Regular embeddings of $K_{n,n}$ where $n$ is a power of 2. I:
Metacyclic case

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Abstract

A 2-cell embedding of a graph in an orientable closed surface is called regular if its automorphism group acts regularly on arcs of the embedded graph. The aim of this and of the associated consecutive paper is to give a classification of regular embeddings of complete bipartite graphs $K_{n,n}$, where $n = 2^e$. The method involves groups $G$ which factorize as a product $XY$ of two cyclic groups of order $n$ so that the two cyclic factors are transposed by an involutory automorphism. In particular, we give a classification of such groups $G$. Employing the classification we investigate automorphisms of these groups, resulting in a classification of regular embeddings of $K_{n,n}$ based on that for $G$. We prove that given $n = 2^e$ (for $e \geq 3$), there are, up to map isomorphism, exactly $2^{e-2} + 4$ regular embeddings of $K_{n,n}$. Our analysis splits naturally into two cases depending on whether the group $G$ is metacyclic or not.

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1. Introduction

One of the central problems in topological graph theory is to classify all the regular embeddings in orientable surfaces of a given class of graphs. Here we study orientably regular embeddings of graphs, by which we mean 2-cell embeddings in oriented surfaces such that the
orientation-preserving automorphism group of the resulting map acts transitively on the directed edges. The classification problem was treated in a general setting in [9]. However, for particular classes of graphs, it has been solved in only a few cases. Most notably, all regular embeddings of complete graphs $K_n$ have been determined by James and Jones [14], showing that the regular embeddings are precisely those discovered by Biggs [1]. This classification has been extended to the ‘cocktail party’ graphs $K_n \otimes K_2$ by Nedela and Škoviera [23]. More recently, the classification has been achieved for $n$-dimensional cubes, $n$ odd, in [8], for complete multipartite graphs $K_n(K_p)$, $p$ prime, in [6], and for merged Johnson graphs in [16].

The aim of this paper is to consider a similar problem for the complete bipartite graphs $K_{n,n}$. It is well known that for each integer $n$, the graph $K_{n,n}$ has at least one regular embedding in an orientable surface, namely the standard embedding, described by Biggs and White [2, Section 5.6.7] as a Cayley map for the group $\mathbb{Z}_{2n}$. Nedela et al. [26] showed that there is exactly one regular embedding of $K_{p,p}$ when $p$ is a prime. This result was generalized by Jones et al. [17], who proved that there is a unique regular embedding of $K_{n,n}$ if and only if $\gcd(n, \phi(n)) = 1$. Note that the set of all integers $n$ for which $K_{n,n}$ admits a unique regular embedding coincides with the set of all integers $n$ for which there exists exactly one group of order $n$ [17]. A different generalization of [26] has been proved in [18], where regular embeddings of $K_{n,n}$ are classified in the case where $n$ is odd and no primes $p$ and $q$ dividing $n$ satisfy $p \equiv 1 \mod(q)$. In particular, there are exactly $p^{e-1}$ regular embeddings of $K_{p^e,p^e}$, where $p$ is an odd prime. Most of these results were obtained by using a purely group-theoretical interpretation of the problem, which will also be used in the present paper. A more combinatorial approach has been taken by Kwak and Kwon in [20,21], where they have determined all the reflexible regular embeddings of $K_{n,n}$. These embeddings had been constructed earlier by Ljšková et al. in [22]. In particular, for $n$ odd the standard embedding is the only reflexible regular embedding of $K_{n,n}$.

Our method uses the fact, proved in [17], that if $\mathcal{M}$ is a regular orientable embedding of $K_{n,n}$ for any $n$, then the group $G = \text{Aut}_0^+ \mathcal{M}$ of automorphisms of $\mathcal{M}$, preserving orientation and vertex-colours, factorizes as a product of two disjoint cyclic groups of order $n$. When $n$ is an odd prime power, a result of Huppert [10] implies that such a group $G$ must be metacyclic, and this fact was used in [18] to classify the possibilities for $G$, and hence for $\mathcal{M}$. When $n$ is a power of 2, however, Huppert’s result does not apply, and indeed for each $e \geq 2$ there are regular embeddings of $K_{n,n}$ ($n = 2^e$) for which $G$ is not metacyclic, so these are not direct analogues of the maps arising when $p$ is odd. Nevertheless, the techniques used in the odd case can also be applied for $n = 2^e$ provided that one is restricted to those embeddings for which $G$ is metacyclic. The purpose of the present paper is to give a classification of regular embeddings of $K_{n,n}$ for $n = 2^e$ in the metacyclic case. The complementary non-metacyclic case will be treated in a second paper [7], using different techniques.

It is well known that the automorphism group $A = \text{Aut}(\mathcal{M})$ of a regular map is generated by a generator $r$ of the stabilizer of a vertex $v$ (which is necessarily cyclic) and by an involution $\ell$ inverting an edge incident with $v$, see [9]. Moreover, the embedding is determined by the group $A$ and the choice of generators $(r, \ell)$ [25,5]. A regular map given by $A = \langle r, \ell \rangle$, with $\ell^2 = 1$, is called an algebraic map $\mathcal{M}(A; r, \ell)$. Two algebraic maps $\mathcal{M}(A; r, \ell)$ and $\mathcal{M}(A; r', \ell')$ are isomorphic if there is a group automorphism in $\text{Aut}(A)$ taking $r \mapsto r'$ and $\ell \mapsto \ell'$. In the particular case where the underlying graph of $\mathcal{M}(A; r, \ell)$ is $K_{n,n}$, the automorphism group $A = \text{Aut}(\mathcal{M})$ contains a subgroup of index 2 that is a product $\langle r \rangle \langle r^\ell \rangle$ of two cyclic groups. Indeed, any $y \in \langle r \rangle \cap \langle r^\ell \rangle$ fixes two adjacent vertices. Since $K_{n,n}$ is a simple graph, regularity of the action on darts gives $y = 1$, so $|G| = |\langle r, r^\ell \rangle| \geq |r||r^\ell| = n^2$. By regularity of the action
Theorem 1.1 solves the classification problem for products of cyclic groups in the case when non-metacyclic case. The groups have been determined in the case where the order of the cyclic factors is an odd prime, and also for many other values of $n$. In all cases where the solution has been achieved the groups $G$ are metacyclic. In fact, the main obstacle to describing the groups that are products of cyclic groups is a lack of understanding of the structure of the groups in the non-metacyclic case.

The following two theorems present the main results of this and the following paper [7]. Theorem 1.1 solves the classification problem for products of cyclic groups in the case when $n = 2^e$ and the two cyclic factors are interchanged by an involutory automorphism. Theorem 1.2 describes the associated regular embeddings of $K_{n,n}$ up to isomorphism.

**Theorem 1.1.** Suppose that $G = \langle a \rangle \langle b \rangle$, where $|a| = |b| = 2^e$, $e \geq 2$, $\langle a \rangle \cap \langle b \rangle = 1$, and $a^\alpha = b$ for some involution $\alpha$ in $\text{Aut}(G)$. Then one of the following cases holds:

1. $G$ is metacyclic and $G$ has a presentation

   $$G = G_1(e, f) = \langle h, g \mid h^{2^e} = g^{2^e} = 1, h^g = h^{1+2^f} \rangle$$

   where $f = 2, \ldots, e$, and we may set $a = g^u$ and $b = g^u h$, where $u$ is odd and $1 \leq u \leq 2^e - f$;

2. $G$ is not metacyclic, $|G'| = 2$, and $G$ has a presentation

   $$G = G_2 = \langle a, b \mid a^4 = b^4 = 1, [b, a] = a^2b^2, [a^2, b] = [b^2, a] = 1 \rangle$$;

3. $G$ is not metacyclic, $|G'| > 2$, and $G$ has presentation

   $$G = G_3(e, k, l) = \langle a, b \mid a^{2^e} = b^{2^e} = 1, c = [b, a] = a^{2+k2^{e-1}b^{-2-k2^{e-1}}}, c^a = c^{-1+l2^{e-2}a^{-4}}, c^b = c^{-1-l2^{e-2}b^{-4}}, \rangle$$

   where $e \geq 3$ and $k, l \in \{0, 1\}$. Moreover, $G_3(e, 0, 1) \cong G_3(e, 1, 1)$.

