Computers and Mathematics with Applications 59 (2010) 1191–1226

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Impulsive differential inclusions with fractional order

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ARTICLE INFO

Keywords:
Fractional differential inclusions
Fractional derivative
Fractional integral
Fractional differential equations
Existence and uniqueness
Fixed point
Decomposable
Continuous selection
Compactness
Relaxation theorem
R_δ
Acyclic
Contractible
Absolute retract
Topology degree
Poincaré operator

ABSTRACT

In this paper, we first present an impulsive version of the Filippov–Ważewski theorem and a continuous version of the Filippov theorem for fractional differential inclusions of the form

$$\begin{array}{ll} D_*^\alpha y(t) \in F(t, y(t)), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \alpha \in (1, 2], \\ y(t_k^+) = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y'(t_k^+) = \bar{I}_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = a, \quad y'(0) = c, & \end{array}$$

where $J = [0, b]$, D_*^α denotes the Caputo fractional derivative, and F is a set-valued map. The functions I_k, $\bar{I}_k$ characterize the jump of the solutions at impulse points t_k ($k = 1, \dots, m$). Additional existence results are obtained under both convexity and nonconvexity conditions on the multivalued right-hand side. The proofs rely on the nonlinear alternative of Leray–Schauder type, a Bressan–Colombo selection theorem, and Covitz and Nadler's fixed point theorem for multivalued contractions. The compactness of the solution set is also investigated. Finally, some geometric properties of solution sets, R_δ sets, acyclicity and contractibility, corresponding to Aronszajn–Browder–Gupta type results, are obtained. We also consider the impulsive fractional differential equations

$$\begin{array}{ll} D_*^\alpha y(t) = f(t, y(t)), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \alpha \in (1, 2], \\ y(t_k^+) = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y'(t_k^+) = \bar{I}_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = a, \quad y'(0) = c, & \end{array}$$

and

$$\begin{array}{ll} D_*^\alpha y(t) = f(t, y(t)), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \alpha \in (0, 1], \\ y(t_k^+) = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = a, & \end{array}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is a single map. Finally, we extend the existence result for impulsive fractional differential inclusions with periodic conditions,

$$\begin{array}{ll} D_*^\alpha y(t) \in \varphi(t, y(t)), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \alpha \in (1, 2], \\ y(t_k^+) = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y'(t_k^+) = \bar{I}_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = y(b), \quad y'(0) = y'(b), & \end{array}$$

where $\varphi : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map. The study of the above problems use an approach based on the topological degree combined with a Poincaré operator.

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doi:10.1016/j.camwa.2009.05.011

1. Introduction

Differential equations with impulses were considered for the first time in the 1960's by Milman and Myshkis [1,2]. A period of active research, primarily in Eastern Europe from 1960–1970, culminated with the monograph by Halanay and Wexler [3].

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume that these perturbations act instantaneously or in the form of "impulses". As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given extensive attention. Works recognized as landmark contributions include [4–7]. There are also many different studies in biology and medicine for which impulsive differential equations are good models (see, for example, [8–10] and the references therein).

In recent years, many examples of differential equations with impulses with fixed moments have flourished in several contexts. In the periodic treatment of some diseases, impulses correspond to administration of a drug treatment or a missing product. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs.

During the last ten years, impulsive ordinary differential inclusions and functional differential inclusions with different conditions have been intensely studied by many mathematicians. At present the foundations of the general theory are already laid, and many of them are investigated in detail in the book of Aubin [11], Benchohra et al. [12], and in the papers of Henderson and Ouahab [13], Graef et al. [14–16], Graef and Ouahab [17–20] and the references therein.

Differential equations with fractional order have recently proved valuable tools in the modeling of many physical phenomena [21–25]. There has also been a significant theoretical development in fractional differential equations in recent years; see the monographs of Kilbas et al. [26], Miller and Ross [27], Podlubny [28], Samako et al. [29], and the papers of Bai and Lu [30], Diethelm et al. [21,31,32], El-Sayed [33–35], El-Sayed and Ibrahim [36], Kilbas and Trujillo [37], Mainardi [24], Momani and Hadid [38], Momani et al. [39], Nakhushev [40], Podlubny et al. [41], and Yu and Gao [42].

Very recently, some basic theory for initial-value problems for fractional differential equations and inclusions involving the Riemann–Liouville differential operator was discussed by Benchohra et al. [43], Lakshmikantham [44], and Lakshmikantham and Vastala [45–47]. El-Sayed and Ibrahim [36] initiated the study of fractional multivalued differential inclusions.

Applied problems requiring definitions of fractional derivatives are those that are physically interpretable for initial conditions containing $y(0), y'(0)$, etc. The same requirements are true for boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann–Liouville type and the Caputo type, see Podlubny [28].

Recently, fractional functional differential equations and inclusions with standard Riemann–Liouville and Caputo derivatives with differences conditions were studied by Benchohra et al. [48,43,49], Henderson and Ouahab [50], and Ouahab [51].

Impulsive differential equations and inclusions with Captuo fractional derivatives when $\alpha \in (0, 2]$ were studied by Agarwal et al. [52,53] and Henderson and Ouahab [54].

In this paper, we shall be concerned with the existence of solutions, Filippov's theorem and the relaxation theorem of impulsive fractional value problems (IVP's for short) for differential inclusions. More precisely, we will consider the following problem,

$$D_*^\alpha y(t) \in F(t, y(t)), \quad \text{a.e. } t \in J = [0, b],\ 1 < \alpha \le 2 \tag{1}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{2}$$

$$y'(t_k^+) = \bar{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{3}$$

$$y(0) = a, \qquad y'(0) = c, \tag{4}$$

where D_*^α is the Caputo fractional derivatives, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact values ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $I_k, \bar{I}_k \in C(\mathbb{R}, \mathbb{R})$ $(k = 1, \ldots, m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ stand for the right and the left limits of $y(t)$ at $t = t_k$, respectively.

The paper is organized as follows. We first collect some background material and basic results from multivalued analysis and fractional calculus in Sections 2 and 3 respectively. Then, we shall be concerned with Filippov's theorem for first-order impulsive differential inclusions with fractional order. This is the aim of Section 4. Section 5 is devoted to the relaxed problem associated with Problem (1)–(4), that is, the problem where we consider the convex hull of the right-hand side. In Section 6, we prove the existence of solutions under both convexity and nonconvexity conditions on the multivalued right-hand side. The proofs rely on the nonlinear alternative of Leray–Schauder type, a Bressan–Colombo selection theorem and Covitz and Nadler's fixed point theorem for multivalued contractions. The compactness of the sets of solutions is also established. In Section 6.4, we present some existence and uniqueness results for impulsive differential equations with fractional order.

Finally, Section 7 is devoted to proving some geometric properties of solution sets such as acyclicity, R_δ, and contractibility. Then, existence results for impulsive differential inclusions with fractional inclusions and periodic conditions are provided in Section 8.

We end the paper with some conclusions and remarks.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $AC^i([0, b], \mathbb{R}^n)$ be the space of functions $y : [0, b] \to \mathbb{R}^n$, i-differentiable and whose ith derivative, $y^{(i)}$, is absolutely continuous.

We take $C(J, \mathbb{R})$ to be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le b\}.$$

$L^1(J, \mathbb{R})$ refers to the Banach space of measurable functions $y : J \longrightarrow \mathbb{R}$ which are Lebesgue integrable; it is normed by

$$|y|_1 = \int_0^b |y(s)| \mathrm{d}s.$$

Let $(X, \|\cdot\|)$ be a separable Banach space, and denote

$$\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\},$$
$$\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\},$$
$$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\},$$
$$\mathcal{P}_{b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\},$$
$$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\},$$
$$\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X).$$

A multivalued map $G : X \longrightarrow \mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded set B of X (i.e., $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$). The map G is *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if, for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. Finally, we say that G is *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). We say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

A multivalued map $G : J \longrightarrow \mathcal{P}_{cl}(X)$ is said to be *measurable* if for each $x \in X$ the function $Y : J \longrightarrow \mathbb{R}_+$ defined by

$$Y(t) = d(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\}$$

is measurable.

Lemma 2.1 (*See [55], Thm 19.7*). *Let E be a separable metric space and G a multivalued map with nonempty closed values. Then G has a measurable selection.*

Lemma 2.2 (*See [56], Lemma 3.2*). *Let $G : [0, b] \to \mathcal{P}(E)$ be a measurable multifunction and $u : [0, b] \to E$ a measurable function. Then for any measurable $v : [0, b] \to \mathbb{R}^+$ there exists a measurable selection g of G such that, for a.e. $t \in [0, b]$,*

$$|u(t) - g(t)| \le d(u(t), G(t)) + v(t).$$

Corollary 2.3. *Let $G : [0, b] \to \mathcal{P}_{cp}(E)$ be a measurable multifunction and $g : [0, b] \to E$ be a measurable function. Then there exists a measurable selection u of G such that*

$$|u(t) - g(t)| \le d(g(t), G(t)).$$

Proof. Let $v_\epsilon : [0, b] \to \mathbb{R}_+$ be defined by $v_\epsilon(t) = \epsilon > 0$. Then from Lemma 2.2 there exists a measurable selection u_ϵ of G such that

$$|u_\epsilon(t) - g(t)| \le d(g(t), G(t)) + \epsilon.$$

We take $\epsilon = \frac{1}{n}$, $n \in \mathbb{N}$; hence, for every $n \in \mathbb{N}$, we have

$$|u_n(t) - g(t)| \le d(g(t), G(t)) + \frac{1}{n}.$$

Using the fact that G has compact values, we may pass to a subsequence if necessary to get that $u_n(.)$ converges to measurable function u in E. Then

$$|u(t) - g(t)| \leq d(g(t), G(t)). \quad \square$$

Definition 2.4. The multivalued map $F : J \times X \longrightarrow \mathcal{P}(X)$ is L^1-Carathéodory if

(i) $t \longmapsto F(t, y)$ is measurable for each $y \in X$;
(ii) $y \longmapsto F(t, y)$ is upper semi-continuous for almost all $t \in J$;
(iii) For each $q > 0$, there exists $\phi_q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t, y) \leq \phi_q(t) \text{ for all } \|y\| \leq q \text{ and for almost all } t \in J\}.$$

Lemma 2.5 ([57]). *Let X be a Banach space. Let $F : [0, b] \times X \longrightarrow \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map, and let Γ be a linear continuous mapping from $L^1([0, b], X)$ to $C([0, b], X)$. Then, the operator*

$$\Gamma \circ S_F : C([0, b], X) \longrightarrow \mathcal{P}_{cp,c}(C([0, b], X)), \qquad y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C([0, b], X) \times C([0, b], X)$.

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space; see [58].

Definition 2.6. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \text{for each } x, y \in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

For more details on multivalued maps we refer to the books by Aubin et al. [59,60], Deimling [61], Gorniewicz [55], Hu and Papageorgiou [62], Kamenskii [63], Kisielewicz [58] and Tolstonogov [64].

3. Fractional calculus

According to the Riemann–Liouville approach to fractional calculus the notation of fractional integral of order α ($\alpha > 0$) is a natural consequence of the well-known formula (usually attributed to Cauchy), that reduces the calculation of the n-fold primitive of a formulion $f(t)$ to a single integral of convolution type. In our notation the Cauchy formula reads

$$I^n f(t) := \frac{1}{(n-1)!}\int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad t > 0, n \in \mathbb{N}.$$

Definition 3.1. The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a, b], \mathbb{R})$ is defined by

$$I^\alpha_{a^+} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\mathrm{d}s,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and for $t \leq 0$, $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function and Γ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \mathrm{e}^{-t}\mathrm{d}t, \quad \alpha > 0.$$

For consistency, $I^0 = Id$ (Identity operator), i.e., $I^0 f(t) = f(t)$. Furthermore, by $I^\alpha f(0^+)$ we mean the limit (if it exists) of $I^\alpha f(t)$ for $t \to 0^+$; this limit may be infinite.

After the notion of fractional integral, that of fractional derivative of order α ($\alpha > 0$) becomes a natural requirement and one is tempted to substitute α with $-\alpha$ in the above formulas. However, this generalization needs some care in order to guarantee the convergence of the integral and preserve the well-known properties of the ordinary derivative of integer order. Denoting by D^n, with $n \in \mathbb{N}$, the operator of the derivative of order n, we first note that

$$D^n I^n = Id, \qquad I^n D^n \neq Id, \quad n \in \mathbb{N},$$

i.e., D^n is the left-inverse (and not the right-inverse) to the corresponding integral operator J^n. We can easily prove that

$$I^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a^+)\frac{(t-a)^k}{k!}, \quad t > 0.$$

As consequence we expect that D^α is defined as the left-inverse to I^α. For this purpose, introducing the positive integer n such that $n-1 < \alpha \leq n$, one defines the fractional derivative of order $\alpha > 0$:

Definition 3.2. For a function f given on interval $[a, b]$, the αth Riemann–Liouville fractional-order derivative of f is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-s)^{-\alpha+n-1} f(s)\mathrm{d}s,$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

Defining, for consistency, $D^0 = I^0 = Id$, then we easily recognize that

$$D^\alpha I^\alpha = Id, \quad \alpha \geq 0, \tag{5}$$

and

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \quad \alpha > 0, \gamma - 1, \ t > 0. \tag{6}$$

Of course, the properties (5) and (6) are a natural generalization of those known when the order is a positive integer.

Note the remarkable fact that the fractional derivative $D^\alpha f$ is not zero for the constant function $f(t) = 1$, if $\alpha \notin \mathbb{N}$. In fact, (6) with $\gamma = 0$ illustrates that

$$D^\alpha 1 = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha > 0, t > 0. \tag{7}$$

It is clear that $D^\alpha 1 = 0$, for $\alpha \in \mathbb{N}$, due to the poles of the gamma function at the points $0, -1, -2, \dots$.

We now observe an alternative definition of fractional derivative, originally introduced by Caputo [65,66] in the late 1960's and adopted by Caputo and Mainardi [67] in the framework of the theory of Linear Viscoelasticity (see a review in [24]).

Definition 3.3. Let $f \in AC^n([a, b])$. The Caputo fractional-order derivative of f is defined by

$$(D_*^\alpha f)(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^n(s)\mathrm{d}s.$$

This definition is of course more restrictive than the Riemann–Liouville definition, in that it requires the absolute integrability of the derivative of order m. Whenever we use the operator D_*^α we (tacitly) assume that this condition is met. We easily recognize that in general

$$D^\alpha f(t) := D^m I^{m-\alpha} f(t) \neq J^{m-\alpha} D^m f(t) := D_*^\alpha f(t), \tag{8}$$

unless the function $f(t)$, along with its first $n-1$ derivatives, vanishes at $t = a^+$. In fact, assuming that the passage of the m-derivative under the integral is legitimate, we recognize that, for $m-1 < \alpha < m$ and $t > 0$,

$$D^\alpha f(t) = D_*^\alpha f(t) + \sum_{k=0}^{m-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a^+), \tag{9}$$

and therefore, recalling the fractional derivative of the power function (6),

$$D^\alpha \left(f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a^+) \right) = D_*^\alpha f(t). \tag{10}$$

The alternative definition, that is, Definition 3.3, for the fractional derivative thus incorporates the initial values of the function and of lower order. The subtraction of the Taylor polynomial of degree $n-1$ at $t = a^+$ from $f(t)$ means a sort of regularization of the fractional derivative. In particular, according to this definition, the relevant property for which the fractional derivative of a constant is still zero, i.e.,

$$D_*^{\alpha} 1 = 0, \quad \alpha > 0. \tag{11}$$

We now explore the most relevant differences between the two fractional derivatives given in Definitions 3.2 and 3.3. From Riemann–Liouville fractional derivatives, we have

$$D^{\alpha}(t-a)^{\alpha-j} = 0, \quad \text{for } j = 1, 2, \ldots, [\alpha]+1. \tag{12}$$

From (11) and (12) we thus recognize the following statements about functions about functions which, for $t > 0$, admit the same fractional derivative of order α, with $n-1 < \alpha \le n$, $n \in \mathbb{N}$,

$$D^{\alpha} f(t) = D^{\alpha} g(t) \Leftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j (t-a)^{\alpha-j}, \tag{13}$$

and

$$D_*^{\alpha} f(t) = D_*^{\alpha} g(t) \Leftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j (t-a)^{n-j}. \tag{14}$$

In these formulas the coefficients c_j are arbitrary constants. For proving all main results we present the following auxiliary lemmas.

Lemma 3.4 ([26]). *Let $\alpha > 0$ and let $y \in L^{\infty}(a,b)$ or $C([a,b])$. Then*

$$(D_*^{\alpha} I^{\alpha} y)(t) = y(t).$$

Lemma 3.5 ([26]). *Let $\alpha > 0$ and $n = [\alpha]+1$. If $y \in AC^n[a,b]$ or $y \in C^n[a,b]$, then*

$$(I^{\alpha} D_*^{\alpha} y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k.$$

For further reading and details on fractional calculus, we refer to the books and papers by Kilbas [26], Podlubny [28], Samko [29], Caputo [65–67].

