CHARACTERISTIC FUNCTIONS

Consider $\mathbf{x} = (x_1, ..., x_n)^T$ which has an $\mathscr{L}_1$-spherical distribution, i.e., $\mathbf{x} \sim S(n, 1; G)$. Then the c.f. of $\mathbf{x}$ is given by

$$E(e^{i\mathbf{t}^T\mathbf{x}}) = E(e^{i\mathbf{t}^T R^* \cdot \mathbf{w}}) = \int_0^\infty \phi_{\mathbf{w}}(r^* t_1, ..., r^* t_n)\, dG(r^*), \tag{3.1}$$

where R^* has c.d.f. G and $\phi_{\mathbf{w}}$ is the c.f. of $\mathbf{w} \sim U(F_n)$. Therefore, it suffices to investigate $\phi_{\mathbf{w}}$. A special feature of this characteristic function is that it is a real function. We have the following main result.

THEOREM 1. *Let $\phi_{\mathbf{w}}(t_1, ..., t_n)$ be the c.f. of $\mathbf{w} \sim U(F_n)$, then*

$$\phi_{\mathbf{w}}(t_1, ..., t_n) = \begin{cases} \Gamma(n) \sum_{k=1}^{n} (-1)^{(n-1)/2}\, t_k^{n-1} \cos(t_k) \cdot \Delta_k(t_1^2, ..., t_n^2), & n \text{ odd}, \\ \Gamma(n) \sum_{k=1}^{n} (-1)^{(n/2)-1}\, t_k^{n-1} \sin(t_k) \cdot \Delta_k(t_1^2, ..., t_n^2), & n \text{ even}, \end{cases} \tag{3.2}$$

where $t_j^2 \neq t_k^2$, for $j, k = 1, ..., n$, $j \neq k$.

Proof. The technique in proving this case is similar to that of Lemma 7.1 in Fang *et al.* (1990, p. 185). Let $x_1, ..., x_n$ be an i.i.d. sample from double-exponential with p.d.f.

$$(2\lambda)^{-1} \exp\{-\lambda^{-1} |x|\}, \qquad \lambda > 0, \quad -\infty < x < \infty,$$

that is, $x_1, ..., x_n \overset{\text{iid}}{\sim} \text{DE}(\lambda)$, $\mathbf{x} = (x_1, ..., x_n)^T$. It follows from Theorem 1.1 in Song and Gupta (1997a, b) that

$$\mathbf{w} = (w_1, ..., w_n)^T \overset{d}{=} \left(x_1 \Big/ \sum_{j=1}^{n} |x_j|, ..., x_n \Big/ \sum_{j=1}^{n} |x_j| \right)^T \sim U(F_n). \tag{3.3}$$

Without loss of generality, we take $\lambda = 1$ and obtain the c.f. of $\mathbf{w}$ as

$$\begin{aligned} \phi_{\mathbf{w}}(t_1, ..., t_n) &= E \exp\{i \mathbf{t}^T \mathbf{w}\} = E \exp\left\{ i \mathbf{t}^T \mathbf{x} \Big/ \sum_{j=1}^{n} |x_j| \right\} \\ &= \int_{R^n} \exp\left\{ i \frac{t_1 x_1 + \cdots + t_n x_n}{|x_1| + \cdots + |x_n|} \right\} 2^{-n} \exp\{-(|x_1| + \cdots + |x_n|)\}\, d\mathbf{x} \\ &= \int_{R_+^n} \exp\{-(x_1 + \cdots + x_n)\} \prod_{j=1}^{n} \cos\left(\frac{t_j x_j}{x_1 + \cdots + x_n} \right) d\mathbf{x}, \end{aligned}$$

where in the last step we have used the symmetry of the integrand. The transformation $y_j = x_j / \sum_{i=1}^n x_i$, $1 \leqslant j \leqslant n-1$, $y_n = \sum_{i=1}^n x_i$, has the Jacobian $J(x_1, ..., x_n \to y_1, ..., y_n) = y_n^{n-1}$. Therefore, we have

$$\phi_{\mathbf{w}}(t_1, ..., t_n) = \Gamma(n) \int_{V_{n-1}} \cos\left[t_n \left(1 - \sum_{j=1}^{n-1} y_j\right)\right] \prod_{j=1}^{n-1} \cos(t_j y_j)\, dy_j. \tag{3.4}$$

Putting $h(u) = \cos(t_n(1-u)) \cdot I_{[0,1]}(u)$ in the CKS formula (2.4), where $I_D(\cdot)$ represents the indicator function of domain D, we have

$$\phi_{\mathbf{w}}(t_1, ..., t_n) = \Gamma(n) \sum_{k=1}^{n-1} \Delta_k(t_1^2, ..., t_{n-1}^2) \cdot B_{n-1}(t_k^2),$$

where

$$B_{n-1}(t) = \begin{cases} (-1)^{(n-2)/2}\, t^{(n-2)/2} (\sqrt{t} \sin(\sqrt{t}) - t_n \sin(t_n))/(t - t_n^2), \\ \qquad n-1 \text{ odd}, \\ (-1)^{(n-3)/2}\, t^{(n-2)/2} \cdot \sqrt{t}(\cos(t_n) - \cos(\sqrt{t}))/(t - t_n^2), \\ \qquad n-1 \text{ even}. \end{cases}$$

Noting the identity

$$\sum_{k=1}^{n} (-1)^{(n-3)/2}\, t_k^{n-1} \Delta_k(t_1^2, ..., t_n^2) = 0, \qquad n \text{ is odd}, \tag{3.5}$$

which can be obtained by taking $N(x) = (-1)^{(n-1)/2}\, x^{(n+1)/2}$ in (2.1) and setting $x = 0$, we have, when n is odd,

$$\phi_{\mathbf{w}}(t_1, ..., t_n) = \Gamma(n) \sum_{k=1}^{n-1} \Delta_k(t_1^2, ..., t_n^2) \cdot (-1)^{(n-3)/2}\, t_k^{n-1}(\cos(t_n) - \cos(t_k)).$$

