Log-concavity and compressed ideals in certain Macaulay posets

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Abstract

Let $B_n$ be the poset of subsets of $\{1,2,\ldots,n\}$ ordered by inclusion and $M_n$ be the poset of monomials in $x_1,x_2,\ldots,x_n$ ordered by divisibility. Both these posets have an additional linear order making them what is called Macaulay posets. We show in this paper that the profiles (also called $f$-vectors) of ideals in $B_n$ and $M_n$ generated by the first elements (relatively to the linear order) of a given rank are log-concave.

Keywords: Log-concavity; Binomial coefficients; Macaulay posets; Unimodality

1. Introduction

In this paper, we will prove that some sequences of nonnegative integers arising from extremal set theory are unimodal or log-concave. The question of unimodality is a natural one concerning a sequence of positive integers; however, it can be very hard to solve. There are a lot of such open problems about integer sequences in combinatorics: see [14,2] for an excellent survey on these problems.

The sequences we are interested in are related to shadow functions $\hat{c}_k^P$ of Macaulay posets $(P, \leq, \preceq)$, where $\leq$ is the partial order on $P$ and $\preceq$ is a linear extension of $\leq$. Macaulay posets were defined to bring a unified point of view on analogous theorems; they are presented in [5, Chapter 8]. We will restrict ourselves to the two most well-known Macaulay posets, namely $B_n$ and $M_n$. $B_n$ is the set of subsets of $\{1,2,\ldots,n\}$ partially ordered by inclusion; it has been proved to have the Macaulay property by Kruskal [9] and Katona [8], independently. $M_n$ is the set of monomials in variables $x_1,x_2,\ldots,x_n$ ordered by divisibility (equivalently: the set of multisets of
\{1,2,\ldots,n\} ordered by multiset inclusion); Macaulay [12] proved that \(M_n\) has that property which is now called Macaulay property. Now there are several simpler proofs of these theorems: see e.g. [4,6], or [1].

The necessary terminology is introduced in Section 2, where it is also shown how Macaulay posets can help to prove unimodality of some sequences of positive integers (Proposition 2.2). In Section 3, we look at the particular case of \(B_n\). The main theorem of the paper is Theorem 3.1: it says that, if \(C(m)\) is the set of the first \(m\) elements of \(B_n\) with respect to the linear order \(\leq\), then the sequence \((f_0,f_1,\ldots,f_n)\) is strongly log-concave, where \(f_k\) is the number of elements of \(C(m)\) of rank \(k\). Section 4 is devoted to the poset \(M_n\). We show in Theorem 4.1 a similar but weaker result than in the case of \(B_n\). Both these theorems can be seen as a particular study of the shadow functions \(\overline{c}_k^B\) and \(\overline{c}_k^M\), or simply as log-concavity properties of sequences of sums of binomial coefficients.

2. Macaulay posets

As usual, \(\mathbb{N}\) is the set of nonnegative integers, \(\mathbb{N}^* := \mathbb{N} - \{0\}\), and \([n] := \{1,2,\ldots,n\}\).

We will be interested in ranked posets only, i.e. posets \((P,\leq)\) with a rank function \(r:P \to \mathbb{N}\) satisfying \(r(p) = 0\) for some minimal element \(p \in P\) and \(r(q)=r(p) + 1\) if \(p<q\) and there exists no \(x \in P\) with \(p<x<q\). The rank of \(P\) is defined by \(r(P) := \max\{r(p): p \in P\} \in \mathbb{N} \cup \{\infty\}\). Such a poset is called graded if all its maximal chains have the same finite cardinality. We define the \(i\)th level of \(P\) by \(N_i(P) := \{p \in P : r(p) = i\}\) and set \(W_i(P) := |N_i(P)|\). We often write \(N_i\) and \(W_i\) for \(N_i(P)\) and \(W_i(P)\).

An ideal \(I \subseteq P\) is a subset such that if \(p \in I\) and \(q < p\) then \(q \in I\); if \(E \subseteq P\) then the subset \(\langle E \rangle := \{p \in P : p \leq x\ \text{for some} \ x \in E\}\) is the ideal generated by \(E\). The sequence \(f(I) := (f_0(I), f_1(I), \ldots, f_r(I))\), with \(f_i(I) = |I \cap N_i(P)|\), is called the profile of the ideal \(I\). The shadow of \(P\) is the set \(A(P) := \{q \in P : q < p \ \text{and} \ r(q) = r(p) - 1\}\) and if \(E \subseteq P\) then \(A(E) := \bigcup_{x \in E} A(x)\). Note that if \(F\) is a subset of an ideal \(I\) then \(A(F) \subseteq I\).