The presentation in case (1) shows that the group $G = \text{Aut}^+ \mathcal{M}$ is a split extension of a normal subgroup $\langle h \rangle \cong C_n$ by a complement $\langle g \rangle \cong C_n$; it is a direct product of these subgroups if and only if $f = e$ (which is equivalent to taking $f = 0$), in which case $\text{Aut}^+ \mathcal{M}$ is the wreath product $C_n \wr C_2$. 


Corresponding to the groups $G = G_1(e, f)$, $G_2$ and $G_3(e, k, l)$ in Theorem 1.1, we use $A_1(e, f, m)$, $A_2$ and $A_3(e, k, l)$ respectively to denote the extension of $G$ by $a$, where $a^2 = 1$ and $a^n = b$, noting that $A_3(e, 0, 1) \not\cong A_3(e, 1, 1)$. Moreover, $A_1(e, f_1, m_1) \cong A_1(e, f_2, m_2)$ if and only if $(f_1, m_1) = (f_2, m_2)$. Now we are ready to state the second main theorem.

**Theorem 1.2.** Let $\mathcal{M}$ be a regular map whose underlying graph is the complete bipartite graph $K_{2e, 2e}$, $e \geq 2$.

1. If $e = 2$ then $\mathcal{M}$ is isomorphic to $\mathcal{M}(A_1(2, 2, 1); a, \alpha)$ or $\mathcal{M}(A_2; a, \alpha)$.
2. If $e \geq 3$ then either $\mathcal{M}$ is isomorphic to $\mathcal{M}(A_1(e, f, u); a, \alpha)$, where $u$ is odd, $1 \leq u \leq 2e-f$ and $f = 2, \ldots, e$, or $\mathcal{M}$ is isomorphic to $\mathcal{M}(A_3(e, k, l); a, \alpha)$, where $k, l \in \{0, 1\}$.

**Corollary 1.3.** The number of regular embeddings of $K_{2e, 2e}$ is $2e^2 - 4$ for each $e \geq 3$.

It follows that for any given $e \geq 3$ there are exactly four regular embeddings of $K_{n,n}$, $n = 2e$, with non-metacyclic groups $G = \text{Aut}^+\mathcal{M}$; the remaining $2e^2 - 4$ regular embeddings of $K_{n,n}$ come from metacyclic groups $G$. The proofs of our main results are given in two consecutive papers. In the present paper we deal with the case where $G$ is metacyclic. The main result proved in this paper is as follows:

**Theorem 1.4.** Let $\mathcal{M}$ be a regular map whose underlying graph is the complete bipartite graph $K_{2e, 2e}$, $e \geq 2$. Let the colour- and orientation-preserving subgroup $G \leq \text{Aut}(\mathcal{M})$ be metacyclic. Then

1. $G$ has a presentation
   $$G = G_1(e, f) = \langle h, g \mid h^{2e} = g^{2e} = 1, h^8 = h^{1+2f} \rangle$$
   where $f = 2, \ldots, e$; moreover, $G = \langle a \rangle\langle b \rangle$, where $a = g^u$, $b = g^uh$ for some odd $u$ such that $1 \leq u \leq 2e-f$, and the assignment $a \mapsto b$ extends to an involutory group automorphism $\alpha$.
2. $\mathcal{M}$ is isomorphic to $\mathcal{M}(A_1(e, f, u); a, \alpha)$, where $a = g^u$, $b = g^uh$ for some odd $u$ such that $1 \leq u \leq 2e-f$.
   Moreover, different choices of the parameters $f$ and $u$ give rise to non-isomorphic maps.

2. **Preliminaries**

We will assume, without further comment, that all maps we consider are orientable. If $\mathcal{M}$ is such an embedding of $K_{n,n}$, let $\text{Aut}^+\mathcal{M}$ be its orientation-preserving automorphism group, and let $\text{Aut}_0^+\mathcal{M}$ be the subgroup of $\text{Aut}^+\mathcal{M}$ preserving the two parts (regarded as sets of black and white vertices). Our method uses the following result:

**Proposition 2.1.** If $G$ is a finite group, then the following are equivalent:

(a) $G \cong \text{Aut}_0^+\mathcal{M}$ for some regular orientable embedding $\mathcal{M}$ of $K_{n,n}$;
(b) $G$ has cyclic subgroups $X = \langle x \rangle$ and $Y = \langle y \rangle$ of order $n$ such that $G = XY$ and $X \cap Y = 1$, and there is an automorphism $\alpha$ of $G$ transposing $x$ and $y$.

Given such a group $G$, two maps $\mathcal{M}$ and $\mathcal{M}'$ corresponding to pairs $(x, y)$ and $(x', y')$ are isomorphic if and only if there is an automorphism of $G$ such that $x \mapsto x'$, $y \mapsto y'$.

In these circumstances, we will say that the group $G$ or the triple $(G, x, y)$ is $n$-isobicyclic, or simply isobicyclic if $n$ is understood. This result transforms our classification problem into
a group-theoretic problem: given \( n \), find all the \( n \)-isobicyclic triples \((G, x, y)\), and in each case find the orbits of \( \text{Aut} G \) on the corresponding pairs \((x, y)\).

Proposition 2.1 is proved in [17], and outlined in [18], so here we will explain simply that the elements \( x \) and \( y \) of \( G = \text{Aut}^+ T_0 \) are the orientation-preserving automorphisms of \( T_0 \) fixing a black vertex \( v \) and a white vertex \( w \), and sending each of their neighbours to the next neighbour by following the orientation around \( v \) or \( w \); the automorphism \( \alpha \) of \( G \) is induced by conjugation by the automorphism of \( T_0 \) which rotates the edge \( vw \) about its midpoint, transposing \( v \) and \( w \).

The map \( T_0 \) in Proposition 2.1 has type \([2m, n]\), where \( m \) is the order of \( xy \) in \( G \). It has \( 2n \) vertices, \( n^2 \) edges and \( n^2/m \) faces, so it has characteristic \( 2n - n^2 + n^2/m \) and genus

\[
g = 1 + \frac{n}{2} \left( n - \frac{n}{m} - 2 \right) .
\]

Wilson’s operation \( H_j \) [28] acts on regular maps by raising the rotations of edges around vertices to their \( j \)th powers, where \( j \) is coprime to the valency. Here it acts on the maps \( T_0 \) corresponding to \( G \) by replacing the pair \((x, y)\) with \((x^j, y^j)\) where \( \gcd(j, n) = 1 \). In particular, the mirror-image \( T_0 \) of \( T_0 \) corresponds to the pair \((x^{-1}, y^{-1})\), so \( T_0 \) is reflexible (has an orientation-reversing automorphism) if and only if \( G \) has an automorphism of order 2 inverting \( x \) and \( y \).

Example 1. For each \( n \geq 1 \) the group

\[
G = \langle x, y \mid x^n = y^n = [x, y] = 1 \rangle = X \times Y \cong C_n \times C_n
\]

is \( n \)-isobicyclic. Here the pair \((x, y)\) is unique up to automorphisms of \( G \), so this group gives rise to a unique regular embedding of \( K_{n,n} \). This is the standard embedding \( S_{n,n} \) of \( K_{n,n} \), in which the two sets of \( n \) vertices can be labelled with the elements of \( \mathbb{Z}_n \) so that the orientation of the surface around each vertex induces the cyclic permutation \((0, 1, \ldots, n - 1)\) of the labels of its neighbours. (The original construction of this embedding, due to Biggs and White [2, Section 5.6.7], was as a Cayley map for the additive group \( \mathbb{Z}_{2n} \) with respect to the generators \( 1, 3, \ldots, 2n - 1 \) in that cyclic order.) We have \( \text{Aut}^+ S_{n,n} \cong C_n \wr C_2 \), the wreath product of \( C_2 \) by \( C_2 \), that is, a split extension of the normal subgroup \( G = XY \) by a complement \( \langle \alpha \rangle \cong C_2 \) which transposes \( X \) and \( Y \). Since \( xy \) has order \( n \), \( S_{n,n} \) has type \([2n, n]\) and genus \((n - 1)(n - 2)/2\); it is invariant under Wilson’s operations \( H_j \), where \( \gcd(j, n) = 1 \), and in particular it is reflexible.

