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Log-concavity and compressed ideals in certain Macaulay posets

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Abstract

Let B_n be the poset of subsets of $\{1,2,\ldots,n\}$ ordered by inclusion and M_n be the poset of monomials in $x_1,x_2,\ldots,x_n$ ordered by divisibility. Both these posets have an additional *linear* order making them what is called Macaulay posets. We show in this paper that the profiles (also called f-vectors) of ideals in B_n and M_n generated by the first elements (relatively to the linear order) of a given rank are log-concave. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we will prove that some sequences of nonnegative integers arising from extremal set theory are unimodal or log-concave. The question of unimodality is a natural one concerning a sequence of positive integers; however, it can be very hard to solve. There are a lot of such open problems about integer sequences in combinatorics: see [14,2] for an excellent survey on these problems.

The sequences we are interested in are related to shadow functions ∂_k^P of Macaulay posets $(P,\leqslant,\preccurlyeq)$, where $\leqslant$ is the partial order on P and $\preccurlyeq$ is a *linear* extension of $\leqslant$. Macaulay posets were defined to bring a unified point of view on analogous theorems; they are presented in [5, Chapter 8]. We will restrict ourselves to the two most well-known Macaulay posets, namely B_n and M_n. B_n is the set of subsets of $\{1,2,\ldots,n\}$ partially ordered by inclusion; it has been proved to have the Macaulay property by Kruskal [9] and Katona [8], independently. M_n is the set of monomials in variables $x_1,x_2,\ldots,x_n$ ordered by divisibility (equivalently: the set of multisets of

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$\{1,2,\ldots,n\}$ ordered by multiset inclusion); Macaulay [12] proved that M_n has that property which is now called Macaulay property. Now there are several simpler proofs of these theorems: see e.g. [4,6], or [1].

The necessary terminology is introduced in Section 2, where it is also shown how Macaulay posets can help to prove unimodality of some sequences of positive integers (Proposition 2.2). In Section 3, we look at the particular case of B_n. The main theorem of the paper is Theorem 3.1: it says that, if $C(m)$ is the set of the first m elements of B_n with respect to the linear order $\preccurlyeq$, then the sequence $(f_0, f_1, \ldots, f_n)$ is strongly log-concave, where f_k is the number of elements of $C(m)$ of rank k. Section 4 is devoted to the poset M_n. We show in Theorem 4.1 a similar but weaker result than in the case of B_n. Both these theorems can be seen as a particular study of the shadow functions $\partial_k^{B_n}$ and $\partial_k^{M_n}$, or simply as log-concavity properties of sequences of sums of binomial coefficients.

2. Macaulay posets

As usual, $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{N}^* := \mathbb{N} - \{0\}$, and $[n] := \{1, 2, \ldots, n\}$. We will be interested in *ranked* posets only, i.e. posets $(P, \leqslant)$ with a *rank function* $r : P \to \mathbb{N}$ satisfying $r(p) = 0$ for some minimal element $p \in P$ and $r(q) = r(p) + 1$ if $p < q$ and there exists no $x \in P$ with $p < x < q$. The *rank of* P is defined by $r(P) :=$ max $\{r(p){:}\ p \in P\} \in \mathbb{N} \cup \{\infty\}$. Such a poset is called *graded* if all its maximal chains have the same finite cardinality. We define the ith *level* of P by $N_i(P) := \{p \in P : r(p) = i\}$ and set $W_i(P) := |N_i(P)|$. We often write N_i and W_i for $N_i(P)$ and $W_i(P)$.

An *ideal* $I \subseteq P$ is a subset such that if $p \in I$ and $q < p$ then $q \in I$; if $E \subseteq P$ then the subset $\langle E \rangle := \{p \in P{:}\ p \leqslant x$ for some $x \in E\}$ is the *ideal generated by* E. The sequence $f(I) := (f_0(I), f_1(I), \ldots, f_{r(P)}(I))$, with $f_i(I) = |I \cap N_i(P)|$, is called the *profile* of the ideal I. The *shadow* of $p \in P$ is the set $\Delta(p) := \{q \in P{:}\ q < p$ and $r(q) = r(p) - 1\}$ and if $E \subseteq P$ then $\Delta(E) := \bigcup_{x \in E} \Delta(x)$. Note that if F is a subset of an ideal I then $\Delta(F) \subseteq I$.

