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On the Hadwiger numbers of starlike disks

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ABSTRACT

The Hadwiger number $H(J)$ of a topological disk J in $\mathbb{R}^2$ is the maximal number of pairwise nonoverlapping translates of J that touch J. It is well known that for a convex disk, this number is 6 or 8. A conjecture of A. Bezdek and K. and W. Kuperberg says that the Hadwiger number of a starlike disk is at most 8. Bezdek proved that this number is at most 75 for any starlike disk. In this note, we prove that the Hadwiger number of a starlike disk is at most 35. Furthermore, we show that the Hadwiger number of a topological disk J such that $(\operatorname{conv} J) \setminus J$ is connected is 6 or 8.

1. Introduction and preliminaries

This paper deals with topological disks in the Euclidean plane $\mathbb{R}^2$. We make use of the linear structure of $\mathbb{R}^2$, and identify a point with its position vector. We denote the origin by o, and the standard orthonormal basis of $\mathbb{R}^2$ by $\{e_x, e_y\}$. For simplicity, we use the notation $(\alpha, \beta) = \alpha e_x + \beta e_y \in \mathbb{R}^2$ for any $\alpha, \beta \in \mathbb{R}$. For a set $X \subset \mathbb{R}^2$, $\operatorname{conv} X$, $\operatorname{card} X$, $\operatorname{int} X$ and $\operatorname{bd} X$ denote the *convex hull*, the *cardinality*, the *interior* and the *boundary* of X, respectively. If $X = \{x_1, x_2, \ldots, x_k\}$ is a finite set, we may use the notation $\operatorname{conv} X = [x_1, x_2, \ldots, x_k]$. In particular, for $p, q \in \mathbb{R}^2$, the closed segment with endpoints p and q is denoted by $[p, q]$. We set $(p, q) = [p, q] \setminus \{p, q\}$, $(p, q] = [p, q] \setminus \{p\}$ and $[p, q) = [p, q] \setminus \{q\}$. The Euclidean norm of a point $p \in \mathbb{R}^2$ is denoted by $\|p\|$.

A *topological disk*, or for short a *disk*, is a compact subset of $\mathbb{R}^2$ with a simple, closed, continuous curve as its boundary. In other words, a disk is a subset of $\mathbb{R}^2$ homeomorphic to the closed unit disk of the plane. Two disks J_1 and J_2 are *nonoverlapping* if their interiors are disjoint. If J_1 and J_2 are nonoverlapping and $J_1 \cap J_2 \neq \emptyset$, then J_1 and J_2 *touch*. A disk S is *starlike relative to a point* p, if, for every $q \in S$, S contains the closed segment $[p, q]$. In particular, a convex disk K is starlike relative to any point $p \in K$.

The *Hadwiger number*, or *translative kissing number*, of a disk J, denoted by $H(J)$, is the maximal number of pairwise nonoverlapping translates of J that touch J. Grünbaum [7] proved that the

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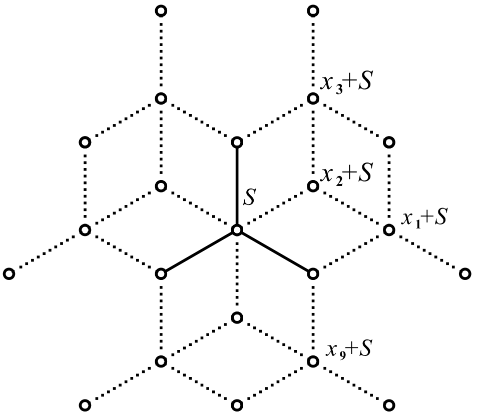

Fig. 1. A starlike set S of segments with $H(S) > 8$.

Hadwiger number of a parallelogram is 8, and that the Hadwiger number of any other convex disk is 6. In [8], the authors showed that the Hadwiger number of any disk is at least 6. Bezdek et al. [2] asked whether $H(J) \leq 8$ for any disk J, or if this is not so, whether there is a universal constant $\kappa \in \mathbb{R}$ such that $H(J) \leq \kappa$ for every disk J (see also Problem 5, p. 95 in the book [3]). They formulated also the conjecture that $H(S) \leq 8$ for any starlike disk S.

In 2007, Cheong and Lee [4] constructed, for every $n > 0$, a disk with Hadwiger number at least n, and thus showed that the answer to the question mentioned above is no. On the other hand, Bezdek proved in [1] that the Hadwiger number of a starlike disk is at most 75. In [10] it is shown that the Hadwiger number of a centrally symmetric starlike disk is at most 12. In the first part of the paper we prove the following theorem.

Theorem 1. *The Hadwiger number $H(S)$ of a starlike disk S is at most 35.*

If a set S is the union of finitely many closed segments meeting at a given point, let us call it a *starlike* set of segments. We may define the Hadwiger number of a starlike set S of segments as the maximal cardinality of a family of translates of S such that each translate contains a point of S and no translate crosses S or any other translate in the family. Clearly, any upper bound on the set of the Hadwiger numbers of starlike sets of segments is an upper bound on the set of the Hadwiger numbers of starlike disks. We note that the estimate 75 of Bezdek holds also for starlike sets of segments (cf. Theorem 2 in [1]). Unfortunately, our proof cannot be generalized to starlike sets of segments. Figure 1 shows that the assertion in Lemma 1 fails if S is not a disk. This figure shows also the existence of a starlike set S of segments with $H(S) > 8$. As far as the author is aware, the configuration in Fig. 1 was found independently by K. Swanepoel and P. Papez.

We ask the following question.

Question 1. Is it true that the Hadwiger number of any starlike set of segments is at most 9?

In the second part we examine disks that are not necessarily starlike. For a disk J, we call the connected components of $(\operatorname{conv} J) \setminus J$ the *pockets* of J. Clearly, a disk is convex if and only if it has no pockets. Our goal is to characterize the Hadwiger numbers of disks with at most one pocket.

Theorem 2. *Let J be a disk with at most one pocket.*

2.1 *If there is a direction $u \in \mathbb{S}^1$ such that the intersections of J, with the two supporting lines of* $\operatorname{conv} J$ *parallel to u, are two segments of the same length $\lambda > 0$, and $\lambda u + J$ touches J, then $H(J) = 8$.*
2.2 *Otherwise, $H(J) = 6$ (cf. Fig. 2).*

We call a disk J that satisfies the conditions in 2.1 a *parallelogram-like disk* (cf. Fig. 3).