The group \( G \) in Example 1 is clearly metacyclic, and Theorem 1.2 gives further regular embeddings with this property. The next two examples demonstrate the existence of regular embeddings for which \( G \) is not metacyclic; Example 2 is a single instance, which is extended in Example 3 to an infinite family.

In order to construct these examples, let \( T \) denote the square tessellation \([4, 4]\) of the complex plane, with the Gaussian integers \( a + bi \in \mathbb{Z}[i] \) as vertices. For each even integer \( n = 2q \geq 2 \), let \( T_n \) be the reflexible torus map \([4, 4]\)_n obtained from \( T \) by identifying points \( n \) steps apart along its Petrie paths [3, Section 8.6]. Equivalently, \( T_n \) is the quotient \([4, 4]_{q,q} \) of \( T \) by the group \( T_n \) of automorphisms of \( T \) generated by the translations \( z \mapsto z + q(1 \pm i) \); see [3, Section 8.3]. We can take the square in \( \mathbb{C} \) with corners \( \pm q, \pm qi \) as a fundamental region for \( T_n \), and by identifying opposite sides we obtain \( T_n \) as an orientable map of type \([4, 4]\) with \( 2q^2 \) vertices, \( 4q^2 \) edges and \( 2q^2 \) faces. Its automorphism group is an extension of an abelian normal subgroup of order \( 2q^2 \), induced by the translations of \( T \) and acting regularly on the vertices, by a dihedral group of order 8 fixing a vertex. The underlying graph is bipartite, with the vertex \( a + bi \) coloured black or white according to whether \( a \) and \( b \) have equal or opposite parity.
Example 2. When \( n = 4 \) the graph underlying \( T_n \) is \( K_{4,4} \), so \( T_4 \) is a regular embedding of \( K_{4,4} \), of type \( \{ 4, 4 \} \) and genus 1; it cannot be the standard embedding, since this has type \( \{ 8, 4 \} \) and genus 3. We can take \( x \) and \( y \) to be the automorphisms of \( T_2 \) induced by the rotations \( z \mapsto iz \) and \( z \mapsto i(z-1)+1 \) of \( T \) around the vertices 0 and 1, so that \( \text{Aut}_0^+ T_4 \) has the form

\[
G = (x, y | x^4 = y^4 = [x^2, y] = [x, y^2] = 1, [x, y] = x^2 y^2),
\]

an extension of its centre \( Z(G) = \langle x^2, y^2 \rangle \cong C_2 \times C_2 \) by \( G/Z(G) \cong C_2 \times C_2 \); then \( \text{Aut}_0^+ T_4 \) is an extension of \( G \) by a complement \( \langle a \rangle \) transposing \( x \) and \( y \), induced by the rotation \( z \mapsto 1-z \) of \( T \) around the point 1/2. Since \( G \) has order 16 and exponent 4, if it is metacyclic then it must be an extension of \( C_4 \) by \( C_4 \); by mapping \( G \) onto \( C_4 \) we see from (2.3) that the only normal subgroups with quotient \( C_4 \) are the normal closures of \( xy \) and of \( xy^{-1} \), and these are not isomorphic to \( C_4 \) since \( xy \) and \( xy^{-1} \) have order 2. Thus \( G \) is not metacyclic.

Example 3. For each even \( n = 2q \geq 2 \), let \( O_n \) be the map obtained by applying Wilson’s opposite operation [19,28] to the torus map \( T_n = \{ 4, 4 \}_n \) described above: thus we cut \( T_n \) along all its edges and then rejoin adjacent pairs of faces along these edges but with the opposite orientation, so that edges and faces are preserved while vertices and Petrie polygons are interchanged. Like \( T_n \), the resulting map \( O_n \) is reflexible, with the same automorphism group, and it is orientable since \( T_n \) is face-bipartite. Now \( T_n \) has two parallel sets of \( n \) Petrie polygons, each polygon having no edges in common with the other polygons in its own set, and one edge in common with each polygon in the other set; it follows that the vertices and edges of \( O_n \) have the same incidence properties, that is, the underlying graph of \( O_n \) is \( K_{n,n} \). By construction, \( O_n \) is a reflexible map of type \( \{ 4, n \} \); in fact, since \( T_n = \{ 4, 4 \}_n \) it follows that \( O_n \) is the map \( \{ 4, n \}_4 = \{ 4, 2q \}_4 \) of genus \( (q-1)^2 = (n-2)^2/4 \), the dual of the map \( \{ 2q, 4 \}_4 \) in Table 8 of [3]. Since its faces are 4-gons, \( O_n \) is a minimum-genus embedding of \( K_{n,n} \).

The automorphisms \( x \) and \( y \) of \( O_n \) correspond to the automorphisms of \( T_n \) induced by the glide-reflections \( z \mapsto \pm iz + 1 \) of \( T \), moving \( T_n \) one step along an orthogonal pair of Petrie polygons. It follows that

\[
G = (x, y | x^n = y^n = 1, (x^2)^y = x^{-2}, (y^2)^x = y^{-2}, (xy)^2 = 1).
\]

The last relation can be written as [\( x, y \) = \( y^{-2} \)^\( x \) \( x^{-2} \)], so an equivalent presentation is

\[
G = (x, y | x^n = y^n = 1, (x^2)^y = x^{-2}, (y^2)^x = y^{-2}, [x, y] = x^{-2} y^2).
\]

Thus \( G \) has a normal subgroup \( N = \langle x^2, y^2 \rangle \cong C_q \times C_q \), with \( G/N \cong C_2 \times C_2 \), so that \( O_n \) is a \( q^2 \)-sheeted covering of the spherical map \( O_n/N \cong O_2 \cong S_{2,2} \), branched over its four vertices. Now \( [x, y] = x^{-2} y^2, [x, y]^2 = x^{-2} y^{-2} \) and \( [x, y]^3 = x^2 y^2 \), so the commutator subgroup \( G' \) is the subgroup \( \langle x^2 y^2, x^{-2} y^2 \rangle \) of index \( \gcd(2, q) \) in \( N \). If \( n > 4 \) then \( G' \) is not cyclic, so \( G \) cannot be metacyclic; this is also true when \( n = 4 \) since \( O_4 \) is isomorphic to the map \( T_4 \) in Example 2. In particular, for each \( n = 2^e \geq 4 \) we obtain a regular embedding of \( K_{n,n} \) for which \( G \) is not metacyclic. Note that the map constructed above is the map \( M(A_3(e, 0, 0); a, \alpha) \) in Theorem 1.2.

3. Normal subgroups

We now start the proof of Theorem 1.4. Suppose that \( G \) is \( n \)-isobicyclic for some \( n \geq 1 \). For each \( m \) dividing \( n \), let \( X_m \) and \( Y_m \) denote the unique subgroups of order \( m \) in \( X \) and \( Y \). Let \( X^* \) denote the core \( \cap_{g \in G} X^g \) of \( X \) in \( G \), the kernel of the action of \( G \) on the black vertices. By a result
of Douglas [4] and Itô [13] (see also [11, VI.10.1(a)]), this normal subgroup of \( G \) is non-trivial, and the same applies to the core \( Y^* = (X^*)^\alpha \) of \( Y \) in \( G \). Since the commutator of two normal subgroups is contained in their intersection, we have \([X^*, Y^*] \leq X^* \cap Y^* \leq X \cap Y = 1\); thus \( X^* \) and \( Y^* \) commute and therefore generate \( X^* \times Y^* \), which is a normal subgroup of \( \text{Aut}^+M \).

