Modal Logics for Qualitative Possibility Theory

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ABSTRACT

Possibilistic logic has been proposed as a numerical formalism for reasoning with uncertainty. There has been interest in developing qualitative accounts of possibility, as well as an explanation of the relationship between possibility and modal logics. We present two modal logics that can be used to represent and reason with qualitative statements of possibility and necessity. These logics have a natural semantics based on a qualitative abstraction of possibility distributions. Within this modal framework, we are able to identify interesting relationships among possibilistic logic, beliefs, and conditionals. In particular, we demonstrate that possibilistic logic naturally induces a notion of belief identical to that of the widely used epistemic logic weak S5, and that current approaches to conditional default reasoning and belief revision can be mapped into possibilistic logic, including the means of conditional reasoning based on high probabilities investigated by Adams [1] and Pearl [2, 3].

KEYWORDS: ***Possibility theory, modal logic, belief logics, conditionals, default reasoning, belief revision, high probability***

1. INTRODUCTION

There has been a great deal of interest in the relationship between numerical and nonnumerical approaches to uncertain reasoning in artificial intelligence (AI). While numerical methods, for instance, those based on classical probability theory, tend to be more general than their qualita-

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tive counterparts (e.g., proposals for symbolic default reasoning), these qualitative systems can be useful when reasoning "numerically" is computationally or expressively intractable, and they help illuminate the underlying structure of such reasoning. Possibility theory has been proposed as one such numerical formalism for reasoning under uncertainty (see Dubois and Prade [4] for an introduction). It is based on the notion of necessity measures, which determine the degree of certainty associated with an item of belief, and the dual possibility measures, determining the degree of surprise associated with (or the willingness to accept) a potential belief. Possibility theory has proven to be remarkably robust and has found application in a number of areas in AI, ranging from default reasoning [5] to reasoning about linguistic quantifiers [6].

Naturally, qualitative accounts of possibility theory have been proposed and shown to capture the underlying structure of possibility and necessity measures [7]. Qualitative possibility theory is presented as a set of postulates that constrain any reasonable ordering of possibility on sentences. Such qualitative characterizations give us the ability to express possibilistic relationships without having to assume particular numerical values, relying only on the relative possibility of propositions. This qualitative relationship is especially important in possibility theory, because changing the absolute magnitude of degrees of possibility has little effect on key inferential relations, as long as the relative magnitudes remain fixed (contrast this with probability theory where such latitude does not exist).

Accepting the utility of qualitative possibility, a logic of qualitative possibility becomes crucial if we wish to derive consequences based on partial information. Given constraints on the relationship between certain propositions (e.g., their relative possibility), certain other constraints may be required to hold on any suitable possibility measure. A logical calculus permits us to specify a partial (qualitative) possibility measure and derive information implicit in the specification. We can thus reason without requiring complete information. This is crucial from the point of view of AI, for reasoning with incomplete information is the norm. Of course, we require a logic with sufficient expressive power to capture the types of constraints that might be required in a knowledge base. One such possibilistic logic is developed in [8]. Others have provided logics in which logical constraints on probabilities can be specified in an analogous fashion (see, e.g., [9, 10], though these retain the quantitative aspects of probabilities).

Given the nature of necessity and possibility measures, the connection to modal logics is also of great interest [4]. Modal logics have been developed, to a large extent, to capture the notions of possibility and necessity [11]; hence, it seems natural to expect some close relationship. In this study, we present a family of modal logics for representing and reasoning with assertions of relative (qualitative) possibility and necessity and other constraints on these qualitative orderings. We will concentrate on two modal logics, CO and CO*, in which we can faithfully represent the notions of qualitative necessity and possibility. These representations will respect the essential qualities of possibility and necessity measures. The expressive power we need to capture possibilistic logic is achieved with two modalities: the usual $\Box$, corresponding to truth at accessible worlds; and the less standard $\overleftarrow{\Box}$, expressing truth at inaccessible worlds. We note that, in contrast to many multimodal logics used in knowledge representation, the additional modality carries no excess semantical baggage. Our semantics is based on the usual Kripke structures for monomodal logics, the added modal operator increasing only our ability to constrain the form of such structures. The correspondence does not use the (perhaps expected) mapping of qualitative necessity and possibility into the operators $\Box$ and $\Diamond$. However, we provide other operators, defined using $\Box$ and $\overleftarrow{\Box}$, that do capture these absolute notions. In particular, we will define in our logics a modal operator for belief and show the tight correspondence between qualitative possibility and the usual notion of epistemic possibility.

Aside from demonstrating that simple modal logics can be used to express qualitative possibility, embedding possibilistic logic into CO and CO* also illustrates important connections to a number of other formalisms for defeasible reasoning. These systems can also be embedded into our logics; among them are conditional approaches to default reasoning [12], ε-semantics [1, 2], belief revision [13], counterfactual logics [14], and autoepistemic logic [15, 16].

In the next section we discuss qualitative possibility and present the logics CO and CO*. We show how these logics may be used to represent qualitative possibility. In section 3, we examine the connections between possibilistic logic and some other systems for defeasible reasoning. One interesting connection is to the modal logic KD45 (weak S5), the most commonly used logic for knowledge and belief in AI. In fact, we demonstrate that possibility theory naturally induces an epistemic logic that is exactly KD45. However, our logic allows us to show even stronger connections to autoepistemic logic [15], the nonmonotonic counterpart of KD45. Also of particular interest is a conditional connective, definable in our logics, that is identical to that proposed by Adams [1] for reasoning with high probabilities. The relationship between possibility theory and Adams's logic has been examined independently by Dubois and Prade [17], but our formulation has independent motivation, and lends itself to a complete calculus of conditionals. Furthermore, the expressive power of our logics allows us to formulate important properties directly in the object language.

2. A MODAL REPRESENTATION OF POSSIBILITY

2.1. Possibilistic Logic

Possibilistic logic has been developed to a considerable extent by Dubois and Prade (see [4] for a survey), and is intended to capture a form of uncertainty or degree of belief associated with the beliefs or facts in a knowledge base *KB*. Assume we have some underlying classical propositional language in which we are able to express these items of belief. A possibility measure Π maps the sentences of this language into the real interval $[0, 1]$. The value $\Pi(A)$ is intended to represent the degree of possibility of A. We take this to capture the amount of surprise associated with adopting A as an epistemic possibility. If $\Pi(A) = 1$ there is no surprise associated with A—it is completely possible (i.e., A is consistent with the agent's beliefs). If $\Pi(A) < 1$, then some degree of surprise is associated with learning A; we take this to mean that $\neg A$ is believed, for A is not an epistemic possibility. At the extreme, $\Pi(A) = 0$ indicates that surprise is maximal (i.e., an agent would never adopt A as a possibility, or would never give up belief in $\neg A$). A possibility measure must satisfy the following three properties:

1. $\Pi(\top) = 1$
2. $\Pi(\bot) = 0$
3. $\Pi(A \vee B) = \max(\Pi(A), \Pi(B))$

A necessity measure N is a similar mapping, associating with A a degree of necessity. We take $N(A)$ to represent the willingness of an agent to give up belief in A (or the degree of entrenchment of A in a belief set; see section 4). Necessity measures are constrained by:

1. $N(\top) = 1$
2. $N(\bot) = 0$
3. $N(A \wedge B) = \min(N(A), N(B))$

If $N(A) = 1$ then A is fully entrenched and can never be given up, while $N(A) = 0$ indicates that A is not believed at all.[1] Naturally, the willingness of an agent to give up a belief A should be related to the degree of surprise associated with accepting $\neg A$ as an epistemic possibility, for giving up A is just accepting $\neg A$ as possible. Indeed, one may define

[1] The constraint $N(\bot) = 0$ has been relaxed by some authors to allow for some degree of inconsistency handling; in other words, we may have that a contradictory sentence is given some positive necessity or "degree of belief." We will not consider this generalization here.

necessity measures using the identity

$$N(A) = 1 - \Pi(\neg A).$$

Possibility and necessity measures can be seen, intuitively, as refining the usual conception of belief and epistemic possibility by allowing one to specify the "extent" to which a sentence is believed or is possible. For example, imagine a knowledge base *KB* containing the information A (Anne will come to the party) and C (Cheryl will come to the party). According to the interpretation provided above, both A and C have a positive degree of necessity. While a standard belief logic provides the same capability, a necessity measure captures the idea that certain beliefs can be held more firmly than others. This information would be crucial if the agent in possession of *KB* were to learn, for instance, that one of Anne or Cheryl will stay at home, that is, $\neg(A \wedge C)$. If degrees of necessity are assigned to the beliefs in *KB*, the agent can decide which of A or C (or both) to retract to permit the incorporation of the new information. If $N(A) = .6$ and $N(C) = .4$ then the belief A is held more firmly and the new knowledge base will reflect this fact by retaining A and giving up C. It might be the case that the consequence of *KB* $A \equiv C$ is held more strongly than either A or C (e.g., $N(A \equiv C) = .8$ and $N(A) = N(C) = .4$). In this case, the belief $A \equiv C$ will be retained and, as a result, both A and C will be retracted. Perhaps Anne and Cheryl only go to parties together. In what follows, we will look at the relationship among possibilistic logic, logics of belief, and belief revision in some detail.