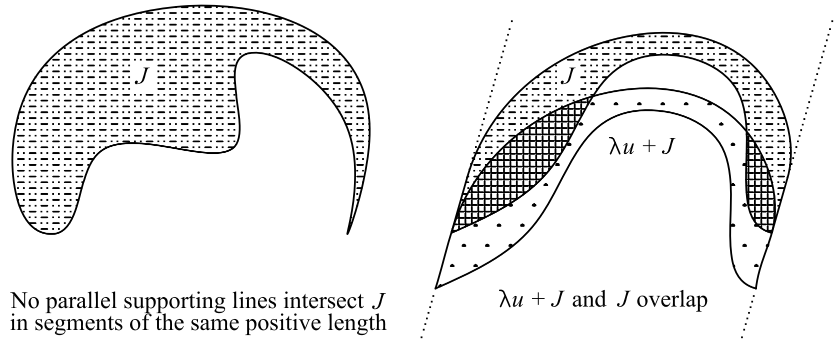

Fig. 2. Disks with one pocket and with Hadwiger number equal to 6.

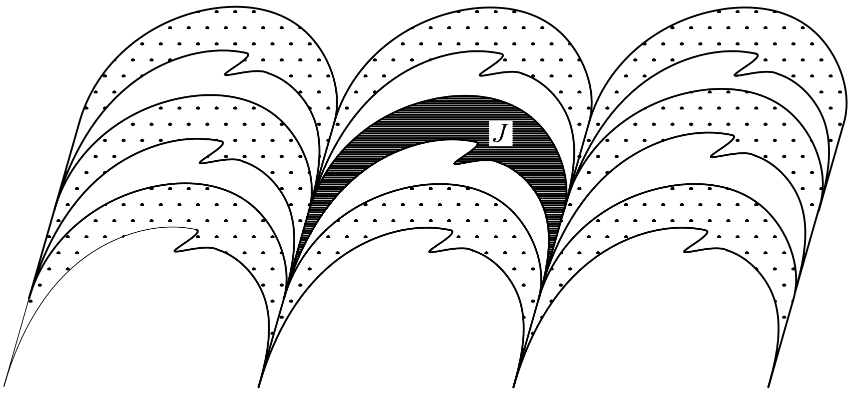

Fig. 3. Pairwise nonoverlapping translates of a parallelogram-like disk.

We note that the disk D_n^m $(m \geq n)$ in [4], with n pairwise nonoverlapping translates touching D_n^m, has $2n+2$ pockets. With reference to this observation and Theorem 2, we ask the following question.

Question 2. Is it true for every positive integer k that there is an integer $N(k)$ such that the Hadwiger number of a disk with at most k pockets is at most $N(k)$?

2. Proof of Theorem 1

Let $S \subset \mathbb{R}^2$ be a disk that is starlike relative to the origin, and let $\mathfrak{F} = \{S_i : i = 1, 2, \ldots, n\}$ be a family of pairwise nonoverlapping translates of S, with $n = H(S)$, such that each $S_i = x_i + S$ touches S. Suppose that $K = \operatorname{conv} S$, $K_i = \operatorname{conv} S_i$ for $i = 1, 2, \ldots, n$, $X = \{x_i : i = 1, 2, \ldots, n\}$, and $C = \operatorname{conv} X$. Furthermore, suppose that $R_i = \{\lambda x_i : \lambda \in \mathbb{R} \text{ and } \lambda \geq 0\}$.

First, we prove a few lemmas that we use in the proof of Theorem 1.

Lemma 1. *We have $o \in \operatorname{int} C$, and $X \subset \operatorname{bd} C$.*

Proof. Note that if $o \in \operatorname{bd} \operatorname{conv}(X \cup \{o\})$, then there is a supporting line $\bar{L}$ of $F = \operatorname{conv}\left(S \cup \left(\bigcup_{i=1}^n S_i\right)\right)$ that passes through a point of S. Thus, there is a translate of S, on the other side of $\bar{L}$, that touches S and does not overlap F. Since $n = H(S)$, we have a contradiction, which proves the first statement.

For a contradiction, suppose that $x_i \in \operatorname{int} C$ for some value of i. Note that if $i \neq j$, then $x_i \notin [o, x_j]$. Thus, there are indices $j \neq k$ such that $x_i \in \operatorname{int}[o, x_j, x_k]$. Since $H(S') = H(S)$ for any affine image S' of S, we may assume that $x_j = e_x$ and $x_k = e_y$.

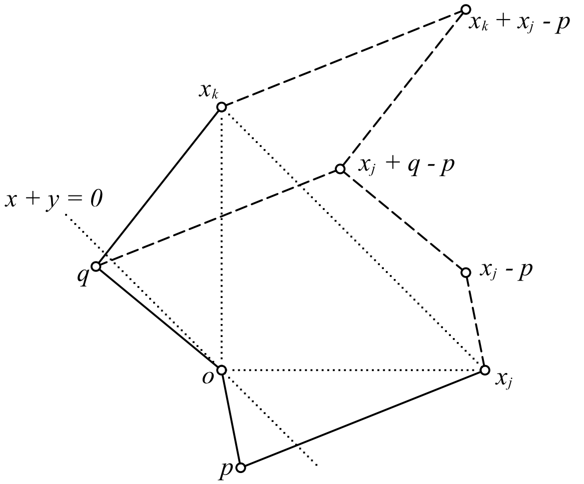

Fig. 4. An illustration for Lemma 1.

Consider points $p \in S_j \cap S$ and $q \in S_k \cap S$, and note that $[o, p]$, $[o, q]$, $[o, p - x_j]$, $[o, q - x_k] \subset S$. Our aim is to show that for any such starlike disk S, S_i overlaps S, S_j or S_k. In our examination, to help the reader follow the arguments, the segments in the figures belonging to S, S_j or S_k are drawn with continuous lines, and all the other lines are dotted or dashed.