A sequence $(a_i)_{i \geqslant 0}$ of nonnegative integers is called *unimodal* if there exists an index $j \geqslant 0$ such that $a_i \leqslant a_{i+1}$ for $0 \leqslant i < j$ and $a_i \geqslant a_{i+1}$ for $i \geqslant j$. It is *log-concave* if $a_i^2 \geqslant a_{i+1} a_{i-1}$ for $i \geqslant 1$ and *strongly log-concave* if $i a_i^2 \geqslant (i+1) a_{i+1} a_{i-1}$ for $i \geqslant 1$. One can easily see that a sequence which is log-concave and *without internal zeros* (i.e. there exist no indices $i < j < k$ with $a_j = 0$ and $a_i \neq 0 \neq a_k$) is unimodal. Note that the profile of an ideal is always without internal zeros; thus, such a log-concave profile is unimodal, in particular.

The problem of minimizing the shadow of a subset of P is important in the theory of finite sets. Macaulay posets are posets on which there exists an extra linear order which solves the problem. The presentation of Macaulay posets follows [5, Chapter 8.1].

Let $(P, \leqslant)$ be a ranked poset and let $\preccurlyeq$ be a linear order on P. If $E \subseteq P$ let $C(m, E)$ be the set of the first m elements of E with respect to the linear order $\preccurlyeq$. If F is finite and $F \subseteq N_k$ then $C(F)$ or CF is defined by $C(|F|, N_k)$ and is called the *compression* of F. A set $E \subseteq P$ is *compressed* if $C(E \cap N_i) = E \cap N_i$ for every $i = 0, \ldots, r(P)$.

In addition, we define the *position functions* $p: P \to \mathbb{N}^*$ (resp. $p_k: N_k(P) \to \mathbb{N}^*$) by $p(x) := |\{z \in P\colon\ z \preccurlyeq x\}|$ (resp. $p_k(x) := |\{z \in N_k(P)\colon\ z \preccurlyeq x\}|$).

Definition. A ranked poset $(P, \leqslant)$ is a *Macaulay poset* if there exists a linear order $\preccurlyeq$ on P such that

$$\Delta(CF) \subseteq C(\Delta F) \quad \text{for every } k \geqslant 1 \text{ and every finite } F \subseteq N_k(P). \tag{1}$$

Hence a Macaulay poset is a triple $(P, \leqslant, \preccurlyeq)$ satisfying the stated property (the linear order $\preccurlyeq$ is of course not unique in general). An equivalent definition can be given as follows:

Proposition 2.1 (Engel [5, Proposition 8.1.1]). *Let $\preccurlyeq$ be a linear order on a ranked poset $(P, \leqslant)$. Then* (1) *holds if and only if*

(i) $|\Delta F| \geqslant |\Delta(CF)|$ *for every finite* $F \subseteq N_k,\ k \geqslant 1$;
(ii) $\Delta(CF) = C(\Delta(CF))$ *for every finite* $F \subseteq N_k,\ k \geqslant 1$.

Given a linear order $\preccurlyeq$ on P, the condition (ii) is easy to check. Hence let us focus on the first. Since $|\Delta(CF)|$ depends on $|F|$ and k only, define the *shadow functions* $\partial_k^P\colon [W_k] \to \mathbb{N}^*$ by $\partial_k^P(m) := |\Delta(CF)|$, with finite $F \subseteq N_k$ and $|F| = m$. An explicit description of these functions is necessary for practical use of shadow-minimization in Macaulay posets; moreover, it is the key to the kind of profiles we will consider, namely ideals of the form $\langle C(m, N_k)\rangle$.

Definition. A Macaulay poset $(P, \leqslant, \preccurlyeq)$ is called *shadow increasing* if $\partial_k^P(m) \leqslant \partial_{k+1}^P(m)$ for all $k \geqslant 1$ and $1 \leqslant m \leqslant \min\{W_k, W_{k+1}\}$.

Proposition 2.2 (Engel [5, Proposition 8.1.6]). *The profile of a graded shadow increasing Macaulay poset is unimodal.*

As a direct consequence of Proposition 2.2 we get that, for a shadow increasing Macaulay poset P, the profile $f(\langle C(m, N_k(P))\rangle)$ is unimodal since $\langle C(m, N_k(P))\rangle$ is obviously a graded shadow increasing Macaulay poset. Thus, here is a way of proving the unimodality of some ideals' profiles. But of course, neither the Macaulay property nor the shadow increasing property are easy to prove. However three families of posets are known to have both properties: chain products $S(k_1, k_2, \ldots, k_s)$ (see [3,4]), star products $T(k_1, k_2, \ldots, k_s)$ (see [10,11]) and their duals $\mathrm{Col}(k_1, k_2, \ldots, k_s)$ (see [5]). In Sections 3 and 4 we will prove stronger versions of Proposition 2.2 for the posets B_n and M_n (without using Macaulay or shadow increasing properties).

The poset B_n. Put the following linear order $\preccurlyeq$ (called *reverse lex*) on B_n: for $A, A' \in B_n$,

$$A \prec A' \quad \text{if } \max\{(A \cup A') \backslash (A \cap A')\} \in A'.$$

Then $(B_n, \leqslant, \preccurlyeq)$ is a Macaulay poset.