4. The Filippov–Ważewski theorem

Let $J_0 = [0, t_1], J_k = (t_k, t_{k+1}]$, $k = 1, \ldots, m$, and let y_k be the restriction of a function y to J_k. In order to define mild solutions for Problem (1)–(4), consider the space

$$\begin{aligned} PC &= \{y: J \to \mathbb{R} \mid y_k \in C(J_k, \mathbb{R}),\ k = 0, \ldots, m, \text{ and } y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k^-) \\ &= y(t_k) \text{ for } k = 1, \ldots, m\}. \end{aligned}$$

Endowed with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_\infty : k = 0, \ldots, m\},$$

this is a Banach space.

Definition 4.1. A function $y \in PC \cap \bigcup_{k=0}^{m} AC(J_k, \mathbb{R})$ is said to be a solution of (1)–(4) if there exists $v \in L^1(J, \mathbb{R})$, with $v(t) \in F(t, y(t))$ for a.e. $t \in J$, such that y satisfies the fractional differential equation $D_*^{\alpha} y(t) = v(t)$ a.e. on J, and the conditions (2)–(4).

5. Relaxation theorem

In this section we examine to what extent the convexification of the right-hand side of the inclusion introduces new solutions. More precisely, we want to find out if the solutions of the nonconvex problem are dense in those of the convex one. Such a result is known in the literature as a relaxation theorem and it has important implications in optimal control theory. It is well known that in order to have optimal state–control pairs, the system has to satisfy certain convexity requirements. If these conditions are not present, then in order to guarantee the existence of optimal solutions we need to pass to an augmented system with convex structure by introducing the so-called relaxed (generalized, chattering) controls. The resulting relaxed problem has a solution. The relaxation theorems tell us that the relaxed optimal state can be approximated by original states, which are generated by a more economical set of controls that are much simpler to build. In particular, "strong relaxation" theorems imply that this approximation can be achieved using states generated by bang–bang controls. For the problem for first-order differential inclusions, fractional differential inclusions and impulsive functional differential inclusions, we refer to [59] (Thm. 2, p. 124) or [60] (Thm. 10.4.4, p. 402), Hu and Papageorgiou [62], Djabli et al. [68], Graef and Ouahab [19], Henderson and Ouahab [102,50] and Ouahab [51]. More precisely, we compare trajectories of (1)–(4) and of the relaxation impulsive differential inclusion

$$D_*^\alpha y(t) \in \overline{co}F(t, y(t)), \quad \text{a.e. } t \in J =: [0, b] \setminus \{t_1, \dots, t_m\}, \tag{15}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{16}$$

$$y'(t_k^+) = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{17}$$

$$y(0) = \bar{a} \qquad y'(0) = \bar{c}, \tag{18}$$

where $\overline{co}\,A$ refers to the closure of the convex hull of the set A. We will need the following auxiliary results in order to prove our main relaxation theorem. The first two are concerned with measurability of multivalued mappings. The third one is due to Mazur, while the last one is a classical fixed point theorem.

Lemma 5.1 (*[69]*). *Let $U: [0, b] \to \mathcal{P}_{cl}(E)$ be a measurable, integrably bounded set-valued map and let $t \mapsto d(0, U(t))$ be an integrable map. Then the integral $\int_0^b U(t)\mathrm{d}t$ is convex, the map $t \mapsto co\,U(t)$ is measurable and, for every $\varepsilon > 0$, and every measurable selection u of $\overline{co}\,U(t)$, there exists a measurable selection $\bar{u}$ of U such that*

$$\sup_{t\in[0,b]} \left| \int_0^t u(s)\mathrm{d}s - \int_0^t \bar{u}(s)\mathrm{d}s \right| \le \varepsilon$$

and

$$\overline{\int_0^b \overline{co}\,U(t)\mathrm{d}t} = \overline{\int_0^b U(t)\mathrm{d}t} = \int_0^b \overline{co}\,U(t)\mathrm{d}t.$$

Lemma 5.2 (*Mazur's Lemma, [70], Theorem 21.4*). *Let E be a normed space and $\{x_k\}_{k\in\mathbb{N}} \subset E$ a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^m \alpha_{mk}x_k$, where $\alpha_{mk} > 0$ for $k = 1, 2, \dots, m$, and $\sum_{k=1}^m \alpha_{mk} = 1$, which converges strongly to x.*

Lemma 5.3 (*Covitz–Nadler, [71]*). *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $\mathrm{Fix}\,N \neq \emptyset$.*

The following hypotheses will be assumed for the remainder of this section. Let $\bar{a}, \bar{c} \in \mathbb{R}$ and $g \in L^1(J, \mathbb{R})$, and let $x \in PC \cap \bigcup_{k=0}^m AC(J_k, \mathbb{R})$ be a solution of the impulsive differential problem with fractional order:

$$\begin{cases} D_*^\alpha x(t) = g(t), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \alpha \in (1, 2], \\ x(t_k^+) = I_k(x(t_k^-)), & k = 1, \dots, m, \\ x'(t_k^+) = \bar{I}_k(x(t_k^-)), & k = 1, \dots, m, \\ x(0) = \bar{a}, \qquad x'(0) = \bar{c}. \end{cases} \tag{19}$$

We will make use of the following pairs of two assumptions:

($\mathcal{H}_1$) The function $F: J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is such that
 (a) for all $y \in \mathbb{R}$, the map $t \mapsto F(t, y)$ is measurable,
 (b) the map $\gamma : t \mapsto d(g(t), F(t, x(t)))$ is integrable.

($\mathcal{H}_2$) There exists a function $p \in L^1(J, \mathbb{R}^+)$ such that

$$H_d(F(t, z_1), F(t, z_2)) \le p(t)|z_1 - z_2|, \quad \text{for all } z_1, z_2 \in \mathbb{R}.$$

($\overline{\mathcal{H}_1}$) The function $F: J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ satisfies
 (a) for all $y \in \mathbb{R}$, the map $t \mapsto F(t, y)$ is measurable,
 (b) the map $t \mapsto d(0, F(t, 0))$ is integrable.

($\overline{\mathcal{H}_2}$) There exist constants $c_k, \bar{c}_k \ge 0$ such that

$$|I_k(u_1) - I_k(u_2)| \le c_k|u_1 - u_2|, \ |\bar{I}_k(u_1) - \bar{I}_k(u_2)| \le \bar{c}_k|u_1 - u_2|, \quad \text{for each } u_1, u_2 \in \mathbb{R}.$$

Remark 5.4. From Assumptions ($\mathcal{H}_1(a)$) and ($\mathcal{H}_2$), it follows that the multifunction $t \mapsto F(t, x_t)$ is measurable, and by two lemmas in [72] (Lemmas 1.4 and 1.5) we deduce that $\gamma(t) = d(g(t), F(t, x(t)))$ is measurable (see also the Remark on p. 400 in [60]).

Let $P(t)=\int_0^t p(s)\mathrm{d}s$, and define the functions η_0 and H_0 by

$$\eta_0(t)=M\delta_0+M\int_0^t [H_0(s)p(s)+\gamma(s)]\,\mathrm{d}s,\quad t\in[0,t_1],$$

where

$$H_0(t)=\delta_0 M\exp\left(M\mathrm{e}^{P(t)}\right)+M\int_0^t \gamma(s)\exp\left(M\mathrm{e}^{P(t)-P(s)}\right)\mathrm{d}s,$$

and where $M=\max\left(1,b,\frac{b^{\alpha-1}}{\Gamma(\alpha)}\right)$ and $\delta_0=|a-\bar{a}|+|c-\bar{c}|$.

Theorem 5.5 *([73]). Suppose that hypotheses $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ are satisfied. Problem (1)–(4) has at least one solution y satisfying, for a.e. $t\in[0,b]$, the estimates*

$$|y(t)-x(t)|\le M\sum_{0\le k<i}\delta_k+M\sum_{0\le t_k<t}\eta_k(t)$$

and

$$|D_*^\alpha y(t)-g(t)|\le Mp(t)\sum_{0<t_k<t}H_k(t)+\sum_{0<t_k<t}\gamma_k(t),$$

where, for $k=1,\dots,m$,

$$\eta_k(t)=M\int_{t_k}^t [H_k(s)p(s)+\gamma(s)]\,\mathrm{d}s,\quad t\in(t_k,t_{k+1}],$$

$$H_k(t)=\delta_k\exp\left(M\mathrm{e}^{P_k(t)}\right)+\int_{t_k}^t \gamma(s)\exp\left(M\mathrm{e}^{P_k(t)-P_k(s)}\right)\mathrm{d}s,$$

and

$$\delta_k:=|I_1(y(t_k))-I_k(x(t_k))|+|\bar{I}_1(y(t_k))-\bar{I}_1(x(t_k))|.$$

Then our main contribution is the following.

Theorem 5.6. *Assume that $(\mathcal{H}_2)$, $(\overline{\mathcal{H}_1})$, $(\overline{\mathcal{H}_2})$ hold. Then Problem (15) has at least one solution. In addition, for all $\varepsilon>0$ and every solution x of Problem (15), Problem (1)–(4) has a solution y defined on $[0,b]$ satisfying*

$$\|x-y\|_{PC}\le\varepsilon.$$

In particular, $S^{co}_{[0,b]}(a,c)=\overline{S_{[0,b]}(a,c)}$, where

$$S^{co}_{[0,b]}(a,c)=\{y\colon y \text{ is a solution to (1)–(4) on } [0,b]\}.$$

Remark 5.7. Notice that the multivalued map $t\mapsto\overline{co}\,F(t,\cdot)$ also satisfies $(\mathcal{H}_2)$.

Proof. The proof consists of two parts, with each part involving multiple steps.

Part 1. $S^{co}_{[0,b]}\neq\emptyset$: For this, we first transform Problem (15) into a fixed point problem and then make use of Lemma 5.3. Consider the problem on the interval $[-r,t_1]$, that is,

$$\begin{cases} D_*^\alpha y(t)\in\overline{co}\,F(t,y(t)), & \text{a.e. } t\in[0,t_1],\\ y(0)=a, \quad y'(0)=c. \end{cases}\tag{20}$$

It is clear that all solutions of Problem (20) are fixed points of the multivalued operator $N:PC_1\to\mathcal{P}(PC_1)$ defined by

$$N(y):=\left\{h\in PC_1: h(t)=a+tc+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)\mathrm{d}s,\ t\in[0,t_1]\right\}$$

where

$$g\in S_{\overline{co}F,y}=\left\{g\in L^1([0,t_1],\mathbb{R}): g(t)\in\overline{co}\,F(t,y(t)) \text{ for a.e. } t\in(0,t_1]\right\}$$

and $PC_1=C([0,t_1],\mathbb{R})$. To show that N satisfies the assumptions of Lemma 5.3, the proof will be given in two steps. In Steps 3, 4, we study the problem on the intervals $(t_k,t_{k+1}]$ for $k=1,\dots,m-1$.

Step 1. $N(y) \in \mathcal{P}_{cl}(PC_1)$ for each $y \in PC_1$: Indeed, let $\{y_n\} \in N(y)$ be such that $y_n \to \tilde{y}$ in PC_1, as $n \to +\infty$. Then $\tilde{y} \in PC_1$ and there exists a sequence $g_n \in S_{\overline{co}\,F,y}$ such that

$$y_n(t) = a + tc + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_n(s)\mathrm{d}s, \quad t \in [0, t_1].$$

Then $\{g_n\}$ is integrably bounded. Since $F(\cdot,\cdot)$ has closed values, let $w(\cdot) \in F(\cdot, 0)$ be such that $|w(t)| = d(0, F(t, 0))$. From $(\overline{\mathcal{H}_1})$ and $(\overline{\mathcal{H}_2})$, we infer that for a.e. $t \in [0, t_1]$,

$$\begin{aligned}|g_n(t)| &\le |g_n(t) - w(t)| + |w(t)| \\ &\le p(t)\|y\|_{PC_1} + d(0, F(t, 0)) := M_*(t), \quad \forall\, n \in \mathbb{N},\end{aligned}$$

that is,

$$g_n(t) \in M(t)B(0, 1), \quad \text{a.e. } t \in [0, t_1].$$

Since $B(0, 1)$ is compact in $\mathbb{R}$, there exists a subsequence still denoted $\{g_n\}$ which converges to g.

Then the Lebesgue dominated convergence theorem implies that, as $n \to \infty$,

$$\|g_n - g\|_{L^1} \to 0 \quad \text{and thus } y_n(t) \to \tilde{y}(t)$$

with

$$\tilde{y}(t) = a + tc + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\mathrm{d}s, \quad t \in [0, t_1],$$

proving that $\tilde{y} \in N(y)$.

Step 2. There exists $\gamma < 1$ such that $H_d(N(y), N(\overline{y})) \le \gamma \|y - \overline{y}\|_{PC_1}$ for each $y, \overline{y} \in PC_1$, where the norm $\|y - \overline{y}\|_{PC_1}$ will be chosen conveniently. Indeed, let $y, \overline{y} \in \Omega([-r, t_1])$ and $h_1 \in N(y)$. Then there exists $g_1(t) \in \overline{co}\,F(t, y(t))$ such that, for each $t \in [0, t_1]$,

$$h_1(t) = a + tc + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_1(s)\mathrm{d}s.$$

Since, for each $t \in [0, t_1]$,

$$H_d(\overline{co}\,F(t, y_t), \overline{co}\,F(t, \overline{y}_t)) \le p(t)|y(t) - \overline{y}(t)|,$$

then there exists some $w(t) \in \overline{co}\,F(t, \overline{y}(t))$ such that

$$|g_1(t) - w(t)| \le p(t)|y(t) - \overline{y}(t)|, \quad t \in [0, t_1].$$

Consider the multimap $U_1 : [0, t_1] \to \mathcal{P}(\mathbb{R})$ defined by

$$U_1(t) = \{w \in \mathbb{R} : |g_1(t) - w| \le p(t)|y(t) - \overline{y}(t)|\}.$$

As in the proof of Theorem 5.5, we can show that the multivalued operator $V_1(t) = U_1(t) \cap \overline{co}\,F(t, \overline{y}(t))$ is measurable and takes nonempty values. Then there exists a function $g_2(t)$, which is a measurable selection for V_1. Thus, $g_2(t) \in \overline{co}\,F(t, \overline{y}(t))$ and

$$|g_1(t) - g_2(t)| \le p(t)|y - \overline{y}|, \quad \text{for a.e. } t \in [0, t_1].$$

For each $t \in [0, t_1]$, let

$$h_2(t) = a + tc + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_2(s)\mathrm{d}s.$$

Therefore, for each $t \in [0, t_1]$, we have

$$\begin{aligned}|h_1(t) - h_2(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g_1(s) - g_2(s)|\,\mathrm{d}s \\ &\le \frac{t_1^\alpha}{\Gamma(\alpha)} \int_0^t p(s)|y(s) - \overline{y}(s)|\mathrm{d}s \\ &\le \frac{t_1^\alpha}{\Gamma(\alpha)} \int_0^t p(s)\mathrm{e}^{\tau \int_0^s p(u)\mathrm{d}u} \left(\sup_{0 \le z \le t_1} \mathrm{e}^{-\tau \int_0^z p(u)\mathrm{d}u} |y(z) - \overline{y}(z)| \right) \mathrm{d}s \\ &\le \frac{t_1^\alpha}{\Gamma(\alpha)\tau} \int_0^t (\mathrm{e}^{\tau \int_0^s p(u)\mathrm{d}u})' \|y - \overline{y}\|_{BPC_1} \mathrm{d}s.\end{aligned}$$

Hence,

$$\|h_1 - h_2\|_{BPC_1} \le \frac{t_1^\alpha}{\tau} \|y - \overline{y}\|_{BPC_1},$$

where

$$\|y\|_{BPC_1} = \sup\left\{ e^{-\tau \int_0^t p(s)ds} |y(t)| : t \in [0, t_1], \tau > \frac{t_1^\alpha}{\Gamma(\alpha)} \right\}.$$

By an analogous relation, obtained by interchanging the roles of y and $\bar{y}$, we find that

$$H_d(N(y), N(\bar{y})) \leq \frac{t_1^\alpha}{\Gamma(\alpha)\tau} \|y - \bar{y}\|_{BPC_1}.$$

Then N is a contraction and hence, by Lemma 5.3, N has a fixed point y_0, which is solution to Problem (20).