Suppose now on that \( n = 2^e \) for some integer \( e \geq 2 \), so \(|G| = n^2 = 2^{2e}\). Since \( X^* \) and \( Y^* \) are non-trivial cyclic 2-groups, they have unique subgroups \( X_2 \) and \( Y_2 \) of order 2; these are central subgroups of \( G \) and generate a subgroup \( Z_1 = X_2 \times Y_2 \) which is central in \( G \) and normal in \( \text{Aut}^+M \). Now \( G/Z_1 \) is \( \mathbb{Z}/2\)-isobicyclic where \( \mathbb{Z}/2 = n/2 \), so factoring out \( Z_1 \) and iterating this argument, we obtain a central series \( 1 = Z_0 < Z_1 < Z_2 < \cdots < Z_e = G \) of subgroups \( Z_i = X_{2i}Y_{2i} \) of order \( 2^{2i} \) in \( G \), where ‘central’ means that \( Z_{i+1}/Z_i \leq \text{Z}(G/Z_i) \) for all \( i \). (Note that for \( i > 1 \), \( Z_i \) need not be the direct product of \( X_{2i} \) and \( Y_{2i} \); see Example 2, for instance, with \( i = 2 \).)

There is an alternative description of these subgroups \( Z_i \). Since \( G \) is a 2-generator 2-group, its Frattini subgroup \( \Phi = \Phi(G) \) (the intersection of its maximal subgroups) has quotient \( G/\Phi \cong C_2 \times C_2 \) [11, III.14, III.15]. Now \( \Phi = G'G^2 \), where \( G' \) is the commutator subgroup and \( G^2 \) is generated by the squares [11, III.14], so \( \Phi \) contains \( X^2Y^2 = X_{2e-1}Y_{2e-1} = Z_{e-1} \); since \(|\Phi| = 2^{2(e-1)} = |Z_{e-1}| \) we have \( \Phi = Z_{e-1} \). Indeed, since \( X^2Y^2 \leq G^2 \leq \Phi \) we also have \( Z_{e-1} = G^2 \). One can apply the same argument to \( \Phi \), which is \( n/2 \)-isobicyclic; iterating, we see that \( Z_{i-1} = \Phi(Z_i) = Z_i^2 \) for all \( i \), so each \( Z_i \) is the subgroup \( G^{2^{i-1}} \) of \( G \) generated by the \( 2^{i-1} \)th powers. In particular, all \( n \)th powers are equal to 1, so \( G \) has exponent dividing \( n \); since \( x \) has order \( n \), the exponent is equal to \( n \). To summarize, we have proved:

**Lemma 3.1.** \( G \) has a central series \( 1 = Z_0 < Z_1 < Z_2 < \cdots < Z_e = G \) of subgroups \( Z_i = X_{2i}Y_{2i} = G^{2^{i-1}} \) of order \( 2^{2i} \), with \( Z_{i-1} = \Phi(Z_i) = Z_i^2 \) and \( Z_i/Z_{i-1} \cong C_2 \times C_2 \) for each \( i = 1, \ldots, e \). \( \Box \)

**Corollary 3.2.** \( G \) has exponent \( n \). \( \Box \)

### 4. The structure of \( G \)

From now on we will assume that \( G \) is metacyclic.

**Proposition 4.1.** \( G \) has a presentation

\[
G = \langle g, h \mid g^n = h^n = 1, h^8 = h^r \rangle
\]

with \( r \) odd.

**Proof.** Proposition 2.1 implies that \(|G| = |X|.|Y| = n^2 = 2^{2e}\), so \( G \) is a 2-group. By our hypothesis, \( G \) is metacyclic. The general form for a metacyclic \( p \)-group is given in [11, III.11.2], so putting \( p = 2 \) we find that \( G \) has a presentation

\[
\langle g, h \mid h^{2^i} = 1, g^{2^j} = h^{2^k}, h^8 = h^r \rangle
\]

where \( 0 \leq k \leq i, r^{2^j} \equiv 1 \mod(2^i) \) and \( 2^k(r - 1) \equiv 0 \mod(2^i) \), so \( G \) has a normal subgroup \( H = \langle h \rangle \cong C_{2^i} \) with \( G/H \cong C_2 \), and \(|G| = 2^{i+j} \). In our case we have \(|G| = n^2 = 2^{2e}\), so \( i + j = 2e \). We showed in Corollary 3.2 that \( G \) has exponent \( n \), and since \( h \) has order \( 2^i \) it follows that \( i \leq e \), so \( i \leq j \). Similarly, since \( g \) has order \( 2^{i+j-k} \) we have \( i + j - k \leq e \), so \( k \geq (i + j)/2 \geq i \); however, \( k \leq i \), so \( k = i \) and hence \((i + j)/2 = i \) giving \( i = j = e \). Thus \( G \) has the presentation given in (4.1). Since \( r^{2^e} \equiv 1 \mod(2^e) \), \( r \) is odd. \( \Box \)
This result shows that $G$ is a split extension of $H = \langle h \rangle$ by $\langle g \rangle$, both of them cyclic groups of order $n$. In order to find a canonical presentation for $G$, and to study its structure more deeply, we need to know the multiplicative orders of certain units in $\mathbb{Z}_n$. Let the notation $2^k \parallel m$ indicate that $2^k$ is the highest power of 2 dividing an integer $m$.

**Lemma 4.2.** If $2^k \parallel u \pm 1$ where $k \geq 2$ then $2^{k+i} \parallel u^{2^i} - 1$ for all $i \geq 1$.

**Proof.** We can write $u = \pm 1 + s2^k$ with $s$ odd and $k \geq 2$. Then $u^2 - 1 = s2^{k+1}(\pm 1 + s2^{2k-1})$, so $2^{k+i} \parallel u^{2^i} - 1$. Iterating this argument gives $2^{k+i} \parallel u^{2^i} - 1$ for all $i \geq 1$. □

**Corollary 4.3.** If $2^f \parallel u - 1$ where $2 \leq f \leq e$, or if $2^f \parallel u + 1$ where $2 \leq f < e$, then $u$ has multiplicative order $2^e - f$ in $\mathbb{Z}_{2^e}$.

**Proof.** Putting $k = f$ in Lemma 4.2 gives $2^{f+i} \parallel u^{2^i} - 1$ for all $i \geq 1$, so if $f < e$ then $u^{2^e-f} \equiv 1 \mod(2^e)$; thus the order of $u$ divides $2^e-f$, and is equal to $2^e-f$ since Lemma 4.2 also gives $u^{2^e-f-1} \not\equiv 1 \mod(2^e)$. The case where $2^f \parallel u - 1$ with $f = e$ is trivial. □

**Proposition 4.4.** Either

$$G = G_f := \langle g, h \mid g^n = h^n = 1, h^g = h^{1+2^f} \rangle \quad (4.3)$$

or

$$G = H_f := \langle g, h \mid g^n = h^n = 1, h^g = h^{-1+2^f} \rangle \quad (4.4)$$

for some $f \in \{2, \ldots, e\}$.