Observe that x_i is not contained in the open parallelograms $P_j = \mathrm{int}[o, x_j, p, x_j - p]$, as otherwise the segment $[x_i, x_i + p]$ crosses $[x_j, p]$, and thus, S_i and S_j overlap (note that this argument is valid also in the case where $p \in [o, x_j]$). Similarly, x_i is not contained in $P_k = \mathrm{int}[o, x_k, q, x_k - q]$, since otherwise S_i and S_k overlap. We set $T = [o, x_j, x_k]$ and $Q = (\mathrm{int}\, T) \setminus \left(P_j \cup P_k\right)$. So far, we have that $x_i \in Q$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f((\alpha, \beta)) = \alpha + \beta$. We show that $0 \le f(p) \le 1$ and $0 \le f(q) \le 1$.

For a contradiction, suppose first that $f(p) < 0$ or $f(q) < 0$. Without loss of generality, we may assume that $f(p) < 0$ and $f(p) \le f(q)$ (cf. Fig. 4), which yields that $Q \subseteq [o, x_j - p) + ([o, q] \cup [q, x_k])$. If $x_i \in \left([o, x_j - p) + [o, q)\right)$, then $[x_i, x_i + x_j - p]$ crosses $[o, q]$ and thus, S_i overlaps S; a contradiction. Similarly, if $x_i \in \left([o, x_j - p) + (q, x_k]\right)$, then $[x_i, x_i + x_j - p]$ crosses $[q, x_k]$, and S_i overlaps S_k. Finally, if $x_i \in [q, q + x_j - p)$, then q lies in the relative interior of a segment in S_i, from which it readily follows that S_i is not a disk; a contradiction.

Next, suppose that $f(p) > 1$ or $f(q) > 1$. Without loss of generality, we may assume that $f(p) > 1$ and that $0 \le f(q) \le f(p)$. Then $Q \subseteq [o, p) + ([o, x_k - q] \cup [x_k - q, x_k])$. From here, the assertion follows by an argument similar to the one in the previous paragraph.

In the following, we denote the line with the equation $x + y = 1$ by L.

Case 1: both the y-coordinate of p and the x-coordinate of q are negative. Without loss of generality, we may assume that $f(q) \ge f(p)$. Then, since f is linear, we have that $f(x_j + q - p) \ge 1$, or in other words, that L separates $x_j + q - p$ from the origin. Note that Q is covered by the union of the sets $U_1 = [x_j, x_j - p) + [o, q)$, $U_2 = [o, q) + [o, x_j - p)$, $U_3 = [x_k, q) + [o, x_j - p)$, $[q, x_j + q - p)$ and $[x_j - p, x_j + q - p)$ (cf. the left-hand side of Fig. 5). If $x_i \in U_1$, then $[x_i, x_i + p]$ and $[x_j, x_j + q]$ cross, and thus, S_i and S_j overlap; a contradiction. If $x_i \in U_2$ or $x_i \in U_3$, then $[x_i, x_i + x_j - p]$ crosses $[o, q]$ or $[q, x_k]$, respectively, and thus, S_i overlaps S or S_k. If $x_i \in [q, x_j + q - p)$, then S and S_k touch each other in a relative interior point of $[x_i, x_i - p]$, which yields that S_i is not a disk; a contradiction. Finally, if $x_i \in [x_j - p, x_j + q - p)$, then S_i meets the segments $[o, q)$ and $[x_j, x_j + q)$ from different sides. Since $[x_j, x_j + q)$ is the translate of $[o, q)$ in S_j, from this it follows that S is not a disk; a contradiction.

Case 2: either the y-coordinate of p or the x-coordinate of q is negative. Without loss of generality, we may assume that the y-coordinate of p is nonnegative and that the x-coordinate of q is negative. First, we examine the case where $f(p) \ge f(q)$, which yields that L separates o and $x_k + p - q$ (cf. the right-hand side of Fig. 5). Then Q is covered by the union of the sets $V_1 = [x_k, x_k - q) + [o, p)$,

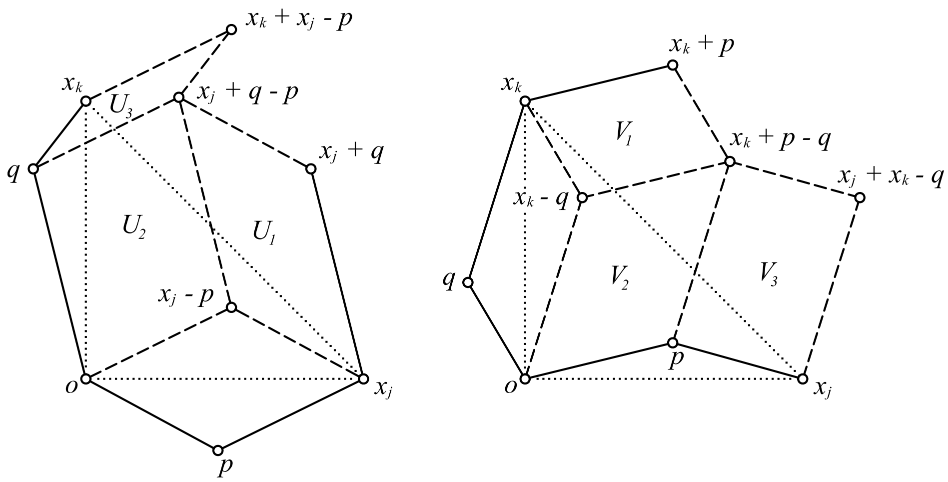

Fig. 5. Illustrations for Cases 1 and 2 of Lemma 1.

$V_2 = [o, x_k - q) + [o, p)$, $V_3 = [x_j, p) + [o, x_k - q)$, $[p, x_k + p - q)$ and $[x_k - q, x_j + p - q)$. If $x_i \in V_1, x_i \in V_2$ or $x_i \in V_3$, then S_i overlaps S_k, S or S_j, respectively. If $x_i \in [p, x_k + p - q)$ or $x_i \in [x_k - q, x_j + p - q)$, then S is not a disk.

If $f(p) \leq f(q)$, then the assertion follows by a similar argument.

Case 3: both the y-coordinate of p and the x-coordinate of q are nonnegative. The proof in this case is similar to the proof in the previous two cases; hence we omit it. □

With reference to Lemma 1, we may relabel the indices of the elements of $\mathfrak{F}$ in such a way that $x_1, x_2, \dots, x_n = x_0$ are in counterclockwise order on bd C.

