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Hermite Positive Definite Solution of a Class of Matrix Equation

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Abstract

In this paper, the Hermite positive definite solutions of the nonlinear matrix equation $X^s + A^* X^{-t} A = Q$ are discussed. A sufficient condition and two necessary and sufficient conditions for the existence of Hermite positive definite solutions for this equation are derived. The existence of minimal Hermite positive definite solution is also studied here, and an iterative method for obtaining the minimal Hermite positive definite solution is given.

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Keywords: Nonlinear matrix equation; Hermitian positive definite solution; Minimal solution; Iterative method

1. Introduction

This paper considers the nonlinear matrix equation

$$X^s + A^* X^{-t} A = Q, \tag{1}$$

where A, Q are $n \times n$ complex matrices and A is nonsingular, Q is Hermite positive definite, A^* stands for the conjugate transpose of the matrix A. s and t are positive real numbers.

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This class of matrix equation often arises in dynamic programming, control theory, stochastic filtering, statistics, and so on. It has been extensively studied by several authors, and some properties of the solutions have been obtained[1-9].

Some authors considered the equation in the case that $s=1$, $t \geq 1$ or $0 < t \leq 1$ [3-5]. Some other authors discussed these equations for particular choices of t and matrix Q, where $s=1$. For example, the case t=1 is studied in [6], and for t=2 and Q=I, see [7]. A more general case is t=n, which is discussed in [8,9].

In this paper, we study the properties of Hermite positive definite solutions and discuss the necessary conditions and sufficient conditions for the existence of Hermite positive definite solutions of Eq.(1). Then we consider the existence of minimal Hermite positive definite solution, and propose an iterative method to obtain the minimal Hermite positive definite solution of Eq.(1).

The following notations are used through out this paper. For $n \times n$ complex matrix A, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ stand for the minimal and maximal eigenvalues of matrix A, respectively. $A > 0$ $(A \geq 0)$ denotes that A is a positive definite(semi-definite) matrix , and $A > B$ $(A \geq B)$ means $A - B$ is positive definite(semi-definite).

2. Main results

Theorem 1 Eq.(1) has a Hermite positive definite solution if and only if there is a nonsingular matrix W, such that $W^*W = WW^*$ ψψψψψψ $A = (W^*W)^{t/2} Z Q^{1/2}$, where $(W^s Q^{-1/2})^*(W^s Q^{-1/2}) + Z^*Z = I$. In this case, Eq.(1) has a Hermite positive definite solutionψ $X = W^*W$.

Proof If X is a Hermite positive definite solution of Eq.(1), then there is unique Hermite positive definite matrix W, such that $X = W^2$ (see [10]). Substituting $X = W^2$ into Eq.(1) gives

$$W^{2s} + A^* W^{-2t} A = Q.$$

Noticing that W and Q are Hermite positive definite, then we have

$$(Q^*)^{-1/2}(W^*)^s W^s Q^{-1/2} + (Q^*)^{-1/2} A^* (W^*)^{-t} W^{-t} A Q^{-1/2} = I,$$

that is,

$$(W^s Q^{-1/2})^*(W^s Q^{-1/2}) + (W^{-t} A Q^{-1/2})^*(W^{-t} A Q^{-1/2}) = I.$$

Let $Z = W^{-t} A Q^{-1/2}$, then $A = W^t Z Q^{1/2} = (W^*W)^{t/2} Z Q^{1/2}$ and

$$(W^s Q^{-1/2})^*(W^s Q^{-1/2}) + Z^*Z = I.$$

Conversely, if there is a nonsingular matrix W, such that $W^*W = WW^*$ ψψandψ $A = (W^*W)^{t/2} Z Q^{1/2}$, where $(W^s Q^{-1/2})^*(W^s Q^{-1/2}) + Z^*Z = I$, let $X = W^*W$ ψthen

$$\begin{aligned} X^s + A^* X^{-t} A &= (W^*)^s W^s + (Q^*)^{1/2} Z^* (W^*W)^{t/2} (W^*W)^{-t} (W^*W)^{t/2} Z Q^{1/2} \\ &= (W^*)^s W^s + (Q^*)^{1/2} Z^* Z Q^{1/2} \\ &= (Q^*)^{1/2}[(Q^*)^{-1/2}(W^*)^s W^s Q^{-1/2} + Z^*Z] Q^{1/2} = Q. \end{aligned}$$

So $X = W^*W$ is a Hermite positive definite solution of Eq.(1).