**Proof.** If we replace $g$ in presentation (4.1) with another generator $g' = g^i$ of $\langle g \rangle$, where $i$ is odd, we obtain the same presentation for $G$, except that in the final relation $r$ is replaced with $r' = r^i$. Two units $r, r' \in \mathbb{Z}_n$ satisfy $r' = r^i$ (where $i$ is odd) if and only if they generate the same subgroup of the multiplicative group $U_n$ of units mod($n$). Since $n = 2^e$ we have $U_n = \langle -1 \rangle \times V_n$, with $V_n = \langle 5 \rangle \cong C_{2^{e-2}}$ consisting of the units congruent to 1 mod(4) [15, Theorem 6.10]. Writing $r = \delta v$ and $r' = \delta' v'$, with $\delta, \delta' \in \langle -1 \rangle$ and $v, v' \in V_n$, we see that $(r') = (r)$ if and only if $\delta' = \delta$ and $(v') = (v)$. Since $V_n$ is cyclic, this last condition is equivalent to $v'$ and $v$ having the same multiplicative order, and since they are both congruent to 1 mod(4), Corollary 4.3 implies that this is equivalent to $v' - 1$ and $v - 1$ being divisible by the same power of 2. By a suitable choice of $i$ we may therefore replace $r$ with $q := \pm 1 + 2^f$ for some $f = 2, \ldots, e$, so that (renaming the generators) we have $h^g = h^q$ and $G = G_f$ or $H_f$ as in (4.3) or (4.4). □

These groups are all non-abelian, apart from $G_e \cong C_n \times C_n$. Note also that $G_f$ is the group denoted by $G_1(e, f)$ in Theorems 1.1 and 1.4; for convenience we will retain the notation $G_f$ from now on. For similar reasons we write $H_f$ instead of $H_1(e, f)$.

If $G = G_f$ then $[h, g] = h^{2^f}$, which generates a normal subgroup of order $2^e-f$; and since $G = \langle g, h \rangle$, $G'$ is the normal closure of $[h, g]$, which therefore coincides with this subgroup. This shows that for a given $n = 2^e$, the different groups $G_f$ can be distinguished from each other by the fact that $|G'_f| = 2^e-f$. The groups $H_f$ are of less interest here since, as we shall eventually prove in Corollary 5.5, they are not isobicyclic, so they do not arise from regular embeddings of $K_{n,n}$.

For each $i = 0, 1, \ldots$, the subgroup $\langle g^{2^i}, h^{2^i} \rangle$ of $G$ is a semidirect product of $\langle h^{2^i} \rangle$ by $\langle g^{2^i} \rangle$, both cyclic of order $2^e-i$, so it has index $2^{2i}$ in $G$. Being generated by $2^i$th powers, it is contained
in \( G^{2i} \); but we saw in Lemma 3.1 that \( G^{2i} \) also has index \( 2^{2i} \), so

\[
\langle g^{2i}, h^{2i} \rangle = G^{2i} = \mathbb{Z}_{e-i}.
\]

Each element of \( G \) can be written uniquely in the standard form \( g^ih^j \) where \( i, j \in \mathbb{Z}_n \). Multiplication of standard forms is given by

\[
g^ih^j.g^kh^l = g^{i+k}(h^j)g^k.h^l = g^{i+k}h^{j+1},
\]

(4.5)
since \( (h^j)g^k = (h^{j+1})^k \).

An element \( g^ih^j \) is central in \( G \) if and only if \( g^i \) commutes with \( h \) and \( h^j \) commutes with \( g \). Since \( h^{q^i} = h^{q^j} \) and \( (h^j)g = h^{j+1} \) this happens if and only if \( q^i \equiv 1 \) and \( jq \equiv 1 \mod(n) \). If \( G = G_f \), this is equivalent to \( 2^{e-f} \mid i \) (by Lemma 4.2) and \( 2^{e-f} \mid j \), so \( G_f \) has centre

\[
Z(G_f) = \langle g^{2e-f}, h^{2e-f} \rangle = Z_f \cong C_{2f} \times C_{2f}.
\]

(4.6)

Powers in \( G \) are given by

\[
(g^ih^j)^m = g^{im}(h^j)^{g(i-1)}(h^j)^{g(j-2)} \cdots (h^j)^{g(i-1)}.
\]

(4.7)

5. Generating pairs for \( G \)

Let \( \mathcal{P} \) denote the set of all pairs \( (x, y) \in G \times G \) such that the subgroups \( X = \langle x \rangle \) and \( Y = \langle y \rangle \) satisfy \( G = XY \) and \( X \cap Y = 1 \), and let \( \mathcal{B} \) be the subset consisting of those pairs where \( x \) and \( y \) are transposed by an automorphism of \( G \). By Proposition 2.1, the regular embeddings of \( K_{n,n} \) associated with \( G \) correspond to the orbits of \( \text{Aut} G \) on \( B \). Our strategy is first to find \( \mathcal{P} \), then by finding the automorphisms of \( G \) to determine \( B \), and finally to find the orbits of \( \text{Aut} G \) on \( B \). The following technical result enables us to determine \( \mathcal{P} \).

**Proposition 5.1.** Let \( m := 2^{e-1} \). Then

\[
(g^ih^j)^m = \begin{cases} 
  g^{im}h^{jm} & \text{if } G = G_f \text{ for } i \text{ is even}, \\
  g^{im} & \text{if } G = H_f \text{ and } i \text{ is odd}.
\end{cases}
\]

**Proof.** By Eq. (4.7) we have

\[
(g^ih^j)^m = g^{im}h^{j(q^m-1)/(q^i-1)}.
\]

Suppose first that \( q^i \equiv 1 \mod(4) \), that is, either \( q = 1 + 2f \) or \( q = -1 + 2f \) and \( i \) is even. Then \( 2^k \parallel q^i - 1 \) for some \( k \geq 2 \), so Lemma 4.2 gives \( 2^{k+e-1} \parallel q^{im} - 1 \), and hence \( 2^{e-1} \parallel (q^m - 1)/(q^i - 1) \); thus the exponent of \( h \) in \( (g^ih^j)^m \) is 0 or \( m \) according to whether \( j \) is even or odd, so \( (g^ih^j)^m = g^{im}h^{jm} \). Now suppose that \( q^i \equiv -1 \mod(4) \), that is, \( q = -1 + 2f \) and \( i \) is odd. Then Lemma 4.2 shows that \( 2^{e+1} \mid q^{im} - 1 \), so \( 2^e \mid (q^m - 1)/(q^i - 1) \) and the exponent of \( h \) in \( (g^ih^j)^m \) is 0. \( \square \)

**Corollary 5.2.** The elements of order \( n \) in \( G \) are those in \( G \setminus \Phi \).

**Proof.** Since \( G \) has exponent \( n = 2^e \), the elements of order \( n \) are those with non-identity \( m \)th powers. The result therefore follows immediately from Proposition 5.1. \( \square \)
In fact, Proposition 5.1 shows that if \( G = G_f \) then taking \( n \)th powers induces an isomorphism 
\[ \mu : G/\Phi \to Z_1 = \langle g^m, h^m \rangle \] given by \( g^i h^j \Phi \mapsto (g^i h^j)^m = g^{im} h^{jm} \); however, \( \mu \) is not a homomorphism if \( G = H_f \). Corollary 5.2 fails if we do not assume that \( G \) is metacyclic, as shown by Example 2, where the element \( xy \in G \setminus \Phi \) has order 2, not 4.

We now consider when the cyclic subgroups generated by two elements of order \( n \) have trivial intersection (or are disjoint in the rather imprecise group-theoretic terminology).

**Corollary 5.3.** Elements \( a \) and \( b \) of order \( n \) in \( G \) generate disjoint subgroups if and only if their disjoint intersection (or are disjoint) in the rather imprecise group-theoretic terminology).

**Proof.** The cyclic subgroups \( A = \langle a \rangle \) and \( B = \langle b \rangle \) are disjoint if and only if their subgroups \( A_2 = A^m \) and \( B_2 = B^m \) of order 2, contained in \( Z_1 \), are disjoint. If \( G = G_f \) then \( \mu : G/\Phi \to Z_1 \) is an isomorphism, so this is equivalent to \( A \) and \( B \) having disjoint images in \( G/\Phi \), that is, to \( a \Phi \) and \( b \Phi \) generating \( G/\Phi \). If \( G = H_f \) the result follows from Corollary 5.2. \( \square \)

Recall that a characteristic subgroup of a group \( G \) is one which is invariant under all automorphisms of \( G \).