Given a fixed $k \in \mathbb{N}^*$, define the *k-binomial representation of an integer* $m \in \mathbb{N}^*$ by

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_1}{1} + 1,$$

where the a_i's are natural numbers and $a_k > a_{k-1} > \cdots > a_1 \geqslant 0$ (such a representation exists and is unique). Then the operator $\partial_k^{B_n}$ is given by

$$\partial_k^{B_n}(m) = 1 + \sum_{i=2}^{k} \binom{a_i}{i-1}$$

if the k-binomial representation of m is $1 + \sum_{i=1}^{k} \binom{a_i}{i}$.

The poset M_n. Put the following linear order $\preccurlyeq$ on M_n:

$$x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \prec x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n} \quad \text{if } a_j < b_j, \text{ where } j = \max\{i\colon\ a_i \neq b_i\}.$$

Then $(M_n, \leqslant, \preccurlyeq)$ is a Macaulay poset. In this case, the operator $\partial_k^{M_n}$ is given by

$$\partial_k^{M_n}(m) = 1 + \sum_{i=2}^{k} \binom{a_i - 1}{i-1}$$

if the k-binomial representation of m is $1 + \sum_{i=1}^{k} \binom{a_i}{i}$.

Details and proofs of the above statements can be found in [5,1,15,7].

3. The result for B_n

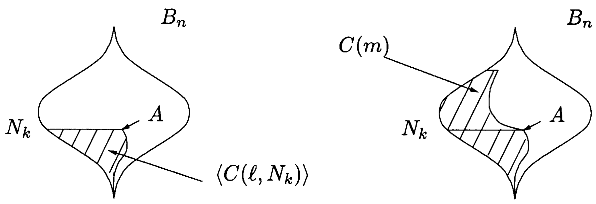
In this section, the compression operator C will always be used relatively to the reverse lex linear order $\preccurlyeq$ introduced at the end of the preceding section. The reverse lex order is such that if $m \leqslant 2^{n_0}$, then $C(m, B_{n_0}) = C(m, B_n)$ for all $n \geqslant n_0$. Hence, we can simplify the notation $C(m, B_n)$ by writing $C(m)$ only and think of n as large enough such that $m \leqslant 2^n$.

We prove the following result:

Theorem 3.1. *The profile $f(C(m))$ is strongly log-concave for every $m \in \mathbb{N}^*$.*

This theorem can be seen as a generalization of the observation that $f(C(2^n)) = \left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right)$ is strongly log-concave. Note that $C(m)$ is an ideal since $A \leqslant A' \Rightarrow A \preccurlyeq A'$, for $A, A' \in B_n$. Theorem 3.1 has been generalized to the poset $\mathrm{Col}(k_1, k_2, \ldots, k_s)$ in [13].

As a particular case of Theorem 3.1, we get that $f(\langle C(\ell, N_k)\rangle)$ is strongly log-concave for $\ell \leqslant W_k$: take $m = p(A)$ with $A \in B_n$ such that $p_k(A) = \ell$. The profile $f(\langle C(\ell, N_k)\rangle)$ is then exactly the restriction of $f(C(m))$ to its $k+1$ first levels. The difference between both statements should be clear considering the following figure (this kind of picture is taken from [15]): the elements of rank i are pictured horizontally, in reverse lex order, and such that N_i is above N_{i-1}. The shaded regions represent respectively $\langle C(\ell, N_k)\rangle$ and $C(m)$.



Let us first compute the coefficients of the profile $f(C(m))$. In the sequel, $\lfloor x\rfloor$ stands for the greatest integer less than or equal to the real number x.

Lemma 3.2. *Let $m\in\mathbb{N}^*$ and assume that $m-1=\sum_{i=1}^k 2^{c_i}$ with $0\leqslant c_1<c_2<\cdots<c_k$. Then*

(i) $\max\{i\colon\ f_i(C(m))>0\}=\lfloor\log_2 m\rfloor$;
(ii) $f_\ell(C(m))=\delta_{k,\ell}+\sum_{i=1}^k\binom{c_i}{\ell-k+i}$, *for every* $\ell\in\mathbb{N}$.

Proof. Observe that the bijection $\{a_1,a_2,\ldots,a_k\}\mapsto 1+\sum_{i=1}^k 2^{a_i-1}$ from B_n to $[2^n]$ preserves linear orders in the sense that the reverse lex order corresponds to the natural order in the integers. In particular, $p(\{a_1,a_2,\ldots,a_k\})=1+\sum_{i=1}^k 2^{a_i-1}$ and $\{c_1+1, c_2+1,\ldots,c_k+1\}$ is the last element of $C(m)$.