Semantically, possibility (and necessity) measures can be understood in terms of possibility distributions over possible worlds [18]. Assume our underlying language is finitely generated and thus corresponds to a finite set of valuations or possible worlds W.[2] A distribution π assigns to each world a degree of possibility from the interval $[0,1]$. A distribution determines a possibility measure Π via the following relationship:

$$\Pi(A) = \max\{\pi(w) \colon w \models A\}.$$

In other words, the degree of possibility of A is just that of the most possible A-world. The concept of assigning degrees of possibility to worlds will be crucial in our qualitative semantics.

Qualitative necessity measures are discussed in [8, 19]. Postulates are proposed constraining the qualitative relationship $A \geq_N B$, which is read as "A is at least as necessary as B." If we define $A \geq_N B$ to be true just

[2] Thus, we have a finite Boolean algebra of propositions or "events," the Lindenbaum algebra of the propositional logic. By imposing additional constraints on possibility measures, this finiteness restriction can be relaxed.

when $N(A) \geq N(B)$ for any necessity measure N, then $\geq_N$ will satisfy the postulates for qualitative necessity (in finite settings), and these relations are the only ones that can be so-defined [7]. A qualitative necessity ordering is any ordering satisfying these postulates.[3]

(N1) $A \geq_N A$

(N2) $A \geq_N B$ or $B \geq_N A$

(N3) If $A \geq_N B$ and $B \geq_N C$ then $A \geq_N C$

(N4) $\top \geq_N \bot$

(N5) $\top \geq_N A$ for all A

(N6) If $B \geq_N C$ then $A \wedge B \geq_N A \wedge C$ for all A

Such an ordering is, in fact, a total preordering (or weak ordering) of the sentences in the language. In other words, the sentences are ranked according to necessity.

Qualitative possibility orderings are defined using related postulates, with $A \geq_\pi B$ meaning A is at least as possible as B, or $\Pi(A) \geq \Pi(B)$. The relationship between these qualitative relationships can be given as $A \geq_N B$ iff $\neg B \geq_\pi \neg A$.

Fariñas and Herzig [8] have axiomatized qualitative possibility in a logic called qualitative possibility logic (QPL), in which the relation $\geq_\pi$ is incorporated as a conditional connective. Using a conditional logic permits a rather natural formulation, allowing one to express sentences of the form $A \geq_\pi B$, or Boolean combinations of such sentences, and draw appropriate conclusions. This appears to be the first logical axiomatization of qualitative possibility and, as such, provides many of the advantages we expect of a logical calculus. Furthermore, they show QPL is equivalent to Lewis's [14] conditional logic VN.

As mentioned previously, there is also a great deal of interest in a purely modal formulation of qualitative possibility. Fariñas and Herzig also make an initial attempt to develop a modal theory of possibility that uses only unary modal operators in place of the conditional $\geq_\pi$. Unfortunately, the resulting logic PL requires an infinite set of modal operators, each corresponding to a unique member of the measure set for Π. Semantically, each operator is evaluated with respect to a separate accessibility relation. This certainly permits the expression of qualitative properties like $A \geq_\pi B$, but doesn't seem to reflect the qualitative nature of QPL or other qualitative postulates. In particular, there is no modal operator corresponding to (some degree of) possibility or necessity.

[3] For any ordering $\geq$ we propose (e.g. $\geq_N$), the corresponding relations $\leq$, $<$, and $>$ are defined in the standard way.

In the remainder of this section, we present a series of modal logics for just this purpose. Aside from demonstrating that qualitative possibility can be captured in a modal logic, the logics we present have considerable expressive power, allowing us to represent constraints and conditions on possibility orderings that cannot be captured in a purely conditional logic formulation.

2.2. Logics CO and CO*

Our possible worlds semantics for qualitative possibility theory takes as a point of departure the concept of a possibility distribution defined previously. Such a distribution assigns degrees of possibility to worlds. Qualitatively, this is simply a ranking of worlds according to their degree of possibility, or how plausible these situations are for an agent.

Our modal structures consist of a set W of possible worlds and a binary ordering relation $\preccurlyeq$ over W. Intuitively, W is the set of situations an agent "considers possible," or assigns some non-zero degree of possibility. We do not intend this to represent epistemic possibility, for there will be worlds among this set that are inconsistent with an agent's beliefs. Rather, these are the set of worlds an agent could possibly consider adopting, even if it changed its mind about certain beliefs it currently possesses. For example, W could be the set of physically or logically possible worlds (for an agent).

We take $\preccurlyeq$ to be a ranking of these worlds according to their degree of possibility or plausibility, the extent to which an agent is willing to accept these worlds as epistemically possible, or consistent with its beliefs. When $w \preccurlyeq v$ we intend that v is at least as possible as w. Intuitively, when v is more possible than w we can think of v as being "more consistent" with an agent's current beliefs than w, or think of v as a preferable, more plausible alternative state of affairs for an agent to adopt should it be forced to choose between the two. We define the relations $\prec$, $\succcurlyeq$, and $\succ$ in terms if $\preccurlyeq$ in the usual way. Such an ordering can be related to a possibility distribution π by equating $w \preccurlyeq v$ with $\pi(w) \leq \pi(v)$.

With this in mind, it should be clear that the relation $\preccurlyeq$ must be a weak ordering, or a total preorder, on W. That is $\preccurlyeq$ is transitive and connected.[4] This imposes the restriction that any two worlds w, v must have comparable degrees of possibility. If neither is more possible than the other, then they are equally possible. In [20], we develop a weaker logic called CT40 that relaxes this condition, requiring only reflexivity and transitivity. We return to this point in the concluding section. We are thus lead to the notion of a *CO-model.* (We assume **P** to be the atomic variables underlying our logical language.)

[4] $\preccurlyeq$ is (totally) connected if $w \preccurlyeq v$ or $v \preccurlyeq w$ for any $v, w \in W$ (this implies reflexivity).

DEFINITION [20, 21] *A CO-model is a triple* $M = \langle W, \preccurlyeq, \varphi \rangle$, *where* W *is a non-empty set (of possible worlds),* $\preccurlyeq$ *is a transitive, connected, binary relation on* W, *and* φ *maps* $\mathbf{P}$ *into* 2^W ($\varphi(A)$ *is the set of worlds where* A *is true).*

A CO-model consists of a set of clusters of possible worlds totally ordered by $\preccurlyeq$, where a cluster is a set of equally possible worlds. Formally, a cluster is any subset $\mathscr{C} \subseteq W$ such that $w \preccurlyeq v$ for all $w, v \in \mathscr{C}$ and for no strict superset of $\mathscr{C}$ does this property hold. In Figure 1, each large circle represents such a cluster of worlds and the arrows point in the direction of increasing possibility. So v is more possible than both u and w ($u \prec v$ and $w \prec v$), while u and w are equally possible.

If we intend the possibility ranking to respect an agent's current belief set, then it ought to be the case that the maximally possible worlds in this ranking be precisely those the agent considers not merely possible, but epistemically possible (i.e., those worlds consistent with its beliefs). Although we do not need to enforce this constraint to deal with possibilistic logic, we will discuss how it can be expressed in section 3, and how beliefs are related to possibility measures.

We will now define a modal language with which we can express qualitative notions of possibility and necessity. Our language $\mathbf{L}$ will be formed from a denumerable set of $\mathbf{P}$ of propositional variables, together with the connectives $\neg$, $\supset$, $\Box$, and $\overleftarrow{\Box}$. This is a standard modal language with one additional modal operator $\overleftarrow{\Box}$. The connectives $\wedge$, $\vee$, and $\equiv$ are defined in terms of these in the usual way. We use $\top$ and $\bot$ to denote the identically true and false propositions, respectively. We denote by $\mathbf{L}_{CPL}$ the propositional sublanguage of $\mathbf{L}$.