Remark 1 When s = 1, the condition $W^*W=WW^*$ in Theorem 1 can be omitted.

Theorem 2 Eq.(1) has a Hermite positive definite solution if and only if there are unitary matrices P, U_2 and diagonal matrices $\Gamma>0$, $\Sigma\geq 0$, such that

$$A=[(Q^*)^{1/2}P^*\Gamma PQ^{1/2}]^{t/(2s)}U_2\Sigma PQ^{1/2},$$

where $\Gamma+\Sigma^2=I$. In this case, $X=((Q^*)^{1/2}P^*\Gamma PQ^{1/2})^{1/s}$ is a Hermite positive definite solution of Eq.(1).

Proof If Eq.(1) has a Hermite positive definite solution, then from Theorem 1, there is a nonsingular matrix W, such that $A=(W^*W)^{t/2}ZQ^{1/2}$, where $(W^sQ^{-1/2})^*(W^sQ^{-1/2})+Z^*Z=I$. Now let

$$G=\begin{pmatrix} W^sQ^{-1/2} \\ Z \end{pmatrix},$$

then G can be expanded into a unitary matrix

$$\begin{pmatrix} W^sQ^{-1/2} & U \\ Z & V \end{pmatrix}.$$

By CS decomposition, there are unitary matrices U_1, U_2, P, V_2 and diagonal matrices $K\geq 0$, $\Sigma\geq 0$ such that

$$\begin{pmatrix} W^sQ^{-1/2} & U \\ Z & V \end{pmatrix}=\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}\begin{pmatrix} K & -\Sigma \\ \Sigma & K \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & V_2 \end{pmatrix},$$

where $K^2+\Sigma^2=I$, then $W^sQ^{-1/2}=U_1KP$, $Z=U_2\Sigma P$. W is non-singular, so $K>0$. Let $\Gamma=K^2>0$, then $\Gamma+\Sigma^2=I$, moreover,

$$\begin{aligned}A&=(W^*W)^{t/2}ZQ^{1/2}=((Q^*)^{1/2}P^*KU_1^*U_1KPQ^{1/2})^{t/(2s)}U_2\Sigma PQ^{1/2}\\ &=[(Q^*)^{1/2}P^*K^2PQ^{1/2}]^{t/(2s)}U_2\Sigma PQ^{1/2}\\ &=[(Q^*)^{1/2}P^*\Gamma PQ^{1/2}]^{t/(2s)}U_2\Sigma PQ^{1/2}.\end{aligned}$$

Conversely, if there are unitary matrices P, U_2 and diagonal matrices $\Gamma>0$, $\Sigma\geq 0$, $\Gamma+\Sigma^2=I$, such that

$$A=[(Q^*)^{1/2}P^*\Gamma PQ^{1/2}]^{t/(2s)}U_2\Sigma PQ^{1/2},$$

let $X=((Q^*)^{1/2}P^*\Gamma PQ^{1/2})^{1/s}$, then

$$\begin{aligned}X^s+A^*X^{-t}A&=(Q^*)^{1/2}P^*\Gamma PQ^{1/2}+(Q^*)^{1/2}P^*\Sigma U_2^*((Q^*)^{1/2}P^*\Gamma PQ^{1/2})^{t/(2s)}\\ &\quad((Q^*)^{1/2}P^*\Gamma PQ^{1/2})^{-t/s}((Q^*)^{1/2}P^*\Gamma PQ^{1/2})^{t/(2s)}U_2\Sigma PQ^{1/2}\\ &=(Q^*)^{1/2}P^*\Gamma PQ^{1/2}+(Q^*)^{1/2}P^*\Sigma U_2^*U_2\Sigma PQ^{1/2}\\ &=(Q^*)^{1/2}P^*(\Gamma+\Sigma^2)PQ^{1/2}=Q.\end{aligned}$$

So $X=((Q^*)^{1/2}P^*\Gamma PQ^{1/2})^{1/s}$ is a Hermite positive definite solution of Eq.(1).

Remark 2 In Theorem 1 and Theorem 2, if we let $s=1$, $t=q\ge 1$, $Q=I$, then the conclusions are just Theorem 1 and Theorem 2 in [5]. That is, the results obtained in this paper are the generalization of [5].

Lemma 1([4]) Let $n\times n$ complex matrices $A\ge B>0$. If $\alpha\in(0,1]$, then $A^{\alpha}\ge B^{\alpha}>0$. And if $\alpha\in[-1,0)$, then $0<A^{\alpha}\le B^{\alpha}$.