Since $p_\ell(\{1,2,\ldots,\ell\})=1$ we get

$$f_\ell(C(m))>0\ \Leftrightarrow\ \{1,2,\ldots,\ell\}\in C(m)\ \Leftrightarrow\ p(\{1,2,\ldots,\ell\})=2^\ell\leqslant m.$$

Hence (i) is proved. Now set $A:=\{a_1,a_2,\ldots,a_k\}$ with $a_i:=c_i+1$ and consider for every $\ell\geqslant 0$ the following partition of $\{D\in N_\ell\colon\ D\prec A\}$:

$$\bigcup_{i=1}^k\{D\in N_\ell\colon\ D\prec A,\ a_i\notin D\ \text{and}\ \{a_{i+1},a_{i+2},\ldots,a_k\}\subseteq D\}$$

where the ith part is of cardinality $\binom{a_i-1}{\ell-k+i}$. Hence, $f_\ell=|\{D\in N_\ell\colon\ D\preccurlyeq A\}|=\delta_{k,\ell}+\sum_{i=1}^k\binom{c_i}{\ell-k+i}$ since, of course, one has to count A itself when $\ell=k$. $\square$

Next we show an upper bound for the quotient $f_\ell(C(m))/f_{\ell-1}(C(m))$.

Proposition 3.3. *Let $m\in\mathbb{N}$, $m>1$; we have*

$$\frac{f_\ell(C(m))}{f_{\ell-1}(C(m))}\leqslant\frac{\lfloor\log_2(m-1)\rfloor-\ell+2}{\ell},\quad \textit{for every } 1\leqslant\ell\leqslant\lfloor\log_2 m\rfloor.$$

Proof. It is well-known that B_n has the dual normalized matching property, i.e. for all $F \subseteq N_\ell(B_n)$ and $0<\ell\leqslant r(P)$, we have $|F|/|\Delta F| \leqslant W_\ell(B_n)/W_{\ell-1}(B_n)$ (see [5, Chapter 4.5]). Let $A=\{a_1,a_2,\ldots,a_k\}$ be such that $a_1<a_2<\cdots<a_k$ and $p(A)=m$. Then $C(m)=\{D\subseteq\mathbb{N}^*\colon\ D\preccurlyeq A\}$ is an ideal of the poset B_{a_k}. Set $F:=C(m)\cap N_\ell(B_{a_k})$. Since $C(m)$ is an ideal, we have $\Delta F\subseteq(C(m)\cap N_{\ell-1}(B_{a_k}))$; hence $|\Delta F|\leqslant f_{\ell-1}(C(m))$. Thus,

$$\frac{f_\ell(C(m))}{f_{\ell-1}(C(m))}\leqslant\frac{|F|}{|\Delta F|}\leqslant\frac{W_\ell(B_{a_k})}{W_{\ell-1}(B_{a_k})}=\frac{a_k-\ell+1}{\ell}.$$

Finally $a_k-1=\lfloor\log_2(m-1)\rfloor$ since $m=p(A)=1+\sum_{i=1}^{k}2^{a_i-1}$. □

Now we find a lower bound for the same quotient:

Proposition 3.4. *Let* $m\in\mathbb{N}$, $m>1$; *we have*

$$\frac{f_\ell(C(m))}{f_{\ell-1}(C(m))}>\frac{\lfloor\log_2(m-1)\rfloor-\ell}{\ell},\quad \textit{for every } 1\leqslant\ell\leqslant\lfloor\log_2 m\rfloor.$$

Proof. We proceed by induction on ℓ, for arbitrary m.

Write $c:=\lfloor\log_2(m-1)\rfloor$, $m':=m-2^c$, $f_i:=f_i(C(m))$ and $f_i':=f_i(C(m'))$.

If $\ell=1$, then $f_0=1$ and $f_1=c+1$, whence $f_1/f_0>c-1$.

Assume $\ell>1$. By Lemma 3.2, $f_i=\binom{c}{i}+f'_{i-1}$ for $i\geqslant 1$. Let us rewrite the statement to be proved in an equivalent form using the f_i''s:

$$\frac{f_\ell}{f_{\ell-1}}>\frac{c-\ell}{\ell}\Leftrightarrow\frac{\binom{c}{\ell}+f'_{\ell-1}}{\binom{c}{\ell-1}+f'_{\ell-2}}>\frac{c-\ell}{\ell}$$

$$\Leftrightarrow\ \ell f'_{\ell-1}+\binom{c}{\ell-1}>(c-\ell)f'_{\ell-2}. \tag{2}$$

If $f'_{\ell-2}=0$ then we are done since $\binom{c}{\ell-1}\geqslant 1$; hence, we can assume that $f'_{\ell-2}>0$.