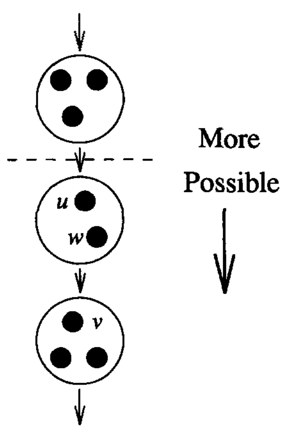

Figure 1. A CO-model.

The satisfaction of formulae at worlds in a model is specified in the following definition.

DEFINITION *Let $M = \langle W, R, \varphi \rangle$ be a CO-model, with $w \in W$. The truth of a formula A at w in M (where $M \models_w A$ means A is true at w) is defined inductively as:*

1. $M \models_w A$ *iff* $w \in \varphi(A)$ *for atomic sentence* A
2. $M \models_w \neg A$ *iff* $M \not\models_w A$
3. $M \models_w A \supset B$ *iff* $M \models_w B$ *or* $M \not\models_w A$
4. $M \models_w \Box A$ *iff for each* v *such that* $w \preccurlyeq v$, $M \models_v A$
5. $M \models_w \overleftarrow{\Box} A$ *iff for each* v *such that not* $w \preccurlyeq v$, $M \models_v A$

If $M \models_w A$ we say that M satisfies A at w. Such worlds are often referred to as A-worlds.

If we think of $\preccurlyeq$ as an accessibility relation in the sense of modal logic, $w \preccurlyeq v$ can be understood as asserting that v is accessible to w, or that w "sees" v. Thus, in this sense, a world sees only those worlds that are equally or more possible than itself. The sentence $\Box A$ then has the usual interpretation: $\Box A$ is true at w iff A holds all worlds accessible to w. Given our reading of $\preccurlyeq$, this means A must be true at all worlds that have a degree of possibility at least equal to that of w. The sentence $\overleftarrow{\Box} A$, in contrast, has a nonstandard interpretation in modal logic: $\overleftarrow{\Box} A$ holds when A is true at all inaccessible worlds, those strictly less possible than w. While the standard operator $\Box$ can force certain (classes of) worlds to be inaccessible, $\overleftarrow{\Box}$ can force certain worlds to be accessible. To illustrate the expressive power of CO, consider again Figure 1. If w satisfies $\Box A$, this means no $\neg A$-worlds can be accessible to w. This forces all such worlds to be inaccessible; in other words, $\Box A$ forces all $\neg A$-worlds to lie above the dashed line. If the same world also satisfies $\overleftarrow{\Box} \neg A$, this means no A-worlds can be inaccessible, so all such worlds must be accessible. This second type of constraint cannot be enforced using $\Box$ alone. When w satisfies $\Box A \wedge \overleftarrow{\Box} \neg A$, essentially a line is drawn across the structure (as in Figure 1), all worlds above it satisfying $\neg A$ and all below it satisfying A. This additional expressive power will prove extremely useful in the following sections.

We define several new connectives as follows:

1. $\Diamond A \equiv_{\mathrm{df}} \neg \Box \neg A$
2. $\overleftarrow{\Diamond} A \equiv_{\mathrm{df}} \neg \overleftarrow{\Box} \neg A$
3. $\overleftrightarrow{\Box} A \equiv_{\mathrm{df}} \Box A \wedge \overleftarrow{\Box} A$
4. $\overleftrightarrow{\Diamond} A \equiv_{\mathrm{df}} \Diamond A \vee \overleftarrow{\Diamond} A$

(Note that $\overleftrightarrow{\Diamond} A$ can also be defined as $\neg \overleftrightarrow{\Box} \neg A$.) It is easy to verify that these connectives have the following truth conditions:

1. $M \models_w \Diamond A$ iff for some v such that $w \preccurlyeq v$, $M \models_v A$

2. $M \models_w \overleftarrow{\Diamond} A$ iff for some v such that not $w \preccurlyeq v$, $M \models_v A$
3. $M \models_w \overleftrightarrow{\Box} A$ iff for all $v \in W$, $M \models_v A$
4. $M \models_w \overleftrightarrow{\Diamond} A$ iff for some $v \in W$, $M \models_v A$

Clearly, $\Diamond A$ asserts that A is true at some situation at least as possible as the current situation, while $\overleftarrow{\Diamond} A$ requires that A be true at some less possible world. $\overleftrightarrow{\Box} A$ and $\overleftrightarrow{\Diamond} A$ state that A is true at all worlds or some world, respectively, whether more or less possible. Validity is defined in a straightforward manner, a sentence A being CO-valid ($\models_{CO} A$) just when every CO-model M satisfies A at every world.

DEFINITION [20, 21] *The conditional logic CO is the smallest* $S \subseteq \mathbf{L}$ *such that* S *contains CPL (and its substitution instances) and the following axiom schemata, and is closed under the following rules of inference*:

K $\Box(A \supset B) \supset (\Box A \supset \Box B)$

K′ $\overleftarrow{\Box}(A \supset B) \supset (\overleftarrow{\Box} A \supset \overleftarrow{\Box} B)$

T $\Box A \supset A$

4 $\Box A \supset \Box\Box A$

S $A \supset \overleftarrow{\Box}\Diamond A$

H $\overleftrightarrow{\Diamond}(\Box A \wedge \overleftarrow{\Box} B) \supset \overleftrightarrow{\Box}(A \vee B)$

Nec *From* A *infer* $\overleftrightarrow{\Box} A$.

MP *From* $A \supset B$ *and* A *infer* B.

Provability and derivability are defined in the usual way, in terms of theoremhood [2].

THEOREM 1 [20, 21] $\vdash_{CO} A$ *iff* $\models_{CO} A$.

We often want to ensure that all logically possible worlds are taken into consideration in our models (for instance in the context of belief revision [23, 24] or autoepistemic reasoning [16, 25]). In our current setting, we will think of the worlds in a model as those that have some non-zero degree of possibility. Ensuring that all situations are captured in a model guarantees that every logically possible world is assigned some positive degree of possibility (and, as we will soon see, that each satisfiable sentence is possible to some degree). For this purpose we introduce the logic CO*, which is based on the class of CO-models in which all propositional valuations are represented (see also [16]). We think of these models as capturing "full" possibility distributions, where no state of affairs is deemed completely impossible if it is logically consistent.

DEFINITION [20, 21] *CO* is the smallest extension of CO closed under all rules of CO and containing the following axioms*:

LP $\overleftrightarrow{\Diamond} A$ *for all satisfiable propositional* A

DEFINITION *Let* $M = \langle W, R, \varphi \rangle$ *be a Kripke model. For all* $w \in W, w^*$ *is defined as the map from* **P** *into* $\{0, 1\}$ *such that* $w^*(A) = 1$ *iff* $w \in \varphi(A)$; *in other words,* w^* *is the valuation associated with* w.

DEFINITION [20, 21] *A CO*-model is any* $M = \langle W, R, \varphi \rangle$, *such that* M *is a CO-model and*

$$\{f\colon f \text{ maps } \mathbf{P} \text{ into } \{0,1\}\} \subseteq \{w^* : w \in W\}.$$

THEOREM 2 [20, 21] $\vdash_{CO*} A$ *iff* $\models_{CO*} A$.

2.3. Expressing Qualitative Possibility

We now turn to the task of expressing relationships of qualitative possibility and necessity in our modal language. Recall that we can equate the relation $v \preccurlyeq w$ with the fact $\pi(w) \geq \pi(v)$ for some possibility distribution of interest. In a finite setting, such a possibility distribution induces a possibility measure Π on sentences via the relationship

$$\Pi(A) = \max\{\pi(w)\colon w \models A\}.$$

In other words, the degree of possibility of A is simply the degree of possibility of the most possible A-worlds.[5] Qualitatively, we are not interested in the absolute degree of possibility of A; rather we want to capture the relative degree of possibility of A compared to that of other sentences.

Clearly, A has at least as high a degree of possibility as B iff $\Pi(A) \geq \Pi(B)$. If w is some maximally possible A-world, and v is some maximally possible B-world, this ensures that $\pi(w) \geq \pi(v)$, or that $v \preccurlyeq w$. However, since v is a maximally possible B-world, it must be that $u \preccurlyeq w$ for any B-world u. Thus, we can state that A is at least as possible as B iff every B-world in W can "see" some A-world. This other words, for each B-world u, there will be some A-world w such that $u \preccurlyeq w$. Notice that this specification does not assume the existence of limiting (or maximally possible) A-worlds or B-worlds, and thus applies to arbitrary propositional languages and CO-models. We express this condition in our bimodal language as $\overleftrightarrow{\Box}(B \supset \Diamond A)$: whenever B holds, there is some more plausible world satisfying A. We refer to this as a qualitative possibility ordering.