A sequence \((a_i)_{i \geq 0}\) of nonnegative integers is called unimodal if there exists an index \(j \geq 0\) such that \(a_i \leq a_{i+1}\) for \(0 \leq i < j\) and \(a_i \geq a_{i+1}\) for \(i \geq j\). It is log-concave if \(a_i^2 \geq a_{i+1}a_{i-1}\) for \(i \geq 1\) and strongly log-concave if \(ia_i^2 \geq (i+1)a_{i+1}a_{i-1}\) for \(i \geq 1\). One can easily see that a sequence which is log-concave and without internal zeros (i.e. there exist no indices \(i < j < k\) with \(a_j = 0\) and \(a_i \neq a_k\)) is unimodal. Note that the profile of an ideal is always without internal zeros; thus, such a log-concave profile is unimodal, in particular.

The problem of minimizing the shadow of a subset of \(P\) is important in the theory of finite sets. Macaulay posets are posets on which there exists an extra linear order which solves the problem. The presentation of Macaulay posets follows [5, Chapter 8.1].

Let \((P,\leq)\) be a ranked poset and let \(\leq\) be a linear order on \(P\). If \(E \subseteq P\) let \(C(m,E)\) be the set of the first \(m\) elements of \(E\) with respect to the linear order \(\leq\). If \(F\) is finite and \(F \subseteq N_k\) then \(C(F)\) or \(CF\) is defined by \(C(F,N_k)\) and is called the compression of \(F\). A set \(E \subseteq P\) is compressed if \(C(E \cap N_i) = E \cap N_i\) for every \(i = 0,\ldots,r(P)\).
In addition, we define the position functions \( p : P \to \mathbb{N}^* \) (resp. \( p_k : N_k(P) \to \mathbb{N}^* \)) by \( p(x) := |\{z \in P: z \preceq x\}| \) (resp. \( p_k(x) := |\{z \in N_k(P): z \preceq x\}| \)).

**Definition.** A ranked poset \((P, \preceq)\) is a Macaulay poset if there exists a linear order \( \preceq \) on \( P \) such that
\[
\Delta(CF) \subseteq C(\Delta F) \quad \text{for every } k \geq 1 \text{ and every finite } F \subseteq N_k(P).
\] (1)

Hence a Macaulay poset is a triple \((P, \preceq, \preceq)\) satisfying the stated property (the linear order \( \preceq \) is of course not unique in general). An equivalent definition can be given as follows:

**Proposition 2.1** (Engel [5, Proposition 8.1.1]). Let \( \preceq \) be a linear order on a ranked poset \((P, \preceq)\).

Then (1) holds if and only if

(i) \( |\Delta F| \geq |\Delta(CF)| \) for every finite \( F \subseteq N_k \), \( k \geq 1 \);

(ii) \( \Delta(CF) = C(\Delta(CF)) \) for every finite \( F \subseteq N_k \), \( k \geq 1 \).

Given a linear order \( \preceq \) on \( P \), the condition (ii) is easy to check. Hence let us focus on the first. Since \( |\Delta(CF)| \) depends on \( |F| \) and \( k \) only, define the shadow functions \( \tilde{\phi}_k^F : [W_k] \to \mathbb{N}^* \) by \( \tilde{\phi}_k^F(m) := |\Delta(CF)| \), with finite \( F \subseteq N_k \) and \( |F| = m \). An explicit description of these functions is necessary for practical use of shadow-minimization in Macaulay posets; moreover, it is the key to the kind of profiles we will consider, namely ideals of the form \( \langle C(m, N_k) \rangle \).

**Definition.** A Macaulay poset \((P, \preceq, \preceq)\) is called shadow increasing if \( \tilde{\phi}_k^F(m) \leq \tilde{\phi}_{k+1}^F(m) \) for all \( k \geq 1 \) and \( 1 \leq m \leq \min\{W_k, W_{k+1}\} \).

**Proposition 2.2** (Engel [5, Proposition 8.1.6]). The profile of a graded shadow increasing Macaulay poset is unimodal.