Step 3. Let $y_2 := y|_{[t_1, t_2]}$ be a possible solution to the problem

$$\begin{cases} D_*^\alpha y(t) \in \overline{co}\, F(t, y(t)), & t \in (t_1, t_2], \\ y(t_1^+) = I_1(y_0(t_1^-)), \\ y'(t_1^+) = \bar{I}_1(y_0(t_1^-)). \end{cases} \tag{21}$$

Then y_2 is a fixed point of the multivalued operator $N : PC_2 \to \mathcal{P}(PC_2)$ defined by

$$N(y) := \left\{ h \in PC_2 : h(t) = \begin{cases} I_1(y_0(t_1)) + (t - t_1)\bar{I}_1(y_0(t_1)) \\ \quad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t - s)^{\alpha-1} g(s)ds, \quad t \in (t_1, t_2] \end{cases} \right\}$$

where

$$g \in S_{\overline{co}\, F, y} = \left\{ g \in L^1([t_1, t_2], \mathbb{R}) : g(t) \in \overline{co}\, F(t, y(t)) \text{ for a.e. } t \in [t_1, t_2] \right\}.$$

Again, we show that N satisfies the assumptions of Lemma 5.3. Clearly, $N(y) \in \mathcal{P}_{cl}(PC_2)$ for each $y \in PC_2$. It remains to show that there exists $0 < \gamma < 1$ such that

$$H_d(N(y), N(\bar{y})) \leq \gamma \|y - \bar{y}\|_{BPC_2},$$

for each $y, \bar{y} \in PC_2$. For this purpose, let $y, \bar{y} \in PC_2$ and $h_1 \in N(y)$. Then there exists $g_1(t) \in \overline{co}\, F(t, y(t))$ such that, for each $t \in [0, t_2]$,

$$h_1(t) = I_1(y_0(t_1)) + (t - t_1)\bar{I}_1(y_0(t_1)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t - s)^{\alpha-1} g_1(s)ds.$$

Since from $(\overline{\mathcal{H}}_2)$

$$H_d(\overline{co}\, F(t, y(t)), \overline{co}\, F(t, \bar{y}_t)) \leq p(t)|y(t) - \bar{y}(t)|, \quad t \in [t_1, t_2],$$

we deduce that there is a $w(\cdot) \in \overline{co}\, F(\cdot, \bar{y}(\cdot))$ such that

$$|g_1(t) - w(t)| \leq p(t)|y(t) - \bar{y}(t)|, \quad t \in [t_1, t_2].$$

Consider the multivalued map $U_2 : [t_1, t_2] \to \mathcal{P}(\mathbb{R})$ defined by

$$U_2(t) = \{ w \in \mathbb{R} : |g_1(t) - w| \leq p(t)|y(t) - \bar{y}(t)| \}.$$

Since the multivalued operator $V_2(t) = U_2(t) \cap \overline{co}\, F(t, \bar{y}(t))$ is measurable with nonempty values, there exists $g_2(t)$ which is a measurable selection for V_2. Then $g_2(t) \in \overline{co}\, F(t, \bar{y}(t))$ and

$$|g_1(t) - g_2(t)| \leq p(t)|y(t) - \bar{y}(t)|, \quad \text{for a.e. } t \in [t_1, t_2].$$

For a.e. $t \in [t_1, t_2]$, define

$$h_2(t) = I_1(y_0(t_1)) + (t - t_1)\bar{I}_1(y_0(t_1)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g_2(s)ds.$$

For some $\tau > Me^{wt_2}$, we have the estimates

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t - s)^{\alpha-1} |g_1(s) - g_2(s)|\, ds \\ &\leq \frac{t_2^\alpha}{\Gamma(\alpha)} \int_{t_1}^t p(s)|y(s) - \bar{y}(s)|ds \\ &\leq \frac{t_2^\alpha}{\Gamma(\alpha)\tau} \|y - \bar{y}\|_{BPC_2}. \end{aligned}$$

By an analogous relation, obtained by interchanging the roles of y and $\bar{y}$, we obtain

$$H_d(N(y), N(\bar{y})) \leq \frac{t_2^\alpha}{\Gamma(\alpha)\tau} \|y - \bar{y}\|_{BPC_2},$$

where

$$\|y\|_{BPC_2} = \sup\left\{ e^{-\tau\int_{t_1}^{t} p(s)ds}|y(t)| : t \in [t_1, t_2] \right\}.$$

Therefore N is a contraction and thus, by Lemma 5.3, N has a fixed point y_1 solution of Problem (21).

Step 4. We continue this process taking into account that $y_m := y|_{[t_m,b]}$ is a solution of the following problem:

$$\begin{cases} D_*^{\alpha}y(t) \in F(t, y(t)), \quad t \in (t_m, b], \\ y(t_m^+) = I_m(y_{m-1}(t_m^-)), \\ y'(t_m^+) = \bar{I}_m(y_{m-1}(t_m^-)). \end{cases} \tag{22}$$

Then a solution y of Problem (15) may be defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, t_2], \\ \dots & \dots \\ y_m(t), & \text{if } t \in (t_m, b]. \end{cases}$$

Part 2. Let x be a solution of Problem (15). Then, there exists $g \in S_{\overline{co}F,x}$ such that

$$x(t) = a + tc + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds, \quad t \in [0, t_1],$$

and

$$x(t) = I_k(x(t_k)) + (t-t_k)\bar{I}_k(x(t_k)) + \frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}g(s)ds, \quad t \in (t_k, t_{k+1}], k = 1, \dots, m,$$

i.e., x is a solution of the problem

$$\begin{cases} D_*^{\alpha}x(t) = g(t), & \text{a.e. } t \in [0, b] \setminus \{t_1, \dots, t_m\}, \\ x(t_k^+) = I_k(x(t_k^-)), & k = 1, \dots, m, \\ x'(t_k^+) = \bar{I}_k(x(t_k^-)), & k = 1, \dots, m. \\ x(0) = a, \quad x'(0) = c. \end{cases} \tag{23}$$

Let $\varepsilon > 0$ and $\delta > 0$ be given by the relation $Me^{\omega b}\varepsilon = \delta L\sum_{k=1}^{m} R_k$, where L, R_k, for $k = 0, 1, \dots, m$, will be defined later on. From Lemma 5.1, there exists a measurable selection f_* of $t \mapsto F(t, x(t))$ such that

$$\begin{aligned} &\sup_{t\in[0,b]}\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_*(s)ds\right| \\ &\quad \le \frac{b^{\alpha}}{\Gamma(\alpha)}\sup_{t\in[0,b]}\left|\int_0^t g(s)ds - \int_0^t f_*(s)ds\right| \le \frac{b^{\alpha}}{\Gamma(\alpha)}\delta. \end{aligned}$$

Let

$$z(t) = a + tc + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_*(s)ds, \quad t \in [0, t_1]$$

and

$$z(t) = I_k(x(t_k)) + (t-t_k)\bar{I}_k(x(t_k)) + \frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}f_*(s)ds, \quad t \in (t_k, t_{k+1}], k = 1, \dots, m.$$

Hence

$$|x(t) - z(t)| \le \frac{b^{\alpha}}{\Gamma(\alpha)}\delta.$$

With assumption $(\overline{\mathcal{H}_2})$, we infer that, for all $u \in \overline{co}F(t, z(t))$,

$$\begin{aligned} \gamma(t) := d(g(t), F(t, x(t))) &\le d(g(t), u) + H_d(F(t, z(t)), F(t, x(t))) \\ &\le H_d(\overline{co}F(t, x(t)), \overline{co}F(t, z(t))) + H_d(F(t, z(t)), F(t, x(t))) \\ &\le 2p(t)|x(t) - z(t)| \le 2\delta p(t). \end{aligned}$$

Since, under $(\overline{\mathcal{H}_1}(a))$ and $(\overline{\mathcal{H}_2})$, γ is measurable (see [60] or [72], Lemma 1.5), by the above inequality, we deduce that $\gamma \in L^1(J, \mathbb{R})$. From Theorem 5.5, Problem (1)–(4) has a solution y which satisfies

$$|y(t) - x(t)| \le \eta_0(t), \quad t \in [0, t_1].$$

It is clear that $\delta_0 = 0$ and

$$\eta_0(t) \le \delta R_0.$$

Hence

$$\|y - x\|_\infty \le \delta R_0,$$

where

$$R_0 = 2\|p\|_{L^1}^2 M^2 \exp\left(Me^{P(t_1)}\right) + 2M\|p\|_{L^1}.$$

For $t \in (t_1, t_2]$, we obtain

$$|y(t) - x(t)| \le M[(c_1 + (t_2 - t_1)\bar{c}_1)]\eta_0(t_1) + M\eta_1(t).$$

Since

$$H_1(t) \le \left[M[1 + (c_1 + (t_2 - t_1)\bar{c}_1)]R_0 + \left((c_1 + (t_2 - t_1)\bar{c}_1)R_0 \exp\left(Me^{P_1(t_2)}\right) + 2\|p\|_{L^1} \exp\left(Me^{P_1(t_2)}\right)\right)\right]\delta,$$

there exists R_1 such that

$$\sup\{|y(t) - x(t)| : t \in [t_1, t_2]\} \le (M[(c_1 + (t_2 - t_1)\bar{c}_1)]R_0 + R_1)\delta.$$

We continue this process until we get that there exists $L > 0$ such that, for all $t \in [0, b]$, we have

$$|x(t) - y(t)| \le L\delta \sum_{k=0}^{m} R_k.$$

Set

$$\delta = \frac{\epsilon}{L \sum_{k=0}^{m} R_k}.$$

Then we obtain

$$\|y_* - y\|_{PC} \le \epsilon,$$

and the proof is complete. □

6. Existence results

6.1. Convex case

For our main consideration of Problem (1)–(4), a nonlinear alternative of Leray–Schauder type is used to investigate the existence of solutions for first-order impulsive fractional differential inclusions.

Theorem 6.1 (*Convex Case*). *Suppose the following hold:*

($\mathcal{B}_1$) *The function $F : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is L^1-Carathéodory,*

($\mathcal{B}_2$) *There exist a function $p \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that*

$$\|F(t, z)\|_{\mathcal{P}} \le p(t)\psi(|z|) \quad \text{for a.e. } t \in J \text{ and each } z \in \mathbb{R},$$

with

$$\int_0^b p(s)\mathrm{d}s < \int_{|a|+t_1|c|}^{\infty} \frac{\mathrm{d}u}{\psi(u)} \cdot .$$

Then the set of solutions for Problem (1)–(4) is nonempty and compact. Moreover, the operator solution $S(\cdot, \cdot) : \mathbb{R}^2 \to \mathcal{P}(PC)$, defined by

$$S(a, c) = \{y \in PC : y \text{ solution of the problem with } y(0) = a, y'(0) = c\},$$

is u.s.c.

Proof. Transform the problem into a fixed point problem. Consider first Problem (1)–(4) on the interval $[0, t_1]$, that is, the problem

$$D_*^\alpha y(t) \in F(t, y(t)), \quad \text{a.e. } t \in [0, t_1], \tag{24}$$

$$y(0) = a, \qquad y'(0) = c. \tag{25}$$

It is clear that the solutions of the Problem (24)–(25) are fixed points of the multivalued operator, $N_0 : C([0, t_1], \mathbb{R}) \to \mathcal{P}(C([0, t_1], \mathbb{R}))$ defined by

$$N_0(y) := \left\{h \in C([0, t_1], \mathbb{R}) : h(t) = a + tc + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} g(s)\mathrm{d}s, \text{ if } t \in [0, t_1]\right\}$$

where

$$g \in S_{F,y} = \Big\{g \in L^1([0,t_1],\mathbb{R}) : g(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,t_1]\Big\}.$$

As in [49–51], we can show that N_0 is completely continuous, with compact and convex values. □

Now we show that N_0 is *upper semi-continuous.* Since N_0 is completely continuous, it suffices to prove that N_0 has a closed graph. Let $y_n \longrightarrow y_*,\ h_n \in N_0(y_n)$ and $h_n \longrightarrow h_*,\ y_n \longrightarrow y_*$ as $n \to \infty$. We will prove that $h_* \in N_0(y_*)$. Now $h_n \in N_0(y_n)$ implies that there exists $g_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$h_n(t) = a + tc + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_n(s)\mathrm{d}s.$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$h_*(t) = a + tc + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_*(s)\mathrm{d}s.$$

Consider the linear operator,

$$\Psi : L^1([0,t_1],\mathbb{R}) \longrightarrow C([0,t_1],\mathbb{R})$$

$$g \longmapsto (\Psi g)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)\mathrm{d}s.$$

Then

$$\|\Psi(g)\|_\infty \le \frac{t_1^\alpha}{\Gamma(\alpha)}\|g\|_{L^1},$$

and this implies that Ψ is continuous. From Lemma 2.5, it follows that $\Psi \circ S_{F,y}$ is a closed graph operator. Moreover, we have that

$$h_n \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 2.5 that

$$h_*(t) = a + tc + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_*(s)\mathrm{d}s$$

for some $g_* \in S_{F,y_*}$.

A priori bounds. *We now show that there exists an open set $U_0 \subseteq C([0,t_1],\mathbb{R})$ with no $y \in \lambda N_0(y)$, for $\lambda \in (0,1)$ and $y \in \partial U_0$.*

Let $y \in C([0,t_1],\mathbb{R})$ and $y \in \lambda N_0(y)$ for some $0 < \lambda < 1$. Thus there exists $g \in S_{F,y}$ such that, for each $t \in [0,b]$, we have

$$y(t) = \lambda\Big[a + tc + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)\mathrm{d}s\Big], \tag{26}$$

and so

$$|y(t)| \le |a| + t_1|c| + \frac{t_1^\alpha}{\Gamma(\alpha)}\int_0^t p(s)\psi(|y(s)|)\mathrm{d}s, \quad t \in [0,t_1].$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : 0 \le s \le t\}, \quad 0 \le t \le t_1.$$

Then

$$\mu(t) \le |a| + t_1|c| + \frac{t_1^\alpha}{\Gamma(\alpha)}\int_0^t p(s)\psi(\mu(s))\mathrm{d}s.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$c_* = v(0) = |a|, \qquad \mu(t) \le v(t), \quad t \in [0,t_1]$$

and

$$v'(t) = \frac{t_1^\alpha}{\Gamma(\alpha)}p(t)\psi(\mu(t)), \quad t \in [0,t_1].$$

Using the nondecreasing character of ψ, we get

$$v'(t) \le \frac{t_1^\alpha}{\Gamma(\alpha)}p(t)\psi(v(t)), \quad t \in [0,t_1].$$

This implies, for each $t \in [0, t_1]$, that

$$\int_{v(0)}^{v(t)} \frac{\mathrm{d}u}{\psi(u)} \leq \frac{t_1^\alpha}{\Gamma(\alpha)} \int_0^{t_1} p(s)\mathrm{d}s.$$

Then there exists $M_0 > 0$ such that

$$\sup_{t\in[0,t_1]} |y(t)| \leq M_0.$$

Set

$$U_0 = \{y \in C([0, t_1], \mathbb{R}) : \|y\|_\infty < M_0 + 1\}.$$

$N_0 : \overline{U}_0 \to \mathcal{P}(C([0, t_1], \mathbb{R}))$ is continuous and completely continuous. From the choice of U_0, there is no $y \in \partial U_0$ such that $y \in \lambda N_0(y)$, for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [74], we deduce that N_0 has a fixed point y_0 in U_0, which is a solution to (24)–(25).

Now, let $y_2 := y|_{(t_1,t_2]}$ be a solution to the problem

$$D_*^\alpha y(t) \in F(t, y(t)), \quad \text{a.e. } t \in (t_1, t_2], \tag{27}$$

$$y(t_1^+) = I_1(y_0(t_1^-)). \tag{28}$$

$$y'(t_1^+) = \bar{I}_1(y_0(t_1^-)). \tag{29}$$

Then y_1 is a fixed point of the multivalued operator $N : PC_1 \to \mathcal{P}(PC_1)$ defined by

$$N_1(y) := \left\{ h \in PC_1 : h(t) = \begin{cases} I_1(y_1(t_1)) + (t - t_1)\bar{I}_1(y_0(t_1)) \\ \quad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-1} g(s)\mathrm{d}s, \quad t \in [t_1, t_2], \end{cases} \right\}$$

where

$$g \in S_{F,y} = \left\{ g \in L^1([t_1, t_2], \mathbb{R}) : g(t) \in F(t, y(t)) \text{ for a.e. } t \in [t_1, t_2] \right\}.$$

Clearly, N_1 is completely continuous, u.s.c., compact and convex valued.

We now show there exists an open set $U_1 \subseteq PC_1$ with no $y \in \lambda N_1(y)$ for $\lambda \in (0, 1)$ and $y \in \partial U_1$.

