ON LINEAR EXTENSIONS OF ORDERED SETS WITH A SYMMETRY

Bernhard GANTER, Gerhart HÄFNER and Werner POGUNTKE

Received 5 May 1986

It is shown that any finite cycle-free ordered set with a non-trivial automorphism contains a pair of elements x and y such that the proportion of linear extensions in which x lies below y is $\frac{1}{2}$.

1. Introduction

A linear extension of a finite ordered set $(P, \prec)$ is an order-preserving bijection $\lambda: P \to \{1, 2, \ldots, |P|\}$, i.e., $x \prec y$ in $P$ implies $\lambda(x) \prec \lambda(y)$.

For any pair $x, y$ of distinct elements in $P$, $p(x \prec y)$ denotes the fraction of linear extensions such that $\lambda(x) < \lambda(y)$. The problem we are dealing with is to show that if $P$ is not a chain, then there is always a pair $x, y$ in $P$ such that $p(x < y)$ is close to $\frac{1}{2}$.

The motivation for this comes from sorting problems. The reader is referred to the survey articles [3] and [7].

The best known general result is the one by Kahn and Saks [4] saying that $\frac{1}{11} \leq p(x < y) \leq \frac{8}{11}$ always holds for some $x$ and $y$. It is a conjecture of Fredman that $\frac{1}{3} < p(x < y) < \frac{2}{3}$ can always be attained. That this is the best possible general result is shown by the three-element ordered set with just one comparability.


The purpose of this paper is to prove the following.

Theorem. Let $(P, \prec)$ be a finite cycle-free ordered set, and let $\alpha$ be a non-trivial automorphism of $(P, \prec)$. Then $p(x < \alpha(x)) = \frac{1}{2}$ for any $x \in P$ with $\alpha(x) \neq x$.

By cycle-free, as usual, we mean that $P$ does not contain a subset which (with the inherited order) is isomorphic to any cycle (cf. Fig. 1).

![Fig. 1. 4-cycle, 6-cycle, 8-cycle, ...](image-url)
2. Ordered sets with an automorphism: two observations

The ordered set \((C, <)\) described in Fig. 2 shows that in the presence of cycles, \(p(x < \alpha(x))\) does not necessarily equal \(\frac{1}{2}\). \((C, <)\) has 1431 linear extensions 720 of which put \(x\) below \(\alpha(x)\).

![Image of ordered set](image)

Fig. 2.

On the other hand, Fredman's conjecture is true for ordered sets with a non-trivial automorphism. We give a short argument which was brought to our attention by Pouzet:

Let \(\alpha\) be an automorphism of the finite ordered set \((P, <)\). Observe that \(P(x < y) = P(\alpha(x) < \alpha(y))\) for any \(x, y \in P\). Now assume there are no \(x, y \in P\) with \(\frac{1}{3} \leq p(x < y) \leq \frac{2}{3}\), and define a relation \(\ll\) on \(P\) by "\(u \ll v\) if \(p(u < v) > \frac{2}{3}\)".

It is easy to see that under the assumptions made, \(\ll\) is transitive and, in fact, a linear order on \(P\). Since \(\alpha\) respects \(\ll\), it follows that \(\alpha\) is the identity on \(P\).

We think, but have not been able to prove, that the \(\frac{1}{3} - \frac{2}{3}\)-bound can be improved for ordered sets with a non-trivial automorphism.

3. Proof of the theorem

Let \(P, \alpha\) and \(x\) be as in the Theorem. By the covering graph of \(P\), we mean the undirected graph \(CP = (P, E)\) with an edge between \(u\) and \(v\), \((u, v) \in E\), if and only if \(u\) is a lower or upper cover of \(v\) in \(P\).

We now define an equivalence relation \(\sim\) on \(P\) which is crucial for the proof: for \(u, v \in P\), \(u \sim v\) if and only if \(u = v\) or there is a path in \(CP\) between \(u\) and \(v\) which contains no fixed point of \(\alpha\). The \(\sim\)-class containing \(u \in P\) is denoted by \([u]\).

Let \(\Lambda(x < \alpha(x))\) be the set of linear extensions \(\lambda\) of \(P\) putting \(x\) below \(\alpha(x)\), i.e., \(\lambda(x) < \lambda(\alpha(x))\); \(\Lambda(\alpha(x) < x)\) consists of those extensions doing the opposite.