As a direct consequence of Proposition 2.2 we get that, for a shadow increasing Macaulay poset \( P \), the profile \( f(\langle C(m, N_k(P)) \rangle) \) is unimodal since \( \langle C(m, N_k(P)) \rangle \) is obviously a graded shadow increasing Macaulay poset. Thus, here is a way of proving the unimodality of some ideals’ profiles. But of course, neither the Macaulay property nor the shadow increasing property are easy to prove. However, three families of posets are known to have both properties: chain products \( S(k_1, k_2, \ldots, k_s) \) (see [3,4]), star products \( T(k_1, k_2, \ldots, k_s) \) (see [10,11]) and their duals \( \text{Col}(k_1, k_2, \ldots, k_s) \) (see [5]). In Sections 3 and 4 we will prove stronger versions of Proposition 2.2 for the posets \( B_n \) and \( M_n \) (without using Macaulay or shadow increasing properties).

The poset \( B_n \). Put the following linear order \( \preceq \) (called reverse lex) on \( B_n \): for \( A, A' \in B_n \),
\[
A \prec A' \quad \text{if} \quad \max\{(A \cup A') \setminus (A \cap A')\} \in A'.
\]
Then \((B_n, \leq, \preceq)\) is a Macaulay poset.

Given a fixed \(k \in \mathbb{N}^*\), define the \(k\)-binomial representation of an integer \(m \in \mathbb{N}^*\) by

\[
m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_1}{1} + 1,
\]

where the \(a_i\)'s are natural numbers and \(a_k > a_{k-1} > \cdots > a_1 \geq 0\) (such a representation exists and is unique). Then the operator \(\partial_k^{B_n}\) is given by

\[
\partial_k^{B_n}(m) = 1 + \sum_{i=2}^{k} \binom{a_i}{i-1}
\]

if the \(k\)-binomial representation of \(m\) is \(1 + \sum_{i=1}^{k} \binom{a_i}{i}\).

The poset \(M_n\). Put the following linear order \(\preceq\) on \(M_n\):

\[x_1^{a_1}x_2^{a_2} \ldots x_n^{a_n} \preceq x_1^{b_1}x_2^{b_2} \ldots x_n^{b_n}\]

if \(a_j < b_j\), where \(j = \max\{i : a_i \neq b_i\}\).

Then \((M_n, \leq, \preceq)\) is a Macaulay poset. In this case, the operator \(\partial_k^{M_n}\) is given by

\[
\partial_k^{M_n}(m) = 1 + \sum_{i=2}^{k} \binom{a_i-1}{i-1}
\]

if the \(k\)-binomial representation of \(m\) is \(1 + \sum_{i=1}^{k} \binom{a_i}{i}\).

Details and proofs of the above statements can be found in [5,1,15,7].

3. The result for \(B_n\)

In this section, the compression operator \(C\) will always be used relatively to the reverse lex linear order \(\preceq\) introduced at the end of the preceding section. The reverse lex order is such that if \(m \leq 2^n\), then \(C(m, B_{n_0}) = C(m, B_n)\) for all \(n \geq n_0\). Hence, we can simplify the notation \(C(m, B_n)\) by writing \(C(m)\) only and think of \(n\) as large enough such that \(m \leq 2^n\).

We prove the following result:

**Theorem 3.1.** The profile \(f(C(m))\) is strongly log-concave for every \(m \in \mathbb{N}^*\).

This theorem can be seen as a generalization of the observation that \(f(C(2^n)) = \left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right)\) is strongly log-concave. Note that \(C(m)\) is an ideal since \(A \leq A' \Rightarrow A \preceq A'\), for \(A, A' \in B_n\). Theorem 3.1 has been generalized to the poset \(\text{Col}(k_1, k_2, \ldots, k_s)\) in [13].

As a particular case of Theorem 3.1, we get that \(f(C(\ell, N_k)))\) is strongly log-concave for \(\ell \leq W_k\); take \(m = p(A)\) with \(A \in B_n\) such that \(p_k(A) = \ell\). The profile \(f(C(\ell, N_k)))\) is then exactly the restriction of \(f(C(m))\) to its \(k + 1\) first levels. The
difference between both statements should be clear considering the following figure (this kind of picture is taken from [15]): the elements of rank $i$ are pictured horizontally, in reverse lex order, and such that $N_i$ is above $N_{i-1}$. The shaded regions represent respectively $\langle C(\ell, N_k) \rangle$ and $C(m)$.

Let us first compute the coefficients of the profile $f(C(m))$. In the sequel, $[x]$ stands for the greatest integer less than or equal to the real number $x$.