**Proposition 5.4.** If \( G \) is non-abelian then \( H \Phi \) is a characteristic subgroup of \( G \).

**Proof.** Any \( \theta \in \text{Aut} G \) must preserve \( \Phi \) and must send the normal subgroup \( H = \langle h \rangle \) to a cyclic normal subgroup \( J \) of order \( n \). If \( J \Phi \neq H \Phi \) then Corollary 5.3 implies that \( H \) and \( J \) are disjoint subgroups of \( G \), so \( |HJ| = |H||J|/|H \cap J| = n^2 = |G| \) and hence \( G = HJ \). Since \( H \) and \( J \) are normal subgroups of \( G \) we have \( |H, J| \leq H \cap J = 1 \) and hence \( G = H \times J \). Thus \( G \) is abelian, against our assumption, so \( J \Phi = H \Phi \) as required. \( \square \)

**Corollary 5.5.** The groups \( H_f \) are not isobicyclic.

**Proof.** If \( H_f \) is isobicyclic then some pair \((x, y) \in \mathcal{P}\) must be transposed by an automorphism \( \alpha \) of \( H_f \). By Corollary 5.3, exactly one of the cosets \( x \Phi \) and \( y \Phi \) coincides with \( h \Phi \). However, \( h \Phi \) is invariant under \( \alpha \) by Proposition 5.4, since \( H_f \) is non-abelian, so \( \alpha \) cannot transpose \( x \) and \( y \). \( \square \)

Having excluded the groups \( H_f \), we may assume from now on that \( G = G_f \) for some \( f = 2, \ldots, e \), so \( q = 1 + 2^f \).

**Proposition 5.6.** We have 
\[ \mathcal{P} = \{(x, y) \in G \times G \mid G/\Phi = \langle x \Phi, y \Phi \rangle \} \]
\[ = \{(g^i h^j, g^k h^l) \in G \times G \mid il - jk \not\equiv 0 \text{ mod}(2)\}. \]

**Proof.** By Corollary 5.2, the cyclic subgroups \( X \) and \( Y \) of order \( n \) in \( G \) are those generated by elements \( x, y \in G \setminus \Phi \). By Corollary 5.3, such subgroups are disjoint if and only if \( x \Phi \) and \( y \Phi \) generate \( G/\Phi \), or equivalently \( x = g^i h^j \) and \( y = g^k h^l \) with \( il - jk \not\equiv 0 \text{ mod}(2) \). For each such pair \( x, y \) we have \( |XY| = |X||Y|/|X \cap Y| = n^2 = |G| \), so \( G = XY \). Thus \( \mathcal{P} \) consists of all such pairs \( (x, y) \). \( \square \)

**Corollary 5.7.** \( |\mathcal{P}| = 3 \cdot 2^{4e-3} \).

**Proof.** We use Proposition 5.6 to count the pairs \((x, y) \in \mathcal{P}\); there are \( |G \setminus \Phi| = 2^{2e} - 2^{2e-2} = 3 \cdot 2^{2e-2} \) choices for \( x \), and each is paired with \( |G \setminus X \Phi| = 2^{2e} - 2^{2e-1} = 2^{2e-1} \) elements \( y \), so \( |\mathcal{P}| = 3 \cdot 2^{4e-3} \). \( \square \)
6. Automorphisms of $G$

Recall that $B$ consists of those pairs $(x, y) \in \mathcal{P}$ where $x$ and $y$ are transposed by an automorphism of $G$, and that the regular embeddings of $K_{n,n}$ associated with $G$ correspond to the orbits of Aut $G$ on $B$. Since $G = \langle x, y \rangle$ for each pair $(x, y) \in \mathcal{P}$, Aut $G$ acts semiregularly on $\mathcal{P}$, and hence on $B$, with orbits of length $|\text{Aut } G|$. First we deal with the case $f = e$.

**Proposition 6.1.** The group $G = G_e$ corresponds to a single regular embedding of $K_{n,n}$, namely the standard embedding $S_{n,n}$.

**Proof.** If $G = G_e \cong C_n \times C_n$ then Aut $G \cong GL_2(\mathbb{Z}_n)$, of order $3 \cdot 2^{4e-3} = |\mathcal{P}|$, so Aut $G$ acts transitively on $\mathcal{P}$; all pairs $(x, y) \in \mathcal{P}$ are transposed by an automorphism (since one pair $(g, h)$ is), so $B = \mathcal{P}$ and $G$ corresponds to a unique regular embedding. Since $\mathcal{M} = S_{n,n}$ has Aut$_0^+ \mathcal{M} \cong C_n \times C_n$, this is the corresponding map. □

By Proposition 6.1, we may assume for the rest of this section that $f \neq e$, so $G = G_f$ for some $f = 2, \ldots, e - 1$. Thus $G$ is non-abelian, and $H \Phi = \langle g^2, h \rangle$ is a characteristic subgroup of $G$ by Proposition 5.4.

We will consider the automorphisms of $G$ through their induced actions as linear transformations of $G/\Phi$, regarded as a two-dimensional vector space over $\mathbb{Z}_2$. Each $\theta \in \text{Aut } G$ sends the pair $g, h$ to some pair $g' = g^i h^j, h' = g^k h^l$, so it is represented on $G/\Phi$ as the matrix $A = \begin{pmatrix} i & j \\ k & l \end{pmatrix} \in GL_2(\mathbb{Z}_2)$ with respect to the basis $(g\Phi, h\Phi)$. (Here we abuse the notation by identifying the exponents $i, j, k, l \in \mathbb{Z}_n$ with their images in $\mathbb{Z}_2$.) Since $A$ is invertible, $il - jk \neq 0 \text{ mod}(2)$. Since $G$ is non-abelian, Proposition 5.4 implies that the one-dimensional subspace $H \Phi/\Phi$ is invariant under Aut $G$, so $k \equiv 0 \text{ mod}(2)$ and hence $i, l \neq 0 \text{ mod}(2)$.

**Proposition 6.2.** The automorphisms of $G$ are the mappings given by $g \mapsto g^i h^j, h \mapsto g^k h^l$ where $i, j, k, l \in \mathbb{Z}_n$ with $i \equiv 1 \text{ mod}(2^{e-f}), k \equiv 0 \text{ mod}(2^{e-f}),$ and $l \neq 0 \text{ mod}(2)$.

**Proof.** Any automorphism $\theta$ of $G$ sends $g$ and $h$ to $g' = g^i h^j$ and $h' = g^k h^l$, where $i \neq 0 \text{ mod}(2)$ and $k \equiv 0 \neq l \text{ mod}(2)$ by Propositions 5.4 and 5.6. Any such pair $g', h'$ generate $G$ by the Burnside Basis Theorem [11, III.3.15], since their images generate $G/\Phi$; they both have order $n$ by Corollary 5.2, so the mapping $g \mapsto g', h \mapsto h'$ determines an automorphism of $G$ if and only if $(h')^q = (h')^q$, corresponding to the final relation in (4.3). Now

$$(h')^{g'} = (g^k h^l)^{g^i h^j} = (g^k)^{g^i h^j} (h^l)^{g^i h^j} = (g^k)^{h^l} (h^l)^{g^i} \in g^k H,$$

and

$$(h')^q = (g^k h^l)^q \in g^k H,$$

so if $\theta$ is an automorphism then $kq \equiv k \text{ mod}(2^e)$; since $q = 1 + 2^f$ this is equivalent to $k \equiv 0 \text{ mod}(2^{e-f})$, that is, $g^k \in Z(G)$ by (4.6). In this case

$$(h')^{g'} = g^k (h^l)^{g'} = g^k h^{lq} \quad \text{and} \quad (h')^q = g^k qh^{lq} = g^k h^{lq},$$

so $\theta$ is an automorphism if and only if $lq \equiv lq \text{ mod}(2^e)$, that is, $q^{-1} \equiv 1 \text{ mod}(2^e)$ since $l$ is odd, or equivalently $i \equiv 1 \text{ mod}(2^{e-f})$ by Corollary 4.3. □

**Corollary 6.3.** $|\text{Aut } G| = 2^{2e+2f-1}$. 

Proof. In Proposition 6.2, the number of choices for each of \(i, j, k\) and \(l\) is \(2^f, 2^e, 2^f\) and \(2^{e-1}\) respectively, and multiplying these gives the number of automorphisms. 