Lemma 2. *Consider points $w_i \in S \cap S_i$ for $i = 1, 2, \dots, n$. Then $w_1, w_2, \dots, w_n$ are in this counterclockwise order around o.*

Proof. Note that as $o \in \operatorname{int} C$, and the points $x_1, x_2, \dots, x_n$ are in this counterclockwise order on bd C, they are in the same order around o. We define the points $\bar{x}_i$ as follows: If $w_i \in \operatorname{int} C$, then $\bar{x}_i = x_i$, and otherwise it is the intersection point of $[o, w_i]$ and bd C. Suppose that $\bar{R}_i = \{\lambda \bar{x}_i : \lambda \in \mathbb{R} \text{ and } \lambda \geq 0\}$, and suppose that $Q_i = \operatorname{int} \operatorname{conv}(R_i \cup \bar{R}_i)$.

First, we show that if $x_j \in Q_i$ for some $j \neq i$, then $x_j \notin [o, w_i, x_i]$, $w_i \in [o, w_j, x_j]$ and $w_j \notin \operatorname{int} C$. Consider some $i \neq j$ with $x_j \in Q_i$. Then $w_i \notin \operatorname{int} C$, as otherwise $R_i = \bar{R}_i$. If $x_j \in \operatorname{int}[o, w_i, x_i]$, then $[x_j, x_j + w_i]$ crosses $[x_i, w_i]$, and S_i and S_j overlap; a contradiction. If $x_j \in (w_i, x_i)$, then $[x_j, x_j + (w_i - x_i)] \subset Sj$, which, since this segment is the translate of $[x_i, w_i]$ by $x_j - x_i$ and since their relative interiors intersect, yields that S is not a disk; a contradiction. If $x_j \notin [o, w_i, x_i]$, then $[w_j, x_j] \cap \bar{R}_i \neq \emptyset$, as otherwise $[o, w_j]$ crosses $[w_i, x_i]$ or $x_i \in \operatorname{int}[o, w_j, x_j]$ (cf. Fig. 6). Thus, in this case $w_i \in [o, w_j, x_j]$, which, as $x_j \in$ bd C, yields that $w_j \notin \operatorname{int} C$.

Next, we show that $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ are in this counterclockwise order around o. To do this, it suffices to show that there are no values of $i \neq j$ such that $\bar{x}_i$, $\bar{x}_j$ and $\bar{x}_{i+1}$ are in this counterclockwise order around o. Suppose for a contradiction that there are such values.

First we consider the case where x_i, w_j and x_{i+1} are in this counterclockwise order around o. Since x_i, x_j and x_{i+1} are *not* in this counterclockwise order, we have $x_i \in Q_j$ or $x_{i+1} \in Q_j$, say $x_i \in Q_j$. Then, clearly, $x_{i+1} \notin Q_j$, and, by the argument in the second paragraph of this proof, we have $x_i \notin [o, x_j, w_j]$, $w_j \notin \operatorname{int} C$ and $w_j \in [o, w_i, x_i]$. Thus, $w_i \notin \operatorname{int} C$, which yields that x_i, w_i and x_{i+1} are in this counterclockwise order. Since x_j, w_i and w_{i+1} are in this counterclockwise order, it follows that so are $\bar{x}_j$, $\bar{x}_i$ and $\bar{x}_{i+1}$.

Now we examine the case where x_i, w_j and x_{i+1} are not in this counterclockwise order. Then, since x_i, x_j and x_{i+1} are not, we have that $w_j \in Q_i$ or $w_j \in Q_{i+1}$, say $w_j \in Q_i$. From this, we obtain that $w_j \in$

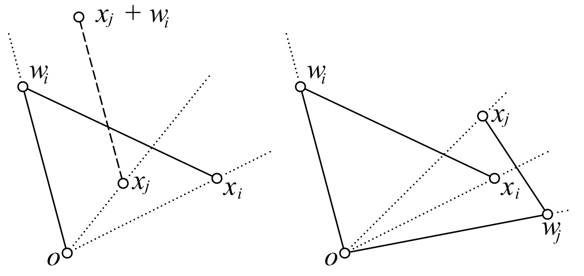

Fig. 6. An illustration for Lemma 2.

$[o, w_i, x_i]$ and as $x_j \notin [o, w_i, x_i]$, we have that $[w_j, x_j]$ intersects both $[o, x_i]$ and $[o, x_{i+1}]$. Since $x_i, x_j \notin [o, w_{i+1}, x_{i+1}]$, this implies that $w_j \in [o, w_{i+1}, x_{i+1}]$ and $w_{i+1} \in [o, w_i, x_i]$. From this, it readily follows that $w_i, w_j, w_{i+1} \notin \operatorname{int} C$, and thus, that $\bar{x}_i, \bar{x}_{i+1}$ and $\bar{x}_j$ are in this counterclockwise order around o.

We have shown that $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ are in this counterclockwise order around o. Since these points are in $\operatorname{bd} C$ and they can be connected to o by mutually noncrossing polygonal curves in $\operatorname{int} C$, their counterclockwise order around o is the same as that of the points $w_1, w_2, \ldots, w_n$. $\square$

We need the next lemma of Bezdek to prove Lemma 4 (cf. Lemma 3 in [1]).

Lemma 3 (*Bezdek*)**.** *For any $i = 1, 2, \ldots, n$, $\operatorname{int} K_i$ contains at most one element of $X \setminus \{x_i\}$.*

We call S_i and S_j *separated* if $x_i \notin \operatorname{int} K_j$, and $x_j \notin \operatorname{int} K_i$.

Lemma 4. *There is a subfamily $\mathfrak{F}'$ of $\mathfrak{F}$, of cardinality at least $\lfloor \frac{n-2}{2} \rfloor$, such that any two elements of $\mathfrak{F}'$ are separated.*

Proof. For $i = 1, 2, \ldots, n$, we choose points $w_i \in S \cap S_i$, and set $\Gamma_i = [o, w_i] \cup [w_i, x_i]$. By Lemma 2, the points $w_1, w_2, \ldots, w_n$ are in counterclockwise order around o.