Set $c':=\lfloor\log_2(m'-1)\rfloor$ and consider 2 cases:

Case (i): $c'\leqslant\ell-2$.

Then $m'\leqslant 2^{\ell-1}$, which yields $f'_{\ell-2}\leqslant\ell-1$. To obtain (2) it is enough to show that

$$\binom{c}{\ell-1}>(c-\ell)(\ell-1),\quad\text{with } 1\leqslant\ell-1\leqslant c.$$

In order to prove it we introduce $f:\mathbb{N}\to\mathbb{N}$, $f(u):=\binom{u}{\ell-1}-(u-\ell)(\ell-1)$. We have $f(\ell-1)=\ell>0$ and, for $u\geqslant\ell-1$,

$$f(u+1)-f(u)=\binom{u+1}{\ell-1}-\binom{u}{\ell-1}+((u-\ell)-(u+1-\ell))(\ell-1)$$

$$= \binom{u}{\ell-2} - (\ell-1) \geqslant \binom{\ell-1}{\ell-2} - (\ell-1) \geqslant 0.$$

Hence, $f(u) > 0$ for $u \geqslant \ell - 1$ and (2) is proved in the first case.

Case (ii): $c' \geqslant \ell - 1$.

In this case $m' > 2^{\ell-1}$ and $f'_{\ell-1} > 0$. By the induction hypothesis, we have $f'_{\ell-1}/f'_{\ell-2} > (c' - (\ell-1))/\ell - 1$. Moreover, Lemma 3.2 gives the upper bound $f'_{\ell-2} \leqslant \sum_{i=0}^{\ell-2} \binom{c'-i}{\ell-2-i} = \binom{c'+1}{\ell-2}$. To show (2), it is then enough to see that

$$\ell \frac{c' - \ell + 1}{\ell - 1} + \frac{\binom{c}{\ell-1}}{\binom{c'+1}{\ell-2}} > c - \ell, \quad \text{with } 1 < \ell \leqslant c' + 1 \leqslant c.$$

Consequently, we have to show (using the substitutions $s := c - c' \geqslant 1$ and $b := c' + 1$)

$$\frac{(b+s-1)!}{(b+s-\ell)!} > \frac{b!}{(b-\ell+2)!}(s(\ell-1) + 1 - b), \quad 1 < \ell \leqslant b, \ s \geqslant 1.$$

As before, let

$$g : \mathbb{N} \to \mathbb{N}, \quad g(u) := \frac{(b+u-1)!}{(b+u-\ell)!} - \frac{b!}{(b-\ell+2)!}(u(\ell-1) + 1 - b).$$

We have $g(1) > 0$ since $\ell \leqslant b$. For $u \geqslant 1$:

$$\begin{aligned} g(u+1) - g(u) &= \frac{(b+u)!}{(b+u-\ell+1)!} - \frac{(b+u-1)!}{(b+u-\ell)!} \\ &\quad - \frac{b!}{(b-\ell+2)!}((u+1)(\ell-1) + 1 - b - (u(\ell-1) + 1 - b)) \\ &= \frac{(b+u-1)!}{(b+u-\ell+1)!}(\ell-1) - \frac{b!}{(b-\ell+2)!}(\ell-1) \geqslant 0. \end{aligned}$$

Hence, $g(u) > 0$ for every $u \geqslant 1$ and (2) is proved in the second case. □

By using both preceding propositions we can prove Theorem 3.1. Before proceeding with the proof, we recall an elementary fact: if $a, c \in \mathbb{N}$ and $b, d \in \mathbb{N}^*$ then

$$\frac{a}{b} < \frac{c}{d} \ \Leftrightarrow \ \frac{a}{b} < \frac{a+c}{b+d} \ \Leftrightarrow \ \frac{a+c}{b+d} < \frac{c}{d}$$

and $(a+c)/(b+d)$ is called the *mediant* of a/b and c/d.

Proof of Theorem 3.1. We show that the *strict* inequalities

$$\ell f_\ell(C(m))^2 > (\ell+1) f_{\ell+1}(C(m)) f_{\ell-1}(C(m))$$

hold for every $m \in \mathbb{N}^*$. As before, we set $c := \lfloor \log_2(m-1) \rfloor$, $m' := m - 2^c$, $c' := \lfloor \log_2(m'-1) \rfloor$, $f_i := f_i(C(m))$ and $f_i' := f_i(C(m'))$.

