Journal of Algebra 321 (2009) 28–85

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Rectangular low level case of modular branching problem for $\mathrm{GL}_n(K)$

Vladimir Shchigolev

Department of Algebra, Faculty of Mathematics, Lomonosov Moscow State University, Leninskiye Gory, Moscow 119899, Russia

ARTICLE INFO

Article history:
Received 27 March 2007
Available online 28 October 2008
Communicated by Leonard L. Scott, Jr.

Keywords:
Modular branching problem
Lowering operators
Restriction of representations of algebraic groups to subgroups

ABSTRACT

In this paper, we find an explicit combinatorial criterion for the existence of a nonzero $\mathrm{GL}_{n-1}(K)$-high weight vector of weight $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - d, \lambda_{i+1}, \ldots, \lambda_{n-1})$, where $d < \mathrm{char}(K)$ and K is an algebraically closed field, in the irreducible rational $\mathrm{GL}_n(K)$-module $L_n(\lambda_1, \ldots, \lambda_n)$ with highest weight $(\lambda_1, \ldots, \lambda_n)$. For this purpose, new modular lowering operators are introduced.

1. Introduction

Let K be an algebraically closed field of characteristic $p > 0$. We denote by $\mathrm{GL}_n(K)$ the general linear group of degree n over K. Let $\mathrm{D}_n(K)$ and $\mathrm{T}_n(K)$ denote the subgroups of $\mathrm{GL}_n(K)$ consisting of all diagonal matrices and all upper triangular matrices, respectively. We call the elements of $\mathbf{Z}^n$ *weights*. Any weight $(\lambda_1, \ldots, \lambda_n)$ will be identified with the character of $\mathrm{D}_n(K)$ that takes $\mathrm{diag}(t_1, \ldots, t_n)$ to $t_1^{\lambda_1} \cdots t_n^{\lambda_n}$. We shall understand the weight spaces of rational $\mathrm{GL}_n(K)$-modules always with respect to the torus $\mathrm{D}_n(K)$. A vector v of a rational $\mathrm{GL}_n(K)$-module is called a $\mathrm{GL}_n(K)$-*high weight vector* if the line $K \cdot v$ is fixed by $\mathrm{T}_n(K)$. We denote by $X^+(n)$ the subset of $\mathbf{Z}^n$ consisting of all weakly decreasing sequences and call the elements of $X^+(n)$ *dominant weights*.

For $n > 1$, the group $\mathrm{GL}_{n-1}(K)$ will be identified with the subgroup of $\mathrm{GL}_n(K)$ consisting of matrices having 0 in the last row and the last column except the position of their intersection, where they have 1. In what follows, $L_n(\lambda)$ denotes the irreducible rational $\mathrm{GL}_n(K)$-module with highest weight $\lambda \in X^+(n)$ and v_λ^+ denotes a nonzero vector of $L_n(\lambda)$ of weight λ, which we fix. Let $[s..t]$, $[s..t)$, $(s..t]$, $(s..t)$ denote the sets $\{x \in \mathbf{Z} \mid s \leqslant x \leqslant t\}$, $\{x \in \mathbf{Z} \mid s \leqslant x < t\}$, $\{x \in \mathbf{Z} \mid s < x \leqslant t\}$, $\{x \in \mathbf{Z} \mid s < x < t\}$, respectively.

E-mail address: shchigolev_vladimir@yahoo.com.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.08.034

To formulate the main result of this paper, we introduce the strict partial order $\dot{<}$ on $\mathbf{Z}^2$ and the subsets $\mathfrak{Y}_d^\lambda(i,n)$ and $\mathfrak{C}^\lambda(i,n)$ of $\mathbf{Z}^2$ and $\mathbf{Z}$ respectively as follows: $(a,b) \dot{<} (x,y)$ holds if and only if $a<x$ and $b<y$; for any $\lambda \in \mathbf{Z}^n$, we put

$$\mathfrak{Y}_d^\lambda(i,n) := \{(t,h) \in (i..n] \times [1..d] \mid t - i + \lambda_i - \lambda_t - h \equiv 0 \pmod p\},$$

$$\mathfrak{C}^\lambda(i,n) := \{s \in (i..n) \mid s - i + \lambda_i - \lambda_s \equiv 0 \pmod p\}.$$