Not every involution in \(\text{Aut} G\) transposes a pair \((x, y) \in \mathcal{P}\): for instance, the inner automorphisms act trivially on \(G/\Phi\), since \(G' \leq \Phi\), and these include involutions since \(G\) is a non-abelian 2-group. For those involutions which do transpose such a pair, we have:

Corollary 6.4. Any involution \(\alpha \in \text{Aut} G\) which transposes a pair \((x, y) \in \mathcal{P}\) is represented on \(G/\Phi\) as the matrix \(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \text{GL}_2(\mathbb{Z}_2)\) with respect to the basis \(\{g, h\Phi\}\).

Proof. Since \(x \Phi\) and \(y \Phi\) generate \(G/\Phi\), \(\alpha\) induces an automorphism of order 2 on \(G/\Phi\). By Proposition 5.4, \(\alpha\) leaves \(H \Phi/\Phi\) invariant, so the result follows. 

7. Proof of the classification theorems, the metacyclic case

We will first count the regular embeddings of \(K_n, n\) corresponding to \(G = G_f\) for each \(f\).

By Proposition 2.1, this is equivalent to counting the orbits of \(\text{Aut} G\) on \(B\). First let \(f < e\).

By Proposition 6.2, the automorphisms \(\theta\) of \(G\) are given by \(g \mapsto g^i h^j, h \mapsto g^k h^l\) where \(i \equiv 1, k \equiv 0 \mod(2^{e-f})\) and \(l \neq 0 \mod(2)\); thus \(g^{i-1}, g^k \in Z = Z(G)\) by (4.6), and hence \(\theta\) sends an arbitrary element \(x = g^u h^v\) of \(G\) to

\[
g^{u'} h^{v'} = (g^{i} h^{j})^u (g^k h^l)^v = g^{(i-1)u + kv} (gh^j)^{u} h^{lv} = g^{iu + kv} h^{j(lq^{u-1} + \ldots + q+1)+lv}\]

(7.1)

by (4.7). If \((x, y) \in B\) for some \(y \in G\) then \(x \not\in H \Phi\) by Proposition 5.4 and Corollary 6.4, or equivalently \(u \not\equiv 0 \mod(2)\). Since \(q \equiv 1 \mod(2)\) we have \(q^{u-1} + \ldots + q + 1 \equiv u \mod(2)\), so if \(x \not\in H \Phi\) then \(q^{u-1} + \ldots + q + 1\) is a unit mod\((n)\). For any such \(x\), by choosing a suitable value of \(j\) we can therefore find some \(\theta \in \text{Aut} G\) so that \(v'\) takes any given value in \(\mathbb{Z}_n\); similarly, with suitable \(i\) and \(k\) we can also have any \(u' \equiv u \mod(2^{e-f})\), so the orbit of \(x\) under \(\text{Aut} G\) is the coset \(xHZ\) of \(HZ = \langle g^{2^{e-f}}, h \rangle\), of size \(|HZ| = 2^{e+f}\).

Let \(\mathcal{B}(x) = \{y \in G \mid (x, y) \in B\}\) and \(\mathcal{P}(x) = \{y \in G \mid (x, y) \in \mathcal{P}\}\), so \(\mathcal{B}(x) \subseteq \mathcal{P}(x)\). By definition of \(\mathcal{B}\), each \(y \in \mathcal{B}(x)\) is an image of \(x\) under \(\text{Aut} G\), so \(\mathcal{B}(x) \subseteq xHZ \cap \mathcal{P}(x)\). Now \(xHZ\) consists of the elements 

\[
y = g^u h^v\]

with \(u' \equiv u \mod(2^{e-f})\); by Proposition 5.6, \((x, y) \in \mathcal{P}\) if and only if \(uv' - vu' \not\equiv 0 \mod(2)\), or equivalently \(v' \not\equiv v \mod(2)\), since \(u' \equiv u \not\equiv 0 \mod(2)\), so

\[
xHZ \cap \mathcal{P}(x) = \{g^{u} h^{v'} \mid u' \equiv u \mod(2^{e-f}), \ v' \not\equiv v \mod(2)\}.\]

(7.2)

Counting values of \(u'\) and \(v'\) we have \(|xHZ \cap \mathcal{P}(x)| = 2^{2f} (2^{e} - 2^{e-1}) = 2^{e+f-1}\). Now \(\text{Aut} G\) has order \(2^{2e+2f-1}\) by Corollary 6.3, and its orbit \(xHZ\) containing \(x\) has length \(2^{e+f}\), so the stabilizer (\(\text{Aut} G\)) of \(x\) in \(\text{Aut} G\) has order \(2^{e+f-1}\); this subgroup acts semi-regularly on \(\mathcal{P}(x)\), and preserves the coset \(xHZ\), so by comparing orders we see that it acts regularly on \(xHZ \cap \mathcal{P}(x)\).

As a representative of the orbit \(xHZ\) of \(\text{Aut} G\) we can choose \(x = g^u\) for some unique odd \(u \in \{1, \ldots, 2^f\}\) (so \(v = 0\)); then the element \(y = g^uh\) is in \(xHZ \cap \mathcal{P}(x)\) by (7.2). By Proposition 6.2, putting \(i = 1, j(q^{u-1} + \ldots + q + 1) = 1, k = 0\) and \(l = -1\) in \(\mathbb{Z}_n\) gives an automorphism \(\alpha\) of \(G\) such that \(g \mapsto gh^j, h \mapsto h^{-1}\), this choice of \(j\) being possible since \(q^{u-1} + \ldots + q + 1\) is a unit mod\((n)\). Then (7.1) gives \(x\alpha = y\) and \(y\alpha = x\), so \(\alpha\) is an involution since \(G = \langle x, y \rangle\). Thus \((x, y) \in B\), so \(B(x)\) is non-empty. Now \(B(x)\) is a subset of \(xHZ \cap \mathcal{P}(x)\), invariant under \(\langle \text{Aut} G \rangle_x\), and \(\langle \text{Aut} G \rangle_x\) acts transitively on \(xHZ \cap \mathcal{P}(x)\), so
\( B(x) = xHZ \cap P(x) \). Since this equation is satisfied by one element \( x = g^u \) of the orbit \( xHZ \) of \( \text{Aut} G \), it is satisfied by every element \( x \) of this orbit. This is true for all odd \( u \), so

\[
B = \{(g^uh^v, g^{u}h^{v'}) \mid u \not\equiv 0 \mod(2), u' \equiv u \mod(2^{e-f}) \text{ and } v' \not\equiv v \mod(2)\}. \tag{7.3}
\]

Moreover, this argument shows that two pairs \((x, y) \in B\) are in the same orbit of \( \text{Aut} G \) if and only if their corresponding values of \( u \) are congruent \( \mod(2^{e-f}) \), and that we can take the pairs \((g^u, g^uh)\) with odd \( u = 1, \ldots, 2^{e-f} \) as representatives of these orbits. It follows that there are \( \phi(2^{e-f}) \) orbits of \( \text{Aut} G \) on \( B \), so by Proposition 2.1 this is the number of regular embeddings of \( K_{n,n} \) associated with \( G = G_f \) for each \( f = 2, \ldots, e-1 \). This is also valid for \( f = e \) by Proposition 6.1, since \( \phi(1) = 1 \). Summing over all \( f \), we have a total of

\[
\sum_{f=2}^{e} \phi(2^{e-f}) = 2^{e-3} + 2^{e-4} + \cdots + 2 + 1 + 1 = 2^{e-2}
\]

regular embeddings of \( K_{n,n} \).