Assume the statement to be false: let m be minimal such that there exists an $\ell \geqslant 1$ with

$$\frac{f_\ell}{f_{\ell-1}} \leqslant \frac{\ell+1}{\ell} \frac{f_{\ell+1}}{f_\ell}. \tag{3}$$

First, we see that we must have $\ell > 1$: otherwise $f_0 = 1$, $f_1 = c+1$, $f_2 = \binom{c}{2} + f_1' \leqslant \binom{c}{2} + c' + 1$ and

$$\frac{2}{1} \frac{f_2}{f_1} \leqslant \frac{c(c-1) + 2(c'+1)}{c+1} \leqslant \frac{c(c-1)+2c}{c+1} = c < \frac{f_1}{f_0}.$$

Thus, Propositions 3.3, 3.4 and inequality (3) yield

$$\frac{c-\ell}{\ell} < \frac{f_\ell}{f_{\ell-1}} \leqslant \frac{\ell+1}{\ell} \frac{f_{\ell+1}}{f_\ell} \leqslant \frac{\ell+1}{\ell} \frac{c-(\ell+1)+2}{\ell+1} = \frac{c-\ell+1}{\ell}. \tag{4}$$

By Lemma 3.2, we know that $f_i = \binom{c}{i} + f_{i-1}'$ for every $i \geqslant 1$. Then we have $f_\ell' > 0$, since otherwise

$$\frac{\ell+1}{\ell} \frac{f_{\ell+1}}{f_\ell} = \frac{\ell+1}{\ell} \frac{\binom{c}{\ell+1}}{\binom{c}{\ell} + f_{\ell-1}'} \leqslant \frac{\ell+1}{\ell} \frac{\binom{c}{\ell+1}}{\binom{c}{\ell}} = \frac{c-\ell}{\ell},$$

contradicting (4). Thus $f_\ell' > 0$, whence $f_{\ell-1}' > 0$ and $f_{\ell-2}' > 0$.

Since

$$\frac{f_\ell}{f_{\ell-1}} = \frac{\binom{c}{\ell} + f_{\ell-1}'}{\binom{c}{\ell-1} + f_{\ell-2}'} \quad \text{and} \quad \frac{f_\ell}{f_{\ell-1}} \leqslant \frac{c-\ell+1}{\ell}$$

by (4) we get

$$f_{\ell-1}'/f_{\ell-2}' \leqslant f_\ell/f_{\ell-1}$$

by mediant's property. The same argument yields

$$\frac{\ell+1}{\ell} \frac{f_\ell'}{f_{\ell-1}'} > \frac{\ell+1}{\ell} \frac{f_{\ell+1}}{f_\ell}$$

since

$$\frac{\ell+1}{\ell} \frac{f_{\ell+1}}{f_\ell} = \frac{(\ell+1)\binom{c}{\ell+1} + (\ell+1) f_\ell'}{\ell \binom{c}{\ell} + \ell f_{\ell-1}'} \quad \text{and} \quad \frac{c-\ell}{\ell} < \frac{\ell+1}{\ell} \frac{f_{\ell+1}}{f_\ell}.$$

Thus, finally:

$$\frac{\ell}{\ell-1} \frac{f_\ell'}{f_{\ell-1}'} - \frac{f_{\ell-1}'}{f_{\ell-2}'} > \frac{\ell+1}{\ell} \frac{f_\ell'}{f_{\ell-1}'} - \frac{f_{\ell-1}'}{f_{\ell-2}'} > \frac{\ell+1}{\ell} \frac{f_{\ell+1}}{f_\ell} - \frac{f_\ell}{f_{\ell-1}} \geqslant 0,$$

which contradicts the minimality of m in (3). $\square$

Remark. Inequalities of Theorem 3.1, which are indeed strict, are sharp in the following sense: consider the ideals $C(2^k+1)$, with $k \geqslant 1$. Their profiles are of the form $\left(1, k+1, \binom{k}{2}, \binom{k}{3}, \ldots\right)$, and $f_2/f_1 - \frac{3}{2} f_3/f_2 = 1/(k+1)$ tends to 0 as k tends to infinity.

Now we know that the profile of $C(m)$ is log-concave — hence unimodal. It arises the question where exactly is the peak of this sequence. The answer is that the profile $f(C(m))$ behaves approximatively like the sequence of binomial coefficients (i.e. $f(C(2^k))$).

Theorem 3.5. *The following inequalities hold, for every* $m > 1$:

$$f_0 < f_1 < \cdots < f_j \geqslant f_{j+1} > f_{j+2} > \cdots > f_{\lfloor \log_2 m \rfloor},$$

where $(f_0, f_1, \ldots, f_{\lfloor \log_2 m \rfloor}) = f(C(m))$ *and*

$$j = \begin{cases} \dfrac{\lfloor \log_2 (m-1) \rfloor}{2} & \textit{if } \lfloor \log_2(m-1) \rfloor \textit{ is even;} \\ \dfrac{\lfloor \log_2(m-1) \rfloor \pm 1}{2} & \textit{if } \lfloor \log_2(m-1) \rfloor \textit{ is odd.} \end{cases}$$