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[5] If our language is not finitely generated, there are an infinite number of worlds; the formulation of possibility measures in terms of such distributions is then problematic, even if we take $\Pi(A)$ to be $\sup\{\pi(w)\colon w \models A\}$. Furthermore, there is nothing in the formulation of CO-models that ensures the existence of most possible A-worlds for arbitrary sentences A. For instance, there may be an infinite chain of more and more possible A-worlds. While there is difficulty in providing numerical possibility distributions in these cases, we will see that a qualitative account has no difficulty with such orderings.

Definition *Let M be a CO-model. The qualitative possibility ordering determined by M is* $\leq_{PM}$ *, given by*

$$A \leq_{PM} B \qquad \text{iff } M \models \overleftrightarrow{\Box}(B \supset \Diamond A).$$

A is at least as possible as B iff $A \leq_{PM} B$.[6]

The dual of such a relationship is a qualitative necessity ordering.

Definition *Let M be a CO-model. The qualitative necessity ordering determined by M is* $\leq_{EM}$ *, given by*

$$B \leq_{EM} A \qquad \text{iff } \neg B \leq_{PM} \neg A.$$

A is at least as necessary as B iff $B \leq_{EM} A$.

It is easy to see that $B \leq_{EM} A$ iff $M \models \overleftrightarrow{\Box}(\neg A \supset \Diamond \neg B)$.

Figure 2 shows a CO-model where A, B, $\neg B$, and C are each more possible than $\neg A$. Every world where $\neg A$ holds is strictly less possible than some world where these other propositions hold. We also see that $A \wedge \neg C$ is more possible than $\neg A \wedge C$. A and C are equally (and maximally) possible, yet A is more necessary than C. This is due to the fact that as we "move up" from the bottom cluster, we find a $\neg C$-world before a $\neg A$-world. $\neg C$ is more readily "accepted" than $\neg A$, so C is less necessary (or less entrenched). Notice, since there are no worlds satisfying (for instance) $\neg A \wedge \neg C$ in the model, we judge all such worlds to have no possibility, $\pi(w) = 0$. Correspondingly, according to our definition of $\leq_{PM}$, $\neg A \wedge \neg C$ is (strictly) less possible than any sentence

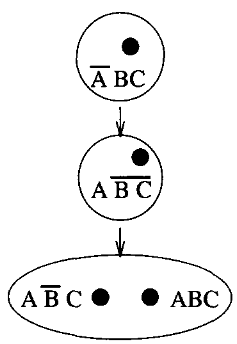

Figure 2. Qualitative necessity and possibility.

[6] We use $\leq_{PM}$ to indicate greater possibility rather than $\geq_{PM}$ to remain consistent with [24, 26] and other expositions (where this operator is related to other concepts).

satisfied in the model. Furthermore, every sentence in the language is at least as possible as $\neg A \wedge \neg C$. That is, $\alpha \leq_{MP} \neg A \wedge \neg C$ for all α (including $\alpha = \perp$). Thus, we see that $\Pi(\neg A \wedge \neg C) = 0$, since in (normalized) possibility measures $\Pi(\perp) = 0$. Correspondingly, the sentence $A \vee C$ is completely necessary: $N(A \vee C) = 1$.

To give a concrete interpretation for this model, imagine that A, B, and C stand for the propositions that Anne, Bill, and Cheryl go to a certain party, respectively. We believe that Anne and Cheryl will go, but we are not sure about Bill—it is completely possible that he will go and completely possible that he will not. While we believe that both Anne and Cheryl will go, if somehow we were convinced that one of them would not, we would say that Cheryl would stay home, since A is more necessary than C. Notice that Bill will stay home then (the belief $B \supset C$ is maximally necessary). Finally, though this would be hard to do, if someone convinced us that Anne would stay home, we would still maintain the belief $A \vee C$ (again, fully necessary) and conclude that Cheryl will go.

As mentioned above, one should think of the $\preccurlyeq$ -maximal worlds in a model (those with maximal possibility) as representing the epistemic state of the agent in question. In other words, each maximally possible world is epistemically possible. In this example, we consider the two lowest worlds to be those consistent with the agent's beliefs, while all other worlds violate some belief. In a (normalized) possibility distribution, these two worlds are assigned degree of possibility 1. Hence, $\Pi(A) = \Pi(B) = \Pi(\neg B) = \Pi(C) = 1$, while $0 < \Pi(\neg A) < \Pi(\neg C) < 1$. In this model, the agent believes $A \wedge C$. In the next section we will see how belief can be expressed at the object level.

We can show two key results concerning this model of qualitative possibility and necessity.

THEOREM 3 *Any qualitative necessity ordering determined by a CO-model M is a qualitative necessity ordering satisfying postulates $(N1)$ through $(N6)$.*

Proof We take $M = \langle W, \preccurlyeq, \varphi \rangle$ to be a CO*-model with worlds w, v, etc. Recall that

$$A \geq_{EM} B \quad \text{means} \quad M \models \overleftrightarrow{\Box}(\neg A \supset \Diamond \neg B).$$

(N1) $\overleftrightarrow{\Box}(\neg A \supset \Diamond \neg A)$ is a simple theorem of CO, so $A \geq_{EM} A$.

(N2) Assume A and B are satisfiable (if not, $A \geq_{EM} B$ or $B \geq_{EM} A$). Suppose $A \not\geq_{EM} B$. Then for some w, $M \models_w \neg A \wedge \Box B$. This means that, for any $\neg B$-world v, $v \preccurlyeq w$ and $M \models_v \Diamond \neg A$. Hence, $M \models \overleftrightarrow{\Box}(\neg B \supset \Diamond \neg A)$ and $B \geq_{EM} A$.

(N3) Suppose $A \geq_{EM} B$ and $B \geq_{EM} C$. Then M satisfies both $\overleftrightarrow{\Box}(\neg A \supset \Diamond \neg B)$ and $\overleftrightarrow{\Box}(\neg B \supset \Diamond \neg C)$; thus, $\overleftrightarrow{\Box}(\neg A \supset \Diamond \neg C)$ holds as well, and $A \geq_{EM} C$.

(N4) Clearly, $\overleftrightarrow{\Box}(\bot \supset \Diamond \top)$ is a theorem of CO; and clearly, $\overleftrightarrow{\Box}(\top \supset \Diamond \bot)$ is unsatisfiable in CO (since $\Diamond \bot$ is unsatisfiable). Thus, $\top >_{EM} \bot$.

(N5) $\overleftrightarrow{\Box}(\bot \supset \Diamond A)$ is a theorem of CO; so for all A, $\top \geq_{EM} A$.

(N6) Suppose $B \geq_{EM} C$; so $M \models \overleftrightarrow{\Box}(\neg B \supset \Diamond \neg C)$. Let $M \models_w \neg(A \wedge B)$. If $M \models_w \neg A$ then $M \models_w \Diamond \neg A$ and $M \models_w \Diamond \neg(A \wedge C)$ If $M \models_w \neg B$ then $M \models_w \Diamond \neg C$ (since $M \models \overleftrightarrow{\Box}(\neg B \supset \Diamond \neg C)$). Thus, $M \models_w \Diamond \neg(A \wedge C)$. Hence, $M \models_w \neg(A \wedge B) \supset \Diamond \neg(A \wedge C)$ for all w; i.e., $A \wedge B \geq_{EM} A \wedge C$. ■

THEOREM 4 *For any ordering $\geq_N$ satisfying postulates $(N1)$ through $(N6)$ there is a CO-model M determining the corresponding qualitative necessity ordering*: $A \geq_{EM} B$ *iff* $A \geq_N B$.

Proof The proof uses the technique of Grove [26]. Let $\geq_\pi$ be the possibility ordering determined by $\geq_N$; i.e., $A \geq_\pi B$ iff $\neg B \geq_N \neg A$. For this ordering $\geq_\pi$, let a cut $\mathscr{C}$ be any set of sentences satisfying the following closure property:

$$\text{If } A \in \mathscr{C} \text{ and } A \geq_\pi B, \text{ then } B \in \mathscr{C}.$$

Thus, a cut contains all sentences with at most a specified degree of possibility.

It is easy to verify that cuts are totally ordered under set inclusion. Let $\mathscr{C}$ and $\mathscr{D}$ be two cuts with $A \in \mathscr{C}$ and $B \in \mathscr{D}$. For any pair of sentences A, B we have $A \geq_\pi B$ or $B \geq_\pi A$, so either $A \in \mathscr{D}$ or $B \in \mathscr{C}$. So $\mathscr{C} \subseteq \mathscr{D}$ or $\mathscr{D} \subseteq \mathscr{C}$.