Let $y \in PC_1$ and $y \in \lambda N_1(y)$ for some $0 < \lambda < 1$. Then,

$$y(t) = \lambda \left[I_1(y_0(t_1)) + (t - t_1)\bar{I}_1(y_0(t_1)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\mathrm{d}s \right],$$

for some $\lambda \in (0, 1)$. And so,

$$|y(t)| \leq |I_1(y_0(t_1))| + (t_2 - t_1)|I_1(y_0(t_1))| + \frac{t_2^\alpha}{\Gamma(\alpha)} \int_{t_1}^t p(s)\psi(|y(s)|)\mathrm{d}s, \quad t \in (0, b]. \tag{30}$$

Set

$$\mu(t) = \sup\{|y(s)| : s \in [t_1, t_2]\}, \quad t \in [t_1, t_2].$$

Then

$$\mu(t) \leq |I_1(y_0(t_1))| + (t_2 - t_1)|I_1(y_0(t_1))| + \frac{t_2^\alpha}{\Gamma(\alpha)} \int_{t_1}^t p(s)\psi(\mu(s))\mathrm{d}s. \tag{31}$$

This implies that there exists $M_1 > 0$ such that

$$\|y\|_{PC_1} \leq M_1.$$

Set

$$U_1 = \{y \in PC_1 : \|y\|_{PC_1} < M_1 + 1\}.$$

$N_1 : \overline{U}_1 \to \mathcal{P}(PC_1)$ is continuous and completely continuous. From the choice of U_1, there is no $y \in \partial U_1$ such that $y \in \lambda N_1(y)$, for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [74], we deduce that N_1 has a fixed point y_1 in U_1, which is a solution to (27)–(29). We continue this process, and taking into account that $y_m := y|_{(t_m,b]}$ is a solution to the problem

$$D_*^\alpha y(t) \in F(t, y(t)), \quad \text{a.e. } t \in (t_m, b], \tag{32}$$

$$y(t_m^+) = I_m(y_{m-1}(t_m^-)), \tag{33}$$

$$y'(t_m^+) = \bar{I}_m(y_{m-1}(t_m^-)). \tag{34}$$

The solution y of Problem (1)–(4) is then defined by

$$y(t)=\begin{cases} y_0(t), & \text{if } t\in[0,t_1],\\ y_1(t), & \text{if } t\in(t_1,t_2],\\ \vdots & \\ y_m(t), & \text{if } t\in(t_m,b]. \end{cases}$$

Using the fact that $F(\cdot,\cdot)\in\mathcal{P}_{cv,cp}(\mathbb{R})$, $F(t,.)$ is u.s.c. and Mazur's lemma, by Ascoli's theorem, we can prove that the solution set of Problem (1)–(4) is compact.

Now, we will show that $S(\cdot,\cdot)$ is u.s.c. by proving that the graph of S,

$$\Gamma(a,c):=\{(a,c,y)\mid y\in S(a,c)\},$$

is closed. Let $(a_n,c_n,y_n)\in\Gamma$, i.e., $y_n\in S(a_n,c_n)$, and let $(a_n,c_n,y_n)\to(a,c,y)$ as $n\to\infty$. Since $y_n\in S(a_n,c_n)$, there exists $v_n\in L^1(J,\mathbb{R})$ such that

$$y_n(t)=\begin{cases} a_n+c_nt+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t(t-s)^{\alpha-1}v_n(s)\mathrm{d}s, & \text{if } t\in[0,t_1],\\ I_1(y_n(t_1))+(t-t_1)\bar{I}_1(y_n(t_1))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t(t-s)^{\alpha-1}v_n(s)\mathrm{d}s, & \text{if } t\in(t_1,t_2],\\ \vdots & \\ y_n(t_m)+I_m(y_n(t_m))+(t-t_m)\bar{I}_m(y_n(t_m))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t(t-s)^{\alpha-1}v_n(s)\mathrm{d}s, & \text{if } t\in(t_m,b]. \end{cases}$$

Since (a_n,c_n,y_n) converges to (a,c,y), there exists $M>0$ such that

$$|a_n|+|c_n|\le M\quad\text{for all } n\in\mathbb{N}.$$

By using $(\mathcal{B}_2)$, we can easily prove that there exist $\overline{M}>0$ such that

$$\|y_n\|_{PC}\le\overline{M}\quad\text{for all } n\in\mathbb{N}.$$

From the definition of y_n, we have $D_*^\alpha y_n(t)=v_n(t)$ a.e. $t\in J$, and so

$$|v_n(t)|\le p(t)\psi(M),\quad t\in J.$$

Thus, $v_n(t)\in p(t)\psi(M)\overline{B}(0,1):=\chi(t)$ a.e. $t\in J$. It is clear that $\chi:J\to\mathcal{P}_{cp,cv}(\mathbb{R})$ is a multivalued map that is integrably bounded. Since $\{v_n(\cdot):n\ge1\}\in\chi(\cdot)$, we may pass to a subsequence if necessary to obtain that v_n converges weakly to v in $L^1_w(J,\mathbb{R})$. From Mazur's lemma, there exists

$$v\in\overline{conv}\{v_n(t):n\ge1\},$$

so there exists a subsequence $\{\bar{v}_n(t):n\ge1\}$ in $\overline{conv}\{v_n(t):n\ge1\}$, such that $\bar{v}_n$ converges strongly to $v\in L^1(J,\mathbb{R})$. Since $F(t,\cdot)$ is u.s.c., for every $\epsilon>0$, there exists $n_0(\epsilon)$ such that, for every $n\ge n_0(\epsilon)$, we have

$$\bar{v}_n(t)\in F(t,y_n(t))\subseteq F(t,\widetilde{y}(t))+\epsilon B(0,1).$$

This implies that $v(t)\in F(t,y(t))$, a.e. $t\in J$. Let

$$z(t)=\begin{cases} a+ct+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t(t-s)^{\alpha-1}v(s)\mathrm{d}s, & \text{if } t\in[0,t_1],\\ I_1(y(t_1))+(t-t_1)\bar{I}_1(y(t_1))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t(t-s)^{\alpha-1}v(s)\mathrm{d}s, & \text{if } t\in(t_1,t_2],\\ \vdots & \\ I_m(y(t_m))+(t-t_m)\bar{I}_m(y(t_m))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t(t-s)^{\alpha-1}v(s)\mathrm{d}s, & \text{if } t\in(t_m,b]. \end{cases}$$

For each $t\in J$, the mapping $\Gamma:L^1(J,\mathbb{R})\to\mathbb{R}$ defined by

$$\Gamma(g)(t)=\int_0^t g(s)\mathrm{d}s$$

is a continuous linear operator from $L^1(J,\mathbb{R})$ into $\mathbb{R}$. It remains continuous if these spaces are endowed with topologies [75]. Therefore, for each $t\in J$, the sequence $\{y_n(t)\}$ converges to $y(t)$ and the continuity of I_k, $\bar{I}_k$ $k=1,\ldots,m$, we have

$$y(t)=\begin{cases} a+ct+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t(t-s)^{\alpha-1}v(s)\mathrm{d}s, & \text{if } t\in[0,t_1],\\ I_1(y(t_1))+(t-t_1)\bar{I}_1(y(t_1))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t(t-s)^{\alpha-1}v(s)\mathrm{d}s, & \text{if } t\in(t_1,t_2],\\ \vdots & \\ I_m(y(t_m))+(t-t_m)\bar{I}_m(y(t_m))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t(t-s)^{\alpha-1}v(s)\mathrm{d}s, & \text{if } t\in(t_m,b].\end{cases}$$

Thus, $y\in S(a,c)$. Now, we show that $S(\cdot,\cdot)$ maps bounded sets into relatively compact sets of PC. Let B be a bounded set in $\mathbb{R}^2$ and let $\{y_n\}\subset S(B)$. Then there exist $\{a_n,c_n\}\subset B$ such that

$$y_n(t)=\begin{cases} a_n+c_nt+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t(t-s)^{\alpha-1}v_n(s)\mathrm{d}s, & \text{if } t\in[0,t_1],\\ I_1(y_n(t_1))+(t-t_1)I_1(y_n(t_1))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t(t-s)^{\alpha-1}v_n(s)\mathrm{d}s, & \text{if } t\in(t_1,t_2],\\ \vdots & \\ I_m(y_n(t_m))+(t-t_m)I_m(y_n(t_m))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t(t-s)^{\alpha-1}v_n(s)\mathrm{d}s, & \text{if } t\in(t_m,b]\end{cases}$$

where $v_n\in S_{F,y_n}$, $n\in\mathbb{N}$. Since $\{a_n,c_n\}$ is a bounded sequence, there exists a subsequence of $\{a_n,c_n\}$ converging to $\{a,c\}$, so from $(\mathcal{B}_2)$, there exist $M_*>0$ such that

$$\|y_n\|_{PC}\leq M_*,\ n\in\mathbb{N}.$$

As in [52,49,51,54], we can show that $\{y_n : n\in\mathbb{N}\}$ is equicontinuous in PC. As a consequence of the Arzelá–Ascoli theorem, we conclude that there exists a subsequence of $\{y_n\}$ converging to y in PC. By a similar argument to the one above, we can prove that

$$y(t)=\begin{cases} a+ct+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t(t-s)^{\alpha-1}v(s)\mathrm{d}s, & \text{if } t\in[0,t_1],\\ I_1(y(t_1))+(t-t_1)\bar{I}_1(y(t_1))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t(t-s)^{\alpha-1}v(s)\mathrm{d}s, & \text{if } t\in(t_1,t_2],\\ \vdots & \\ I_m(y(t_m))+(t-t_m)\bar{I}_m(y(t_m))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t(t-s)^{\alpha-1}v(s)\mathrm{d}s, & \text{if } t\in(t_m,b]\end{cases}$$

where $v\in S_{F,y}$. Thus, $y\in S(a,c)$, and this implies that $S(\cdot,\cdot)$ is u.s.c.

Remark 6.2. For the proof that $S(.,.)$ is compact we can used the compactness of $B(0,1)$ in $\mathbb{R}$ and the Lebesgue dominated convergence theorem.

6.2. Nonconvex case

In this subsection, we present a second result for Problem (1)–(4) with a nonconvex valued right-hand side. We will make use of some new conditions.

(A1) $F:[0,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathcal{P}_{cp}(\mathbb{R})$; $t\longmapsto F(t,x)$ is measurable for each $x\in\mathbb{R}$.

(A2) There exists a function $p\in L^1([0,b],\mathbb{R}^+)$ such that, for a.e. $t\in[0,b]$ and all $x,y\in\mathbb{R}$,

$$H_d(F(t,x),F(t,y))\leq p(t)|x-y|,$$

and

$$d(0,F(t,0))\leq p(t)\quad\text{for a.e. } t\in[0,b].$$

Theorem 6.3. *Suppose that hypotheses* (A1) *and* (A2) *are satisfied. Then the IVP* (1)–(4) *has at least one solution.*

Proof. For the proof, see Theorem 5.6. $\square$

Lemma 6.4. *Assume that the conditions of Theorem 6.3 are satisfied and $F : [0, b] \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$. Then the solution set of Problem (1)–(4) is compact.*

Proof. Using the fact the $F(\cdot, \cdot) \in \mathcal{P}_{cv}(\mathbb{R})$ and Mazur's lemma, by Ascoli's theorem, we can prove that the solution set of Problem (1)–(4) is compact.

We now lay some groundwork for this subsection, where F is not necessarily convex valued.
Let E be a Banach space and A a be subset of $J \times E$.

Definition 6.5. A is called $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $I \times D$, where I is Lebesgue measurable in J and D is Borel measurable in E.

Definition 6.6. A subset $A \subset L^1(J, E)$ is decomposable if, for all $u, v \in A$ and for every Lebesgue measurable set $I \subset J$, we have

$$u\chi_I + v\chi_{J\setminus I} \in A,$$

where χ stands for the characteristic function. The family of all nonempty closed and decomposable subsets of $L^1(J, \mathbb{R}^n)$ is denoted by $\mathcal{D}$.

Let $F : J \times E \to \mathcal{P}(E)$ be a multivalued map with nonempty closed values. Assign to F the multivalued operator $\mathcal{F} : C(J, E) \to \mathcal{P}(L^1(J, E))$ defined by $\mathcal{F}(y) = S_{F,y}$ and let $\mathcal{F}(t, y) = S_{F,y}(t),\ t \in J$. The operator $\mathcal{F}$ is called the *Nemyts'kĭi operator* associated to F.

Definition 6.7. Let $F : J \times E \to \mathcal{P}(E)$ be a multivalued function with nonempty compact values. We say that F is of lower semi-continuous type (*l.s.c.* type) if its associated Nemyts'kĭi operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Proposition 6.8 ([76]). *Consider an l.s.c. multivalued map $G : S \to \mathcal{D}$ and assume that $\phi : S \to L^1(J, \mathbb{R}^n)$ and $\psi : S \to L^1(J, \mathbb{R}^+)$ are continuous maps, and for every $s \in S$, the set*

$$H(s) = \overline{\{u \in G(s) : |u(t) - \phi(s)(t)| < \psi(s)(t)\}}$$

is nonempty. Then the map $H : S \to \mathcal{D}$ is l.s.c., and so it admits a continuous selection.

We now provide the main result of this subsection.

Theorem 6.9. *Assume the multivalued map $F : J \times \mathbb{R} \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$ satisfies $(\mathcal{B}_2)$ and*

$$(\mathcal{B}_4) \quad \begin{cases} \text{(a)} \quad (t, x) \mapsto F(t, x) \text{ is } \mathcal{L} \otimes \mathcal{B} \text{ measurable;} \\ \text{(b)} \quad x \mapsto F(t, x) \text{ is lower semi-continuous for a.e. } t \in J. \end{cases}$$

Then Problem (1)–(4) has at least one solution.

(*The following two lemmas will be fundamental in the proof of* Theorem 6.9.)

Lemma 6.10 (*See [59,77,61,62]*). *Let X be a separable metric space and let E be a Banach space. Then every l.s.c. multivalued operator $N : X \to \mathcal{P}_{cl}(L^1(J, E))$ with closed decomposable values has a continuous selection, i.e., there exists a continuous single-valued function $f : X \to L^1(J, E)$ such that $f(x) \in N(x)$ for every $x \in X$.*

Lemma 6.11 (*See [61,78]*). *Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be an integrably bounded multivalued function satisfying $(\mathcal{H}_2)$. Then F is of lower semi-continuous type.*

Proof of Theorem 6.9. From Lemmas 6.10 and 6.11, there exists a continuous selection function $f : PC \to L^1(J, \mathbb{R})$ such that $f(y)(t) \in \mathcal{F}(t, y)$ for every $y \in PC$ and a.e. $t \in J$. Next, consider the following impulsive fractional problem,

$$\begin{cases} D^\alpha_* y(t) = f(y)(t), \quad \text{a.e. } t \in J, \\ y(t_k^+) = I_k(y(t_k)), \quad k = 1, \dots, m, \\ y'(t_k^+) = \bar{I}_k(y(t_k)), \quad k = 1, \dots, m \\ y(0) = a, \qquad y'(0) = c. \end{cases} \tag{35}$$

Clearly, if $y \in PC$ is a solution of Problem (35), then y is a solution to Problem (1)–(4).
The remainder of the proof will be given in several steps.

Step 1: Consider the problem

$$D^\alpha_* y(t) = f(y)(t), \quad \text{a.e. } t \in [0, t_1], \tag{36}$$

$$y(0) = a, \qquad y'(0) = c. \tag{37}$$

Transform Problem (36)–(37) into a fixed point problem. Consider the operator $N : C([0, t_1], \mathbb{R}) \longrightarrow C([0, t_1], \mathbb{R})$, defined by

$$N(y)(t) = a + tc + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(y)(s)\mathrm{d}s, \quad t \in [0, t_1].$$

Since $\alpha \in (1, 2]$, we can easily prove that N is completely continuous; see Henderson and Ouahab [13] and Ouahab [51,79]. As in Theorem 6.1, if y is a possible solution of the equation $y = \lambda N(y)$, for some $\lambda \in (0, 1)$, there exists $K_0 > 0$ such that

$$\|y\|_\infty \le K_0.$$

Set

$$U_0 = \{y \in C([0, t_1], \mathbb{R}) : \sup\{|y(t)| : 0 \le t \le t_1\} < K_0 + 1\}.$$

As a consequence of the nonlinear alternative of Leray–Schauder type [74], we deduce that N has a fixed point y in U_0, which is a solution to Problem (36)–(37). Denote this solution by y_0.

Step 2: Consider now the problem

$$D_*^\alpha y(t) = f(y)(t), \quad \text{a.e. } t \in (t_1, t_2], \tag{38}$$

$$y(t_1^+) = I_1(y_0(t_1)), \tag{39}$$

$$y'(t_1^+) = \bar{I}_1(y_0(t_1)). \tag{40}$$

Let

$$PC_1 = \{y \in C((t_1, t_2], \mathbb{R}) : y(t_1^+) \text{ exists}\}.$$

Consider the operator $N_1 : PC_1 \to PC_1$ defined by

$$N_1(y)(t) = I_1(y_0(t_1^-)) + (t - t_1)\bar{I}_1(y_0(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f(y)(s)\mathrm{d}s, \quad t \in (t_1, t_2].$$

As in [13,51,79] and Theorem 6.1, we can show that N_1 is continuous and completely continuous, and if z is a possible solution of the equations $y = \lambda N_1(y)$ for some $\lambda \in (0, 1)$, there exists $K_1 > 0$ such that

$$\|y\|_{PC_1} \le K_1.$$

Set

$$U_1 = \{y \in PC_1 : \|z\|_{PC_1} < K_1 + 1\}.$$

As a consequence of the nonlinear alternative of Leray–Schauder type [74], we deduce that N_1 has a fixed point y which is a solution to Problem (38)–(40). Denote this solution by y_1.