Proof. Set $c := \lfloor \log_2(m-1) \rfloor$; by Propositions 3.3 and 3.4, we have

$$f_\ell / f_{\ell-1} \in ((c-\ell)/\ell,\ (c-\ell+2)/\ell)].$$

But

$$f_j/f_{j-1} > 1 \Rightarrow (c-j+2)/j > 1 \Leftrightarrow j < (c+2)/2$$

and

$$f_{j+1}/f_j \leqslant 1 \Rightarrow (c-(j+1))/(j+1) < 1 \Leftrightarrow j > (c-2)/2.$$

Hence

$$j \in \left(\frac{c}{2} - 1,\ \frac{c}{2} + 1 \right),$$

which proves the statement. $\square$

Remark. One can easily prove by induction the following improvement:
If $c := \lfloor \log_2(m-1) \rfloor$ *is odd, then we have*

$$j = \begin{cases} \dfrac{c-1}{2} & \textit{if } m < \frac{2}{3}(2^{c+1}-1); \\ \dfrac{c+1}{2} & \textit{if } m \geqslant \frac{2}{3}(2^{c+1}-1). \end{cases}$$

4. The result for M_n

Recall that the compression operator C is now relative to the linear order $\preccurlyeq$ which makes M_n a Macaulay poset. As this poset is infinite and the linear order $\preccurlyeq$ is such that $x_1^k \prec x_2$ for all k, the question of log-concavity of the profile $f(C(m,M_n))$ is trivial. In fact, we see immediately that $f(C(m,M_n)) = f(\langle C(1,N_{m-1})\rangle) = (1,1,\dots,1)$ is log-concave, but not strongly. Hence, it is reasonable to restrict ourselves to the following statement, which will turn out to be a corollary of Theorem 3.1.

Theorem 4.1. *Let $k \in \mathbb{N}^*$ and $m \in [W_k(M_n)]$; then $f(\langle C(m,N_k)\rangle)$ is log-concave.*

Proof. Let $(f_0, f_1, \dots, f_k)$ be the profile considered (with $f_k = m$). We have to show that $f_{\ell-1}^2 \geqslant f_\ell f_{\ell-2}$ for every $\ell = 2,3,\dots,k$. Write the ℓ-binomial representation of f_ℓ:

$$f_\ell = 1 + \sum_{i=1}^{\ell} \binom{a_i}{i} \quad \text{with} \quad 0 \leqslant a_1 < a_2 < \cdots < a_\ell.$$

As we saw in Section 2, we have

$$f_{\ell-1} = \partial_\ell^{M_n}(f_\ell) = 1 + \sum_{i=2}^{\ell} \binom{a_i - 1}{i-1}, f_{\ell-2} = \partial_{\ell-1}^{M_n}(f_{\ell-1}) = 1 + \sum_{i=3}^{\ell} \binom{a_i - 2}{i-2}.$$

Now, set

$$g_\ell := f_\ell - f_{\ell-1} = 1 + \sum_{i=1}^{\ell} \binom{a_i - 1}{i},$$

$$g_{\ell-1} := f_{\ell-1} - f_{\ell-2} = 1 + \sum_{i=2}^{\ell} \binom{a_i - 2}{i-1},$$

$$h_\ell := g_\ell - g_{\ell-1} = 1 + \sum_{i=1}^{\ell} \binom{a_i - 2}{i}$$

and observe that

$$f_{\ell-1}^2 \geqslant f_\ell f_{\ell-2} \;\Leftrightarrow\; (f_{\ell-2} + g_{\ell-1})^2 \geqslant f_{\ell-2}(f_{\ell-2} + g_{\ell-1} + g_\ell)$$
$$\Leftrightarrow g_{\ell-1}^2 \geqslant h_\ell f_{\ell-2}.$$

We conclude the proof by applying Theorem 3.1. In order to do it, we consider three cases, depending if the above expressions for h_ℓ and $g_{\ell-1}$ are binomial representations or not. Let $A \in B_t$ (with t large enough) such that

(i) $p_\ell(A) = h_\ell$ if $a_1 \geqslant 2$ or
(ii) $p_{\ell-1}(A) = g_{\ell-1}$ if $a_1 < 2$ and $a_2 \geqslant 2$ or
(iii) $p_{\ell-2}(A) = f_{\ell-2}$ if $a_1 = 0$ and $a_2 = 1$

and consider the ideal $C(p(A)) \subseteq B_t$. Denote by $(f'_0, f'_1, \ldots, f'_t)$ its profile and look at each case separately.