By Lemma 3, $\operatorname{int} K_i$ contains at most one point of X different from x_i. Hence, if $X \cap \operatorname{int} K_i \subset \{x_{i-1}, x_i, x_{i+1}\}$ for every value of i, the assertion immediately follows with $\mathfrak{F}' = \{S_{2m} : m = 1, 2, \ldots, \lfloor n/2 \rfloor\}$. Thus, it suffices to show that $X \cap \operatorname{int} K_i \not\subset \{x_{i-1}, x_i, x_{i+1}\}$ for at most two values of i, as in this case, after removing these elements of $\mathfrak{F}$, we may choose the elements of $\mathfrak{F}'$ like in the previous case.

Consider the case where $x_j \in \operatorname{int} K_i$ for some $j \notin \{i-1, i, i+1\}$. Without loss of generality, suppose that $i = 2$. Since $o \in \operatorname{int} C$, we have that the line $L_2 = R_2 \cup (-R_2)$ separates x_1 and x_3 (recall the definition of R_i from the first paragraph of Section 2). Without loss of generality, we may assume that x_j and x_3 lie in the same closed half-plane H bounded by L_2, which yields that $L_3 = R_3 \cup (-R_3)$ separates x_2 and x_j.

Since $x_j \in \operatorname{int} K_2$, there are points $p, q \in S$ such that $x_j \in \operatorname{int}[x_2, x_2+p, x_2+q]$. Note that by Lemma 3, we have that $x_3 \notin \operatorname{int}[x_2, x_2+p, x_2+q]$. For a contradiction, suppose that $o \notin \operatorname{int}[x_2, x_2+p, x_2+q]$. Considering the cases where the line passing through x_2+p and x_2+q separates x_j from o, x_3 or neither, it readily follows that at least one of $[x_2, x_2+p]$ or $[x_2, x_2+q]$ crosses both $[o, x_3]$ and the ray emanating from x_3 and passing through x_j. Since Γ_3 does not cross $[x_2, x_2+p]$ and $[x_2, x_2+q]$, we obtain that $x_j \in [o, x_3, w_3]$ or $x_2 \in [o, x_3, w_3]$, which, like in the proof of Lemma 2, immediately yields that S_j or S_2 overlaps S_3; a contradiction. Hence, we obtain that $o \in \operatorname{int}[x_2, x_2+p, x_2+q]$. Without loss of generality, we may choose our notation such that $q \in H$, which implies that R_3 crosses the segment $[x_2, x_2+q]$.

Suppose that $Q = \operatorname{int}([x_2, o, x_2+p] \cup [x_2, o, x_2+q])$, and consider the case where $w_1 \in Q$. Since S_1 and S do not overlap, we have that $x_1 \notin \operatorname{int}[q, p+q, -q, p-q]$. By Lemma 3, $x_1 \notin \operatorname{int}[x_2, x_2+p, x_2+q]$, from which it readily follows that $x_1 = \alpha p + \beta q$ with $\alpha \geq 1$. Thus, the segment $[x_2, x_2+q]$ crosses $[o, w_1 - x_1]$ (cf. Fig. 7). As $[o, w_1 - x_1] \subset S$, it follows that S and S_2 overlap; a contradiction. We may show similarly that $w_3 \notin Q$.

We obtained that, from $x_j \in \operatorname{int} K_2$ and $x_j \notin \{x_1, x_2, x_3\}$, it follows that the angle $\angle(w_1, o, w_3)$ measured from $[o, w_1]$ to $[o, w_3]$ in the counterclockwise direction is strictly greater than π.

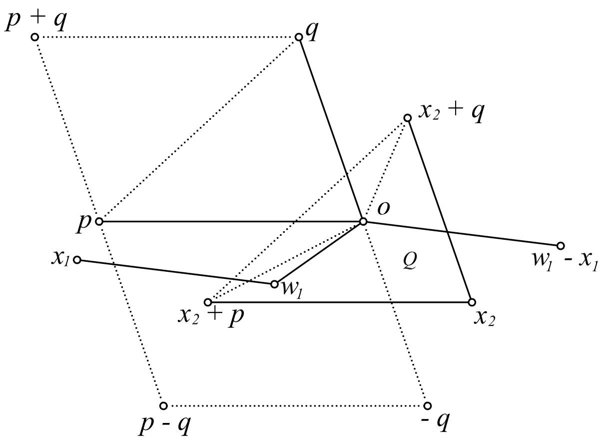

Fig. 7. An illustration for Lemma 4.

Note that $\angle(w_k, o, w_{k+2}) \le \pi$ if $k \notin \{n, 1, 2\}$ and that $\angle(w_n, o, w_2) \le \pi$ or $\angle(w_2, o, w_4) \le \pi$. Thus, $\angle(w_{i-1}, o, w_{i+1}) > \pi$ holds for at most two values of i, and hence the assertion immediately follows. $\square$

Now we are ready to prove Theorem 1. In the proof we use the following notion. Let $D \subset \mathbb{R}^2$ be a convex disk, and suppose that $p, q \in \mathbb{R}^2$. Let $r, s \in D$ be such that $[r, s]$ and $[p, q]$ are parallel, and there are no points r', s' in D such that $[r', s']$ is parallel to $[p, q]$ and $[r', s']$ is longer than $[r, s]$. The *D-distance* of p and q (cf. [11]) is defined as

$$\mathrm{dist}_D(p, q) = \frac{2\|p - q\|}{\|r - s\|}.$$

It is easy to see that $\mathrm{dist}_D(p, q)$ is equal to the distance of p and q in the norm whose unit disk is the central symmetral $\frac{1}{2}(D - D)$ of D. We observe that, for any two convex disks $D \subset D'$ and points $p, q \in \mathbb{R}^2$, we have $\mathrm{dist}_{D'}(p, q) \le \mathrm{dist}_D(p, q)$.

Note that since S_i touches S for every i, K and $x_i + K$ intersect. Thus, $\mathrm{dist}_K(o, x_i) = \mathrm{dist}_{\bar{K}}(o, x_i) \le 2$, where $\bar{K} = \frac{1}{2}(K - K)$. In other words, we have $X \subset 2\bar{K} = K - K$, which yields that $\mathrm{dist}_C(x_i, x_j) \ge \mathrm{dist}_{2\bar{K}}(x_i, x_j) = \frac{1}{2}\mathrm{dist}_{\bar{K}}(x_i, x_j)$ for any $i \ne j$.