This implies the first expression of (3.2) by virtue of (3.5). Likewise the second expression of (3.2) can be reached. ∎

Remark 1. (i) If $t_j^2 = t^2$, for $j = 1, ..., n$, then we have

$$\phi_{\mathbf{w}}(t, ..., t) = \Gamma(n)\, t^{-(n-1)} \delta_n(t), \tag{3.6}$$

where the function $\delta_n(t)$ is given by (2.11). In fact, if $t = 0$, of course, $\phi_{\mathbf{w}}(0, ..., 0) = 1$. Since $\phi_{\mathbf{w}}(-t, ..., -t) = \phi_{\mathbf{w}}(t, ..., t)$, we may assume without loss of generality that $t > 0$. Formula (3.6) follows immediately by using (3.4) and (2.11).

(ii) In principle, the c.f. of $\mathbf{w} \sim U(F_n)$ for other cases can be obtained by taking appropriate limits of (3.2) because of uniform continuity in real space R^n. For example, let n be odd. Then from (3.2) we have, when $t_1^2 = \cdots = t_r^2 = t^2$, $2 \leqslant r < n$, and $t_{r+1}^2, ..., t_n^2$, t^2 are distinct,

$$\frac{\phi_{\mathbf{w}}(t_1, ..., t_n)}{\Gamma(n)(-1)^{(n-1)/2}} = \Omega_r \cdot \prod_{j=r+1}^{n} (t^2 - t_j^2)^{-1} + \sum_{k=r+1}^{n} \frac{t_k^{n-1} \cos(t_k)}{(t_k^2 - t^2)^r} \Delta_k(t_{r+1}^2, ..., t_n^2),$$

where

$$\Omega_r \triangleq \lim_{t_1 \to t, ..., t_r \to t} \sum_{k=1}^{r} t_k^{n-1} \cos(t_k) \, \Delta_k(t_1^2, ..., t_r^2).$$

If we could find a general expression for Ω_r, other cases follow from (3.2). In fact, this idea will be employed in showing Theorems 2 and 3. However, there is a technical difficulty for the present theorem and its corollary. For any specific r, one can work out Ω_r. For example,

$$\Omega_2 = \tfrac{1}{2}\{(n-1)\, t^{n-3} \cos(t) - t^{n-2} \sin(t)\},$$

$$\Omega_3 = \tfrac{1}{8}\{(-t^{n-3} + (n-1)(n-3)\, t^{n-5}) \cos(t) - ((n-1)\, t^{n-4} + (n-2)\, t^{n-5}) \sin(t)\}.$$

But it is not easy to find the recursive pattern of Ω_r, hence induction cannot be used in this situation. A direct attack on the general pattern of Ω_r without induction would be even harder.

Next, let us consider the uniform distribution inside the $\mathscr{L}_1$-sphere in R^n,

$$E_n = \left\{ \mathbf{x} = (x_1, ..., x_n)^T : \mathbf{x} \in R^n, \sum_{j=1}^{n} |x_j| \leqslant 1 \right\}.$$

If $\mathbf{v} \sim U(E_n)$, then the joint p.d.f. of $\mathbf{v}$ is $2^{-n} n! \cdot I_{E_n}(\mathbf{v})$ (see Gupta and Song, 1997a, Example 2.7). In this case, we can represent it as $\mathbf{v} \stackrel{d}{=} R^* \cdot \mathbf{w}$ with $R^* \sim \mathrm{Be}(n, 1)$ being independent of $\mathbf{w} \sim U(F_n)$; that is, $\mathbf{v} \sim S(n, 1; G)$ with $G(x) = x^n \cdot I_{[0,1]}(x)$. We use $\mathrm{Be}(a, b)$ to denote the beta distribution with parameters a and b. Applying (3.1) and Theorem 1, we immediately obtain

COROLLARY 1. *Let $\psi_{\mathbf{v}}(t_1, ..., t_n)$ be the c.f. of $\mathbf{v} \sim U(E_n)$, then*

$$\psi_{\mathbf{v}}(t_1, ..., t_n) = \begin{cases} n! \sum_{k=1}^{n} (-1)^{(n-1)/2}\, t_k^{n-2} \sin(t_k) \cdot \Delta_k(t_1^2, ..., t_n^2), & n \text{ odd}, \\ n! \sum_{k=1}^{n} (-1)^{n/2}\, t_k^{n-2} \cos(t_k) \cdot \Delta_k(t_1^2, ..., t_n^2), & n \text{ even}, \end{cases}$$

where $t_j^2 \neq t_k^2$, for $j, k = 1, ..., n$, $j \neq k$.

Now let us consider the c.f. of an $\mathscr{L}_1$-norm symmetric distribution. Let $\mathbf{z} = (z_1, ..., z_n)^T$ have such a distribution, i.e., $\mathbf{z} \sim L(n, 1; G)$, then the c.f. of $\mathbf{z}$ is given by

$$E(e^{i\mathbf{t}^T\mathbf{z}}) = E(e^{i\mathbf{t}^T R^* \cdot \mathbf{u}}) = \int_0^{\infty} \Phi_{\mathbf{u}}(r^* t_1, ..., r^* t_n)\, dG(r^*),$$

where R^* has c.d.f. G and $\Phi_{\mathbf{u}}$ is the c.f. of $\mathbf{u} \sim U(T_n)$. If the moment generating function (m.g.f.) $E(e^{\mathbf{s}^T\mathbf{u}})$ of random vector $\mathbf{u}$ is available, we can obtain the c.f. of $\mathbf{u}$ by replacing $\mathbf{s}$ with $i\mathbf{t}$. For this reason, we derive the m.g.f. of $U(T_n)$ to signify that all moments exist.