Since \( xy \) has order \( n \) (being an element of \( G \setminus \phi \)), these maps \( M \) all have type \( \{2n, n\} \) and hence have genus \((n-1)(n-2)/2 \). In each case, \( \text{Aut}^+ M \) is a split extension of a normal subgroup \( G = \text{Aut}_X^+ M \) by a complement \( \langle \alpha \rangle \cong C_2 \), where the automorphism \( \alpha \) of \( M \) reverses the edge joining the vertices fixed by \( x \) and \( y \); thus \( \alpha \) acts by conjugation on \( G \) as the automorphism \( \alpha \in \text{Aut} G \) transposing \( x = g^u \) and \( y = g^uh \), so we obtain the description of \( \text{Aut}^+ M \) in Theorem 1.4(1). Since \( \langle \text{Aut} G \rangle_x \) acts transitively on \( B(x) \), there is a single conjugacy class of involutions in \( \text{Aut} G \) transposing pairs \((x, y) \in B \), so the groups \( \text{Aut}^+ M \) corresponding to the \( 2^{e-f-1} \) maps \( M \) associated with each \( G_f \) are all isomorphic.

Wilson’s operations \( H_j \) \((j \in U_n)\) permute the regular embeddings of \( K_{n,n} \), preserving \( G \) and replacing \((x, y)\) with \((x^j, y^j)\). For each \( f \), if we let \( M_u \) denote the regular embedding associated with the pair \((g^u, g^uh)\) then \( H_j \) sends \( M_u \) to \( M_{ju} \), where we regard these subscripts as units \mod(2^{e-f}). \) For a given \( f \) these \( \phi(2^{e-f}) \) maps \( M_u \) therefore form a single orbit under the operations \( H_j \), which confirms that their groups \( \text{Aut}^+ M \) are all isomorphic. The map \( M_u \) is reflexible if and only if it is isomorphic to \( H_{-1}(M_u) = M_{-u} \), or equivalently \( u \equiv -u \mod(2^{e-f}) \); this happens only if \( f = e \) (corresponding to \( M_u = S_{n,n} \)), or \( f = e-1 \), whereas for \( f < e-1 \) the maps \( M_u \) are all chiral. \( \square \)

8. Comments on the classification

The involution \( \alpha \) centralizes and inverts the elements \( r = xy \) and \( x^{-1}y \) of order \( n \), giving cyclic and dihedral subgroups \( R = \langle \alpha, r \rangle \) of order \( 2n \) in \( \text{Aut}^+ M \). Corollary 5.3 implies that each \( R \) is disjoint from \( X \), so it has an orbit of length \( |R : R \cap X| = 2n \) on the vertices and thus permutes them regularly. Each \( M_u \) is therefore a Cayley map for both the cyclic and dihedral groups \( R \) of order \( 2n \) with respect to the generating sets \( R \setminus \langle r \rangle \).

The normal subgroup \( H = \langle x^{-1}y \rangle \) of \( \text{Aut} M \) acts semi-regularly on the vertices and edges of \( M \), and has two orbits of length \( n/2 \) on the faces, so \( M/H \) has two vertices, \( n \) edges and two faces. This map is the regular embedding \( M(n, 1) \) of a dipole \( D_n \) in the notation of [23, 24]. The regular covering \( M \to M/H \) is branched over its two face-centres, and is unbranched elsewhere.

The group \( \text{Ex} M \) of exponents of a map \( M \) of valency \( n \) consists of those \( j \in U_n \) such that \( H_j(M) \cong \text{M} \) [25]. Here, writing \( M = M_u \) we see that \( j \in \text{Ex} M \) if and only if \( ju \equiv u \mod(2^{e-f}) \), or equivalently \( j \equiv 1 \mod(2^{e-f}) \). If \( f = e \) or \( e-1 \) then \( \text{Ex} M = U_n \),
whereas if $2 \leq f \leq e - 2$ then $\text{Ex} M$ is a proper subgroup of index $2^{e-1-f}$ and order $2^f$ in $U_n$. Thus if $e \geq 3$ then $K_{n,n}$ has $e - 2$ non-isomorphic exponent groups for its regular embeddings; as in the odd prime-power case [18] this solves Problem 3 of [25], which asks whether the exponent group for a given graph is unique.

The Petrie dual $P(M)$ of a map $M$ has the same underlying graph, but its faces are bounded by the Petrie polygons of $M$ [3,19,28]. Here, the operation $P$ corresponds to replacing $(x, y) \in B$ with $(x, y^{-1})$. If $f = e$ or $e - 1$ then it follows from (7.3) that $(x, y^{-1}) \in B$, so $P(M)$ is a regular embedding of $K_{n,n}$, isomorphic to $M$ by the uniqueness of $M$ for these values of $f$. If $f < e - 1$, however, $P(M)$ is not a regular map since $(x, y^{-1}) \notin B$ by (7.3), though it is an edge-transitive orientable embedding of $K_{n,n}$.

If, instead of regular embeddings of $K_{n,n}$, we look for those for which the group $G = \text{Aut}^+_0 M$ acts transitively (and thus regularly) on the edges, then a simple modification of Proposition 2.1 (omitting the automorphism $\alpha$) shows that the maps $M$ associated with $G$ correspond to the orbits of $\text{Aut} G$ on $P$. Moreover, the analysis in Sections 3 and 4 shows that the only metacyclic groups $G$ which can occur are the groups $G_f$ and $H_f$ defined in Proposition 4.4 for $f = 2, \ldots, e$: the existence of $\alpha$ is used only to eliminate the groups $H_f$ in the regular case (Corollary 5.5). It follows from Corollaries 5.7 and 6.3 that if $f < e$ then $G_f$ contributes

$$\frac{3 \cdot 2^{4e-3}}{2e+2f-1} = 3 \cdot 2^{(e-f-1)}$$

orbits; of these, $2^{e-f-1}$ are in $B$, each corresponding to a single regular embedding, while the remaining $(3 \cdot 2^{e-f-1} - 1)2^{e-f-1}$ orbits in $P \setminus B$ correspond in pairs (transposed by reversing the vertex-colours) to $(3 \cdot 2^{e-f-1} - 1)2^{e-f-2}$ non-regular embeddings, giving a total of $(3 \cdot 2^{e-f-1} + 1)2^{e-f-2}$ embeddings. The proof of Proposition 6.1 shows that $G_e$ contributes just one orbit, corresponding to the standard embedding. In the case of $H_f$, it follows from Corollaries 5.2 and 5.3 that $|P| = 2^{4e-2}$, while an argument similar to that in Proposition 6.2 and Corollary 6.3 (but with $g = -1 + 2^f$) shows that $|\text{Aut} G| = 2^{e+f}$ or $2^{3e-1}$ as $f < e$ or $f = e$; thus $H_f$ contributes $2^{e-f-2}$ or $2^{e-1}$ orbits respectively, corresponding in pairs to $2^{e-f-3}$ or $2^{e-2}$ non-regular embeddings. Summing over $f = 2, \ldots, e$ we see that the groups $G_f$ and $H_f$ respectively contribute

$$1 + 3 \left( \frac{1}{2} + 2 + 8 + \cdots + 2^{2e-7} \right) + \left( \frac{1}{2} + 1 + 2 + \cdots + 2^{e-4} \right) = 2^{e-3}(2^{e-2} + 1)$$

and

$$2^{e-2} + (2^{e-2} + 2^{e-1} + \cdots + 2^{2e-5}) = 2^{2e-4}$$

embeddings, so the total number with metacyclic groups $G$ is $2^{e-3}(2^{e-1}+2^{e-2}+1)$. For instance, if $e = 2$ then there are two embeddings: $G_2 = C_4 \times C_4$ contributes the standard embedding, while

$$H_2 = \langle g, h \mid g^4 = h^4 = 1, h^8 = h^{-1} \rangle$$

contributes a second embedding of type $\{8, 4\}$ and genus 3, which is not regular; this can be formed by labelling the black and white vertices $1, 2, 3, 4$, and letting the rotation of neighbours about vertices of one colour be $(1, 2, 3, 4)$, but alternately $(1, 2, 3, 4)$ and its inverse around successive vertices of the other colour.
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