Now we define a model $M = \langle W, \preccurlyeq, \varphi \rangle$ where W is (as usual) the set of all maximal consistent sets of propositional sentences over our fixed language, φ is given by set membership of atoms, and $\preccurlyeq$ is defined as:

$$w \preccurlyeq v \text{ iff for every cut } \mathscr{C}, v \cap \mathscr{C} \neq \emptyset \text{ implies } w \cap \mathscr{C} \neq \emptyset.$$

Clearly, $\preccurlyeq$ is reflexive and transitive, and since cuts are nested, it is easy to see $\preccurlyeq$ is connected as well.

If B is unsatisfiable, by (N5), $A \geq_\pi B$ for all A, and in CO* we have that $\overleftrightarrow{\Box}(B \supset \Diamond A)$ is valid. So $\geq_\pi$ and $\leq_{PM}$ correspond for any such B. Now assume B is satisfiable.

1. $A \geq_\pi B$ implies $A \leq_{PM} B$: Let $A \geq_\pi B$ and $M \models_w B$. We will show each such w sees some A-world (so $M \models \overleftrightarrow{\Box}(B \supset \Diamond A)$). Let the family of cuts intersecting w be

$$\mathscr{S} = \{\mathscr{C} : \mathscr{C} \cap w \neq \emptyset\},$$

Then $\mathscr{C} = \cap\mathscr{S}$, the intersection of all cuts in $\mathscr{S}$, is clearly a cut, and $\mathscr{C} \cap w \neq \varnothing$ (since cuts are nested).

Case 1: Consider the set $\{\neg D: D \in \mathscr{C}\}$. If this set is consistent with A, it can be extended to a maximal set v which includes A. Clearly, $\mathscr{C} \cap v = \varnothing$. As $\mathscr{C} \cap w \neq \varnothing$, any cut which intersects v contains $\mathscr{C}$, and intersects w as well. Hence, $w \preccurlyeq v$. (Moreover, $v \not\preccurlyeq w$.)

Case 2: Suppose $\{\neg D: D \in \mathscr{C}\}$ is inconsistent with A. For some $D_1, \ldots, D_n \in \mathscr{C}$,

$$\vdash \neg D_1 \wedge \cdots \wedge \neg D_n \supset \neg A.$$

In other words,

$$\vdash A \supset D_1 \vee \cdots \vee D_n.$$

Using (N2), we see that $D_1 \geq_\pi A$ or $\ldots D_n \geq_\pi A$ This means $A \in \mathscr{C}$. Now let $\mathscr{D}$ be any cut smaller than $\mathscr{C}$, i.e., any $\mathscr{D} \subset \mathscr{C}$. Now consider the set $\{\neg D: D \in \mathscr{D}\}$. As shown above, if this set is consistent with A, it can be extended maximally to include A, determining a world v such that $\mathscr{D} \cap v = \varnothing$. If it is inconsistent with A, as discussed above, it must be that $A \in \mathscr{D}$. But since $A \geq_\pi B$, this implies $B \in \mathscr{D}$, hence that $\mathscr{D} \cap w \neq \varnothing$, contradicting the fact that $\mathscr{D} \subset \mathscr{C}$. Hence, there exists a world v satisfying A such that whenever $\mathscr{D} \subset \mathscr{C}$ it must be that $\mathscr{D} \cap v = \varnothing$. Thus $w \preccurlyeq v$.

In both cases, for any B-world $w \in W$, there exists an A-world v such that $w \preccurlyeq v$. Hence, $M \models \overleftrightarrow{\Box}(B \supset \Diamond A)$.

2. $A \leq_{PM} B$ implies $A \geq_\pi B$: Let $A \leq_{PM} B$. This means for each w such that $M \models_w B$ there is some A-world v such that $w \preccurlyeq v$. Suppose $a \in \mathscr{C}$ for some cut $\mathscr{C}$. If $B \notin \mathscr{C}$, $\{B\} \cup \{\neg D: D \in \mathscr{C}\}$ is consistent (as described in Case 2). Thus, this set can be extended to a maximal consistent set w, such that $M \models_w B$ and $w \cap \mathscr{C} = \varnothing$. Now any v such that $M \models_v A$ has a non-empty intersection with $\mathscr{C}$, and by definition of accessibility in M is such that w cannot see v. That is, the B-world w sees no A-worlds, contradicting the fact that $M \models \overleftrightarrow{\Box}(B \supset \Diamond A)$. Therefore, $B \in \mathscr{C}$ for any cut $\mathscr{C}$ containing A; so $A \geq_\pi B$. ■

These results show that necessity orderings satisfying the postulates and the qualitative necessity orderings determined by CO-models are exactly the same. It immediately follows that the space of qualitative possibility orderings determined by CO-models corresponds precisely to the set of qualitative possibility orderings proposed by Dubois and Prade.

As discussed by Dubois [7], for finitary languages a qualitative possibility measure is compatible with a mapping of sentences into the interval [0, 1] iff the mapping is a possibility measure (where compatibility means $A \geq_{\pi} B$ iff $\Pi(A) \geq \Pi(B)$). Hence, plausibility orderings in CO are precise qualitative counterparts of possibility measures, and entrenchment orderings correspond to necessity measures. Our treatment of possibility and necessity generalizes that of Dubois and Prade by permitting qualitative orderings on infinite languages (with CO-models serving as adequate representation structures).

3. BELIEFS AND CONDITIONALS

3.1. Beliefs

We have seen how to express qualitative possibility and necessity measures using two modal operators. More importantly, the relationship to CO allows us to exhibit connections between possibility theory and other forms of defeasible reasoning. We begin by explaining the connections to belief logics and belief revision. The ordering $\geq_{EM}$ determined by some CO-model turns out to be an expectation ordering in the sense of Gärdenfors and Makinson [27]. These are weakenings of orderings of epistemic entrenchment [13]. Assuming an agent to possess a deductively closed belief set K, an entrenchment relation is an ordering of the elements of K reflecting the extent to which the agent is willing to give up those beliefs. For instance, suppose A and B are in K and B is more entrenched than A. If the agent learns $\neg(A \wedge B)$, contradicting these earlier beliefs, the agent will reject A in favor of the more entrenched belief B.[7]

We denote by $A \leq_E B$ the fact that B is at least as entrenched as A. The ordering on beliefs can be extended to all sentences in $\mathbf{L}_{CPL}$ by simply requiring that nonbeliefs, those $A \notin K$, have the least degree of entrenchment. It is easy to see the intuitive correspondence now to necessity orderings. An entrenchment ordering must satisfy the following postulates [13]:

(E1) If $A \leq_E B$ and $B \leq_E C$ then $A \leq_E C$

(E2) If $A \vdash B$ then $A \leq_E B$

(E3) If $A, B \in K$ then $A \leq_E A \wedge B$ or $B \leq_E A \wedge B$

(E4) If $K \neq Cn(\perp)$ then $A \notin K$ iff $A \leq_E B$ for all B

(E5) If $B \leq_E A$ for all B then $\vdash A$

[7] We will see in the next section how an entrenchment ordering determines a natural theory of belief revision.

Dubois and Prade [9] show a partial correspondence between qualitative necessity and entrenchment. For any necessity ordering $\geq_N$, they define the set of beliefs associated with $\geq_N$ to be

$$K = \{A: A >_N \perp\}$$

Assuming that $N(\perp) = 0$ for any necessity measure used to "generate" the necessity ordering, this means $N(A) > 0$. Thus, A is believed just when it has some degree of necessity. Entrenchment and qualitative necessity correspond if we ignore (N4) and (E5). Entrenchment orderings fail to satisfy (N4) only when every sentence is equally entrenched (including $\perp$); that is, when we are dealing with the inconsistent belief set. We will ignore this case and assume that entrenchment orderings are nontrivial, satisfying (N4).[8]

Qualitative necessity fails to satisfy (E5) because certain nontautologous beliefs are allowed to be certain or completely necessary (i.e., $N(A) = 1$). For example, the CO-model in Figure 2 captures the necessity ordering where the belief $B \supset C$ is completely necessary; that is, it is as necessary as the tautologous belief $\top$. This is due to the fact that no $B \wedge \neg C$-world is accorded positive possibility. In general, necessity orderings determined by CO-models will not satisfy (E5). But if we consider only CO*-models, every logically consistent A has some degree of possibility, and every contingent sentence will be less certain than $\top$. In this sense, CO*-models are full, for all logically possible situations (thus, all consistent sentences) are accorded some degree of possibility.