THEOREM 2. *Let $\Phi_{\mathbf{u}}(s_1, ..., s_n)$ be the m.g.f. of $\mathbf{u} \sim U(T_n)$.*

(i) *If $s_j \neq s_k$, for $j, k = 1, ..., n$, $j \neq k$, then*

$$\Phi_{\mathbf{u}}(s_1, ..., s_n) = (n-1)! \sum_{k=1}^{n} e^{s_k} \Delta_k(s_1, ..., s_n). \tag{3.7}$$

(ii) *If $s_j = s$, for $j = 1, ..., n$, then $\Phi_{\mathbf{u}}(s, ..., s) = \exp(s)$.*

(iii) *If $s_1 = \cdots = s_r = s$, $2 \leqslant r < n$, and $s, s_{r+1}, ..., s_n$ are distinct, then*

$$\Phi_{\mathbf{u}}(s_1, ..., s_n) = (n-1)! \left\{ \frac{e^s}{(r-1)!} \prod_{j=r+1}^{n} (s - s_j)^{-1} + \sum_{k=r+1}^{n} e^{s_k} (s_k - s)^{-r} \Delta_k(s_{r+1}, ..., s_n) \right\}. \tag{3.8}$$

Proof. (i) Since the joint p.d.f. of $\mathbf{u} \sim U(T_n)$ is $(n-1)!/\sqrt{n} \cdot I_{T_n}(\mathbf{u})$, then the m.g.f. of $\mathbf{u}$ is given by

$$\Phi_{\mathbf{u}}(s_1, ..., s_n) = E(e^{\mathbf{s}^T\mathbf{u}}) = \frac{(n-1)!}{\sqrt{n}} \int_{T_n} \exp\left\{ \sum_{j=1}^{n} s_j u_j \right\} d\mathbf{u}.$$

Applying the HG formula (2.7) to the function $g(t) = \exp(t)$, we obtain (3.7) immediately.

(ii) It is trivial.

(iii) Considering the limit of (3.7), we have

$$\frac{\Phi_{\mathbf{u}}(s_1, ..., s_n)}{(n-1)!} = \Omega_r^* \cdot \prod_{j=r+1}^{n} (s - s_j)^{-1} + \sum_{k=r+1}^{n} e^{s_k} (s_k - s)^{-r} \Delta_k(s_{r+1}, ..., s_n), \tag{3.9}$$

where

$$\Omega_r^* \triangleq \lim_{s_1 \to s, ..., s_r \to s} \sum_{k=1}^{r} e^{s_k} \Delta_k(s_1, ..., s_r).$$

In particular, we can obtain

$$\Omega_2^* = e^s, \qquad \Omega_3^* = \frac{e^s}{2!}, \qquad \Omega_4^* = \frac{e^s}{3!}, \ ... \ .$$

Therefore $\Omega_r^* = e^s/(r-1)!$. By substituting this into (3.9), we obtain (3.8). ∎

Now we turn to the c.f. of $U(V_n)$. Again, we can do more by deriving the m.g.f. of $U(V_n)$.

THEOREM 3. *Let $\Psi_{\mathbf{y}}(s_1, ..., s_n)$ be the m.g.f. of $\mathbf{y} \sim U(V_n)$.*

(i) *If $s_j \neq s_k$, for $j, k = 1, ..., n$, $j \neq k$, then*

$$\Psi_{\mathbf{y}}(s_1, ..., s_n) = n! \sum_{k=1}^{n} s_k^{-1}(e^{s_k} - 1)\, \Delta_k(s_1, ..., s_n). \tag{3.10}$$

(ii) *If $s_j = s$, for $j = 1, ..., n$, then*

$$\Psi_{\mathbf{y}}(s, ..., s) = n!(-s)^{-n} - n!\, e^s \sum_{k=1}^{n} (-s)^{-k}/(n-k)!. \tag{3.11}$$

(iii) *If* $s_1 = \cdots = s_r = s$, $2 \leqslant r < n$, *and* $s, s_{r+1}, ..., s_n$ *are distinct, then*

$$\Psi_{\mathbf{y}}(s_1, ..., s_n)$$

$$= n! \left\{ \left((-s)^{-r} - e^s \sum_{k=1}^{r} (-s)^{-k}/(r-k)! \right) \prod_{j=r+1}^{n} (s-s_j)^{-1} \right.$$

$$\left. + \sum_{k=r+1}^{n} s_k^{-1}(e^{s_k}-1)(s_k-s)^{-r} \Delta_k(s_{r+1}, ..., s_n) \right\}. \tag{3.12}$$

Proof. As the joint p.d.f. of $\mathbf{y} \sim U(V_n)$ is $n! \cdot I_{V_n}(\mathbf{y})$, we have

$$E(e^{\mathbf{s}^T\mathbf{y}}) = n! \int_{V_n} \exp\left\{ \sum_{j=1}^{n} s_j y_j \right\} d\mathbf{y}. \tag{3.13}$$

(i) If all $s_1, ..., s_n$ are distinct, we see from (3.13) and (2.5) that

$$E(e^{\mathbf{s}^T\mathbf{y}}) = n! \sum_{k=1}^{n} \Delta_k(s_1, ..., s_n) \int_0^1 \exp(y s_k)\, dy$$

$$= n! \sum_{k=1}^{n} \Delta_k(s_1, ..., s_n)\, s_k^{-1}(e^{s_k}-1),$$

which implies (3.10).

(ii) If all $s_j = s$, for $j = 1, ..., n$, we have from (3.13) and (2.6)

$$E(e^{\mathbf{s}^T\mathbf{y}}) = n! \int_{V_n} \exp\left\{ s \sum_{j=1}^{n} y_j \right\} d\mathbf{y} = \int_0^1 e^{sx} \cdot n x^{n-1}\, dx = \xi_n(1; s).$$

So we obtain (3.11) by Lemma 1.