Step 3: We continue this process, and taking into account that $y_m := y|_{[t_m, b]}$ is a solution to the problem

$$D_*^\alpha y(t) = f(y)(t), \quad \text{a.e. } t \in (t_m, b], \tag{41}$$

$$y(t_m^+) = I_m(y_{m-1}(t_m^-)), \tag{42}$$

$$y'(t_m^+) = \bar{I}_m(y'_{m-1}(t_m^-)). \tag{43}$$

The solution y of Problem (1)–(4) is then defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in [0, t_1], \\ y_1(t), & \text{if } t \in (t_1, t_2], \\ \dots \\ y_m(t), & \text{if } t \in (t_m, b]. \quad \square \end{cases}$$

6.3. Filippov's theorem

Our main result in this section is contained in the following theorem.

Theorem 6.12. *Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be a multivalued map. Assume that, in addition to $(\mathcal{B}_4)$, the following also hold:*

$(\mathcal{H}_1)$ *There exists a function $p \in L^1(J, \mathbb{R}^+)$ such that*

$$H_d(F(t, z_1), F(t, z_2)) < p(t)|z_1 - z_2| \quad \text{for all } z_1, z_2 \in \mathbb{R}^n.$$

($\mathcal{H}_2$) *There exists the continuous mapping $g(\cdot) : PC \to L^1(J, \mathbb{R})$ and $x \in PC$ such that*

$$\begin{cases} D^{\alpha}_{*}x(t) = g(x)(t), & a.e.\ t \in J \setminus \{t_1, \dots, t_m\}, \\ x(t_k^+) = I_k(x(t_k^-)), & k = 1, \dots, m, \\ x'(t_k^+) = \bar{I}_k(x(t_k^-)), & k = 1, \dots, m, \\ x(0) = \bar{a}, \quad x'(0) = \bar{c}, \end{cases} \tag{44}$$

and $d(g(x)(t), F(t, x(t))) < p(t)$, a.e. $t \in [0, b]$. If $\frac{2b^{\alpha}\|p\|_{L^1}}{\Gamma(\alpha)} < 1$, then Problem (1)–(4) has at least one solution y satisfying the estimates

$$\|x - y\|_{PC} \le \sum_{k=0}^{m} \frac{2\delta_k \widetilde{H}_k \|p\|_{L^1}}{\Gamma(\alpha)} + m\|p\|_{L^1}$$

and

$$|D^{\alpha}_{*}y(t) - g(x)(t)| \le 2p(t)\widetilde{H}_k + p(t), \quad t \in J_k, t \in [0, b],$$

where

$$\widetilde{H}_k = \frac{\delta_k \Gamma(\alpha)}{\Gamma(\alpha) - 2b^{\alpha}\|p\|_{L^1}} + \frac{b^{\alpha}\|p\|_{L^1}}{\Gamma(\alpha) - 2b^{\alpha}\|p\|_{L^1}}, \quad k = 0, \dots, m, \delta_0 = |a - \bar{a}| + b|c - \bar{c}|,$$

and

$$\delta_k = |I_k(x(t_k)) - I_k(y(t_k))| + (t_k - t_{k+1})|\bar{I}_k(x(t_k)) - \bar{I}_k(y(t_k))|, \quad k = 1, \dots, m.$$

Proof. We are going to study Problem (1)–(4) in the respective intervals $[0, t_1], (t_1, t_2], \dots, (t_m, b]$. The proof will be given in three steps and then continued by induction.

Step 1. In this first step, we construct a sequence of functions $(y_n)_{n\in\mathbb{N}}$ which will be shown to converge to some solution of Problem (1)–(4) on the interval $[0, t_1]$, namely to

$$\begin{cases} D^{\alpha}_{*}y(t) \in F(t, y(t)), & t \in J_0 = [0, t_1], \alpha \in (1, 2], \\ y(0) = a, & y'(0) = c. \end{cases} \tag{45}$$

Let $f_0(y_0)(t) = g(x)(t)$, $t \in [0, t_1]$, and

$$y_0(t) = \bar{a} + \bar{c}t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f_0(y_0)(s)ds.$$

Let $G_1 : C([0, t_1], \mathbb{R}) \to \mathcal{P}(L^1([0, t_1], \mathbb{R}))$ be given by

$$G_1(y) = \{v \in L^1([0, t_1], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e. } t \in [0, t_1]\},$$

and $\widetilde{G}_1 : C([0, t_1], \mathbb{R}) \to \mathcal{P}(L^1([0, t_1], \mathbb{R}))$ be defined by

$$\widetilde{G}_1(y) = \overline{\{v \in G_1(y) : |v(t) - g(y_0)(t)| < p(t)|y(t) - y_0(t)| + p(t)\}}.$$

Since $t \to F(t, y(t))$ is measurable multifunction, and from Corollary 2.3, there exists a function v_1 which is a measurable selection of $F(t, y(t))$, a.e. $t \in [0, t_1]$, and such that

$$\begin{aligned} |v_1(t) - g(y_0)(t)| &\le d(g(y_0)(t), F(t, y(t))) \\ &< p(t) + p(t)|y(t) - y_0(t)|. \end{aligned}$$

Then $v_1 \in \widetilde{G}_1(y) \ne \emptyset$. By Lemma 6.11, F is of lower semi-continuous type. Then $y \to G_1(y)$ is l.s.c. and has decomposable values. So $y \to \widetilde{G}_1(y)$ is l.s.c. with decomposable values from $C([0, t_1], \mathbb{R}) \to \mathcal{P}(L^1([0, t_1], \mathbb{R}))$.

Then from Lemma 6.10 and Proposition 6.8, there exists a continuous function $f_1 : C([0, t_1], \mathbb{R}) \to L^1([0, t_1], \mathbb{R})$ such that $f_1(y) \in \widetilde{G}_1(y)$ for all $y \in PC_1$. Consider the problem

$$D^{\alpha}_{*}y(t) = f_1(y)(t), \quad t \in [0, t_1], \tag{46}$$

$$y(0) = a, \qquad y'(0) = c. \tag{47}$$

From Theorem 6.9, Problem (46)–(47) has at least one solution, which we denote by y_1.

Hence

$$y_1(t) = a + ct + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f_1(y_1)(s)ds, \quad t \in [0, t_1],$$

where y_1 is a solution of Problem (46)–(47). For every $t \in J$, we have

$$\begin{aligned}|y_1(t) - y_0(t)| &\le |a - \bar{a}| + t_1|c - \bar{c}| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f_1(y_1)(s) - f_0(y_0)(s)|\, ds \\ &\le \delta_0 + \frac{b^\alpha}{\Gamma(\alpha)} \int_0^t p(s)|y_1(s) - y_0(s)| ds + \frac{b^\alpha}{\Gamma(\alpha)} \|p\|_{L^1}.\end{aligned}$$

Then,

$$\|y_1 - y_0\|_\infty \le \frac{\delta_0}{1 - \frac{b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)}} + \frac{b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)\left(1 - \frac{b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)}\right)}.$$

Define the set-valued map $G_2 : C([0, t_1], \mathbb{R}) \to \mathcal{P}(L^1([0, t_1], \mathbb{R}))$ by

$$G_2(y) = \{v \in L^1([0, t_1], \mathbb{R}) : v(t) \in F(t, y(t)),\ \text{a.e. } t \in [0, t_1]\},$$

and

$$\widetilde{G}_2(y) = \overline{\{v \in G_2(y) : |v(t) - f_1(y_1)(t)| < p(t)|y(t) - y_1(t)| + p(t)|y_0(t) - y_1(t)|\}}.$$

Since $t \to F(t, y(t))$ is measurable, and from Corollary 2.3, there exists a function $v_2 \in \widetilde{G}_2$ which is a measurable selection of $F(t, y_1(t))$, a.e. $t \in J$, and such that

$$\begin{aligned}|v_2(t) - f_1(y_1)(t)| &\le d(f_1(y_1)(t), F(t, y(t))) \\ &\le H_d(f_1(y_1)(t), F(t, y(t))) \\ &\le p(t)|y_1(t) - y(t)| \\ &< p(t)|y(t) - y_1(t)| + p(t)|y_1(t) - y_0(t)|.\end{aligned}$$

Then $v_2 \in \widetilde{G}_2(y) \neq \emptyset$. Using the above method, we can prove that $\widetilde{G}_2$ has at least one continuous selection, denoted by f_2. Then there exists $y_2 \in C([0, t_1], \mathbb{R})$ such that

$$y_2(t) = a + tc + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(y_2)(s) ds, \quad t \in [0, t_1],$$

and y_2 is a solution of the problem

$$\begin{cases} D_*^\alpha y(t) = f_2(y)(t), & \text{a.e. } t \in [0, t_1], \\ y(0) = a, \quad y'(0) = c. \end{cases} \tag{48}$$

We then have

$$\begin{aligned}|y_2(t) - y_1(t)| &\le \frac{b^\alpha}{\Gamma(\alpha)} \int_0^t |f_2(y_2)(s) - f_1(y_1)(s)|\, ds \\ &\le \frac{b^\alpha}{\Gamma(\alpha)} \int_0^t p(s)|y_2(s) - y_1(s)| + p(s)|y_1(s) - y_0(s)| ds.\end{aligned}$$

Thus,

$$\|y_2 - y_1\|_\infty \le \frac{\delta_0 b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)\left(1 - \frac{b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)}\right)^2} + \frac{b^{2\alpha} \|p\|_{L^1}^2}{\Gamma^2(\alpha)\left(1 - \frac{b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)}\right)^2}.$$

Let

$$G_3(y) = \{v \in L^1([0, t_1], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e. } t \in [0, t_1]\},$$

and

$$\widetilde{G}_3(y) = \overline{\{v \in G_3(y) : |v(t) - f_2(y_2)(t)| < p(t)|y(t) - y_2(t)| + p(t)|y_1(t) - y_2(t)|\}}.$$

Arguing as we did for $\widetilde{G}_2$ shows that $\widetilde{G}_3$ is an l.s.c. type multivalued map with nonempty decomposable values, so there exists a continuous selection $f_3(y) \in \widetilde{G}_3(y)$, for all $y \in PC$. Consider

$$y_3(t) = a + tc + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_3(y_3)(s) ds, \quad t \in [0, t_1],$$

where y_3 is a solution of the problem

$$\begin{cases} D_*^\alpha y(t) = f_3(y)(t), & \text{a.e. } t \in [0, t_1], \\ y(0) = a, \quad y'(0) = c. \end{cases} \tag{49}$$

We have

$$|y_3(t)-y_2(t)| \le \frac{b^{\alpha}}{\Gamma(\alpha)}\int_0^t |f_3(y_3)(s)-f_2(y_2)(s)|\,\mathrm{d}s.$$

Hence, from the estimates above, we have

$$\|y_3-y_2\|_{\infty} \le \frac{\delta_0 b^{2\alpha}\|p\|_{L^1}^2}{\Gamma^2(\alpha)\left(1-\frac{b^{\alpha}\|p\|_{L^1}}{\Gamma(\alpha)}\right)^3}+\frac{b^{3\alpha}\|p\|_{L^1}^3}{\Gamma^3(\alpha)\left(1-\frac{b^{\alpha}\|p\|_{L^1}}{\Gamma(\alpha)}\right)^3}.$$

Repeating the process for $n=1,2,\ldots,$ we arrive at the bound

$$\|y_n-y_{n-1}\|_{\infty} \le \frac{\delta_0 b^{(n-1)\alpha}\|p\|_{L^1}^{n-1}}{\Gamma^{n-1}(\alpha)\left(1-\frac{b^{\alpha}\|p\|_{L^1}}{\Gamma(\alpha)}\right)^n}+\frac{b^{n\alpha}\|p\|_{L^1}^n}{\Gamma^n(\alpha)\left(1-\frac{b^{\alpha}\|p\|_{L^1}}{\Gamma(\alpha)}\right)^n}. \tag{50}$$

By induction, suppose that (50) holds for some n. Let

$$\widetilde{G}_{n+1}(y)=\overline{\{v\in G_{n+1}(y):|v(t)-f_n(y_n)(t)|<p(t)|y(t)-y_n(t)|+p(t)|y_n(t)-y_{n-1}(t)|\}}.$$

Since again $\widetilde{G}_{n+1}$ is an l.s.c. type multifunction, there exists a continuous function $f_{n+1}(y)\in\widetilde{G}_{n+1}(y)$ which allows us to define

$$y_{n+1}(t)=a+tc+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_{n+1}(y_{n+1})(s)\mathrm{d}s,\quad t\in[0,t_1]. \tag{51}$$

Therefore,

$$|y_{n+1}(t)-y_n(t)| \le \frac{b^{\alpha}}{\Gamma(\alpha)}\int_0^t |f_{n+1}(y_{n+1})(s)-f_n(y_n)(s)|\,\mathrm{d}s.$$

Thus, we arrive at

$$\|y_{n+1}-y_n\|_{\infty} \le \frac{\delta_0 b^{n\alpha}\|p\|_{L^1}^{n}}{\Gamma^{n}(\alpha)\left(1-\frac{b^{\alpha}\|p\|_{L^1}}{\Gamma(\alpha)}\right)^{n+1}}+\frac{b^{(n+1)\alpha}\|p\|_{L^1}^{n+1}}{\Gamma^{n+1}(\alpha)\left(1-\frac{b^{\alpha}\|p\|_{L^1}}{\Gamma(\alpha)}\right)^{n+1}}. \tag{52}$$

Hence, (50) holds for all $n\in\mathbb{N}$, and so $\{y_n\}$ is a Cauchy sequence in $C([0,t_1],\mathbb{R})$, converging uniformly to a function $y\in C([0,t_1],\mathbb{R})$. Moreover, from the definition of $U_n,\ n\in\mathbb{N}$, we have, for a.e. $t\in[0,t_1]$,

$$|f_{n+1}(y_{n+1})(t)-f_n(y_n)(t)| \le p(t)|y_{n+1}(t)-y_n(t)|+p(t)|y_n(t)-y_{n-1}(t)|.$$

Therefore, for almost every $t\in[0,t_1]$, $\{f_n(y_n)(t):n\in\mathbb{N}\}$ is also a Cauchy sequence in $\mathbb{R}$ and so converges almost everywhere to some measurable function $f(\cdot)$ in $\mathbb{R}$. Moreover, since $f_0=g$, we have

$$\begin{aligned}
|f_n(y_n)(t)| &\le |f_n(y_n)(t)-f_{n-1}(y_{n-1})(t)|+|f_{n-1}(y_{n-1})(t)-f_{n-2}(y_{n-2})(t)|+\cdots\\
&\quad+|f_2(y_2)(t)-f_1(y_1)(t)|+|f_1(y_1)(t)-f_0(y_0)(t)|+|f_0(y_0)(t)|\\
&\le 2\sum_{k=1}^{n}p(t)|y_k(t)-y_{k-1}(t)|+|f_0(y_0)(t)|+p(t)\\
&\le 2p(t)\sum_{k=1}^{\infty}|y_k(t)-y_{k-1}(t)|+|g(x)(t)|+p(t)\\
&\le 2\widetilde{H}_0p(t)+|g(x)(t)|+p(t).
\end{aligned}$$

Then, for all $n\in\mathbb{N}$,

$$|f_n(y_n)(t)| \le 2\widetilde{H}_0p(t)+g(x)(t)+p(t)\quad \text{a.e. } t\in[0,t_1]. \tag{53}$$

From (53) and the Lebesgue dominated convergence theorem, we conclude that $f_n(y_n)$ converges to $f(y)$ in $L^1([0,t_1],\mathbb{R})$. Passing to the limit in (51), the function

$$y(t)=a+tc+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(y)(s)\mathrm{d}s,\quad t\in[0,t_1],$$

is a solution to Problem (1)–(4).