PROPOSITION 5 *Let M be a CO*-model. Then for all satisfiable $A \in \mathbf{L}_{CPL}$, we have $A <_{PM} \perp$.*

COROLLARY 6 *Let M be a CO*-model. Then for all falsifiable $A \in \mathbf{L}_{CPL}$, we have $A <_{EM} \top$. Thus (E5) is satisfied by the full qualitative necessity ordering determined by any CO*-model.*

THEOREM 7 [24] *Any qualitative necessity ordering determined by a CO*-model satisfies (E1)–(E5).*

Proof The proof proceeds exactly as that for Theorem 4 with the obvious modifications for (E5). ■

THEOREM 8 [24] *For any entrenchment ordering $\leq_E$ there is a CO*-model M determining the corresponding qualitative necessity ordering: $A \leq_{EM} B$ iff $A \leq_E B$.*

[8] We can capture the trivial ordering by considering the empty "CO-model" as a model for entrenchment. Axiomatically we can express the ordering using the inconsistent theory $\{\perp\}$.

Proof The proof proceeds exactly as that for Theorem 4 with the obvious modifications for (E5). ■

Of course, the real reason for examining logics of qualitative necessity and possibility is to provide a method of expressing and reasoning with qualitative constraints on necessity and possibility (i.e., premises) without relying on complete knowledge of a possibility ordering or measure. Given certain constraints we can determine through logical deduction what must be true in all measures or orderings satisfying these constraints. Thus, one may express a set of qualitative constraints on the relative necessity of beliefs in a knowledge base, or on the plausibility of arbitrary sentences, or even arbitrary Boolean combinations (e.g., negations or disjunctions) of such constraints. These might then be used to determine the relative strength of other beliefs, how one would revise the belief set or determine appropriate default conclusions (see the next section).

The expressive power of CO and CO* can also be used to capture notions that are not amenable to direct analysis using a simple language of qualitative necessity or possibility (e.g., Fariñas and Herzig's QPL). Naturally, we'd like to express relationships of qualitative possibility. In QPL one may assert $A \geq_\pi B$, while in CO we say $\overleftrightarrow{\Box}(B \supset \Diamond A)$ to indicate that A is as possible as B. Absolute concepts such as belief, disbelief, possibility, and necessity are important as well. Recall that we take sentences with maximal possibility ($\Pi(A) = 1$) to be the epistemic possibilities of an agent. Sentence A is believed just when $\neg A$ is epistemically impossible. Thus, belief can be expressed in QPL; for example, "A is believed" is just $\top >_\pi \neg A$. Semantically, in CO, belief in A means that A holds at the set of most possible worlds. This can be expressed in our language as the sentence $\overleftrightarrow{\Diamond} \Box A$, which simply asserts that, at some point in the model, A holds at all more possible worlds. Thus A is forced to hold at the most possible worlds in the structure (i.e., $\neg A$ is epistemically impossible). We can define an epistemic logic based on CO by defining a belief modality in this way:

DEFINITION *The connective* B *is defined in CO as*

$$\mathsf{B}A \equiv_{df} \overleftrightarrow{\Diamond} \Box A.$$

If we consider those valid sentences of CO where the occurrence of modal operators is restricted to conform to the pattern in this definition, we have a modal logic of belief. This logic is precisely the modal logic KD45 (or weak S5), one of the most commonly used logics of belief in AI.

DEFINITION *Let* $\mathbf{L}_B$ *be the language constructed in the usual way from propositional connectives and the defined connective* B. *Let PB be the set of* *sentences* $CO \cap \mathbf{L}_B$ *(translating occurrences of* $\overleftrightarrow{\Diamond}\Box$ *to* B*). We think of PB as the belief logic induced by possibility theory, with modal belief operator* B.

THEOREM 9 *PB is the modal logic KD45.*

Proof To show that PB $\subseteq$ KD45, we demonstrate that each of the axioms and rules of inference of KD45 are valid in CO. A standard axiomatization of KD45 is based on propositional logic plus the following [22]:

K $\mathsf{B}(A \supset B) \supset (\mathsf{B}A \supset \mathsf{B}B)$

D $\mathsf{B}A \supset \neg\mathsf{B}\neg A$

4 $\mathsf{B}A \supset \mathsf{BB}A$

5 $\neg\mathsf{B}A \supset \mathsf{B}\neg\mathsf{B}A$

Nec From A infer $\mathsf{B}A$

We show (semantically) that each of these is valid in CO (treating $\mathsf{B}A$ as $\overleftrightarrow{\Diamond}\Box A$). Let $M = (W, \preccurlyeq, \varphi)$ be a CO-model with worlds w, v, etc. We note that due to the nature of the connective $\overleftrightarrow{\Diamond}$, $M \models_w \overleftrightarrow{\Diamond}A$ for some $w \in W$ iff $M \models_w \overleftrightarrow{\Diamond}A$ for each $w \in W$. Thus, the sentence $\mathsf{B}A$ holds at some world in M iff it holds at all worlds in M.

K: Suppose $M \models \mathsf{B}(A \supset B)$ and $M \models \mathsf{B}A$. Then, for some $w, v \in W$, $M \models_w \Box(A \supset B)$ and $M \models_v \Box A$. Since either $w \preccurlyeq v$ or $v \preccurlyeq w$, we must have that either $M \models_w \Box(A \wedge (A \supset B))$ or $M \models_v \Box(A \wedge (A \supset B))$. Clearly, then $M \models \mathsf{B}B$.

D: If $M \models \mathsf{B}A$ then $M \models_v \Box A$ for some v. Now, for all $w \in W$, either $w \preccurlyeq v$ or $v \preccurlyeq w$. If $w \preccurlyeq v$ then clearly $M \not\models_w \Box\neg A$ (since $M \models_v A$). If $v \preccurlyeq w$, then $M \not\models_w \Box\neg A$ (since $M \models_v \Box A$). Thus, $M \models \neg\mathsf{B}\neg A$.

4: If $M \models \mathsf{B}A$ then $M \models_w \mathsf{B}A$ for each $w \in W$. Thus, $M \models \Box\mathsf{B}A$ and $M \models \overleftrightarrow{\Diamond}\Box\mathsf{B}A$ (i.e., $M \models \mathsf{BB}A$).

5: Proof as for axiom **4**.

Nec: Obvious.

Thus, each theorem of KD45 is also a theorem when translated into CO (i.e., a theorem of PB).

To show that KD45 $\subseteq$ PB, we simply need to show that each nontheorem of KD45 is not a theorem of CO. We proceed by constructing, for each sentence $A \notin KD45$, a CO-model that falsifies A.

Suppose A is not a theorem of KD45. Then there is a KD45-model M' that falsifies A. From [16], we take a KD45-model to be a structure $\langle W, w\rangle$ where W is a set of worlds and w is a world. Propositional formulae are evaluated in the standard fashion at world w, while belief formulae (and subformulae) of the form $\mathsf{B}A$ are satisfied by the model $\langle W, w\rangle$ iff the model $\langle W, v\rangle$ satisfies A for each $v \in W$.

Let $M' = \langle W, w\rangle$ be a KD45-model falsifying A. We construct a CO-model that verifies (and falsifies) exactly the same sentences, thus demonstrating that A is not CO-valid. The required model is $M = \langle W \cup \{w\}, \preccurlyeq \rangle$ (we assume the valuation function is specified by the worlds themselves), were $v \preccurlyeq u$ iff $u \in W$. In other words, M consists of two clusters of worlds, W and $\{w\}$, where the worlds in W are equally possible, but more possible than w. It is easy to verify that $M \models_w \overleftarrow{\Diamond} \Box A$ iff $M' \models \mathsf{B}A$ (and more generally that $M \models_w A$ iff $M' \models A$ for any sentence A). Thus each KD45-falsifiable sentence is falsifiable in CO under the prescribed translation. ■

Thus, the natural epistemic logic induced by our interpretation of qualitative possibility theory is exactly what one might hope. KD45 has some of the following interesting properties. It verifies the usual introspection axioms:

$$\mathsf{B}A \supset \mathsf{BB}A \quad \text{and} \quad \neg\mathsf{B}A \supset \mathsf{B}\neg\mathsf{B}A$$

Furthermore, $\mathsf{B}A$ and $\mathsf{B}\neg A$ are mutually inconsistent, so an agent's beliefs must be consistent. But A and $\mathsf{B}\neg A$ are generally consistent, so a sentence can be believed even though it is (actually) false. We list here some of the various epistemic attitudes an agent can hold towards a proposition, and the corresponding constraints these induce on a possibility measure.