(iii) Considering the limit of (3.10), we have

$$\frac{\Psi_{\mathbf{y}}(s_1, ..., s_n)}{n!} = \Omega_r^{**} \cdot \prod_{j=r+1}^{n} (s-s_j)^{-1}$$
$$+ \sum_{k=r+1}^{n} s_k^{-1}(e^{s_k}-1)(s_k-s)^{-r} \Delta_k(s_{r+1}, ..., s_n), \tag{3.14}$$

where

$$\Omega_r^{**} \triangleq \lim_{s_1 \to s, ..., s_r \to s} \sum_{k=1}^{r} s_k^{-1}(e^{s_k}-1)\, \Delta_k(s_1, ..., s_r).$$

In particular, we have

$$\Omega_2^{**} = s^{-2}(se^s - e^s + 1), \qquad \Omega^{**}_3 = s^{-3}(\tfrac{1}{2}s^2e^s - se^s + e^s - 1),$$
$$\Omega_4^{**} = s^{-4}(\tfrac{1}{6}s^3e^s - \tfrac{1}{2}s^2e^s + se^s - e^s + 1), \ldots .$$

The general pattern of Ω_r^{**} is not quite obvious. Noting that the limit of (3.10) is (3.11), i.e.,

$$\lim_{s_1 \to s, \ldots, s_n \to s} \sum_{k=1}^{n} s_k^{-1}(e^{s_k} - 1)\, \Delta_k(s_1, \ldots, s_n)$$
$$= (-s)^{-n} - e^s \sum_{k=1}^{n} (-s)^{-k}/(n-k)!. \tag{3.15}$$

We have $\Omega_r^{**} = (-s)^{-r} - e^s \sum_{k=1}^{r} (-s)^{-k}/(r-k)!$ by replacing n with r in (3.14). By substituting of Ω_r^{**} into (3.14), we obtain (3.12). ▌

4. DENSITY FUNCTIONS FOR LINEAR FORMS

We first consider the linear function associated with $U(F_n)$ and the double-exponential distribution. Let $x_1, \ldots, x_n \overset{\text{iid}}{\sim} \mathrm{DE}(1)$ and $\mathbf{x} = (x_1, \ldots, x_n)^T$. From (3.3), we know that $\mathbf{w} = \mathbf{x}/\sum_{k=1}^{n} |x_k| \sim U(F_n)$. We are often interested in the exact p.d.f. of linear function such as

$$w = \mathbf{a}^T\mathbf{w} = \sum_{k=1}^{n} a_k w_k \qquad \text{and} \qquad x = \mathbf{a}^T\mathbf{x} = \sum_{k=1}^{n} a_k x_k. \tag{4.1}$$

Since $-\mathbf{x} \overset{d}{=} \mathbf{x}$ and $-\mathbf{w} \overset{d}{=} \mathbf{w}$, we can assume without loss of generality that all $a_1, \ldots, a_n$ are positive. The configuration classifications for $\{a_1, \ldots, a_n\}$ are as follows.

Case 1. All a_k are different, say, $0 < a_1 < \cdots < a_n$.

Case 2. All a_k are equal, say, $a_1 = \cdots = a_n = 1$.

Case 3. At least two of a_k are equal. For example, $(a_1, \ldots, a_5)^T = (0.5, 0.5, 1, 4, 4)^T$;

For Cases 1 and 2, we have the following results.

THEOREM 4. *Case* 1. *The p.d.f.'s of* $w = \mathbf{a}^T\mathbf{w}$ *and* $x = \mathbf{a}^T\mathbf{x}$ *are respectively given by*

$$f_1(w) = \sum_{k=1}^{n} \tau_{nk} \cdot \frac{n-1}{2a_k}\left(1 - \frac{|w|}{a_k}\right)^{n-2} I_{[-a_k,\, a_k]}(w), \tag{4.2}$$

$$g_1(x) = \sum_{k=1}^{n} \tau_{nk} \cdot \frac{1}{2a_k} \exp\left(-\frac{|x|}{a_k}\right), \qquad -\infty < x < +\infty, \tag{4.3}$$

where

$$\tau_{nk} = a_k^{2(n-1)} \Delta_k(a_1^2, ..., a_n^2), \qquad k = 1, ..., n. \tag{4.4}$$

Case 2. *The p.d.f.'s of* $w = \sum_{k=1}^n w_k$ *and* $x = \sum_{k=1}^n x_k$ *are respectively given by*

$$f_2(w) = \sum_{k=1}^{n} \rho_{nk} \cdot \frac{|w|^{k-1}(1-|w|)^{n-k-1}}{2B(k, n-k)}, \qquad |w| \leqslant 1, \tag{4.5}$$

$$g_2(x) = \sum_{k=1}^{n} \rho_{nk} \cdot \frac{|x|^{k-1}\exp(-|x|)}{2\Gamma(k)}, \qquad -\infty < x < +\infty, \tag{4.6}$$

where

$$\rho_{nk} = \binom{2n-k-1}{n-1} \Big/ 2^{2n-k-1}, \qquad k = 1, ..., n. \tag{4.7}$$

Proof.