Next, we give estimates for $|D_*^{\alpha}y(t)-g(x)(t)|$ and $|x(t)-y(t)|$. We have

$$\begin{aligned}
|D_*^{\alpha}y(t)-g(x)(t)| &= |f(y)(t)-f_0(x)(t)|\\
&\le |f(y)(t)-f_n(y_n)(t)|+|f_n(y_n)(t)-f_0(x)(t)|
\end{aligned}$$

$$\le |f(y)(t) - f_n(y_n)(t)| + \sum_{k=1}^{n} |f_k(y_k)(t) - f_{k-1}(y_{k-1})(t)|$$

$$\le |f(y)(t) - f_n(y_n)(t)| + 2\sum_{k=1}^{n} p(t)|y_k(t) - y_{k-1}(t)| + p(t).$$

Using (52) and passing to the limit as $n \to +\infty$, we obtain

$$\begin{aligned}|D_*^\alpha y(t) - g(x)(t)| &\le 2p(t)\sum_{k=1}^{\infty} |y_{k-1}(t) - y_k(t)| \\ &\le 2p(t)\sum_{k=1}^{\infty} \frac{\delta_0 (b^\alpha \|p\|_{L^1})^{k-1}}{\Gamma^{k-1}(\alpha)\left(1 - \frac{b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)}\right)^k} + 2p(t)\sum_{k=1}^{\infty} \frac{(b^\alpha \|p\|_{L^1})^{k}}{\Gamma^{k}(\alpha)\left(1 - \frac{b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)}\right)^k} + p(t),\end{aligned}$$

so

$$|D_*^\alpha y(t) - g(x)(t)| \le 2\widetilde{H}_0 p(t) + p(t), \quad t \in [0, t_1].$$

Similarly,

$$\begin{aligned}|x(t) - y(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(x)(s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(y)(s)\mathrm{d}s\right| \\ &\le \frac{b^\alpha}{\Gamma(\alpha)}\int_0^{t_1} |f(y)(s) - f_n(y_n)(s)|\mathrm{d}s + \frac{b^\alpha}{\Gamma(\alpha)}\int_0^{t_1} |f_n(y_n)(s) - f_0(y_0)(s)|\mathrm{d}s.\end{aligned}$$

As $n \to \infty$, we arrive at

$$\|x - y\|_\infty \le 2\frac{\widetilde{H}_0 b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)} + \|p\|_{L^1}.$$

The obtained solution is denoted by $y_1 := y_{|[0,t_1]}$.

Step 2: Consider now Problem (1)–(4) on the second interval $(t_1, t_2]$, i.e.,

$$\begin{cases} D_*^\alpha y(t) \in F(t, y(t)), & \text{a.e. } t \in (t_1, t_2], \\ y(t_1^+) = I_1(y_1(t_1)), \\ y'(t_1^+) = \bar{I}_1(y_1(t_1)). \end{cases} \tag{54}$$

Let $f_0 = g$, and set

$$y^0(t) = I_1(x(t_1)) + (t - t_1)\bar{I}_1(x(t_1)) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1} f_0(y_0)(s)\mathrm{d}s, \quad t \in (t_1, t_2].$$

Let

$$PC_1 = \{y\colon y \in C(t_1, t_2] \text{ and } y(t_1^+) \text{ exists}\}.$$

As in Step 1, let the multivalued map $G_1\colon PC_1 \to \mathcal{P}(L^1([t_1, t_2], \mathbb{R}))$ be given by

$$G_1(y) = \{v \in L^1([t_1, t_2], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\}.$$

Then it has a continuous selection in $\widetilde{G}_1 : PC_1 \to \mathcal{P}(L^1([t_1, t_2], \mathbb{R}))$, defined by

$$\widetilde{G}_1(y) = \overline{\{v \in G_1(y) : |v(t) - g(y_0)(t)| < p(t)|y(t) - y_0(t)| + p(t)\}}.$$

Define

$$y^1(t) = I_1(y_1(t_1)) + (t - t_1)\bar{I}_1(y_1(t_1)) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1} f_1(y^1)(s)\mathrm{d}s, \quad t \in (t_1, t_2].$$

Next define $G_2 : PC_1 \to \mathcal{P}(L^1([t_1, t_2], \mathbb{R}))$ by

$$G_2(y) = \{v \in L^1([t_1, t_2], \mathbb{R}) : v(t) \in F(t, y(t)), \text{ a.e. } t \in [t_1, t_2]\}$$

and

$$\widetilde{G}_2(y) = \overline{\left\{v \in G_2(y) : |v(t) - f_1(y_1)(t)| < p(t)|y(t) - y^1(t)| + p(t)|y^0(t) - y^1(t)|\right\}}.$$

It has a continuous selection $f_2(y) \in \widetilde{G}_2(y)$. Repeating the process of selection as in Step 1, we can define by induction a sequence of multivalued maps

$$\widetilde{G}_{n+1}(y) = \overline{\{v \in G_{n+1}(y) : |v(t) - f_n(y_n)(t)| < p(t)|y(t) - y^n(t)| + p(t)|y^n(t) - y^{n-1}(t)|\}}.$$

Since again $\widetilde{G}_{n+1}$ is an l.s.c. type multifunction, there exists a continuous function $f_{n+1}(y) \in \widetilde{G}_{n+1}(y)$ which allows us to define

$$y^{n+1}(t) = I_1(y_1(t_1)) + (t - t_1)\bar{I}_1(y_1(t_1)) \tag{55}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f_{n+1}(y^{n+1})(s)\mathrm{d}s, \quad t \in (t_1, t_2], \tag{56}$$

and we can easily prove that

$$|y^{n+1}(t) - y^n(t)| \leq \frac{\delta_1 b^{n\alpha} \|p\|_{L^1}^n}{\Gamma^n(\alpha)\left(1 - \frac{b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)}\right)^{n+1}} + \frac{b^{(n+1)\alpha} \|p\|_{L^1}^{n+1}}{\Gamma^{n+1}(\alpha)\left(1 - \frac{b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)}\right)^{n+1}}, \quad t \in (t_1, t_2],$$

where

$$\delta_1 = |I_1(x(t_1)) - I_1(y_1(t_1))| + (t_2 - t_1)|\bar{I}_1(x(t_1)) - \bar{I}_1(y_1(t_1))|.$$

As in Step 1, we can prove that the sequence $\{y^n\}$ converges to some $y \in PC_1$, a solution to Problem (54) such that, for a.e. $t \in (t_1, t_2]$, we have

$$|D_*^\alpha y(t) - g(x)(t)| := |f(y)(t) - f_0(y_0)(t)| \leq 2\widetilde{H}_1 p(t) + p(t).$$

and

$$|x(t) - y(t)| \leq \frac{2\widetilde{H}_1 b^\alpha \delta_1 \|p\|_{L^1}}{\Gamma(\alpha)} + \|p\|_{L^1}.$$

Denote the restriction $y_{|(t_1,t_2]}$ by y_2.

Step 3: We continue this process until we arrive at the function $y_{m+1} := y|_{(t_m,b]}$, a solution of the problem

$$\begin{cases} D_*^\alpha y(t) \in F(t, y(t)), & \text{a.e. } t \in (t_m, b], \\ y(t_m^+) = I_m(y_{m-1}(t_m)), \\ y'(t_m^+) = \bar{I}_m(y(t_m)). \end{cases}$$

Then, for a.e. $t \in (t_m, b]$, the following estimates are easily derived:

$$|D_*^\alpha y(t) - g(t)| \leq 2\widetilde{H}_m p(t) + p(t), \quad t \in (t_1, t_2]$$

and

$$|x(t) - y(t)| \leq \frac{2\widetilde{H}_m b^\alpha \|p\|_{L^1}}{\Gamma(\alpha)} + \|p\|_{L^1}.$$

Step 4: Summarizing, a solution y of Problem (1)–(4) can be defined as

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, t_2], \\ \ldots & \ldots \\ y_{m+1}(t), & \text{if } t \in (t_m, b]. \end{cases}$$

From Steps 1 to 3, we have that, for a.e. $t \in [0, t_1]$,

$$|x(t) - y(t)| \leq \frac{2\delta_0 b^\alpha \widetilde{H}_0 \|p\|_{L^1}}{\Gamma(\alpha)} + \|p\|_{L^1} \quad \text{and} \quad |D_*^\alpha y(t) - g(x)(t)| \leq 2\widetilde{H}_0 p(t) + p(t),$$

as well as the estimates, valid for $t \in (t_1, b]$,

$$|x(t) - y(t)| \leq \sum_{k=1}^{m} \frac{2\delta_k b^\alpha \widetilde{H}_k \|p\|_{L^1}}{\Gamma(\alpha)} + m\|p\|_{L^1}.$$

Similarly,

$$|D_*^\alpha y(t) - g(x)(t)| \leq 2p(t)\widetilde{H}_k + p(t), \quad t \in J_k, \; k = 1, \ldots m.$$

The proof of Theorem 6.12 is complete. □

6.4. Appendix

The reasoning used in [13,80,51,81] combined with Section 6 can be applied to obtain existence and uniqueness results for the following fractional impulsive problem:

$$D_*^\alpha y(t) = f(t, y(t)), \quad \text{a.e. } t \in J := [0, b] \setminus \{t_1, \ldots, m\}, \; \alpha \in (1, 2], \tag{57}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, 2, \ldots, m, \tag{58}$$

$$y'(t_k^+) = \bar{I}_k(y(t_k^-)), \quad k = 1, 2, \ldots, m, \tag{59}$$

$$y(0) = a, \qquad y'(0) = c, \tag{60}$$

where $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a given function and $I_k, \bar{I}_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, \ldots, m$.

Theorem 6.13. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Assume the condition*

($\mathcal{U}_1$) *There exist a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ and $p \in L^1([0, b], \mathbb{R}_+)$ such that*

$$\|f(t, x)\| \le p(t)\psi(|x|) \quad \text{for a.e. } t \in [0, b] \text{ and each } x \in \mathbb{R},$$

with

$$\int_0^b p(s)\mathrm{d}s < \int_{\bar{c}}^{\infty} \frac{\mathrm{d}x}{\psi(x)},$$

where $\bar{c} = a + b|c|$.

Then the initial-value problem (57)–(60) has at least one solution.

Proof. For the proof, we can restrict Problem (57)–(60) in the respective intervals $[0, t_1]$, $(t_1, t_2], \ldots, (t_m, b]$, and then apply the same method as in [43] or the method in the proof of Theorem 6.9. □

We next introduce some additional conditions that lead to the uniqueness of the solution of (57)–(60).

Theorem 6.14. *Assume that there exists $l \in L^1([0, b], \mathbb{R}_+)$ such that*

$$|f(t, x) - f(t, \bar{x})| \le l(t)|x - \bar{x}|, \quad \text{for all } x, \bar{x} \in \mathbb{R} \text{ and } t \in J.$$

Then the IVP (57)–(60) has a unique solution.

Proof. We consider the problem (57)–(60) on $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \ldots, J_m = (t_m, b]$. Successively applying the condition of the theorem, we can prove the existence and uniqueness of a solution in each of the intervals J_k, $k = 0, \ldots, m$, and by the continuity of the functions $I_k, \bar{I}_k$, we conclude the existence and uniqueness on $[0, b]$. For more details, see Benchohra et al. [43] and Henderson and Ouahab [13]. □

7. Geometric structure of solution sets

7.1. Background in geometric topology

First, we start with some elementary notions and notations from geometric topology. For details, we recommend [82,74,83,84,55,85,86]. In what follows, (X, d) and (Y, d') stand for two metric spaces.

Definition 7.1. A set $A \in \mathcal{P}(X)$ is called a contractible space provided there exists a continuous homotopy $h : A \times [0, 1] \to A$ and $x_0 \in A$ such that

(a) $h(x, 0) = x$, for every $x \in A$,
(b) $h(x, 1) = x_0$, for every $x \in A$,

i.e., if the identity map $A \longrightarrow A$ is homotopic to a constant map (A is homotopically equivalent to a point).

Note that if $A \in \mathcal{P}_{cv,cl}(X)$, then A is contractible. Also the class of contractible sets is much larger than the class of closed convex sets.

Definition 7.2. A compact nonempty space X is called an R_δ set provided there exists a decreasing sequence of compact nonempty contractible spaces $\{X_n\}$ such that $X = \bigcap_{n=1}^{\infty} X_n$.

Definition 7.3. A space X is called an absolute retract (in short $X \in AR$) provided that, for every space Y, every closed subset $B \subseteq Y$ and any continuous map $f : B \to X$, there exists a continuous extension $\widetilde{f} : Y \to X$ of f over Y, i.e., $\widetilde{f}(x) = f(x)$ for every $x \in B$. In other words, for every space Y and for any embedding $f : X \longrightarrow Y$, the set $f(X)$ is a retract of Y.

From Proposition 2.15 in [87], if $X \in AR$, then it is a contractible space.

Definition 7.4. A space A is called acyclic if

(a) $H_0(A) = \mathbb{Q}$,
(b) $H_n(A) = 0$, for every $n > 0$,

where $H_* = \{H_n\}_{n\geq 0}$ is the Čech-homology functor with compact carriers and coefficients in the field of rationals $\mathbb{Q}$. In other words, a space A is acyclic if the map $j : \{p\} \to X,\ j(p) = x_0 \in A$, induces an isomorphism $j_* : H_*(\{p\}) \to H_*(A)$.

From the continuity of Čech-homology functors, we have:

Lemma 7.5 (*[83]*). *Let X be a compact metric space. Then X is an acyclic space whose structure corresponds to one of the following types:*

1. *X is convex,*
2. *X is contractible,*
3. *X is AR,*
4. *X is an R_δ set.*

Next, we present a result about the topological structure of the set of solutions of some nonlinear functional equations due to Aronszajn and developed by Browder and Gupta in [88] (see also Theorem 1.2 in [87]).

Theorem 7.6. *Let X be a space, $(E, \|\cdot\|)$ a Banach space and $f : X \to E$ a proper map, i.e., f is continuous, and for every compact $K \subset E$, the set $f^{-1}(K)$ is compact. Assume further that, for each $\varepsilon > 0$, a proper map $f_\varepsilon : X \to E$ is given and the following two conditions are satisfied:*

(a) *$\|f_\varepsilon(x) - f(x)\| < \varepsilon$, for every $x \in X$,*
(b) *for every $\varepsilon > 0$ and $u \in E$ in a neighborhood of the origin such that $\|u\| \leq \varepsilon$, the equation $f_\varepsilon(x) = u$ has exactly one solution, x_k.*

Then the set $S = f^{-1}(0)$ is an R_δ set.

The following Lasota–Yorke approximation theorem (see [55]) will be needed in this section.

Lemma 7.7. *Let E be a normed space, X a metric space and $f : X \to E$ be a continuous map. Then, for each $\varepsilon > 0$, there is a locally Lipschitz map $f_\varepsilon : X \to E$ such that*

$$\|f(x) - f_\varepsilon(x)\| < \varepsilon, \quad \text{for every } x \in X.$$

7.2. Application

Consider the first-order impulsive single-valued problem:

$$\begin{cases} D_*^\alpha y(t) = f(t, y(t)), & \text{a.e. } t \in J = [t_0, b] \setminus \{t_1, \dots, t_m\}, \\ y(t_k^-) = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y'(t_k^+) = \bar{I}_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = a, \qquad y'(0) = c, \end{cases} \tag{61}$$

where f, I_k and $\bar{I}_k$ are given functions.

Denote by $S(f, \phi)$ the set of all solutions of Problem (61). We are in a position to state and prove an Aronszajn type result for this problem. First, we list two assumptions:

($\mathcal{C}_1$) $f : J \times \mathbb{R} \to \mathbb{R}$ is an Carathéodory function.
($\mathcal{C}_2$) There exist a function $p \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\rho : [0, \infty) \to [0, \infty)$ such that

$$|f(t, x)| \leq p(t)\rho(|x|) \quad \text{for a.e. } t \in J \text{ and each } x \in \mathbb{R}$$

with

$$\int_0^b p(s)\mathrm{d}s < \int_{|a|+t_1|c|}^\infty \frac{\mathrm{d}u}{\rho(u)}.$$

($\mathcal{C}_3$) There exist constants $r_k > 0$ and continuous functions ϕ_k: $\mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|\bar{I}_k(x)|,\ |I_k(x)| \leq r_k\phi_k(|x|) \quad \text{for each } x \in \mathbb{R}, k = 1, \dots, m.$$

Then, the main first result in this section is

Theorem 7.8. *Assume that Assumptions ($\mathcal{C}_1$)–($\mathcal{C}_3$) hold. Then the set $S(f, a, c)$ is an R_δ set, and hence an acyclic space.*

Proof. Let $F : \Omega \to \Omega$ be defined by

$$F(y)(t) = \begin{cases} F_0(y)(t), & t \in [0, t_1], \\ F_1(y)(t), & t \in (t_1, t_2], \\ \vdots \\ F_k(y)(t), & t \in (t_k, t_{k+1}], \\ \vdots \\ F_m(y)(t), & t \in (t_m, b], \end{cases}$$

where

$$F_0(y)(t) = a + tc + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(y)(s)\mathrm{d}s, \quad t \in [0, t_1],$$

and

$$\begin{aligned} F_k(y)(t) &= I_k(F_{k-1}(y)(t_k)) + (t - t_k)\bar{I}_k(F_{k-1}(y)(t_k)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} f(y)(s)\mathrm{d}s, \quad t \in (t_k, t_{k+1}],\ k = 1, \ldots, m. \end{aligned}$$

From $(\mathcal{C}_1)$–$(\mathcal{C}_3)$ we can easily prove that F has at least one fixed point which is a solution of Problem (61).