For \(\lambda \in \Lambda(x < \alpha(x))\), let \(\Phi(\lambda): P \to \{1, 2, \ldots, |P|\}\) be defined

\[
\Phi(\lambda)(p) = \begin{cases} 
\lambda(\alpha(p)) & \text{if } p \in [x], \\
\lambda(\alpha^{-1}(p)) & \text{if } p \in [\alpha(x)], \\
\lambda(p) & \text{else.}
\end{cases}
\]

Our proof will be completed when we have shown that \(\Phi\) is a bijection between \(\Lambda(x < \alpha(x))\) and \(\Lambda(\alpha(x) < x)\).
The first thing we have to check is that $\Phi(\lambda)$ is indeed a mapping. This amounts to showing

**Claim 1.** $x \not\in \alpha(x)$ (or, equivalently, $[x] \cap [\alpha(x)] = \emptyset$).

**Proof.** If there is no path (in $CP$) connecting $x$ and $\alpha(x)$, there is nothing to show. So let $S$ be the set of points on a path connecting $x$ and $\alpha(x)$. Let $T$ be the smallest subset of $P$ containing $S$ and being closed under $\alpha$, i.e., $T = \bigcup_{i=1}^{\infty} \alpha^i(S)$. $T$ is connected (and still cycle-free) and mapped into itself by $\alpha$. By a result of Rival [6], we know there is a fixed point of $\alpha$ in $T$, $f \in T$ with $\alpha(f) = f$. Since $\alpha$ is an automorphism, there also has to be a fixed point of $\alpha$ in $S$, and this proves the Claim. □

The only thing which is left to be proved is that $\Phi(\lambda)$ is order-preserving. The next Claim is the key to get this.

**Claim 2.** If $u \preceq v$, then $u < v$ implies that any of $\{u, \alpha(u), \alpha^{-1}(u)\}$ lies below each of $\{v, \alpha(v), \alpha^{-1}(v)\}$.

**Proof.** Assume $u \preceq v$ and $u < v$. By the definition of $\sim$, there has to be a fixed point $f$ of $\alpha$ with $u \preceq f \preceq v$. Applying $\alpha$ or $\alpha^{-1}$ to each (or both) of these two inequalities gives all the desired relations.

Now let $\lambda \in \Lambda(x < \alpha(x))$ and $p < q$ in $P$. We have to show $\Phi(\lambda)(p) < \Phi(\lambda)(q)$.

If $p, q \notin [x] \cup [\alpha(x)]$, this is clear since $\Phi(\lambda)(p) = \lambda(p)$ and $\Phi(\lambda)(q) = \lambda(q)$. Also, if $p, q \in [x]$, we are done because $\alpha(p) < \alpha(q)$ and hence $\Phi(\lambda)(p) < \Phi(\lambda)(q)$.

The case $p, q \in [\alpha(x)]$ works by symmetry (using $\alpha^{-1}$). □

Let $p \in [x]$ and $q \in [\alpha(x)]$. Application of Claim 2 yields $\alpha(p) < \alpha^{-1}(q)$ and thus $\Phi(\lambda)(p) = \lambda(\alpha(p)) < \lambda(\alpha^{-1}(q)) = \Phi(\lambda)(q)$. A symmetric argument works in case $p \in [\alpha(x)]$ and $q \in [x]$.

The remaining cases, e.g. $p \in [x]$ and $q \notin [x] \cup [\alpha(x)]$, now follow in a similar way using Claim 2.

This completes the proof of the Theorem. □

4. Remarks

(1) The proof of the Theorem shows that the following stronger statement is true: if $(P, <)$ is a finite ordered set with a proper automorphism $\alpha$, and if $x \in P$ with $\alpha(x) \neq x$ is such that there is a fixed point of $\alpha$ on every path connecting $x$ and $\alpha(x)$ in the covering graph of $(P, <)$, then $p(x < \alpha(x)) = \frac{1}{2}$. (If $(P, <)$ has no cycles, then any $x$ with $\alpha(x) \neq x$ has the required property.)

(2) A slight modification of the argument given in the second ‘Observation’
shows that if $\alpha$ is an automorphism of $(P, <)$, then each orbit of $\alpha$ contains a pair $x, y$ with $\frac{1}{2} < p(x < y) < \frac{3}{2}$.

(3) As already mentioned, we have not been able to improve in general the $\frac{1}{2} - \frac{3}{2}$-bound for ordered sets with a proper automorphism. Also, we did not succeed in extending $(C, <)$ of Fig. 2 in such a way as to get much 'worse' examples. For instance, let $C_n$ be the $n$-cycle with an extra element on every other edge; in this notation, $C$ is just $C_6$ (see Fig. 3). If $\alpha_n$ denotes the automorphism as indicated in Fig. 3, it turns out that with growing $n$, $p(x < \alpha_n(x))$ seems to converge to a number close to 0.5033; on the other hand, the fraction for the 'best pair' that can be found in $C_n$ approaches 0.5.

(4) In connection with voting paradoxes, Fishburn considered the following relation $\ll$ on a finite ordered set $(P, <)$: "$u \ll v$ if $p(u < v) > \frac{1}{2}$" (see [1, 2]). The nine-element ordered set $(C, <)$ of Fig. 2 provides a small example in which there is a 'cycle' $x \ll y \ll z \ll x$.

References