Case 1. It suffices to consider the situation when n is odd. From (3.2), the c.f. of $w = \mathbf{a}^T\mathbf{w}$ *is*

$$\begin{aligned}\varphi_w(t) &= E(e^{itw}) = E(e^{it\mathbf{a}^T\mathbf{w}}) = \phi_{\mathbf{w}}(ta_1, ..., ta_n) \\ &= \Gamma(n) \sum_{k=1}^{n} (-1)^{(n-1)/2} a_k^{n-1} \Delta_k(a_1^2, ..., a_n^2) \cdot t^{-(n-1)} \cos(ta_k).\end{aligned}$$

In terms of the inversion theorem, the p.d.f. of w is given by

$$\begin{aligned}f_1(w) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itw}\varphi_w(t)\,dt \\ &= \frac{1}{2\pi}\cdot\Gamma(n)\sum_{k=1}^{n}(-1)^{(n-1)/2} a_k^{n-1}\Delta_k(a_1^2, ..., a_n^2)\cdot I_1,\end{aligned} \tag{4.8}$$

where

$$I_1 \triangleq \int_{-\infty}^{+\infty} e^{-itw} \cdot t^{-(n-1)} \cos(ta_k)\, dt = \int_{-\infty}^{+\infty} \frac{e^{-itw}(e^{ita_k} + e^{-ita_k})}{2t^{n-1}}\, dt.$$

By means of residue theorem in complex analysis, Stuart and Ord (1987, p. 362) derived the complex integral

$$\int_{-\infty}^{+\infty} \frac{e^{ibz}}{z^n}\, dz = \begin{cases} -2\pi i^n b^{n-1}/(n-1)!, & \text{if} \quad b \leqslant 0, \\ 0, & \text{if} \quad b > 0, \end{cases} \tag{4.9}$$

which can be employed to give

$$I_1 = \begin{cases} \pi(-1)^{(n-1)/2}\,(a_k - |w|)^{n-2}/(n-2)!, & |w| \leqslant a_k, \\ 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

By substituting (4.10) into (4.8), we obtain (4.2).

Define $\zeta \triangleq \sum_{k=1}^n |x_k|$, then $\zeta \sim Ga(n, 1)$ and independent $\mathbf{w}$ (Gupta and Song, 1997a, Theorem 1.1). So $x = w \cdot \zeta$ and ζ is independent of w. The p.d.f. of $x = \mathbf{a}^T\mathbf{x}$ is given by

$$\begin{aligned} g_1(x) &= \int_0^\infty f_1\left(\frac{x}{t}\right)\frac{1}{t} \cdot \frac{t^{n-1}e^{-t}}{\Gamma(n)}\, dt \\ &= \sum_{k=1}^n \tau_{nk} \cdot \frac{n-1}{2a_k} \int_{|x|/a_k}^\infty \left(1 - \frac{|x|}{ta_k}\right)^{n-2} \frac{t^{n-2}e^{-t}}{\Gamma(n)}\, dt, \end{aligned}$$

which implies (4.3).

Case 2. It can be shown similarly by using the inversion theorem from c.f. to p.d.f. ▮

Some insights into (4.2)–(4.6) are given in Remark 2 below. We recall some distributions related to the beta distribution. Denote by $x \sim \text{Be}(p, q; a)$, the beta distribution with scale a, if x has p.d.f.

$$\frac{1}{a \cdot B(p, q)} \left(\frac{x}{a}\right)^{p-1} \left(1 - \frac{x}{a}\right)^{q-1}, \qquad 0 \leqslant x \leqslant a, \quad a > 0.$$

The p.d.f.'s of the symmetric beta distribution and the symmetric beta distribution with scale a are respectively defined as

$$\frac{1}{2 \cdot B(p, q)} |x|^{p-1} (1-|x|)^{q-1}, \qquad |x| \leqslant 1,$$

$$\frac{1}{2a \cdot B(p, q)} \left|\frac{x}{a}\right|^{p-1} \left(1-\frac{|x|}{a}\right)^{q-1}, \qquad |x| \leqslant a, \quad a>0,$$

and they are symbolized by Sbe(p, q) and Sbe$(p, q; a)$.

Remark 2. (i) We know that each component w_k of $\mathbf{w}$ has the same symmetric beta distribution with parameters 1 and $n-1$, i.e., $w_k \sim$ Sbe$(1, n-1)$, thereby, $a_k w_k \sim$ Sbe$(1, n-1; a_k)$. Equation (4.2) indicates that the distribution for the sum of n dependent variables with Sbe$(1, n-1; a_k)$ is the mixture of Sbe$(1, n-1; a_k)$. Likewise, (4.3) implies that the distribution for the sum of n independent variables with DE(a_k) $(a_k x_k \sim$ DE$(a_k))$ is the mixture of DE(a_k).

(ii) Formula (4.5) and (4.6) denote the mixtures of the symmetric beta distribution Sbe$(k, n-k)$ and the symmetric gamma distribution SGa$(k, 1)$ respectively. The latter coincides with the result listed on p. 24 of Johnson and Kotz (1970).

Now let us consider the exact distribution of linear function $y = \mathbf{a}^T \mathbf{y} = \sum_{k=1}^n a_k y_k$, where $\mathbf{y} = (y_1, ..., y_n)^T \sim U(V_n)$. The relationship between the c.f. of $y = \mathbf{a}^T \mathbf{y}$ and the c.f. of $\mathbf{y}$ is $\varphi_y(t) = \Psi_{\mathbf{y}}(ta_1, ..., ta_n)$. By means of (3.10), we have

$$\varphi_y(t) = n! \sum_{k=1}^n \sigma_{nk} \cdot \frac{\exp(ita_k) - 1}{(ita_k)^n}, \tag{4.11}$$

where

$$\sigma_{nk} = a_k^{n-1} \Delta_k(a_1, ..., a_n), \qquad k = 1, ..., n. \tag{4.12}$$

In analogy with (4.8), the p.d.f. of $y = \mathbf{a}^T \mathbf{y}$ is given by

$$h_1(y) = \frac{n!}{2\pi} \sum_{k=1}^n \sigma_{nk} \frac{1}{(ia_k)^n} \cdot I_2,$$

where

$$I_2 = \int_{-\infty}^{+\infty} \frac{e^{it(-y+a_k)}}{t^n} \, dt - \int_{-\infty}^{+\infty} \frac{e^{-ity}}{t^n} \, dt.$$

By (4.9), we have for some fixed a_k,

$$I_2 = \begin{cases} -2\pi i^n (y-a_k)^{n-1} \cdot (-1)^n/(n-1)!, & \text{if} \quad a_k > 0, \quad 0 \leqslant y \leqslant a_k, \\ -2\pi i^n (-y+a_k)^{n-1}/(n-1)!, & \text{if} \quad a_k < 0, \quad a_k \leqslant y \leqslant 0, \end{cases}$$

that is,

$$I_2 = \frac{2\pi i^n (a_k - y)^{n-1}}{(n-1)!} \cdot \mathrm{sgn}(a_k) \cdot I_{[\min(0,\, a_k),\, \max(0,\, a_k)]}(y).$$

These results are summarized into the following theorem.