Theorem 3 **For** $0<s\le 1$, $t\ge 1$, **if** Eq.(1) has a Hermite positive definite solution, then it must have a minimal Hermite positive definite solution X_S, moreover, consider the iterative method

$$X_0=0,\ X_{k+1}=(A(Q-X_k^s)^{-1}A^*)^{1/t},\ k=0,1,2,\cdots$$

the iterative sequence $\{X_k\}$ converges to the solution X_S.

Proof If X is an arbitrary Hermite positive definite solution of Eq.(1), then $A^*X^{-t}A<Q$, with Lemma 1 we have $X>(AQ^{-1}A^*)^{1/t}$. Consider the iterative sequence $\{X_k\}$.

$$X_0=0<X,$$

$$X_1=(A(Q-X_0^s)^{-1}A^*)^{1/t}=(AQ^{-1}A^*)^{1/t}<X,$$

assume that $X_k<X$, then

$$X_{k+1}=(A(Q-X_k^s)^{-1}A^*)^{1/t}<(A(Q-X^s)^{-1}A^*)^{1/t}=X,$$

so $X_{k+1}<X,\ k=0,1,2,\cdots$.

Now we will prove that the iterative sequence $\{X_k\}$ is monotonically increasing.

$$X_1=(AQ^{-1}A^*)^{1/t}>X_0=0,$$

$$X_2=(A(Q-X_1^s)^{-1}A^*)^{1/t}>(A(Q-X_0^s)^{-1}A^*)^{1/t}=X_1,$$

assume that $X_k>X_{k-1}$, then

$$X_{k+1}=(A(Q-X_k^s)^{-1}A^*)^{1/t}>(A(Q-X_{k-1}^s)^{-1}A^*)^{1/t}=X_k.$$

That is, the sequence $\{X_k\}$ is monotonically increasing and bounded above by X. Thus it converges to a Hermite positive definite matrix X_S, which is a solution of Eq.(1). For an arbitrary Hermite positive definite solution X, we have obtained that $X_k<X,\ k=0,1,2,\cdots$, then $X_S\le X$. That is, X_S is the minimal Hermite positive definite solution of Eq.(1).

Theorem 4 **For** $0<t\le 1$, $s\ge 1$, if there is a real number $\alpha(\alpha>1)$, for which

$$A^*Q^{-t/s}A<\frac{\alpha^s-1}{\alpha^{s+t}}Q,$$

then Eq.(1) has a Hermite positive definite solution.

Proof Consider the following iterative method

$$X_{k+1}=(Q-A^*X_k^{-t}A)^{1/s},\ k=0,1,2,\cdots.$$

Choosing $X_0=Q^{1/s}$, with Lemma 1 and the condition of this theorem we have

$$X_0=Q^{1/s}>\frac{1}{\alpha}Q^{1/s},$$

$$X_1=(Q-A^*X_0^{-t}A)^{1/s}=(Q-A^*Q^{-t/s}A)^{1/s}<Q^{1/s}=X_0,$$

$$X_1 = (Q - A^* Q^{-t/s} A)^{1/s} > (Q - \frac{\alpha^s - 1}{\alpha^{s+t}} Q)^{1/s} > (Q - \frac{\alpha^s - 1}{\alpha^s} Q)^{1/s} = \frac{1}{\alpha} Q^{1/s},$$

assume that $\frac{1}{\alpha} Q^{1/s} < X_k < X_{k-1}$, then

$$X_{k+1} = (Q - A^* X_k^{-t} A)^{1/s} < (Q - A^* X_{k-1}^{-t} A)^{1/s} = X_k,$$

$$X_{k+1} = (Q - A^* X_k^{-t} A)^{1/s} > (Q - \alpha^t A^* Q^{-t/s} A)^{1/s} > (Q - \alpha^t \frac{\alpha^s - 1}{\alpha^{s+t}} Q)^{1/s} = \frac{1}{\alpha} Q^{1/s},$$

by inductive method, we obtain

$$\frac{1}{\alpha} Q^{1/s} < X_{k+1} < X_k, k = 0,1,2,\cdots.$$

Hence the matrix sequence $\{X_k\}$ is monotonically decreasing, and is bounded below by $(1/\alpha)Q^{1/s}$, so $\{X_k\}$ converges to a Hermite positive definite solution of Eq.(1).

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