By Lemma 4, we may choose a subfamily $\mathfrak{F}'$ of at least $\lfloor \frac{n-2}{2} \rfloor$ pairwise separated elements of $\mathfrak{F}$. Let X' denote the set of the translation vectors of the members of $\mathfrak{F}'$. Note that if $u + S$ and $v + S$ are separated, then $u + \frac{1}{2}K$ and $v + \frac{1}{2}K$ are nonoverlapping. In other words, we have $\mathrm{dist}_K(u, v) \ge 1$ for any distinct $u, v \in X'$, which yields that $\mathrm{dist}_{\bar{C}}(u, v) = \mathrm{dist}_C(u, v) \ge \frac{1}{2}$, with $\bar{C} = \frac{1}{2}(C - C)$.

Gołąb [6] proved that the circumference of every centrally symmetric convex disk measured in its norm is at least 6 and at most 8 (for a more accessible reference, cf. [13]). Fáry and Makai [5] proved that, in any norm, the circumference of any convex disk C and that of its central symmetral $\frac{1}{2}(C - C)$ are equal. Thus, the circumference of C measured in the norm with unit ball $\bar{C}$ is at most 8. Since X' is a set of points in $\mathrm{bd}\, C$ at pairwise $\bar{C}$-distances at least $\frac{1}{2}$, we have $\lfloor \frac{n-2}{2} \rfloor \le \mathrm{card}\, X' \le 16$, from which the assertion immediately follows.

3. Proof of Theorem 2

Suppose that $K = \mathrm{conv}\, J$ and $H(J) = n$. Consider a family $\mathfrak{F} = \{J_i : i = 1, 2, \ldots, n\}$ of pairwise nonoverlapping translates such that each member $J_i = x_i + J$ of $\mathfrak{F}$ touches J, and set $K_i = x_i + K$. If $J = K$, then our result follows from a result of Grünbaum in [7]. Thus, we examine the case where J has exactly one pocket. Note that by Halberg et al. [8], $n \ge 6$, and if J is parallelogram-like, then $n \ge 8$. Thus, it is sufficient to prove that $n \le 8$, and if J is not parallelogram-like, then $n \le 6$.

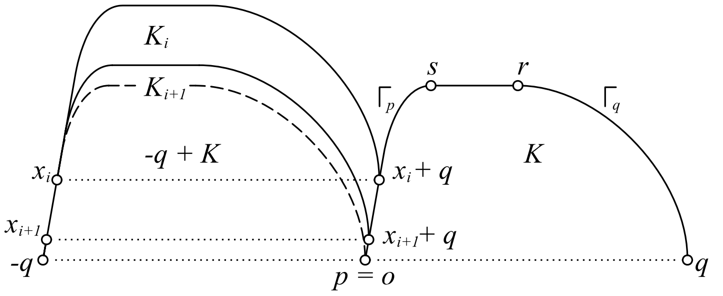

Fig. 8. An illustration for the proof of Theorem 2.

We choose the indices of the elements of $\mathfrak{F}$ in such a way that $J_1 \cap J, J_2 \cap J, \ldots, J_n \cap J$ are in counterclockwise order on $\operatorname{bd} J$. Let p and q denote the endpoints of the longest segment in $\operatorname{bd} K$ that contains $(\operatorname{bd} K) \setminus J$.

Case 1: there is no chord of K, parallel to $[p, q]$, that is longer than $[p, q]$. Then K has two distinct parallel supporting lines, passing through p and q. Since the Hadwiger number of J does not change under an affine transformation, we may assume that $p = o$, $q = e_x$, and that the y-axis and the line $x = 1$ support K.

Consider a translate $x + J$ of J that overlaps K but not J. We show that there is no other translate of J that touches J, overlaps K and does not overlap $x + J$. Suppose for a contradiction that $y + J$ touches J and it overlaps K but not $x + J$. Then $x + K$ and $y + K$ do not overlap, as otherwise $x + ((\operatorname{bd} K) \cap J)$ and $y + ((\operatorname{bd} K) \cap J)$ cross. Without loss of generality, we may assume that $x + K$ and $y + K$ touch each other, $[p, q] \cap (x + K)$ is closer to p than $[p, q] \cap (y + K)$, and the y-coordinate of x is not greater than the y-coordinate of y. Suppose that $[u, v] = [p, q] \cap (x + K)$, where the notation is chosen such that u is closer to p than v. First, observe that $v + (q - p)$ is a point of $x + (q - p) + K$, and that $\|v + (q - p)\| > \|q - p\|$. Furthermore, by the convexity of K, for any $z \in [x + q - p, v]$, $z + K$ has a point, on the positive half of the y-axis, not closer to p than $v + q - p$. Thus, as y is a point of the closed arc of $\operatorname{bd}(x + K)$ between $x + q - p$ and v that does not contain x, $y + K$ also has a point on the positive half of the y-axis, farther from p than $\|q - p\|$, which clearly contradicts our assumptions that $y + J$ overlaps K and does not overlap J. Hence, we have obtained that there is at most one member of $\mathfrak{F}$ that overlaps K. We may show similarly that there is at most one member J_i of $\mathfrak{F}$ such that K_i overlaps J.

Let L be the supporting line of K, parallel to and not containing $[p, q]$. Suppose that $[r, s] = L \cap J$, and note that $[r, s]$ may degenerate to a single point. We choose our notation in such a way that p, q, r and s are in counterclockwise order on $\operatorname{bd} J$. Let H^+ be the closed half-plane with $[p, q] \subset \operatorname{bd} H^+$ and $K \subset H^+$, and suppose that $H^- = \mathbb{R}^2 \setminus H^+$. Let Γ_p denote the open arc of $\operatorname{bd} J$ with endpoints p and s that does not contain q, and let Γ_q denote the open arc of $\operatorname{bd} J$ with endpoints q and r that does not contain p. Clearly, if $[p, q, r, s]$ is a parallelogram, then K is a parallelogram and no two translates of K overlap, and thus, the assertion follows immediately from [7]. Hence, in the following, we deal with the case where $[p, q, r, s]$ is not a parallelogram.

Consider the case where $x_i + q, x_{i+1} + q \in \Gamma_p \cup [s, r)$ for some value of i. Then, clearly, $x_i, x_{i+1} \in (p - q + K) = -q + K$ (cf. Fig. 8), which yields that $x_i \in K_{i+1}$. On the other hand, since J_i and J_{i+1} do not overlap, we have that $x_i \notin \operatorname{int} K_{i+1}$, This implies that $-q, x_{i+1}$ and x_i are collinear, which yields that J is not a disk; a contradiction. Thus, we obtain that $x_i + q \in \Gamma_p \cup [s, r)$ for at most one value of i, and, by a similar argument, that $x_i \in \Gamma_q \cup [r, s)$ for at most one value of i.