**Lemma 3.2.** Let $m \in \mathbb{N}^*$ and assume that $m - 1 = \sum_{i=1}^{k} 2^{c_i}$ with $0 \leqslant c_1 < c_2 < \cdots < c_k$. Then

(i) $\max \{i: f_\ell(C(m)) > 0\} = [\log_2 m]$;

(ii) $f_\ell(C(m)) = \delta_{k, \ell} + \sum_{i=1}^{k} \left( \frac{c_i}{\ell-k+i} \right)$, for every $\ell \in \mathbb{N}$.

**Proof.** Observe that the bijection $\{a_1, a_2, \ldots, a_k\} \mapsto 1 + \sum_{i=1}^{k} 2^{a_i-1}$ from $B_n$ to $[2^n]$ preserves linear orders in the sense that the reverse lex order corresponds to the natural order in the integers. In particular, $p(\{a_1, a_2, \ldots, a_k\}) = 1 + \sum_{i=1}^{k} 2^{a_i-1}$ and $\{c_1 + 1, c_2 + 1, \ldots, c_k + 1\}$ is the last element of $C(m)$.

Since $p_\ell(\{1, 2, \ldots, \ell\}) = 1$ we get

$$f_\ell(C(m)) > 0 \iff \{1, 2, \ldots, \ell\} \in C(m) \iff p(\{1, 2, \ldots, \ell\}) = 2^\ell \leqslant m.$$

Hence (i) is proved. Now set $A := \{a_1, a_2, \ldots, a_k\}$ with $a_i := c_i + 1$ and consider for every $\ell \geqslant 0$ the following partition of $\{D \in N_\ell: D \prec A\}$:

$$\bigcup_{i=1}^{k} \{D \in N_\ell: D \prec A, a_i \notin D \text{ and } \{a_{i+1}, a_{i+2}, \ldots, a_k\} \subseteq D\}$$

where the $i$th part is of cardinality $\left\lfloor \frac{a_i-1}{\ell-k+i} \right\rfloor$. Hence, $f_\ell = |\{D \in N_\ell: D \prec A\}| = \delta_{k, \ell} + \sum_{i=1}^{k} \left( \frac{c_i}{\ell-k+i} \right)$ since, of course, one has to count $A$ itself when $\ell = k$. \hfill \square

Next we show an upper bound for the quotient $f_\ell(C(m))/f_{\ell-1}(C(m))$.

**Proposition 3.3.** Let $m \in \mathbb{N}$, $m > 1$; we have

$$\frac{f_\ell(C(m))}{f_{\ell-1}(C(m))} \leqslant \frac{[\log_2(m - 1)] - \ell + 2}{\ell}, \text{ for every } 1 \leqslant \ell \leqslant [\log_2 m].$$
In order to prove it we introduce $f$. It is well-known that $B_n$ has the dual normalized matching property, i.e. for all $F \subseteq N(B_n)$ and $0 < \ell \leq r(F)$, we have $|F|/|\Delta F| \leq W_{\ell}(B_n)/W_{\ell-1}(B_n)$ (see [5, Chapter 4.5]). Let $A = \{a_1, a_2, \ldots, a_k\}$ be such that $a_1 < a_2 < \cdots < a_k$ and $p(A) = m$. Then $C(m) = \{D \subseteq \mathbb{N}^*: D \subseteq A\}$ is an ideal of the poset $B_n$. Set $F := C(m) \cap N(B_n)$. Since $C(m)$ is an ideal, we have $\Delta F \subseteq (C(m) \cap N_{\ell-1}(B_n))$; hence $|\Delta F| \leq f_{\ell-1}(C(m))$. Thus,

$$\frac{f_\ell(C(m))}{f_{\ell-1}(C(m))} \leq \frac{|F|}{|\Delta F|} \leq \frac{W_{\ell}(B_n)}{W_{\ell-1}(B_n)} = \frac{a_k - \ell + 1}{\ell}. $$

Finally $a_k - 1 = [\log_2(m - 1)]$ since $m = p(A) = 1 + \sum_{i=1}^{k} 2^{a_i - 1}$. □

Now we find a lower bound for the same quotient:

**Proposition 3.4.** Let $m \in \mathbb{N}$, $m > 1$; we have

$$\frac{f_\ell(C(m))}{f_{\ell-1}(C(m))} > \frac{\log_2(m - 1) - \ell}{\ell}, \quad \text{for every } 1 \leq \ell \leq [\log_2 m].$$

**Proof.** We proceed by induction on $\ell$, for arbitrary $m$.

Write $c := [\log_2(m - 1)]$, $m' := m - 2^c$, $f_i := f_i(C(m))$ and $f_i' := f_i(C(m'))$.