Moreover, we call a map $\varphi : A \to B$, where $A, B \subset \mathbf{Z}^2$, *strictly decreasing* if $\varphi(\alpha) \dot{<} \alpha$ for any $\alpha \in A$. *Column* t of the plane $\mathbf{Z}^2$ is the subset $\{(t,k) \mid k \in \mathbf{Z}\}$. We shall also consider the weight $\alpha(s,t) := (0,\dots,0,1,0,\dots,0,-1,\dots,0)$, where 1 is at position s, -1 is at position t and $1 \leqslant s < t \leqslant n$.

Theorem 1.1. *Let $\lambda \in X^+(n)$, $1 \leqslant i < n$ and $1 \leqslant d < p$. Then the module $L_n(\lambda)$ contains a nonzero $\mathrm{GL}_{n-1}(K)$-high weight vector of weight $\lambda - d\alpha(i,n)$ if and only if for each subset Δ of $\mathfrak{Y}_d^\lambda(i,n)$ whose points are incomparable with respect to $\dot{<}$, there exists a strictly decreasing injection from Δ to $\mathfrak{C}^\lambda(i,n) \times \{0\}$.*

The existence of the above mentioned injection can be checked using only subsets of $\mathbf{Z}$. In what follows, let $\pi_1 : \mathbf{Z}^2 \to \mathbf{Z}$ denote the projection to the first component. The set Δ, as well as $\mathfrak{Y}_d^\lambda(i,n)$, contains at most one point in each column, since $d<p$. Therefore, there exists a strictly decreasing injection from Δ to $\mathfrak{C}^\lambda(i,n) \times \{0\}$ if and only if there exists a weakly decreasing injection from $\pi_1(\Delta) - 1$ to $\mathfrak{C}^\lambda(i,n)$. One only needs to devise an algorithm to generate all (maximal) subsets of $\mathfrak{Y}_d^\lambda(i,n)$ whose points are incomparable with respect to $\dot{<}$. If $d=1$ then all points of $\mathfrak{Y}_1^\lambda(i,n)$ are automatically incomparable with respect to $\dot{<}$ and we recover the criterion of [K].

In Section 3, we introduce certain elements (elementary expressions) of the hyperalgebra. Their multiplication rules (3.10)–(3.12) represent the ground arithmetics behind our approach. Elementary expressions are similar to products of the Carter–Lusztig lowering operators introduced in [CL]. In particular, the substitution $u_{i+1},\dots,u_{n-1} \mapsto 0$ takes $S_{i,n,\mathcal{M}}^{(d)}(\emptyset,\dots,\emptyset)$ to the product $S_{i,m_1}^d S_{m_1,m_2}^d \cdots S_{m_k,n}^d$, where $\mathcal{M} = \{m_1 < \cdots < m_k\}$ is a subset of $(i..n)$ and $S_{k,l}$ is the operator defined in [CL, 2.9]. The remarkable fact is that nothing prevents us to view an elementary expression as a similar product in the general situation. This idea leads us to the introduction of the algebra of formal operators $F_{i,n}^{(d)}$ in Section 6.2. As a ring $F_{i,n}^{(d)}$, is simply a (commutative) polynomial algebra over a field. However, it is endowed with the operators $\rho_l = \rho_l^{(1)} + \rho_l^{(2)} + \rho_l^{(3)}$ and $\sigma_{l,m}$. The operator ρ_l represents the left multiplication by E_l. The operators $\sigma_{l,m}$ serve to determine integral elements, which appear in Corollaries 6.12 and 6.14. These corollaries assert that certain elements of $F_{i,n}^{(d)}$ have images in the hyperalgebra and not just in its localization. Another approach to integrality is given in [K, Lemma 2.4].