Note that for any translate $J_i \subset H^+$, at least one of the following holds: $x_i = -q$, $x_i = q$, $x_i + q \in \Gamma_p \cup [s, r)$, $x_i \in \Gamma_q \cup [r, s)$ or $[r, s] \subset K_i$. Hence, by the second and fourth paragraphs in Case 1, we obtain that H^+ contains at most five members of $\mathfrak{F}$. Furthermore, we observe that any two members of $\mathfrak{F}$ that have a point in H^- have nonoverlapping convex hulls, and thus, this may hold for at most three members of $\mathfrak{F}$. Thus, $\operatorname{card} \mathfrak{F} \leq 8$.

To finish the proof in Case 1, we examine the case where J is not parallelogram-like, and show that then $\operatorname{card} \mathfrak{F} \leq 6$. First, we show that at most four elements of $\mathfrak{F}$ intersect H^+. Suppose for a contradiction that this is not the case, or in other words, that for some value of i we have $J_{i-2} = q + J$, $x_{i-1} \in \Gamma_q \cup [r, s)$, $[r, s] \subset K_i$, $x_{i+1} + q \in \Gamma_p \cup [s, r)$ and $J_{i+2} = -q + J$.

Since J_i and J_{i+1} do not overlap, we obtain that $x_{i-1}, x_{i+1} \notin (r, s)$. If $x_{i+1} + q = s$ and $x_{i-1} = r$, then J is parallelogram-like, and hence, we have that $x_{i+1} + q \neq s$ or $x_{i-1} \neq r$, say $x_{i+1} + q \neq s$. Then, as J_{i+1} and J_{i+2} do not overlap, we obtain that $x_{i+1} + q + J$ touches J at $x_{i+1} + q$. Furthermore, if x is on the closed arc of $\Gamma_p \cup \{s\}$ between $x_{i+1} + q$ and s, then $x + J$ does not overlap J, as the region $x + K \cap K$ strictly decreases, in terms of containment, when we move x away from $x_{i+1} + q$. This yields that x_i is a point of this closed arc. On the other hand, $x + K$ touches $J_{i-2} = q + J$ at $x + q$. Thus, J_{i-1} touches J only if $x_i + q = x_{i-1}$, from which it immediately follows that J is parallelogram-like; a contradiction.

We observe that the convex hulls of any two elements of $\mathfrak{F}$ having a point in H^- are nonoverlapping, and that at most three elements of $\mathfrak{F}$ have points in H^-. Thus, to show that card $\mathfrak{F} \leq 6$, it is sufficient to examine the case where, for some value of i, each of J_{i-1}, J_i and J_{i+1} has a point in H^-.

Let L_0 and L_1 denote the y-axis and the line $x = 1$, respectively.

Subcase 1.1: $L_0 \cap \operatorname{int} J_{i-1} \neq \emptyset$ or $L_1 \cap \operatorname{int} J_{i+1} \neq \emptyset$. Without loss of generality, suppose that $L_0 \cap \operatorname{int} J_{i-1} \neq \emptyset$, which yields that p is a common point of J and J_{i-1}.

Consider the case where the y-coordinate of x_i is not smaller than that of x_{i-1}. Note that, by an argument similar to those used in the previous paragraphs, if J_i is contained in $K \cup (q + K_{i-1})$, then $x_i = q + x_{i-1}$, and thus, J_i has a point in the open half-plane $x > 1$. Since this clearly follows also if $J_i \not\subset K \cup (q + K_{i-1})$, and it implies that J_{i+1} has no point in H^-, which is a contradiction, we obtain that the y-coordinate of x_i is smaller than that of x_{i-1}. Similarly, if $q \in J_i$, then J_{i+1} has no point in H^-, and thus, it follows that $q \notin J_i$.

Next, consider the case where $J_{i-2} = -q + J$. Observe than then $q + J_{i-1}$ touches J. Thus, since the y-coordinate of x_i is smaller than that of x_{i-1}, any point of J_i with nonnegative y-coordinate is contained in $q + K_{i-1}$. Thus, J_i touches J at a boundary point u of $q + K_{i-1}$, which is clearly a boundary point of K_i also. Since $x_i - (x_{i-1} + q)$ translates J_i to $q + J_{i-1}$, this vector moves u to another boundary point u' of $q + J_{i-1}$; or in other words, $(x_{i-1} + q) - x_i$ moves a boundary point u' of $q + J_{i-1}$ to u. On the other hand, the translation by $x_{i-1} + q - x_i$ does not move any point of $[u, u']$ to a point of K outside $q + K_{i-1}$. Hence, $u \in [p, q]$, which, as $q \notin J_i$, yields that $u \in (p, q)$ and $[p, u] \subset J$.

We obtained that there is a point $u \in (p, q)$ with $[p, u] \subset J$. We show that it yields card $\mathfrak{F} \leq 6$. Observe that $q \in J_{i+1}$. Clearly, if $q \in [x_{i+1} + r, x_{i+1} + s]$, then J_i does not touch J. Thus, $\operatorname{int} J_{i+1} \cap H^+ \neq \emptyset$, which, by $[p, u] \subset J$, implies that $q + J \notin \mathfrak{F}$.

Now we have card $\mathfrak{F} \leq 7$. Then we can have card $\mathfrak{F} > 6$ only if $x_{i-3} + q \in \Gamma_p \cup [s, r)$, $[r, s] \subset J_{i+3}$ and $x_{i+2} \in \Gamma_q \cup [r, s)$. Since $[p, u] \subset J$, we have that $x_{i-3} + q = s$, and $x_{i+3} = s$. As the point $s + q$ is on $\operatorname{bd}(q + J)$, and $s + q \notin \operatorname{int} K_{i+2}$, we have that x_{i+2} is in the boundary of both J and $q + J$. Thus, x_{i+2} is on the line $x = 1$, and s is on the y-axis, which implies that x_{i+1} is on the line L_1, and that $L_1 \cap J_{i+1}$ is a translate of $[p, s]$. On the other hand, the distance between x_{i+2} and the closest point of $K_i \cap K_1$ is clearly less than that between p and s; a contradiction.