If $\ell = 1$, then $f_0 = 1$ and $f_1 = c + 1$, whence $f_1/f_0 > c - 1$.

Assume $\ell > 1$. By Lemma 3.2, $f_i = \binom{c}{\ell} + f_{i-1}'$ for $i \geq 1$. Let us rewrite the statement to be proved in an equivalent form using the $f_i'$'s:

$$\frac{f_\ell}{f_{\ell-1}} > \frac{c - \ell}{\ell} \iff \frac{\binom{c}{\ell} + f_{\ell-1}'(c)}{f_{\ell-2}'(c)} > \frac{c - \ell}{\ell} \iff \ell f_{\ell-1}'(c) + \binom{c}{\ell} > (c - \ell)f_{\ell-2}'(c).$$

(2)

If $f_{\ell-2}' = 0$ then we are done since $\binom{c}{\ell-1} \geq 1$; hence, we can assume that $f_{\ell-2}' > 0$.

Set $c' := [\log_2(m' - 1)]$ and consider 2 cases:

**Case (i):** $c' \leq \ell - 2$.

Then $m' \leq 2^{\ell-1}$, which yields $f_{\ell-2}' \leq \ell - 1$. To obtain (2) it is enough to show that

$$\binom{c}{\ell - 1} > (c - \ell)(\ell - 1), \quad \text{with } 1 \leq \ell - 1 \leq c.$$

In order to prove it we introduce $f: \mathbb{N} \to \mathbb{N}$, $f(u) := \binom{u}{\ell-1} - (u - \ell)(\ell - 1)$. We have $f(\ell - 1) = \ell > 0$ and, for $u \geq \ell - 1$,

$$f(u + 1) - f(u) = \binom{u + 1}{\ell - 1} - \binom{u}{\ell - 1} + ((u - \ell) - (u + 1 - \ell))(\ell - 1).$$
\[
\frac{u}{\ell - 2} - (\ell - 1) \geq \frac{\ell - 1}{\ell - 2} - (\ell - 1) \geq 0.
\]

Hence, \( f(u) > 0 \) for \( u \geq \ell - 1 \) and (2) is proved in the first case.

**Case (ii):** \( c' \geq \ell - 1 \).

In this case \( m' > 2^{\ell - 1} \) and \( f'_{\ell - 1} > 0 \). By the induction hypothesis, we have \( f'_{\ell - 1}/f'_{\ell - 2} > (c' - (\ell - 1))/\ell - 1 \). Moreover, Lemma 3.2 gives the upper bound \( f'_{\ell - 2} \leq \sum_{i=0}^{\ell - 2} \binom{c' - 1}{i} = \binom{c' + 1}{\ell - 2} \). To show (2), it is then enough to see that

\[
\ell \frac{c' - \ell + 1}{\ell - 1} + \frac{(c' - 1)}{\binom{c' + 1}{\ell - 2}} c - \ell, \quad \text{with } 1 < \ell \leq c' + 1 \leq c.
\]

Consequently, we have to show (using the substitutions \( s := c - c' \geq 1 \) and \( b := c' + 1 \))

\[
\frac{(b + s - 1)!}{(b + s - \ell)!} > \frac{b!}{(b - \ell + 2)!} (s(\ell - 1) + 1 - b), \quad 1 < \ell \leq b, \ s \geq 1.
\]

As before, let

\[
g : \mathbb{N} \to \mathbb{N}, \quad g(u) := \frac{(b + u - 1)!}{(b + u - \ell)!} - \frac{b!}{(b - \ell + 2)!} (u(\ell - 1) + 1 - b).
\]

We have \( g(1) > 0 \) since \( \ell \leq b \). For \( u \geq 1 \):

\[
g(u + 1) - g(u) = \frac{(b + u)!}{(b + u - \ell + 1)!} - \frac{(b + u - 1)!}{(b + u - \ell)!} - \frac{b!}{(b - \ell + 2)!} ((u + 1)(\ell - 1) + 1 - b - (u(\ell - 1) + 1 - b))
\]

\[
= \frac{(b + u - 1)!}{(b + u - \ell + 1)!} (\ell - 1) - \frac{b!}{(b - \ell + 2)!} (\ell - 1) \geq 0.
\]

Hence, \( g(u) > 0 \) for every \( u \geq 1 \) and (2) is proved in the second case. \( \square \)

By using both preceding propositions we can prove Theorem 3.1. Before proceeding with the proof, we recall an elementary fact: if \( a,c \in \mathbb{N} \) and \( b,d \in \mathbb{N}^* \) then

\[
\frac{a}{b} < \frac{c}{d} \iff \frac{a}{b} < \frac{a + c}{b + d} \iff \frac{a + c}{b + d} < \frac{c}{d}
\]

and \((a + c)/(b + d)\) is called the mediant of \( a/b \) and \( c/d \).