The algebra of formal operators $F_{i,n}^{(d)}$ turns out to be a very useful tool to express the behavior of the lowering operators. We define the principal object of our study, the operators $\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)$, as elements of $F_{i,n}^{(d)}$ (see Definition 6.3). The operators $\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)$ are defined similarly to Kleshchev's lowering operators (see [K, Definition 2.3]) and enjoy similar properties given in Lemmas 6.5–6.7 (see [K, Lemma 2.13] or [B, Lemma 4.11]).

To some but not all elements of $F_{i,n}^{(d)}$ (including $\mathcal{T}_{i,n,\mathcal{M}}^{(d)}(I,J)$), there correspond elements of the hyperalgebra (see Section 6.6). However, we want to underline that working directly in the hyperalgebra would be very problematic. Another argument in favor of $F_{i,n}^{(d)}$ is that the raising coefficients introduced in Section 4 are already products over blocks and conform with the factorization procedure described in Section 6.

Expectedly, the present paper exploits the geometry of the integer plane instead of the geometry of the integer line used in [K,B]. The corresponding constructions are given in Section 5, where we define such universal concepts as "interior point," "boundary point," "cone" and "snake."

Our main operator $\mathfrak{T}_{i,n,M}^{(d)}(I)$ in the hyperalgebra over a field is defined in Section 7.1. This is a two-dimensional analog of Kleshchev's lowering operator $T_{r,s}(M)$ defined in [K, §3]. Moreover, Kleshchev's operator is a special case of our operator corresponding to $d = 1$ and $I = \emptyset$. Notice that M is a subset of the integer line in Kleshchev's operator and is a subset of the integer plane in our operator. Generally, the use of $\mathcal{T}^{(d)}_{i,n,M}(I)$ is very similar to the use of Kleshchev's operators. Our approach is especially close to [B] and [BKS]. Lemma 6.17 show how to calculate $\mathcal{T}^{(d)}_{i,n,M}(I)$ without resorting to the algebra of formal operators $F^{(d)}_{i,n}$.

Finally, let us notice how this paper is connected with the author's previous paper [Sh]. We have $\mathcal{T}^{(d)}_{i,n,M\times\{0\}}(\emptyset) = T^{(d)}_{i,n}(M,1)$ (see [Sh, Definition 3.2, Section 6]), which shows that in this paper we recover all the lowering operators that were repeatedly applied to v^+_λ in [Sh] to obtain $\mathrm{GL}_{n-1}(K)$-high weight vectors of $L_n(\lambda)$. However, we do not know how to apply the operators $\mathcal{T}^{(d)}_{i,n,M}(I)$ repeatedly as in [Sh]. The main obstacle here is the raising coefficients for an arbitrary $\mathrm{GL}_{n-1}(K)$-high weight vector $f_{\mu,\lambda}$. These coefficients had a very complex form in [Sh, Section 4]. Overcoming this obstacle would yield the strongest known to date algorithm for generating nonzero $\mathrm{GL}_{n-1}(K)$-high weight vectors in the irreducible module $L_n(\lambda)$.

Finally, notice that the restriction $d < p$ is necessary in Theorem 1.1. The remark at the end of Section 7.5 gives a simple counterexample. Moreover, our construction of the operators $\mathcal{T}^{(d)}_{i,n,M}(I)$ does not include the case where M has more than one point in same column. Indeed, the definition of τ by (7.1) is meaningless in that case. The restriction $d < p$ ensures that the set $\mathcal{X}^\lambda_d(i,n)$ contains at most one point in each column and therefore so does any set M in Sections 7.3 and 7.5. In all our calculations, we assume that $d < p$ and consider $d! \cdot 1_K$ as an invertible element of K.