THEOREM 5. *Let* $\mathbf{y} = (y_1, ..., y_n)^T \sim U(V_n)$. *If all* $a_k(\neq 0)$ *are different, then the p.d.f. of* $y = \mathbf{a}^T \mathbf{y} = \sum_{k=1}^n a_k y_k$ *is*

$$h_1(y) = \sum_{k=1}^n \sigma_{nk} \cdot \frac{n}{a_k} \left(1 - \frac{y}{a_k}\right)^{n-1} \cdot \mathrm{sgn}(a_k) \cdot I_{[\min(0,\, a_k),\, \max(0,\, a_k)]}(y), \qquad (4.13)$$

where weights $\{\sigma_{nk}, k = 1, ..., n\}$ *are given by* (4.12) *and* $\mathrm{sgn}(\cdot)$ *denotes the sign function.*

Remark 3. (i) When $a_k > 0$, the formula (5.5.4) in David (1981, p. 103) coincides with (4.13), which indicates that $h_1(y)$ is the mixture of $\mathrm{Be}(1, n; a_k)$, the beta distribution with scale a_k.

(ii) If $a_1 = \cdots = a_n = 1$, it is easy to see $y = \sum_{k=1}^n y_k \sim \mathrm{Be}(n, 1)$ by viewing (3.11).

Corresponding to Theorem 5, we have

THEOREM 6. *Let* $\mathbf{u} = (u_1, ..., u_n)^T \sim U(T_n)$. *If all* $a_k(\neq 0)$ *are different, then the p.d.f. of* $u = \mathbf{a}^T \mathbf{u} = \sum_{k=1}^n a_k u_k$ *is*

$$h_2(u) = \sum_{k=1}^n \sigma_{nk} \cdot \frac{n-1}{a_k} \left(1 - \frac{u}{a_k}\right)^{n-1} \cdot \mathrm{sgn}(a_k) \cdot I_{[\min(0,\, a_k),\, \max(0,\, a_k)]}(u), \qquad (4.14)$$

where weights $\{\sigma_{nk}, k = 1, ..., n\}$ *are given by* (4.12).

Finally, we present a unified approach to linear functions of variables. We shall adopt the following notations for samples and their order statistics:

$$\mathbf{y} = (y_1, ..., y_n)^T \sim U(V_n), \qquad y_{(1)} \leqslant \cdots \leqslant y_{(n)},$$

$$\mathbf{u} = (u_1, ..., u_n)^T \sim U(T_n), \qquad u_{(1)} \leqslant \cdots \leqslant u_{(n)},$$

$$\mathbf{z} = (z_1, ..., z_n)^T \sim L(n, 1; G), \qquad z_{(1)} \leqslant \cdots \leqslant z_{(n)},$$

$$\boldsymbol{\xi} = (\chi_1, ..., \xi_{n+1})^T \overset{\text{iid}}{\sim} E(1), \qquad \xi_{(1)} \leqslant \cdots \leqslant \xi_{(n+1)},$$

$$\boldsymbol{\eta} = (\eta_1, ..., \eta_n)^T \overset{\text{iid}}{\sim} U[0, 1], \qquad \eta_{(1)} \leqslant \cdots \leqslant \eta_{(n)},$$

where $E(\lambda)$ denotes the exponential with p.d.f.: $\lambda^{-1}\exp(-\lambda^{-1}\xi)$, $\lambda>0$, $\xi>0$. We shall illustrate that all distributions of the linear functions

$$\sum_{k=1}^{n} b_k y_{(k)}, \quad \sum_{k=1}^{n} b_k u_{(k)}, \quad \sum_{k=1}^{n} b_k z_{(k)}, \quad \sum_{k=1}^{n} b_k \xi_{(k)}, \quad \sum_{k=1}^{n} b_k \eta_{(k)},$$

$$\sum_{k=1}^{n} c_k z_k, \quad \text{and} \quad \sum_{k=1}^{n} c_k \xi_k$$

can be reduced to the distribution of $\sum_{k=1}^{n} a_k y_k$ as given by (4.13) or to that of $\sum_{k=1}^{n} a_k u_k$ as given by (4.14).

(i) $\sum_{k=1}^{n} b_k y_{(k)} \to \sum_{k=1}^{n} a_k y_k$. From Example 5.1 of Fang *et al.* (1990, p. 121), we know that $\mathbf{y} \sim U(V_n)$ belongs to the class of the $\mathscr{L}_1$-norm symmetric distribution. Therefore, the conclusion stated in Theorem 5.12 of Fang *et al.* (1990, p. 126) is also available to $\mathbf{y}$. Define $y_k^* \mathrel{\hat{=}} (n-k+1)(y_{(k)} - y_{(k-1)})$, $y_{(0)}=0$, $k=1, ..., n$, called normalized spacings of $\mathbf{y}$, then $(y_1^*, ..., y_n^*)^T \stackrel{d}{=} \mathbf{y}$, which implies

$$\sum_{k=1}^{n} b_k y_{(k)} \stackrel{d}{=} \sum_{k=1}^{n} a_k y_k, \quad \text{where} \quad a_k = \sum_{j=k}^{n} b_j/(n-k+1). \tag{4.15}$$