**Proof of Theorem 3.1.** We show that the strict inequalities

\[
\ell f_{\ell}(C(m))^2 > (\ell + 1)f_{\ell + 1}(C(m))f_{\ell - 1}(C(m))
\]
hold for every \( m \in \mathbb{N}^* \). As before, we set 
\[ c := \left\lfloor \log_2(m - 1) \right\rfloor, \quad m' := m - 2^c, \]
\[ c' := \left\lfloor \log_2(m' - 1) \right\rfloor, \quad f_i := f_i(C(m)) \text{ and } f'_i := f_i(C(m')). \]

Assume the statement to be false: let \( m \) be minimal such that there exists an \( \ell \geq 1 \) with
\[
\frac{f_\ell}{f_{\ell-1}} \leq \frac{\ell + 1}{\ell} \frac{f'_{\ell+1}}{f'_\ell}. \tag{3}
\]

First, we see that we must have \( \ell > 1 \): otherwise \( f_0 = 1, \quad f_1 = c + 1, \quad f_2 = \binom{c}{2} + f'_1 \leq \binom{c}{2} + c' + 1 \) and
\[
\frac{2}{1} \frac{f_2}{f_1} \leq \frac{c(c - 1) + 2(c' + 1)}{c + 1} \leq \frac{c(c - 1) + 2c}{c + 1} = c < \frac{f_1}{f_0}.
\]

Thus, Propositions 3.3, 3.4 and inequality (3) yield
\[
\frac{c - \ell}{\ell} < \frac{f_\ell}{f_{\ell-1}} \leq \frac{\ell + 1}{\ell} \frac{f'_{\ell+1}}{f'_\ell} \leq \frac{\ell + 1}{\ell} \frac{f_{\ell+1}}{f_\ell} \leq \frac{\ell + 1}{\ell} \frac{c - (\ell + 1) + 2}{\ell + 1} = \frac{c - \ell + 1}{\ell}. \tag{4}
\]

By Lemma 3.2, we know that \( f_i = \binom{c}{i} + f'_{i-1} \) for every \( i \geq 1 \). Then we have \( f'_i > 0 \), since otherwise
\[
\ell + 1 \frac{f'_{\ell+1}}{f'_\ell} = \ell + 1 \frac{\binom{c}{\ell+1}}{\binom{c}{\ell} + f'_{\ell-1}} \leq \ell + 1 \frac{\binom{c}{\ell+1}}{\binom{c}{\ell}} \leq \frac{c - \ell}{\ell},
\]
contradicting (4). Thus \( f'_i > 0 \), whence \( f'_{i-1} > 0 \) and \( f'_{i-2} > 0 \).

Since
\[
\frac{f_\ell}{f_{\ell-1}} = \frac{\binom{c}{\ell} + f'_{\ell-1}}{\binom{c}{\ell-1} + f'_{\ell-2}} \quad \text{and} \quad \frac{f_\ell}{f_{\ell-1}} = \frac{c - \ell + 1}{\ell}
\]
by (4) we get
\[
f'_{\ell-1}/f'_{\ell-2} \leq f_\ell/f_{\ell-1}
\]
by mediant’s property. The same argument yields
\[
\frac{\ell + 1}{\ell} \frac{f'_{\ell}}{f'_{\ell-1}} > \frac{\ell + 1}{\ell} \frac{f_{\ell+1}}{f_\ell}
\]
since
\[
\frac{\ell + 1}{\ell} \frac{f_{\ell+1}}{f_\ell} = \frac{\ell + 1}{\ell} \left( \binom{c}{\ell} + (\ell + 1)f'_{\ell} \right) \quad \text{and} \quad \frac{c - \ell}{\ell} < \frac{\ell + 1}{\ell} \frac{f_{\ell+1}}{f_\ell}.
\]