2. Notation and definitions

2.1. Generalities

Throughout the paper, we fix integers i, n and d such that $1 \leqslant i < n$ and $d \geqslant 1$. We define the following sequences of length n: $\alpha_0 := (-1, 0, \dots, 0)$; $\alpha_t := (0, \dots, 0, 1, -1, \dots, 0)$, where 1 is at position t and $t = 1, \dots, n-1$. Thus $\alpha(s,t) = \alpha_s + \dots + \alpha_{t-1}$. The elements of $\mathbf{Z}^n$ are ordered as follows: $\lambda \geqslant \mu$ if $\lambda - \mu$ is an integral linear combination of α_t with nonnegative coefficients. The descending factorial power $x^{\underline{n}}$ equals $x(x-1)\cdots(x-n+1)$ if $n \geqslant 0$ and equals $1/(x+1)\cdots(x-n)$ if $n < 0$. We refer to [GKP, Chapters 2 and 5] for the definitions and relations for descending factorial powers and binomial coefficients. Following the standard agreement, we interpret any expression a^m as the sequence of length m whose every entry is a if this notation does not cause confusion. A formula $A \sqcup B = C$ will mean $A \cup B = C$ and $A \cap B = \emptyset$. For any sequence λ, we denote by λ_j its jth entry. For a set S, $|S|$ denotes the cardinality of S. For any condition $\mathcal{P}$, let $\delta_{\mathcal{P}}$ be 1 if $\mathcal{P}$ is true and 0 if $\mathcal{P}$ is false.

Let $UT(n)$ denote the set of integer $n \times n$ matrices N such that $N_{a,b} = 0$ unless $a < b$ and $UT^{\geqslant 0}(n)$ denote the subset of $UT(n)$ consisting of matrices with nonnegative entries. Let $e_{s,t}$, where $1 \leqslant s < t \leqslant n$, denote the element of $UT(n)$ with 1 at the intersection of row s and column t and 0 elsewhere.

For a matrix $N \in UT(n)$, we denote

- $N_t := \sum_{a=1}^n N_{a,t}$ the sum of elements in column t of N, where $t = 1, \dots, n$;
- $N^s := \sum_{b=1}^n N_{s,b}$ the sum of elements in row s of N, where $s = 1, \dots, n$;
- $N(k) := \sum_{1 \leqslant a \leqslant k < b \leqslant n} N_{a,b}$, where $k = 0, \dots, n$ (summation over a and b).

We shall often use the formula

$$N(l-1) - N(l) = N_l - N^l \quad \text{for any } N \in UT(n) \text{ and } l = 1, \dots, n. \tag{2.1}$$

Let $\lambda \in \mathbf{Z}^n$. For two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of $[1..n] \times \mathbf{Z}$, we define

$$\mathrm{dist}_\lambda(x, y) := y_1 - x_1 + \lambda_{x_1} - \lambda_{y_1} + x_2 - y_2.$$

One can easily check that $\mathrm{dist}_\lambda(x, y) + \mathrm{dist}_\lambda(y, z) = \mathrm{dist}_\lambda(x, z)$ and $\mathrm{dist}_\lambda(x, x) = 0$. This notation allows us to write

$$\mathfrak{Y}_d^\lambda(i, n) = \bigl\{x \in (i..n] \times [1..d] \bigm| \mathrm{dist}_\lambda\bigl((i, 0), x\bigr) \equiv 0 \pmod p\bigr\},$$
$$\mathfrak{C}^\lambda(i, n) = \bigl\{s \in (i..n) \bigm| \mathrm{dist}_\lambda\bigl((i, 0), (s, 0)\bigr) \equiv 0 \pmod p\bigr\}.$$

We shall also consider the set

$$\mathcal{X}_d^\lambda(i, n) := \bigl\{x \in (i..n] \times [0..d] \bigm| \mathrm{dist}_\lambda\bigl((i, 0), x\bigr) \equiv 0 \pmod p\bigr\}.$$

Obviously, $\mathcal{X}_d^\lambda(i, n) = \mathfrak{Y}_d^\lambda(i, n) \sqcup (\mathfrak{C}^\lambda(i, n) \times \{0\}) \sqcup \mathfrak{x}$, where $\mathfrak{x} = \{(n, 0)\}$ if $n - i + \lambda_i - \lambda_n \equiv 0 \pmod p$ and $\mathfrak{x} = \emptyset$ otherwise.