We have shown that $-q+J \in \mathfrak{F}$ yields that card $\mathfrak{F} \leq 6$. Thus, to show that card $\mathfrak{F} \leq 6$, it is sufficient to consider the case where $x_{i+2} = q$, $x_{i+3} \in \Gamma_q \cup [r, s)$, $[r, s] \subset J_{i-3}$, $x_{i-2} + q \in \Gamma_p \cup [s, r)$.

If $L_1 \cap \operatorname{int} J_{i+1} \neq \emptyset$, then we may apply an argument similar to that in the previous paragraphs. Hence, we obtain that $L_1 \cap \operatorname{int} J_{i+1} = \emptyset$, which yields that x_{i+1} is on L_1, and that $(x_{i+1}, q) \subset (x_{i+1} + \Gamma_p)$. Since $L_0 \cap \operatorname{int} J_{i-1} \neq \emptyset$ and J_i touches J, we obtain that the y-coordinate of x_i is less than that of x_{i+1}, and that there is a point $u \in (p, q)$ such that $[u, q] \subset J$. Thus, $x_{i+3} \notin \Gamma_q$, from which we have that $x_{i+3} = r$ and $x_{i-3} = r - q$. But this implies that J_{i-2} does not touch J; a contradiction.

Subcase 1.2: $L_0 \cap \operatorname{int} J_{i-1} \cap = \emptyset$ and $L_1 \cap \operatorname{int} J_{i+1} = \emptyset$. In this case $J \cap L_1 = [w, q]$ and $L_0 \cap J = [z, p]$ for some points w and z with $z \neq p$ and $w \neq q$. Observe that if neither $q + J$ nor $-q + J$ belongs to $\mathfrak{F}$, then card $\mathfrak{F} \leq 6$. Hence, it suffices to consider the case where at least one of them, say $-q + J$, belongs to $\mathfrak{F}$, which yields that $x_{i-1} = -z - q$, and $\|w - q\| \geq \|z - p\|$.

Note that $\|w - q\| > \|z - p\|$, as otherwise J is parallelogram-like. Since $q + J \in \mathfrak{F}$ yields that $\|w - q\| = \|z - p\|$, we have also that $q + J \notin \mathfrak{F}$. Thus, if card $\mathfrak{F} > 6$, then $x_{i-3} + q \in \Gamma_p \cup \{s\}$, $[r, s] \subset K_{i+3}$ and $x_{i+2} \in \Gamma_q \cup \{r\}$. Without loss of generality, we may assume that $x_{i+1} = q - z$. Since J is not parallelogram-like, $L_0 \cap \operatorname{int} J_{i-3} \neq \emptyset$. Hence, since J_{i+3} touches J, $L_0 \cap \operatorname{int} J_{i+3} \neq \emptyset$, which yields that L_1 supports J_{i+2}. But then $w \in J_{i+2}$, and J_{i+2} and J_{i+3} overlap; a contradiction. This finishes the proof in Case 1.

Case 2: there is a chord in K, parallel to $[p, q]$, which is longer than $[p, q]$. Clearly, in this case J is not parallelogram-like, and hence, we need to prove that card $\mathfrak{F} \le 6$. Since the proof is similar to that of Case 1, we just sketch it.

First, we prove the following lemma that we need in the proof. We note that the proof of this lemma is included in the proof of Theorem 7 in [9].

Lemma 5. *Let K be a convex disk. If $\bar{K} = \frac{1}{2}(K - K)$ is a parallelogram, then K is a translate of $\bar{K}$.*

Proof. First, observe that $\bar{K} = \frac{1}{2}(K-K)$ if and only if K is a convex disk of constant width 2 in the norm of $\bar{K}$. Let $\bar{K} = [a_1, a_2, a_3, a_4]$ be a parallelogram, where a_1, a_2, a_3 and a_4 are in this counterclockwise order in $\operatorname{bd} \bar{K}$. Let L_1 and L_2 be the supporting lines of K parallel to $[a_1, a_2]$ such that the translate of L_1 by $a_4 - a_1$ is L_2. Let $[b_1, b_2] = K \cap L_1$ and $[b_3, b_4] = K \cap L_2$ be such that $b_2 - b_1$ and $b_3 - b_4$ are positive multiples of $a_2 - a_1$. Suppose that $c_3 = b_1 + (a_3 - a_1)$ and $c_4 = b_2 + (a_4 - a_2)$. Since K is of constant width 2, we have $[c_3, c_4] \subset [b_3, b_4]$. Observe that $\|c_4 - c_3\| = 2\|a_2 - a_1\| - \|b_2 - b_1\|$. As $\|b_4 - b_3\| \le \|a_2 - a_1\|$, this implies that $\|b_2 - b_1\| = \|b_4 - b_3\| = \|a_2 - a_1\|$, $c_3 = b_3$ and $c_4 = b_4$. Hence $K' = [b_1, b_2, b_3, b_4]$ is a translate of $\bar{K}$. As $K' \subset K$ and K is a convex disk of constant width 2 in the norm of $\bar{K}$, we have $K' = K$. $\square$

Now we return to the proof of our theorem. Suppose for a contradiction that card $\mathfrak{F} \ge 7$. We leave it to the reader to show, by methods similar to those used in Case 1, that the convex hulls of the elements of $\mathfrak{F}$ are mutually nonoverlapping (though they may overlap K), and that the points $x_1, x_2, \dots, x_n$ are in convex position. Let Θ denote the closed polygonal curve with endpoints $x_1, x_2, \dots, x_n$ in counterclockwise order. Observe that $\Theta \subset K - K$, and the sides of Θ are of at least unit length in the norm defined by the difference body $K - K$ of K. By Theorem 2 of [12], there is a convex n-gon P, inscribed in $K - K$, such that the sides of P are of at least unit length in the norm of $K - K$. In other words, there are n mutually nonoverlapping translates of $\bar{K} = \frac{1}{2}(K - K)$, each of which touches $\bar{K}$. Clearly, K is not a parallelogram. Thus Lemma 5 yields that $\bar{K}$ is not a parallelogram, and hence, by [7], we have that $n \le H(\bar{K}) = 6$; a contradiction.

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