Thus $\mathit{Fix}\, F = S(f, a, c) \neq \emptyset$ and there exists $\overline{M} > 0$ such that

$$\|y\|_{PC} \leq \overline{M}, \quad \text{for every } y \in S(f, a, c).$$

Define

$$\widetilde{f}(t, y(t)) = \begin{cases} f(t, y(t)), & \text{if } |y(t)| \leq \overline{M}, \\ f\left(t, \dfrac{\overline{M} y(t)}{|y(t)|}\right), & \text{if } |y| \geq \overline{M}. \end{cases}$$

Since f is L^1-Carathéodory, the function $\widetilde{f}$ is Carathéodory and is integrably bounded by $(\mathcal{C}_2)$. So there exists $h \in L^1(J, \mathbb{R}^+)$ such that

$$\|\widetilde{f}(t, x)\| \leq h(t), \quad \text{a.e. } t \text{ and all } x \in \mathbb{R}. \tag{62}$$

Consider the modified problem

$$\begin{cases} D_*^\alpha y(t) = \widetilde{f}(t, y(t)), & \text{a.e. } t \in J \setminus \{t_1, \ldots, t_m\}, \\ y(t_k^+) = I_k(y(t_k^-)), & k = 1, \ldots, m, \\ y'(t_k^-) = \bar{I}_k(y(t_k^-)), & k = 1, \ldots, m, \\ y(0) = a, \quad y'(0) = c. \end{cases}$$

We can easily prove that $S(f, a, c) = S(\widetilde{f}, a, c) = \mathit{Fix}\widetilde{F}$, where $\widetilde{F} : \Omega \to \Omega$ is as defined by

$$\widetilde{F}(y)(t) = \begin{cases} \widetilde{F}_0(y)(t), & t \in [0, t_1], \\ \widetilde{F}_1(y)(t), & t \in (t_1, t_2], \\ \vdots \\ \widetilde{F}_k(y)(t), & t \in (t_k, t_{k+1}], \\ \vdots \\ \widetilde{F}_m(y)(t), & t \in (t_m, b]. \end{cases}$$

By the inequality (62) and the continuity of I_k and $\bar{I}_k$, we deduce that there exists $R > 0$ such that

$$\|\widetilde{F}(y)\|_{PC} \leq R.$$

Then $\widetilde{F}$ is uniformly bounded. As in Theorem 6.1, we can prove that $\widetilde{F} : PC \to PC$ is compact, which allows us to define the compact perturbation of the identity $\widetilde{G}(y) = y - \widetilde{F}(y)$ which is a proper map. From the compactness of $\widetilde{F}$ and the Lasota–Yorke approximation theorem, we can easily prove that all conditions of Theorem 7.6 are met. Therefore the solution set $S(\widetilde{f}, a, c) = \widetilde{G}^{-1}(0)$ is an R_δ set, and hence an acyclic space by Lemma 7.5. $\square$

7.3. σ-selectionable multivalued maps

The following definitions and the result can be found in [55,89] (see also p. 86 in [59]). Let (X, d) and (Y, d') be two metric spaces.

Definition 7.9. We say that a map $F : X \to \mathcal{P}(Y)$ is σ-Ca-selectionable if there exists a decreasing sequence of compact-valued u.s.c. maps $F_n : X \to Y$ satisfying

(a) F_n has a Carathéodory selection, for all $n \geq 0$ (F_n are called Ca-selectionable),
(b) $F(x) = \bigcap_{n\geq 0} F_n(x)$, for all $x \in X$.

Definition 7.10. (a) A single-valued map $f : [0, a] \times X \to Y$ is said to be measurable-locally-Lipschitz (mLL) if $f(\cdot, x)$ is measurable for every $x \in X$ and, for every $x \in X$, there exists a neighborhood V_x of $x \in X$ and an integrable function $L_x : [0, a] \to [0, \infty)$ such that

$$d'(f(t, x_1), f(t, x_2)) \leq L_x(t) d(x_1, x_2) \quad \text{for every } t \in [0, a] \text{ and } x_1, x_2 \in V_x.$$

(b) A multivalued mapping $F : [0, a] \times X \to \mathcal{P}(Y)$ is mLL-selectionable if it has an mLL-selection.

Definition 7.11. We say that a multivalued map $\phi : [0, a] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ with closed values is upper-Scorza–Dragoni if, given $\delta > 0$, there exists a closed subset $A_\delta \subset [0, 1]$ such that the measure $\mu([0, a] \setminus A_\delta) \leq \delta$ and the restriction ϕ_δ of ϕ to $A_\delta \times \mathbb{R}^n$ is u.s.c.

Theorem 7.12 (*See Theorem 19.19 in [55]*). *Let E, E_1 be two separable Banach spaces and let $F : [a, b] \times E \to \mathcal{P}_{cp,cv}(E_1)$ be an upper-Scorza–Dragoni map. Then F is σ-Ca-selectionable, the maps $F_n : [a, b] \times E \to \mathcal{P}(E_1) (n \in \mathbb{N})$ are almost upper semi-continuous, and we have*

$$F_n(t, e) \subset \overline{conv}(\cup_{x\in E} F_n(t, x)).$$

Moreover, if F is integrably bounded, then F is σ-mLL-selectionable.

Let $S(F, a, c)$ denote the set of all solutions of Problem (1)–(4). Now, we are in position to state and prove another characterization of the geometric structure of $S(F, a, c)$. Let us introduce the following hypothesis:

($\overline{\mathcal{H}}_3$) There exist constants $c_k, \bar{c}_k > 0$ such that

$$|I_k(u) - I_k(\overline{u})| \leq c_k|u - \overline{u}|, \qquad |\bar{I}_k(u) - \bar{I}_k(\overline{u})| \leq \bar{c}_k|u - \overline{u}| \quad k = 1, \ldots, m, \text{ for all } u, \overline{u} \in \mathbb{R}.$$

Theorem 7.13. *Let $F: J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ be a Carathéodory and an mLL-selectionable multivalued map which satisfies Conditions ($\mathcal{B}_2$) and ($\overline{\mathcal{H}}_3$), with $\sum_{k=1}^{k=m}[c_k + (t_{k+1} - t_k)\bar{c}_k] < 1$. Then, for every $a, c \in \mathbb{R}$, the set $S(F, a, c)$ is contractible.*

Proof. Let $f \subset F$ be a measurable, locally Lipschitz selection and consider the single-valued problem

$$\begin{cases} D_*^\alpha y(t) = f(t, y(t)), & \text{a.e. } t \in J \setminus \{t_1, t_2, \ldots, t_m\}, \\ y(t_k^+) = I_k(y(t_k^-)), & k = 1, \ldots, m, \\ y'(t_k^+) = \bar{I}_k(y(t_k^-)), & k = 1, \ldots, m, \\ y(0) = a, \quad y(0) = c. \end{cases} \tag{63}$$

Since $\alpha \in (1, 2]$, as in [49] combined with [13,51], we can prove that Problem (63) has exactly one solution for every $a, c \in \mathbb{R}$. From Theorem 6.1, $S(F, a, c)$ is compact. Define the homotopy $h : S(F, a, c) \times [0, 1] \to S(F, a, c)$ by

$$h(y, \beta)(t) = \begin{cases} y(t), & \text{for } 0 \leq t \leq \beta b, \\ \overline{x}(t), & \text{for } \beta b < t \leq b, \end{cases}$$

where $\overline{x} = S(f, a, c)$ is the unique solution of Problem (63). In particular,

$$h(y, \alpha) = \begin{cases} y, & \text{for } \alpha = 1, \\ \overline{x}, & \text{for } \alpha = 0. \end{cases}$$

We prove that h is a continuous homotopy. Let $(y_n, \alpha_n) \in S(F, a, c) \times [0, 1]$ be such that $(y_n, \beta_n) \to (y, \beta)$, as $n \to \infty$. We shall prove that $h(y_n, \beta_n) \to h(y, \beta)$. We have

$$h(y_n, \beta_n)(t) = \begin{cases} y_n(t), & \text{for } t \in [0, \beta_n b], \\ \overline{x}(t), & \text{for } t \in (\beta_n b, b]. \end{cases}$$

(a) If $\lim_{n\to\infty}\beta_n = 0$, then

$$h(y,0)(t) = \bar{x}(t), \quad \text{for } t \in [0,b].$$

Hence

$$\|h(y_n,\beta_n) - h(y,\beta)\|_{PC} \le \|y_n - \bar{x}\|_{[0,\beta_n b]},$$

which tends to 0 as $n \to +\infty$. The case when $\lim_{n\to\infty}\beta_n = 1$ is treated similarly.

(b) If $\beta_n \neq 0$ and $0 < \lim_{n\to\infty}\beta_n = \beta < 1$, then we may distinguish between two sub-cases:

(i) If $t \in [0,\beta b]$, then for every $n \in \mathbb{N}$, $y_n \in S(F,a,c)$ implies the existence of $v_n \in S_{F,y_n}$ such that, for $t \in [0,\beta_n b]$,

$$y_n(t) = \begin{cases} F_0(v_n)(t), & t \in [0,t_1],\\ F_1(v_n)(t), & t \in (t_1,t_2],\\ \vdots & \\ F_k(v_n)(t), & t \in (t_k,t_{k+1}],\\ \vdots & \\ F_m(v_n)(t), & t \in (t_m,b], \end{cases}$$

where

$$F_0(v_n)(t) = a + tc + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_n(s)\mathrm{d}s, \quad t \in [0,t_1],$$

and

$$\begin{aligned} F_k(y)(t) &= I_k(F_{k-1}(v_n)(t_{k-1})) + (t-t_k)\bar{I}_k(F_{k-1}(v_n)(t_k))\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1} v_n(s)\mathrm{d}s, \quad t \in (t_k,t_{k+1}],\ k = 1,\dots,m. \end{aligned}$$

Now $\{y_n\}$ converges to y; then some $R > 0$ exists and satisfies

$$\|y_n\|_{PC} \le R.$$

Then, Assumption $(\mathcal{B}_2)$ implies that

$$v_n(t) \in p(t)\psi(R)B(0,1), \quad \text{a.e. } t \in [0,b].$$

Using the fact that $B(0,1)$ is compact in $\mathbb{R}$, then there exists a subsequence of v_n that converges to v. Since $F(t,\cdot)$ is u.s.c., then for every $\varepsilon > 0$, there exists $n_0 \ge 0$ such that, for any $n \ge n_0$, we have

$$v_n(t) \in F(t,y_n(t)) \subset F(t,y(t)) + \varepsilon B(0,1), \quad \text{a.e. } t \in [0,\beta b].$$

In addition, $F(\cdot,\cdot) \in \mathcal{P}_{cp,c}(\mathbb{R})$. Hence

$$v(t) \in F(t,y(t)) + \varepsilon B(0,1), \quad \forall\,\varepsilon > 0.$$

Therefore

$$v(t) \in F(t,y(t)), \quad \text{a.e. } t \in [0,\beta b] \Rightarrow |v(t)| \le p(t)\psi(R).$$

It follows that

$$v \in L^1([0,b],\mathbb{R}) \Rightarrow v \in S_{F,y}.$$

Using the continuity of I_k and $\bar{I}_k$ and the Lebesgue dominated convergence theorem, we find that, for $t \in [0,b]$,

$$y(t) = \begin{cases} F_0(y)(t), & t \in [0,t_1],\\ F_1(y)(t), & t \in (t_1,t_2],\\ \vdots & \\ F_k(y)(t), & t \in (t_k,t_{k+1}],\\ \vdots & \\ F_m(y)(t), & t \in (t_m,b]. \end{cases}$$

(ii) If $t \in (\beta_n b, b]$, then

$$h(y_n,\beta_n)(t) = h(y,\beta)(t) = \bar{x}(t).$$

Thus

$$\|h(y_n,\beta_n) - h(y,\beta)\|_{PC} \to 0, \quad \text{as } n \to \infty.$$

Therefore h is a continuous function, proving that $S(F,a,c)$ is contractible to the point $\bar{x} = S(f,a,c)$. $\square$

7.4. More appendix

In this section under weaker conditions on the functions I_k, we present some results on the existence and uniqueness of impulsive differential equations with fractional order. More precisely, we consider the following problem:

$$D_*^\alpha y(t) = f(t, y(t)), \quad \text{a.e. } t \in J := [0, b] \setminus \{t_1, \dots, m\},\ \alpha \in (0, 1], \tag{64}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, 2, \dots, m \tag{65}$$

$$y(0) = a, \tag{66}$$

where $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a given function.

Essential for the main results of this section, we state a generalization of Gronwall's lemma for singular kernels (Lemma 7.1.1 in [90]).

Lemma 7.14. *Let $v : [0, b] \to [0, \infty)$ be a real function, and $w(\cdot)$ be a nonnegative, locally integrable function on $[0, b]$, and suppose there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \le w(t) + a \int_0^t \frac{v(s)}{(t-s)^\alpha} \mathrm{d}s.$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \le w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^\alpha} \mathrm{d}s,$$

for every $t \in [0, b]$.

We will need the following auxiliary result in order to prove our main existence theorems. Lemmas 3.4 and 3.5 will be employed to prove this auxiliary result.

Lemma 7.15. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then y is the unique solution of the problem*

$$D_*^\alpha y(t) = f(y(t)), \quad t \in J, t \ne t_k, k = 1, \dots, m\ \alpha \in (0, 1), \tag{67}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k)), \quad k = 1, \dots, m, \tag{68}$$

$$y(0) = a, \tag{69}$$

if and only if

$$y(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(y(s)) \mathrm{d}s + \sum_{0<t_k<t}^{m} I_k(y(t_k)), \quad t \in [0, b]. \tag{70}$$

Theorem 7.16. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be an continuous function. Assume the condition*

($\mathcal{U}_3$) *There exist $p \in L^1([0, b], \mathbb{R}_+)$ and $\bar{M} > 0$ such that*

$$\|f(t, x)\| \le \bar{M}|x| + p(t) \quad \textit{for a.e. } t \in [0, b] \textit{ and each } x \in \mathbb{R},$$

with

$$\sup\left\{\int_0^t (t-s)^{\alpha-1} p(s) \mathrm{d}s : t \in [0, b]\right\} < \infty.$$

Then the initial-value problem (64)–(66) has at least one solution.

Proof. For the proof, we can restrict Problem (57)–(60) in the respective intervals $[0, t_1]$, $(t_1, t_2]$, $\dots$, $(t_m, b]$ and use the same method as in [43] or the method in Theorem 6.9.

Transform Problem (64)–(66) into a fixed point problem. Consider the operator $\overline{N} : PC \longrightarrow PC$, defined by

$$\overline{N}(y)(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) \mathrm{d}s, \quad \in [0, b],$$

and now, we prove only that all solutions of the problem are a priori bounded. Let y be a possible solution of the equation $y = \lambda \overline{N}(y)$, for some $\lambda \in (0, 1)$. Then

$$y(t) = \lambda \left[a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) \mathrm{d}s\right], \quad \text{for all } t \in [0, t_1].$$

Hence

$$|y(t)| \leq |a| + \frac{M_0}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M |y(s)| \mathrm{d}s. \tag{71}$$

Lemma 7.14 implies

$$|y(t)| \leq |a| + \frac{M_0}{\Gamma(\alpha)} + K(\alpha) M \left(|a| + \frac{M_0}{\Gamma(\alpha)} \right) \int_0^t (t-s)^{\alpha-1} \mathrm{d}s.$$

Hence

$$\|y\|_\infty \leq C + \frac{K(\alpha) M C}{\Gamma(\alpha+1)} := \widetilde{M}_0,$$

where

$$C := |a| + \frac{M_0}{\Gamma(\alpha)}.$$

- Let $t \in (t_1, t_2]$; then

  $$y(t) = \lambda \left[y(t_1) + I_1(y(t_1)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f(s, y(s)) \mathrm{d}s \right]$$

  and

  $$y(t_1^+) = y(t_1) + I_1(y(t_1)).$$

  Thus

  $$\begin{aligned} |y(t_1^+)| &\leq |y(t_1)| + |I_1(y(t_1))| \\ &\leq \widetilde{M}_0 + \sup\{|I_1(u)| : |u| \leq \widetilde{K}_0\}. \end{aligned}$$

  Thus by analogies of above proofs, we can show that there exists $\widetilde{M}_1 > 0$ such that

  $$\sup\{|y(t)| : t \in [t_1, t_2]\} \leq \widetilde{M}_1.$$

- We continue this process and also take into account that

  $$y(t) = \lambda \left[y_{m-1}(t_m) + I_m(y_{m-1}(t_m)) + \frac{1}{\Gamma(\alpha)} \int_{t_m}^t f(s, y(s)) \mathrm{d}s \right], \quad t \in (t_m, b].$$

  We obtain that there exists a constant $\widetilde{M}_m$ such that

  $$\sup\{|y(t)| : t \in [t_m, b]\} \leq \widetilde{M}_m.$$

  Consequently, for each possible solution y to $y = \lambda \overline{N}(y)$, for some $\lambda \in (0, 1)$, we have

  $$\|y\|_\infty \leq \max\{\widetilde{M}_i : i = 0, \ldots, m\} := \overline{M}.$$

  Set

  $$U = \{y \in PC : \|y\|_\infty < \overline{M} + 1\}.$$

  In analogy to [43] (see Theorem 3.3 there), we can easily prove that $\overline{N} : \overline{U} \to PC$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda \overline{N}(y)$, for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [74], we deduce that $\overline{N}$ has a fixed point y in U, which is a solution to Problem (64)–(66), and the proof is complete. □

  We next introduce some additional conditions that lead to the uniqueness of the solution of (64)–(66).