(ii) $\sum_{k=1}^{n} b_k u_{(k)} \to \sum_{k=1}^{n} a_k u_k$. Since $\mathbf{u} \sim U(T_n)$ also belongs to the family of the $\mathscr{L}_1$-norm symmetric distribution, similar to (4.15), we obtain

$$\sum_{k=1}^{n} b_k u_{(k)} \stackrel{d}{=} \sum_{k=1}^{n} a_k u_k, \quad \text{where} \quad a_k = \sum_{j=k}^{n} b_j/(n-k+1).$$

(iii) $\sum_{k=1}^{n} b_k z_{(k)}$ and $\sum_{k=1}^{n} c_k z_k \to \sum_{k=1}^{n} a_k y_k$. Now $\mathbf{z} = (z_1, ..., z_n)^T \sim L(n, 1; G)$ which implies $\mathbf{z} \stackrel{d}{=} R^* \cdot \mathbf{u}$, where $R^* \sim G(\cdot)$ independent of $\mathbf{u} \sim U(T_n)$, hence

$$\sum_{k=1}^{n} b_k z_{(k)} \stackrel{d}{=} R^* \cdot \sum_{k=1}^{n} b_k u_{(k)}, \quad \sum_{k=1}^{n} c_k z_k \stackrel{d}{=} R^* \cdot \sum_{k=1}^{n} c_k u_k.$$

(iv) $\sum_{k=1}^{n} b_k \xi_{(k)}$ and $\sum_{k=1}^{n} c_k \xi_k \to \sum_{k=1}^{n} a_k y_k$. Because

$$(y_1, ..., y_n)^T \stackrel{d}{=} \left(\xi_1 \Big/ \sum_{k=1}^{n+1} \xi_k, ..., \xi_n \Big/ \sum_{k=1}^{n+1} \xi_k \right)^T,$$

then

$$\sum_{k=1}^{n} b_k \xi_{(k)} \stackrel{d}{=} \sum_{k=1}^{n} c_k \xi_k, \qquad \text{where} \quad c_k = \sum_{j=k}^{n} b_j/(n-k+1),$$

$$\sum_{k=1}^{n} c_k \xi_k \stackrel{d}{=} \left(\sum_{k=1}^{n+1} \xi_k\right) \cdot \sum_{k=1}^{n} a_k y_k, \qquad \text{where} \quad a_k = c_k,$$

where $\sum_{k=1}^{n+1} \xi_k \sim Ga(n+1, 1)$ independent of $\sum_{k=1}^{n} a_k y_k$.

(v) $\sum_{k=1}^{n} b_k \eta_{(k)} \to \sum_{k=1}^{n} a_k y_k$. Now $y_k \stackrel{d}{=} (\eta_{(k)} - \eta_{(k-1)})$, $\eta_{(0)} = 0$, $k = 1, ..., n$, then

$$\sum_{k=1}^{n} b_k \eta_{(k)} \stackrel{d}{=} \sum_{k=1}^{n} a_k y_k, \qquad \text{where} \quad a_k = \sum_{j=k}^{n} b_j.$$

5. APPLICATIONS

In this section, we shall demonstrate the usefulness of the preceding results with three important examples: predicting the reliability of components in a system, the random weighting method, and serial correlation.

EXAMPLE 1 (Prediction Problem in Reliability). Consider the nonparametric problem of predicting $\mathbf{a}^T \mathbf{x}_{(2)}$ based on the first m observations $\mathbf{x}_{(1)}$, where $\mathbf{a}$ is an $(n-m) \times 1$ scalar vector and $\mathbf{x} = (\mathbf{x}_{(1)}^T, \mathbf{x}_{(2)}^T)^T \sim S(n, 1; G)$. It is easy to see that $t(\mathbf{x}) = \mathbf{a}^T \mathbf{x}_{(2)} / \sum_{k=1}^{m} |x_k|$ is scale-invariant (see, Gupta and Song, 1997a, Theorem 6.2). Hence we can take $\mathbf{x} = (x_1, ..., x_n)^T$, $x_1, ..., x_n \stackrel{\text{iid}}{\sim} \text{DE}(1)$. If all components of $\mathbf{a}$ are 1, then the p.d.f. of $x \triangleq \mathbf{a}^T \mathbf{x}_{(2)} = \sum_{k=m+1}^{n} x_k$ from (4.6) is given by

$$g_2(x) = \sum_{k=1}^{n-m} \rho_{n-m,k} \cdot \frac{|x|^{k-1} \exp(-|x|)}{2\Gamma(k)}, \qquad -\infty < x < \infty.$$

Since $\sum_{k=1}^{m} |x_k| \sim Ga(m, 1)$ and is independent of $\mathbf{a}^T \mathbf{x}_{(2)}$, the p.d.f. of $t(\mathbf{x})$ is

$$\begin{aligned} h(u) &= \int_0^{\infty} g_2(uv) \cdot \frac{1}{\Gamma(m)} v^{m-1} e^{-v} v \, dv \\ &= \sum_{k=1}^{n-m} \rho_{n-m,k} \cdot \frac{1}{2B(k, m)} \cdot \frac{|u|^{k-1}}{(1+|u|)^{k+m}}. \end{aligned} \tag{5.1}$$

In this case we can obtain the prediction interval $[L_1, U_1]$ of $\mathbf{a}^T \mathbf{x}_{(2)} = \sum_{k=m+1}^n x_k$ for a given confidence coefficient $1-\alpha$,

$$1-\alpha = P\left\{L_1 \leqslant \sum_{k=m+1}^{n} x_k \leqslant U_1\right\}$$
$$= P\left\{\frac{L_1}{\sum_{k=1}^m |x_k|} \leqslant t(\mathbf{x}) \leqslant \frac{U_1}{\sum_{k=1}^m |x_k|}\right\}.$$

Hence

$$[L_1, U_1] = \left[\left(\sum_{k=1}^{m} |x_k|\right) \cdot L_2, \left(\sum_{k=1}^{m} |x_k|\right) \cdot U_2\right], \tag{5.2}$$

where L_2 and U_2 are determined by virtue of (5.1) by

$$1-\alpha = P\{L_2 \leqslant t(\mathbf{x}) \leqslant U_2\} = \int_{L_2}^{U_2} h(u)\, du. \tag{5.3}$$

In the same fashion we may also get the prediction interval of $\mathbf{a}^T \mathbf{x}_{(2)}$ when $a_i \neq a_j$, $i \neq j$, by means of (4.3).