Thus, finally:
\[
\frac{\ell}{\ell - 1} \frac{f'_\ell}{f'_{\ell-1}} - \frac{f'_{\ell-1}}{f'_{\ell-2}} > \frac{\ell + 1}{\ell} \frac{f'_\ell}{f'_{\ell-1}} - \frac{f'_{\ell-1}}{f'_{\ell-2}} > \frac{\ell + 1}{\ell} \frac{f_{\ell+1}}{f_\ell} - \frac{f_\ell}{f_{\ell-1}} \geq 0,
\]
which contradicts the minimality of \( m \) in (3). \( \square \)
Remark. Inequalities of Theorem 3.1, which are indeed strict, are sharp in the following sense: consider the ideals \( C(2^k + 1) \), with \( k \geq 1 \). Their profiles are of the form \( \left( 1, k + 1, \binom{k}{2}, \binom{k}{3}, \ldots \right) \), and \( f_2/f_1 - \frac{3}{2} f_3/f_2 = 1/(k + 1) \) tends to 0 as \( k \) tends to infinity.

Now we know that the profile of \( C(m) \) is log-concave — hence unimodal. It arises the question where exactly is the peak of this sequence. The answer is that the profile \( f(C(m)) \) behaves approximately like the sequence of binomial coefficients (i.e. \( f(C(2^k)) \)).

**Theorem 3.5.** The following inequalities hold, for every \( m > 1 \):

\[
f_0 < f_1 < \ldots < f_j > f_{j+1} > f_{j+2} > \ldots > f_{\lfloor \log_2 m \rfloor},
\]

where \( (f_0, f_1, \ldots, f_{\lfloor \log_2 m \rfloor}) = f(C(m)) \) and

\[
j = \begin{cases} \frac{\lfloor \log_2 (m-1) \rfloor}{2} & \text{if } \lfloor \log_2 (m-1) \rfloor \text{ is even;} \\ \frac{\lfloor \log_2 (m-1) \rfloor \pm 1}{2} & \text{if } \lfloor \log_2 (m-1) \rfloor \text{ is odd.} \end{cases}
\]

**Proof.** Set \( c := \lfloor \log_2 (m-1) \rfloor \); by Propositions 3.3 and 3.4, we have

\[
f_\ell/f_{\ell-1} \in ((c-\ell)/\ell, (c-\ell+2)/\ell)].
\]

But

\[
f_j/f_{j-1} > 1 \Rightarrow (c - j + 2)/j > 1 \iff j < (c + 2)/2
\]

and

\[
f_{j+1}/f_j \leq 1 \Rightarrow (c - (j + 1))/(j + 1) < 1 \iff j > (c - 2)/2.
\]

Hence

\[
j \in \left( \frac{c}{2} - 1, \frac{c}{2} + 1 \right),
\]

which proves the statement. \( \Box \)

**Remark.** One can easily prove by induction the following improvement:

If \( c := \lfloor \log_2 (m-1) \rfloor \) is odd, then we have

\[
j = \begin{cases} \frac{c - 1}{2} & \text{if } m < \frac{3}{4}(2^{c+1} - 1) \\ \frac{c + 1}{2} & \text{if } m \geq \frac{3}{4}(2^{c+1} - 1). \end{cases}
\]
4. The result for $M_n$

Recall that the compression operator $C$ is now relative to the linear order $\preceq$ which makes $M_n$ a Macaulay poset. As this poset is infinite and the linear order $\preceq$ is such that $x_k^+ \prec x_2$ for all $k$, the question of log-concavity of the profile $f(C(m,M_n))$ is trivial. In fact, we see immediately that $f(C(m,M_n)) = f((C(1,N_{m-1}))) = (1,1,\ldots,1)$ is log-concave, but not strongly. Hence, it is reasonable to restrict ourselves to the following statement, which will turn out to be a corollary of Theorem 3.1.

**Theorem 4.1.** Let $k \in \mathbb{N}^*$ and $m \in [W_k(M_n)]$; then $f((C(m,N_k)))$ is log-concave.

**Proof.** Let $(f_0, f_1, \ldots, f_k)$ be the profile considered (with $f_k = m$). We have to show that $f_{\ell-1}^2 \geq f_{\ell} f_{\ell-2}$ for all $\ell = 2, 3, \ldots, k$. Write the $\ell$-binomial representation of $f_{\ell}$:

$$f_{\ell} = 1 + \sum_{i=1}^{\ell} \binom{a_i}{i}$$

with $0 \leq a_1 < a_2 < \cdots < a_\ell$.