Theorem 7.17. *Assume that there exists $l \in L^1([0, b], \mathbb{R}_+)$ such that*

$$|f(t, x) - f(t, \overline{x})| \leq l(t)|x - \overline{x}|, \quad \textit{for all } x, \overline{x} \in \mathbb{R} \textit{ and } t \in J.$$

If

$$\sup \left\{ \int_0^t (t-s)^{\alpha-1} l(s) \mathrm{d}s : t \in [0, b] \right\} < \Gamma(\alpha),$$

then the initial-value problem (64)–(66) has a unique solution.

Proof. We consider Problem (64)–(66) on $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \ldots, J_m = (t_m, b]$. By the condition in the theorem, we can prove the existence and uniqueness in each interval J_k, $k = 1, \ldots, m$, and by the continuity of the functions I_k, we conclude the uniqueness on $[0, b]$. For more details, see Benchohra [43] and Henderson and Ouahab [13]. □

8. Periodic solutions

In this section, we consider the impulsive periodic problem

$$D_y^{\alpha}(t) \in \varphi(t, y(t)), \quad \text{a.e. } t \in J \setminus \{t_1, \ldots, t_m\}, \tag{72}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{73}$$

$$y'(t_k^+) = \bar{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{74}$$

$$y(0) = y(b), \qquad y'(0) = y'(b), \tag{75}$$

where $\varphi : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multifunction.

In cases where $\alpha = 1$ or $\alpha = 2$, a number of papers have been devoted to the study of initial and boundary value problems for impulsive differential inclusions. Some basic results in the theory of periodic boundary value problems for first-order impulsive differential equations and inclusions may be found in [18,91–93,80] and the references therein. Our goal in this section is to give an existence result for the above problem by using topological degree combined with a Poincaré operator.

8.1. Poincaré translation operator

By Poincaré operators we mean the translation operator along the trajectories of the associated differential system, and the first return (or section) map defined on the cross section of the torus by means of the flow generated by the vector field. The translation operator is sometimes also called the Poincaré–Andronov or Levinson operator, or simply the T-operator. In the classical theory (see [94–99] and the references therein), both these operators are defined to be single valued, when assuming, among other things, the uniqueness of solutions of initial-value problems. In the absence of uniqueness, it is often possible to approximate the right-hand sides of the given systems by locally Lipschitzian ones (implying uniqueness already), and then apply a standard limiting argument. This might, however, be rather complicated, and is impossible for discontinuous right-hand sides. On the other hand, set-valued analysis allows us to handle effectively such classically troublesome situations. For additional background details, see [100,55].

Let $\varphi : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a Carathéodory map. We define a multivalued map

$$S_\varphi : \mathbb{R}^2 \to \mathcal{P}(PC^1)$$

by

$$S_\varphi(a, c) = \{y \mid y(\cdot, x) \text{ is a solution of the problem satisfying } y(0, a) = a, y'(0, c) = c\}.$$

Consider the operator P_t defined by $P_t = \Psi \circ S_\varphi$, where

$$P_t : \mathbb{R}^2 \xrightarrow{S_\varphi} \mathcal{P}(PC^1) \xrightarrow{\Psi_t} \mathcal{P}(\mathbb{R}^2)$$

and

$$\Psi_t(y) = (y(0) - y(t), y'(0) - y'(t)),$$

where $PC^1(J, \mathbb{R}^n) = \{y \in PC(J, \mathbb{R}^n) : y'(t)$ is continuous at $t \neq t_k, y'(t_k^+), y'(t_k^-)$ exist, and $k = 1, 2, \ldots, m\}$, is a Banach space with norm

$$\|y\|_{PC^1} = \|y\|_{PC} + \|y'\|_{PC}.$$

Here, P_t is called the Poincaré translation map associated with the Cauchy problem

$$y'(t) \in \varphi(t, y(t)), \quad \text{a.e. } t \in J \setminus \{t_1, \ldots, t_m\}, \tag{76}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{77}$$

$$y'(t_k^+) = \bar{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{78}$$

$$y(0) = a \in \mathbb{R}, \qquad y'(0) = c \in \mathbb{R}. \tag{79}$$

The following lemma is easily proved.

Lemma 8.1. *Let $\varphi : J \times \mathbb{R} \to \mathcal{P}_{cv,cp}(\mathbb{R})$ be a Carathéodory multifunction. Then the periodic problem (72)–(75) has a solution if and only if for some $(a, c) \in \mathbb{R}^2$ we have $0 \in P_b(a, c)$, where P_b is the Poincaré map associated with (76)–(79).*

Set

$$K^n(r) = K^n(x, r), \qquad S^{n-1}(r) = \partial K^n(r), \quad \text{and} \quad P^n = \mathbb{R}^n \setminus \{0\},$$

where $K^n(r)$ is a closed ball in $\mathbb{R}^n$ with center x and radius r, and $\partial K^n(r)$ stands for the boundary of $K^n(r)$ in $\mathbb{R}^n$. For any $X \in ANR$-space, we set

$$J(K^n(r), X) = \{F : X \to \mathcal{P}(X) \mid F \quad \text{u.s.c. with } R_\delta\text{-values}\}.$$

Moreover, for any continuous $f : X \to \mathbb{R}^n$, where $X \in ANR$, we set

$$J_f(K^n(r), X) = \{\varphi : K^n(r) \to \mathcal{P}(X) \mid \varphi = f \circ F \text{ for some } F \in J(K^n(r), X) \text{ and } \varphi(S^{n-1}(r)) \subset P^n\}.$$

Finally, we define

$$CJ(K^n(r), \mathbb{R}^n) = \cup\{J_f(K^n(r), \mathbb{R}^n) \mid f : X \to \mathbb{R}^n \text{ is continuous and } X \in ANR\}.$$

It is well known (see [55]) that, for the multivalued maps in this class, the notion of topological degree is available. To define it, we need an appropriate concept of homotopy in $CJ(K^n(r), \mathbb{R}^n)$.

Definition 8.2. Let $\phi_1, \phi_2 \in CJ(K^n(r), \mathbb{R}^n)$ be two maps of the form

$$\phi_1 = f_1 \circ F_1 : K^n(r) \xrightarrow{F_1} \mathcal{P}(X) \xrightarrow{f_1} \mathbb{R}^n$$
$$\phi_2 = f_2 \circ F_2 : K^n(r) \xrightarrow{F_2} \mathcal{P}(X) \xrightarrow{f_2} \mathbb{R}^n.$$

We say that ϕ_1 and ϕ_2 are homotopic in $CJ(K^n(r), \mathbb{R}^n)$ if there exist a u.s.c. R_δ-valued homotopy $\chi : [0, 1] \times K^n(r) \to \mathcal{P}(X)$ and a continuous homotopy $h : [0, 1] \times X \to \mathbb{R}^n$ satisfying

(i) $\chi(0, u) = F_1(u)$, $\chi(1, u) = F_2(u)$ for every $u \in K^n(r)$,
(ii) $h(0, x) = f_1(x)$, $h(1, x) = f_2(x)$ for every $x \in X$,
(iii) for every $(u, \lambda) \in [0, 1] \times S^{n-1}(r)$ and $x \in \chi(\lambda, u)$, we have $h(x, \lambda) \neq 0$.

The map $H : [0, 1] \times K^n(r) \to \mathcal{P}(\mathbb{R}^n)$ given by

$$H(\lambda, u) = h(\lambda, \chi(\lambda, u))$$

is called a homotopy in $CJ(K^n(r), \mathbb{R}^n)$ between ϕ_1 and ϕ_2.

Theorem 8.3 ([55]). *There exists a map $Deg : CJ(K^n(r), \mathbb{R}^n) \to \mathbb{Z}$, called the topological degree function, satisfying the following properties:*

($\mathcal{C}_1$) *If $\varphi \in CJ(K^n(r), \mathbb{R}^n)$ is of the form $\varphi = f \circ F$ with F single valued and continuous, then $Deg(\varphi) = deg(\varphi)$, where $deg(\varphi)$ stands for the ordinary Brouwer degree of the single valued continuous map $\varphi : K^n(r) \to \mathbb{R}^n$.*
($\mathcal{C}_2$) *If $Deg(\varphi) = 0$, where $\varphi \in CJ(K^n(r), \mathbb{R}^n)$, then there exists $u \in K^n(r)$ such that $0 \in \varphi(u)$.*
($\mathcal{C}_3$) *If $\varphi \in CJ(K^n(r), \mathbb{R}^n)$ and $\{u \in K^n(r) | 0 \in \varphi(u)\} \subset Int\, K^n(r_0)$ for some $0 < r_0 < r$, then the restriction φ_0 of φ to $K^n(r_0)$ is in $CJ(K^n(r), \mathbb{R}^n)$ and $Deg(\varphi_0) = Deg(\varphi)$.*

Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$; $CJ_0(A, B)$ will denote the class of mappings

$$CJ_0(K^n(r), \mathbb{R}^n) = \{\varphi : A \to \mathcal{P}(B) \mid \varphi = f \circ F, F : A \to \mathcal{P}(X), F \text{ is u.s.c. with } R_\delta\text{-values and } f : X \to B \text{ is continuous}\},$$

where $X \in ANR$. We also need the following lemma.

Lemma 8.4. *Let $\varphi : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ be σ-LL-selectionable. Assume that $(\mathcal{B}_1)$–$(\mathcal{B}_2)$ and $(\overline{H}_3)$ hold. Then the set S_φ is an R_δ set.*

Proof. Since φ is σ-LL-selectionable, there exists a decreasing sequence of multivalued maps $F_k : [0, b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ $(k \in \mathbb{N})$ which have Carathéodory selections such that

$$\varphi_{k+1}(t, u) \subset \varphi_k(t, x) \quad \text{for almost all } t \in [0, b],\ x \in \mathbb{R}$$

and

$$\varphi(t, y) = \bigcap_{k=0}^{\infty} \varphi_k(t, y), \quad y \in \mathbb{R}.$$

Then

$$S_\varphi(a, c) = \bigcap_{k=0}^{\infty} S_{\varphi_k}(, a, c).$$

From Theorem 7.13, the set $S_\varphi(a, c)$ is contractible for each $k \in \mathbb{N}$. Hence $S_\varphi(a, c)$ is an R_δ set. □

The following theorem due to Gorniewicz [55] is critical in the proof of the main result in this section.

Theorem 8.5 (*Nonlinear Alternative*). *Assume that $\varphi \in CJ_0(K^n(r), \mathbb{R}^n)$. Then φ has at least one of the following properties:*

(i) *$Fix(\varphi) \neq \emptyset$,*
(ii) *there is an $x \in S^{n-1}(r)$ with $x \in \lambda\varphi(x)$ for some $0 < \lambda < 1$.*

Lemma 8.6. *If $\varphi : X \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is a u.s.c. multivalued map, then φ is σ-LL-selectionable.*

Theorem 8.7. *Let $\varphi: \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ be a u.s.c. multifunction. Assume the condition*

$(\mathcal{R}_1)$ there exist $p \in L^1$ and $M > 0$ such that

$$\|\varphi(t,x)\|_{\mathcal{P}} \leq p(t), \quad a.e.\ t \in [0,b], \quad \sup_{t\in[0,b]} \int_0^t (t-s)^{\alpha-2} p(s)ds < \infty,$$

and

$$|I_k(x)| \leq M, \qquad |\bar{I}_k(x)| \leq M, \quad \forall x \in \mathbb{R}.$$

Then Problem (72)–(75) has at least one solution.

Proof. From Lemma 8.6, φ is σ-LL-selectionable, and so by Lemma 8.4, $S_\varphi(a,c)$ is R_δ, for every $(a,c) \in \mathbb{R}^2$, and from Theorem 6.1, $S_\varphi(\cdot,\cdot)$ is u.s.c. Set $A = B = \mathbb{R}^2$ and $X = PC^1 \in ANR$. We will prove that

$$\Psi : PC^1 \to \mathbb{R}^2 \quad \text{defined by} \quad y \to \Psi(y) = (y(\cdot) - y(0), y'(\cdot) - y(0))$$

is a continuous map. Let $\{y_n, y_n'\}$ be a sequence such that $y_n',\ y_n \to y,\ y'$ in PC^1. Then,

$$|\Psi(y_n)(t) - \Psi(y)(t)| \leq 2\|y_n - y\|_{PC} + 2\|y_n' - y'\|_{PC} \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$\Psi_b \in CJ_0(K^2(r), \mathbb{R}^2).$$

Therefore the Poincaré translation operator $P_t(a,c) = \Psi_t \circ S_\varphi(a,c)$. Let $(a,c) \in \lambda\Psi_t \circ S_\varphi(a,c)$ for some $\lambda \in (0,1)$. Then, there exists $y \in PC^1$ such that $y \in S_\varphi(a,c)$. This implies $y(0) = a$, $y'(0) = c$ and $a = \lambda(a - y(t))$, $c = \lambda(c - y'(t))$, $(a,c) \in S^1(r)$. For $t \in J$, we have

$$a = -\lambda \begin{cases} a + ct + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in [0,t_1], \\ I_1(y(t_1)) + (t-t_1)\bar{I}_1(y(t_1)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in (t_1,t_2], \\ \vdots & \\ I_m(y(t_m)) + (t-t_m)\bar{I}_m(y(t_m)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in (t_m,b], \end{cases}$$

and

$$c = -\lambda \begin{cases} \dfrac{\alpha-1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-2} v(s)ds, & \text{if } t \in [0,t_1], \\ \bar{I}_1(y(t_1)) + \dfrac{\alpha-1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-2} v(s)ds, & \text{if } t \in (t_1,t_2], \\ \vdots & \\ \bar{I}_m(y(t_m)) + \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-2} v(s)ds, & \text{if } t \in (t_m,b]. \end{cases}$$

Hence, from $(\mathcal{R}_1)$ there exists $\widetilde{K}$ such that

$$|a| + |c| \leq \widetilde{K}.$$

Set

$$K^2(\widetilde{K}+1) = \{(a,c) \in \mathbb{R}^2 : |a| + |c| \leq \widetilde{K} + 1\}.$$

From Theorem 6.1, $S_\varphi(\cdot,\cdot)$ is u.s.c., and by Lemma 8.4, $S_\varphi(\cdot,\cdot)$ is R_δ. Since Ψ is continuous, then $P_b \in CJ_0(K^2(\widetilde{K}+1), \mathbb{R}^2)$. As a consequence of the nonlinear alternative of Leray–Schauder type [55], we conclude that $Fix\, P_b \neq \emptyset$. This completes the proof of the theorem. $\square$

9. Concluding remarks

In this paper, we have investigated Problem (1)–(4) under various assumptions on the multivalued hand-side nonlinearity, and we have obtained a number of new results regarding existence of solutions. We first proved a Filippov–Ważewski theorem and continuous versions of Filippov's results to impulsive differential inclusions with fractional order.

The main assumptions on the nonlinearity are the Carathéodory and the Lipschitz conditions with respect to the Hausdorf distance in generalized metric spaces.

In 1976, Lasry and Robert [86] proved that, if the nonlinearity F is compact, convex valued, u.s.c. and bounded, then the set of all solutions for first-order differential inclusions with nonlinearity F is a compact and acyclic set. In 1986, Górniewicz [84] discussed the topological structure of the set of solutions (contractibility and acyclic contractibility) when F is *ML*- or σ-selectionable.

When the multivalued nonlinearity is further σ-Ca- or σ-*mLL*-selectionable, based on Aronszajn type results, we investigated the geometric properties of the solution set, proving that it enjoys R_δ, contractible and acyclicity properties. Also, the existence and uniqueness results for impulsive differential equations with fractional order were established.

Very recently, Djebali et al. [68] discussed the topological structure, Filippov's theorem and the Filippov–Ważewski theorems for impulsive semilinear functional differential inclusions with finite delay.

In the case where $\alpha \geq 1$, the integrable equations involved with impulsive differential equations and inclusions lack singularity, and by the same methods used in [59,4,82,61,68,83,5,89,101,13,80,62–64], we can establish the existence and uniqueness results for differential equations, differential inclusions, impulsive differential equations, and impulsive differential inclusions of fractional order.

Some problems considered in this paper can be improved under weaker conditions on the functions I_k and $\bar{I}_k$.

This paper contributes within the domain of impulsive fractional differential equations and inclusions.

Acknowledgements

The second author thanks Professors Smail Djabli, Lench Górniewicz and Mouffak Benchohra for encouragement and support.

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