EXAMPLE 2 (Random Weighting Method). Since Efron's (1979) well-known paper appeared there has been considerable work on resampling methods. Among all of these techniques, the bootstrap is the simplest and most attractive one, and the random weighting method is an alternative which is aimed at estimating the error distribution of estimators. Let

$$x_k = \mu + e_k, \qquad k = 1, 2, ..., \tag{5.4}$$

be a measure model, where $\{e_k, k=1, 2, ...\}$ are random errors of measurements. It is assumed that $\{e_1, e_2, ...\}$ are i.i.d. with a common distribution function $F(x)$ satisfying $\int x\, dF(x) = \mu$ and $\int (x-\mu)^2\, dF(x) = \sigma^2 > 0$, and that μ and σ^2 are unknown. The common estimator for μ is the sample mean $\bar{x}$, with sample size n. To construct a confidence interval for μ, we need to know the distribution of the error $\bar{x} - \mu$. The main idea of the random weighting method is to construct a distribution based on samples $x_1, ..., x_n$, to mimic the distribution of $\bar{x} - \mu$. Let $\mathbf{u} = (u_{n1}, ..., u_{nn})^T \sim U(T_n)$ be independent of $x_1, ..., x_n$, and define

$$D_n^* = \sqrt{n} \sum_{k=1}^{n} (x_k - \bar{x})\, u_{nk}, \tag{5.5}$$

which is the weighted mean of $\sqrt{n}(x_k - \bar{x})$ with random weight u_{nk}. Zheng (1987, 1992) shows that $D_n^* \mid (x_1, x_2, ...)$, the conditional distribution of D_n^*

given $(x_1, x_2, ...)$, is close to the distribution of the error $\sqrt{n}(\bar{x}-\mu)$ when n is large, i.e., with probability one,

$$D_n^* \mid (x_1, x_2, ...) \xrightarrow{\mathscr{L}} N(0, \sigma^2), \tag{5.6}$$

where the notation $\xrightarrow{\mathscr{L}}$ stands for *convergence in law*, provided that in model (5.4) the errors $\{e_k, k=1, 2, ...\}$ are i.i.d. with $E(e_k)=0$ and $\mathrm{Var}(e_k)=\sigma^2<\infty$.

Our interest here is to find the exact (conditional) p.d.f. of $D_n^* \mid (x_1, x_2, ...)$. In fact, the (conditional) p.d.f. of $D_n^* \mid (x_1, x_2, ...)$ is a mixture of beta distributions with scale by virtue of (4.14) with $a_k=\sqrt{n}(x_k-\bar{x})$.

EXAMPLE 3 (Serial Correlation Problem). Consider the following model of time series (Johnson and Kotz, 1970, p. 233)

$$X_t=\rho X_{t-1}+Z_t, \qquad |\rho|<1, \quad t=1, 2, ..., \tag{5.7}$$

where the Z_t's are mutually independent unit normal variables, and further, Z_t is independent of all X_k for $k<t$. Define a modified noncircular serial correlation coefficient as

$$R_{1,1}=\frac{\sum_{k=1}^{n-1}(X_k-\bar{X}_1)(X_{k+1}-\bar{X}_1)+\sum_{k=n+1}^{2n-1}(X_k-\bar{X}_2)(X_{k+1}-\bar{X}_2)}{\sum_{k=1}^{n}(X_k-\bar{X}_1)^2+\sum_{k=n+1}^{2n}(X_k-\bar{X}_2)^2}, \tag{5.8}$$

where

$$\bar{X}_1=n^{-1}\sum_{k=1}^{n} X_k; \qquad \bar{X}_2=n^{-1}\sum_{k=n+1}^{2n} X_k.$$

The exact distribution of $R_{1,1}$ has been obtained by Pan (1968) for the case when the correlation between X_i and X_j is ρ for $|i-j|=1$, and is 0 otherwise. In this case it can be shown that $R_{1,1}$ is distributed as $(\sum_{k=1}^{n-1}\lambda_k\xi_k)/(\sum_{k=1}^{n-1}\xi_k)$, where the ξ's are mutually independent variables, each distributed as standard exponential, i.e., $(\xi_1, ..., \xi_n)^T \overset{\text{iid}}{\sim} E(1)$, and $\lambda_{2k-1}=\cos(2k\pi/(n+1))$, $k=1, 2, ..., [n/2]$, while $\lambda_2, \lambda_4, ..., \lambda_{2[(n-1)/2]}$ are roots of the equation

$$(1-\lambda)^{-2}\{(-\tfrac{1}{2})^n-\tfrac{1}{2}D_{n-1}(\lambda)-(n+1-n\lambda)\,D_n(\lambda)\}$$

with

$$D_n(\lambda)=\left(-\frac{1}{2}\lambda\right)^n \sum_{k=1}^{[n/2+1]} \binom{n+1}{2k-1} (-1)^{k-1} (\lambda^2-1)^{k-1}.$$

It's easy to see that $R_{1,1}$ has the same distribution as $\sum_{k=1}^{n-1} \lambda_k u_k$ whose p.d.f. is given by (4.14) with $(u_1, ..., u_{n-1})^T \sim U(T_{n-1})$.

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