Case (i): if $a_1 \geqslant 2$ then $1+\sum_{i=1}^{\ell} \binom{a_i-2}{i}$ is the ℓ-binomial representation of $h_\ell = f'_\ell$. Hence, $f'_{-1} = \partial_\ell^{B_t}(f'_\ell) = g_{\ell-1}$ and $f'_{\ell-2} = \partial_{\ell-1}^{B_t}(f'_{\ell-1}) = f_{\ell-2}$. By applying Theorem 3.1 to $C(p(A))$ we get $g_{\ell-1}^2 \geqslant h_\ell f_{\ell-2}$ in particular.

Case (ii): if $a_2 \geqslant 2$ then $1+\sum_{i=2}^{\ell} \binom{a_i-2}{i-1}$ is the $(\ell-1)$-binomial representation of $g_{\ell-1} = f'_{\ell-1}$. Hence, $f'_{\ell-2} = \partial_{\ell-1}^{B_t}(f'_{\ell-1}) = f_{\ell-2}$ and by Lemma 3.2

$$f'_\ell = \sum_{i=2}^{\ell} \binom{a_i-2}{i} \geqslant 1 + \binom{a_1-2}{1} + \sum_{i=2}^{\ell} \binom{a_i-2}{i} = h_\ell.$$

Theorem 3.1 yields ${f'_{\ell-1}}^2 \geqslant f'_\ell f'_{\ell-2}$, whence $g_{\ell-1}^2 \geqslant h_\ell f_{\ell-2}$.

Case (iii): Finally, $1+\sum_{i=3}^{\ell} \binom{a_i-2}{i-2}$ is the $(\ell-2)$-binomial representation of $f_{\ell-2} = f'_{\ell-2}$ since $a_3 \geqslant 2$. By Lemma 4

$$f'_{\ell-1} = \sum_{i=3}^{\ell} \binom{a_i-2}{i-1} = 1 + \binom{-1}{1} + \sum_{i=3}^{\ell} \binom{a_i-2}{i-1} = g_{\ell-1},$$

$$f'_\ell = \sum_{i=3}^{\ell} \binom{a_i-2}{i} = 1 + \binom{-2}{1} + \binom{-1}{2} + \sum_{i=3}^{\ell} \binom{a_i-2}{i} = h_\ell$$

and Theorem 3.1 yields the result. □

Remark. In the case of the poset M_n, the unimodality of $f(\langle C(m, N_k)\rangle)$ is trivial since this profile is indeed an increasing sequence: $\partial_k^{M_n}(m) \leqslant m$ for every $k \geqslant 1$ and $m \in [W_k]$ by definition of the operator $\partial_k^{M_n}$.

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References

[1] I. Anderson, Combinatorics of Finite Sets, Oxford University Press, Oxford, 1987.
[2] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics and geometry: an update, Contemp. Math. 178 (1994) 71–89.
[3] G.F. Clements, More on the generalized Macaulay theorem II, Discrete Math. 18 (1977) 253–264.
[4] G.F. Clements, B. Lindström, A generalization of a combinatorial theorem of Macaulay, J. Combin. Theory Ser. A 7 (1969) 230–238.

[5] K. Engel, Sperner Theory, Cambridge University Press, Cambridge, 1997.
[6] P. Frankl, N. Tokushige, The Kruskal–Katona theorem, some of its analogues and applications, in: P. Frankl et al. (Eds.), Extremal Problems for Finite Sets, Bolyai Society of Mathematical Studies, Vol. 3, Janos Bolyai Math. Soc., Budapest, 1994, pp. 229–250.
[7] C. Greene, D.J. Kleitman, Proof techniques in the theory of finite sets, in: G.C. Rota (Ed.), Studies of Combinatorics, Vol. 17, Mathematical Association of America Studies in Mathematics, Math. Assoc. America, Washington, DC, 1978, pp. 22–79.
[8] G.O.H. Katona, A theorem of finite sets, In: P. Erdös, G. Katona (Eds.), Theory of Graphs, Akademie Kiadó and Academic Press, Budapest and New York, 1968, pp. 187–207.
[9] J.B. Kruskal, The number of simplices in a complex, in: R. Bellman (Ed.), Mathematical Optimization Techniques, University of California Press, Berkeley, Los Angeles, 1963, pp. 251–278.
[10] U. Leck, Extremalprobleme für den schatten in posets, Ph.D. Thesis, Freie Universität Berlin, Berlin, 1995.
[11] U. Leck, A property of colored complexes and their duals, Discrete Math. 213 (2000) 245–251.
[12] F.S. Macaulay, Some properties of enumeration in the theory of modular systems, Proc. London Math. Soc. 26 (1927) 531–555.
[13] P. Pitteloud, A theorem on log-concavity or inequalities for elementary symmetric polynomials, to appear.
[14] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics and geometry, Ann. New York Acad. Sci. 576 (1989) 500–535.
[15] G.M. Ziegler, Lectures on Polytopes, Springer, Berlin, 1995.