As we saw in Section 2, we have

$$f_{\ell-1} = \mathcal{C}^{M_\ell}_t(f_{\ell}) = 1 + \sum_{i=2}^{\ell} \binom{a_i - 1}{i - 1}, f_{\ell-2} = \mathcal{C}^{M_\ell}_{\ell-1}(f_{\ell-1}) = 1 + \sum_{i=3}^{\ell} \binom{a_i - 2}{i - 2}.$$  

Now, set

$$g_{\ell} := f_{\ell} - f_{\ell-1} = 1 + \sum_{i=1}^{\ell} \binom{a_i - 1}{i},$$

$$g_{\ell-1} := f_{\ell-1} - f_{\ell-2} = 1 + \sum_{i=2}^{\ell} \binom{a_i - 2}{i - 1},$$

$$h_{\ell} := g_{\ell} - g_{\ell-1} = 1 + \sum_{i=1}^{\ell} \binom{a_i - 2}{i}$$

and observe that

$$f_{\ell-1}^2 \geq f_{\ell} f_{\ell-2} \iff (f_{\ell-2} + g_{\ell-1})^2 \geq f_{\ell-2}(f_{\ell-2} + g_{\ell-1} + g_{\ell})$$

$$\iff g_{\ell-1}^2 \geq h_{\ell} f_{\ell-2}.$$  

We conclude the proof by applying Theorem 3.1. In order to do it, we consider three cases, depending if the above expressions for $h_{\ell}$ and $g_{\ell-1}$ are binomial representations or not. Let $A \in B_{\ell}$ (with $t$ large enough) such that

(i) $p_{\ell}(A) = h_{\ell}$ if $a_1 \geq 2$ or

(ii) $p_{\ell-1}(A) = g_{\ell-1}$ if $a_1 < 2$ and $a_2 \geq 2$ or

(iii) $p_{\ell-2}(A) = f_{\ell-2}$ if $a_1 = 0$ and $a_2 = 1$
and consider the ideal $C(p(A)) \subseteq B_t$. Denote by $(f'_0, f'_1, \ldots, f'_t)$ its profile and look at each case separately.

**Case (i):** if $a_1 \geq 2$ then $1 + \sum_{i=1}^{t} \binom{a_i - 2}{i}$ is the $\ell$-binomial representation of $h'_\ell = f'_\ell$. Hence, $f'_{\ell-1} = \partial^{B_t}(f'_\ell) = g'_{\ell-1}$ and $f'_{\ell-2} = \partial^{B_t}(f'_{\ell-1}) = f'_{\ell-2}$. By applying Theorem 3.1 to $C(p(A))$ we get $g^2_{\ell-1} \geq h'f'_{\ell-2}$ in particular.

**Case (ii):** if $a_2 \geq 2$ then $1 + \sum_{i=2}^{t} \binom{a_i - 2}{i-1}$ is the $(\ell - 1)$-binomial representation of $g'_{\ell-1} = f'_{\ell-1}$. Hence, $f'_{\ell-2} = \partial^{B_t}(f'_{\ell-1}) = f'_{\ell-2}$ and by Lemma 3.2

$$f'_\ell = \sum_{i=2}^{t} \binom{a_i - 2}{i} = 1 + \binom{a_1 - 2}{1} + \sum_{i=2}^{t} \binom{a_i - 2}{i} = h'_\ell.$$  

Theorem 3.1 yields $f'_{\ell-1} \geq f'_{\ell-2}^2$, whence $g^2_{\ell-1} \geq h'_\ell f'_{\ell-2}$.

**Case (iii):** Finally, $1 + \sum_{i=3}^{t} \binom{a_i - 2}{i-2}$ is the $(\ell - 2)$-binomial representation of $f'_{\ell-2} = f'_{\ell-2}$ since $a_3 \geq 2$. By Lemma 4

$$f'_{\ell-1} = \sum_{i=3}^{t} \binom{a_i - 2}{i-1} = 1 + \binom{-1}{1} + \sum_{i=3}^{t} \binom{a_i - 2}{i-1} = g'_{\ell-1},$$

$$f'_\ell = \sum_{i=3}^{t} \binom{a_i - 2}{i} = 1 + \binom{-2}{1} + \binom{-1}{2} + \sum_{i=3}^{t} \binom{a_i - 2}{i} = h'_\ell$$

and Theorem 3.1 yields the result.  

**Remark.** In the case of the poset $M_n$, the unimodality of $f((C(m, N_k)))$ is trivial since this profile is indeed an increasing sequence: $\partial^M_k(m) \leq m$ for every $k \geq 1$ and $m \in \mathbb{W}_k$ by definition of the operator $\partial^M_k$.

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**References**