- When A is believed it must have some degree of necessity ($N(A) > 0$). In CO this is expressible as $\mathsf{B}A$. The model in Figure 2 satisfies $\mathsf{B}A$ and $\mathsf{B}C$. The inconsistency of $\mathsf{B}A$ and $\mathsf{B}\neg A$ corresponds to the fact that at least one of $\Pi(A)$ or $\Pi(\neg A)$ must equal 1 (or equivalently, one of $N(A)$ or $N(\neg A)$ must equal 0).
- Disbelief is expressed as $\neg\mathsf{B}A$. This is true just when $N(A) = 0$, or $\Pi(\neg A) = 1$. In other words, A is disbelieved iff its negation is accorded epistemic possibility. Notice that $\neg\mathsf{B}A$ and $\neg\mathsf{B}\neg A$ are mutually consistent. This corresponds to the fact that possibility measures allow both A and $\neg A$ to have possibility 1.
- If A has some degree of possibility ($\Pi(\neg A) > 0$), $\neg A$ cannot be certain. This holds exactly when $\overleftrightarrow{\Diamond} A$ is verified by a model (A is true at some world with a nonzero degree of possibility).
- Sentence A is completely necessary ($N(A) = 1$) exactly when $\overleftrightarrow{\Box} A$ holds. The model in Figure 2 satisfies $\overleftrightarrow{\Box}(A \vee B)$ since $A \vee B$ holds at each world (it is completely necessary). $\neg A \wedge \neg B$ is accorded no possibility at all. Notice that in CO* only tautologies are completely necessary.

- Some degree of necessity is assigned to A ($N(A) > 0$) just when it is believed; that is, $\mathsf{B}A$ is true. So, in the example of Figure 2, B and $\neg B$ have a necessity measure of zero (since neither is believed). A and C are accorded some (less than absolute) degree of necessity, with A being more necessary (or entrenched) than C.

To reason effectively with a logic of belief, we need not only the ability to express what is believed, but also what is not. The logics CO and CO* allow us to express the concept of only knowing. To only know (or only believe) a sentence A is to believe A and to believe nothing more than is required by A [16]. For example, given a (finite) knowledge base KB, we usually intend that KB is all that is believed.[9] If $KB \models A$ then A is believed; if $KB \not\models A$ then A is not believed. In usual epistemic logics, merely asserting $\mathsf{B}(KB)$ does not carry this force. Indeed, $\mathsf{B}(KB)$ does not preclude the possibility of $\mathsf{B}A$ when $KB \cup \{A\}$ is consistent, even if $KB \not\models A$. In other words, from $\mathsf{B}(KB)$ we cannot derive $\neg \mathsf{B}A$.

To express that KB is believed in QPL, we need only assert that $\top >_{\pi} \neg KB$ (or that $N(KB) > 0$ in a quantitative setting). But there are no convenient and systematic means of asserting that these are the only beliefs, or that these are the only sentences that have some positive degree of necessity. For example, if we have a knowledge base consisting of only the propositions A and C (Anne and Cheryl will go to the party), then intuitively we expect our knowledge base to answer that B (Bill will go) is possible. No information rules out B (or $\neg B$ for that matter). However, just as with the usual belief logics, simply asserting that A and C have some degree of necessity does not allow one to conclude that B has some degree of possibility.

In CO*, we can express the fact that KB is all that is believed using the sentence

$$O(KB) \equiv_{df} \overleftrightarrow{\Box}\big(KB \supset (\Box\, KB \wedge \overleftarrow{\Box} \neg KB)\big)$$

This sentence ensures that only KB-worlds are most possible (since $\overleftrightarrow{\Box}(KB \supset \Box\, KB)$) and that all KB-worlds are equally (hence, most) possible (since $\overleftrightarrow{\Box}(KB \supset \overleftarrow{\Box} \neg KB)$). Thus, any model satisfying $O(KB)$ is such that the set of KB-worlds form the maximal (most possible) cluster of worlds. This guarantees that any sentence A is disbelieved if $KB \not\models A$. To see this, notice that, since $KB \cup \{\neg A\}$ is consistent, there must be some $\neg A$-world in the maximal cluster, assigned a possibility of 1.

[9] We will often use KB as if it were the conjunction of the elements (a sentence). For a more complete discussion of only knowing see [16, 24, 25].

Typically, we consider only CO*-models when discussing "all that is known," for this terminology suggests that no logical possibilities should be excluded from consideration. When *KB* is believed, only *KB*-worlds can be accepted as epistemically possible. When *KB* is all that is believed, not only should *KB* be believed, but every *KB*-world should be accepted as epistemically possible. If some world is not accepted, then there should be some belief that excludes this world from consideration, some belief falsified by that world. If a world satisfies *KB*, there is no such belief when *KB* is all that is believed.

For a purely propositional *KB* we have that

$$O(KB) \models_{CO*} \mathsf{B}A \qquad \text{iff } KB \models A$$

$$O(KB) \models_{CO*} \neg\mathsf{B}A \qquad \text{iff } KB \not\models A$$

(see [24, 25] for details). In particular, the only sentences assigned a degree of necessity greater than 0 are those entailed by *KB*. In a natural and convenient fashion we can summarize what would require an infinite set of sentences in QPL, (or an unwieldy number for finite languages). The model in Figure 2 satisfies $O(A \wedge C)$, assuming a language with only three atoms (and ignoring the fact that this is not a CO*-model). Thus, $O(A \wedge C)$ is a concise way of expressing that only the consequences of $A \wedge C$ are assigned a positive degree of necessity. Again, this is crucial since, when one specifies some knowledge base *KB*, it is usually intended that only those sentences derivable from *KB* are believed.

The expressive power of CO goes beyond this, however, Nothing prevents the occurrence of nonpropositional sentences in *KB*. We can have belief sentences in *KB*, and even sentences of an autoepistemic nature. In fact, in [25] we show that CO* subsumes autoepistemic logic. For example, applying the *O* operator to a *KB* containing just

$$\mathtt{bird} \wedge \neg\mathsf{B}\neg\mathtt{fly} \supset \mathtt{fly},\ \mathtt{bird}$$

allows one to conclude B(`fly`). Furthermore, with the ability to express only knowing at the object-level, appropriate nonmonotonic conclusions can be reached without appeal to metalogical notions like extensions. So we can think of CO* as adding to qualitative possibility logic the ability to express autoepistemic reasoning. With this connection, of course, degrees of possibility or necessity can be interpreted as generalizing autoepistemic logic as well. We now turn our attention to this generalization.

3.2. Conditional Beliefs.

One problem with epistemic logics is their categorical nature. Sentences can be believed or disbelieved, but no "degrees" of belief can be associated with sentences. Even autoepistemic logic does not actually allow one to represent the defeasibility of beliefs: its nonmonotonicity is essentially due to its indexical nature [15].

Possibility theory allows certain beliefs to be held more firmly than others. Those beliefs with higher degree of necessity are less readily given up than those that are less necessary. This distinction between more or less necessary beliefs is captured semantically in CO by the relative possibility of the "epistemically impossible" worlds. By equating necessity with entrenchment of beliefs, it becomes clear that qualitative possibility theory determines a theory of belief revision.

Gärdenfors [13] describes how an entrenchment ordering determines a method of belief revision. As discussed above, revision is constrained by the general rule stating that more entrenched beliefs should be retained while less entrenched beliefs are sacrificed. This process can be modeled in CO and CO* as described in [23, 24]. Assume an agent possesses a belief set K such that $K \vdash \neg A$. This epistemic state is reflected in a CO-model (or a qualitative possibility ordering) where the set of K-worlds is accorded the highest degree of possibility; in other words, the agent only knows K. Should the agent come to learn A, some revision of K must take place. In fact, since $\neg A$ holds at all K-worlds, all K-worlds must become epistemically impossible given acceptance of A: the new belief set, denoted K_A^*, must consist of only A-worlds. To "minimally" change K, we simply require that the set of most possible A-worlds represent the agent's new epistemic state. To express these considerations semantically, we define a new connective $\Rightarrow$ [23, 24]. We read $A \Rightarrow B$ as "revising by A results in belief in B." This is true just when B is true at each world in the set of most possible A-worlds. When this set of A-worlds is considered to represent an agent's new epistemic state after revising by A, the B is believed in this new state. This can be expressed as follows, giving rise to a modal definition of the conditional connective:

$$A \Rightarrow B \equiv_{df} \overleftrightarrow{\Box} \neg A \vee \overleftrightarrow{\Diamond}(A \wedge \Box(A \supset B)).$$

It should be easy to see that this notion of revision respects the degree of entrenchment of beliefs in *KB*. In fact, in [23, 24] we show that model of belief revision defined in this way is equivalent to the classic AGM postulates of Alchourrón, Gärdenfors, and Makinson [28]. Thus, a qualitative possibility ordering naturally determines a theory of revision equivalent to the AGM theory. This connective can be related to our possibility ordering as follows:

THEOREM 10 *Let M be a CO-model. Then $M \models A \Rightarrow B$ iff $A \wedge B <_{PM} A \wedge \neg B$ or $M \models \overleftrightarrow{\Box} \neg A$.*

Proof If $M \models \overleftrightarrow{\Box} \neg A$ then the equivalence holds trivially. So assume $M \models \overleftrightarrow{\Diamond} A$. Then we have that

$$
\begin{aligned}
A \wedge B <_{PM} A \wedge \neg B \\
&\text{iff for some } w \in W,\ M \models_w (A \wedge B) \wedge \Box \neg (A \wedge \neg B) \\
&\text{iff } M \models_w (A \wedge B) \wedge \Box (A \supset B) \\
&\text{iff } M \models_w A \wedge \Box (A \supset B) \\
&\text{iff } M \models \overleftrightarrow{\Diamond} (A \wedge \Box (A \supset B)) \\
&\text{iff } M \models A \Rightarrow B.
\end{aligned}
$$

■

COROLLARY 11 *$M \models A \Rightarrow B$ iff $A \supset \neg B <_{EM} A \supset B$ or $M \models \overleftrightarrow{\Box} \neg A$.*

In other words, $A \Rightarrow B$ holds just when $A \wedge B$ is more possible than $A \wedge \neg B$ (or A is impossible, $\Pi(A) = 0$).

CO and CO* have been also used as conditional logics for representing default rules. In [20, 29] we use the same conditional connective $\Rightarrow$, reading $A \Rightarrow B$ as "A normally implies B." We can show that CO*, used in this way, captures Lehmann's [12] rational consequence relations and the nonstandard probabilistic semantics of Adams [1] and Goldszmidt and Pearl [2, 30]. In fact, as we describe in [24, 31], one may view this approach to default reasoning as a form of belief revision, in particular, as revision of a theory of expectations (see also Gärdenfors and Makinson [27]). Thus we can define an inferential relation on conditional sentences (or default rules) using qualitative possibility, and it will be equivalent to a number of other systems of defeasible inference.

EXAMPLE Let A, S, E stand for "adult," "grad student," and "employed," respectively, and consider the following set of premises (a standard example from the default reasoning literature):

$$\{A \Rightarrow E, S \Rightarrow A, S \Rightarrow \neg E\}$$

Our conditionals are exception-allowing since $A \wedge \neg E$ is consistent with this theory. Preference for more specific defaults is automatically incorporated into the definition of $\Rightarrow$ as well. From this theory we can derive $S \wedge A \Rightarrow \neg E$ using consequence in CO, but we cannot derive $S \wedge A \Rightarrow E$. Also derivable are constraints on permissible possibility assignments, for example, it must be that $A >_{\pi} S$. It is more plausible that someone is simply an adult than a grad student. Also if we assert $\mathsf{B}A$ we can derive $\mathsf{B}E$.

The relationship with Adams's and Pearl's ε-semantics holds particular interest since its semantic foundations rely on probabilistic notions. Adams proposed that conditional sentences can be interpreted as making statements regarding arbitrarily high conditional probabilities. Pearl [2] has suggested that such conditionals can be interpreted as default rules. Let $A \rightarrow B$ denote such a conditional sentence. The meaning of a set T of such rules (a default theory) is given a nonstandard probabilistic interpretation. The semantics (dubbed ε-semantics) can be presented briefly as follows.

DEFINITION [30] *Let T be a default theory. T is ε-consistent if, for each $\varepsilon > 0$, there is a proper probability assignment P such that $P(B|A) \geq 1 - \varepsilon$ for all $A \rightarrow B \in T$.*

We will say such an assignment P satisfies T to degree ε.

DEFINITION [30] *Let T be an ε-consistent default theory and $A \rightarrow B$ be a conditional rule. T ε-entails $A \rightarrow B$ iff there exists a proper probability assignment for $T \cup \{A \rightarrow B\}$ and, for all $\varepsilon > 0$, there exists some $\delta > 0$ such that for any proper assignment P that satisfies T to degree δ, $P(B|A) \geq 1 - \varepsilon$.*

By presenting qualitative possibility and ε-semantics within our modal framework we can show the following equivalence. We assume $A \rightarrow B$ is some abstract conditional, being interpreted either as $\Rightarrow$ in CO or as a default rule in the sense of ε-semantics (we assume A is satisfiable).

THEOREM 12 [20] *Let T be a finite conditional theory consisting of sentences of the form $A \rightarrow B$, where $A, B \in \mathbf{L}_{CPL}$. Then $T \vdash_{CO} A \rightarrow B$ iff T ε-entails $A \rightarrow B$.*

This connection has also been examined by Dubois and Prade [17]. Using CO as the intermediate framework between ε-semantics and qualitative possibility allows us to see the underlying semantic commonality in these systems. Adams's [1] construction for determining the ε-consistency of conditional theories can be interpreted as ranking possible worlds according to their degree of probability. Given this ranking, it is easy to ensure that the conditional probabilities of the statements in the theory are as high as they need to be. But this ranking can also be construed as a simple CO-model. Our interpretation of CO-models in this paper equates this ranking with the degree of possibility of worlds. On either interpretation of the models, the same conclusions are derivable from simple conditional theories. Hence, degrees of possibility can readily be associated with arbitrarily high probabilities. That is, if $\Pi(A) > \Pi(B)$ then $P(A)$ can be made higher (to at least degree $1 - \varepsilon$) than $P(B)$, for all such A and B.

The results of Boutilier [20, 29] also show that ε-semantics can be modeled in the monomodal logic S4. Thus for the purely conditional fragment of qualitative possibility theories, representation and inference can be performed using S4 (and conversely, if S4 is restricted to its simple "conditional" fragment).[10]

Once we allow Boolean combinations of conditionals, it is not clear that the intuitions underlying Adams's approach remain viable. Our semantics for qualitative possibility is more compelling in this case. We must also contrast our approach with the model of conditional possibility adopted by Dubois and Prade [17]. They provide a semantics for (some) Boolean combinations of conditionals defined in terms of possibility measures. Unfortunately, they equate the "weak negation" of a conditional $\neg(A \Rightarrow B)$ with the "strong negation" $A \Rightarrow \neg B$. This is certainly not the case on our definition of conditionals. Merely denying a conditional is no reason to accept that the antecedent justifies acceptance of the negation of the consequent. It should be quite reasonable to say "My door is not (normally) open or closed." In CO, the following is consistent:

$$\neg(A \Rightarrow B) \wedge \neg(A \Rightarrow \neg B).$$

Our extension of the conditional language is much more compelling in this respect.

4. CONCLUDING REMARKS

We have presented two modal logics for reasoning about orderings of qualitative necessity and possibility. The expressive power of CO and CO* can be used to express constraints on possibility measures in a natural and concise fashion (for example, through only knowing). As pointed out by a number of people, the numbers attached to propositions by possibility measures are perhaps of less importance than the ranking of the propositions. We are able to exploit this fact in developing a simple semantic account of qualitative possibility. This view allows us to exhibit the connection between possibility theory and a number of other forms of defeasible reasoning. Furthermore, these modal possibilistic logics provide a means of representing very general constraints on possibility measures, since we allow arbitrary Boolean combinations of formulae.

[10] We note that S4 structures are precisely CO-models without the requirement of connectedness. While this relaxation is not appropriate in general, simple conditional theories cannot express the distinction between the two types of structures. Thus the simple fragment of the (mono-) modal logic S4.3 (characterized by the class of connected, or CO, models) is also equivalent to these logics. See [20] for details.

The simple modal semantics adopted allows us to explore extensions to possibility theory quite easily. For instance, Benferhat, Debois, and Prade [5] show how one might circumvent various problems in conditional default reasoning using possibilistic logic, and how Pearl's [3] System Z (and extensions of it) can be represented. The expressive power of CO* allows one to capture the required assumptions directly in an object-level theory [21]. We have also begun investigations into the process of revising a possibility or entrenchment ordering in the CO-framework [32, 33], rather than simply revising the beliefs based on such an ordering. This suggests a method for accommodating new possibilistic information without requiring actual belief change.

A number of avenues remain to be explored. By generalizing the logic CO, we can explore weaker types of possibilistic semantics. For example, by dropping the requirement of connectedness (obtaining the logic CT40 of [20]) we are in essence modeling partially ordered possibilistic measure sets. A number of other interesting relationships are brought to light by this work as well. Possibilistic logic has strong ties to Shafer's belief functions [19]. This suggests a link to the forms of defeasible reasoning discussed in the last section, a connection we have yet to explore.

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