2.2. Multisets

A *multiset* is an unordered sequence. We shall use for multisets the same notation as for sets with the difference that instead of the braces $\{,\}$ we shall use the angle brackets $\langle , \rangle$. For example, $\langle a, a, b\rangle = \langle a, b, a\rangle \neq \langle a, b\rangle$, $\langle a^3, b^2\rangle = \langle a, a, a, b, b\rangle$, $\langle 1, 1, 3\rangle \cup \langle 1, 3\rangle = \langle 1, 1, 1, 3, 3\rangle$, $\langle 1, 1, 1, 3\rangle \cap \langle 1, 1, 3, 3\rangle = \langle 1, 1, 3\rangle$, $\langle 1, 1, 1, 2, 2, 2\rangle \setminus \langle 1, 1, 2\rangle = \langle 1, 2, 2\rangle$ and $\langle m^2 \mid m = -1, 0, 1\rangle = \langle 0, 1, 1\rangle$.

For a multiset I and a set S, let $|I|^S$ denote the number of elements (taking into account their multiplicities) of I that belong to S. Let $|I|$ denote the total number of all elements of I. We call $|I|$ the *length* of I.

For a multiset I and an element x occurring in it at least once, we denote by $I_{x \mapsto y}$ the multiset obtained from I by replacing exactly one x with y. For a multiset I with integer entries, we shall use the equalities

$$|I_{q+1\mapsto q}|^{(-\infty..t)} = |I|^{(-\infty..t)} + \delta_{t=q+1} \quad \text{if } q+1 \text{ occurs in } I;$$
$$|I_{q-1\mapsto q}|^{\{t-1\}} = |I|^{\{t-1\}} - \delta_{t=q} + \delta_{t=q+1} \quad \text{if } q-1 \text{ occurs in } I. \tag{2.2}$$

Let $m \in \mathbf{Z}$. For a multiset $J = \langle j_1, \ldots, j_k\rangle$ with integer entries, we put

$$\mathcal{L}_m(J) := \bigl\langle \min\{j_1, m-1\}, \ldots, \min\{j_k, m-1\}\bigr\rangle,$$
$$\mathcal{R}_m(J) := \langle j_s \mid s = 1, \ldots, k,\ j_s \geqslant m-1\rangle.$$

For a multiset $I = \langle i_1, \ldots, i_l\rangle$ with integer entries, we put

$$\mathcal{L}^m(I) := \langle i_s \mid s = 1, \ldots, l,\ i_s \leqslant m-1\rangle,$$
$$\mathcal{R}^m(I) := \bigl\langle \max\{i_1, m-1\}, \ldots, \max\{i_l, m-1\}\bigr\rangle.$$

We obviously have $|\mathcal{L}_m(J)|^{\{m-1\}} = |\mathcal{R}_m(J)|$, $|\mathcal{R}^m(I)|^{\{m-1\}} = |\mathcal{L}^m(I)|$. Moreover, for $m \leqslant m'$, we have

$$\mathcal{L}_m\bigl(\mathcal{L}_{m'}(J)\bigr) = \mathcal{L}_m(J), \qquad \mathcal{L}^m\bigl(\mathcal{L}^{m'}(I)\bigr) = \mathcal{L}^m(I),$$
$$\mathcal{R}_{m'}\bigl(\mathcal{R}_m(J)\bigr) = \mathcal{R}_{m'}(J), \qquad \mathcal{R}^{m'}\bigl(\mathcal{R}^m(I)\bigr) = \mathcal{R}^{m'}(I),$$
$$\mathcal{R}_m\bigl(\mathcal{L}_{m'}(J)\bigr) = \mathcal{L}_{m'}\bigl(\mathcal{R}_m(J)\bigr), \qquad \mathcal{R}^m\bigl(\mathcal{L}^{m'}(I)\bigr) = \mathcal{L}^{m'}\bigl(\mathcal{R}^m(I)\bigr). \tag{2.3}$$

Below, we give the table that is meant to help the reader to keep track of the calculations in Sections 6 and 7.

	$x<m-1$	$x=m-1$	$x>m-1$
$\mathcal{L}_m(J\cup\langle x\rangle)$	$\mathcal{L}_m(J)\cup\langle x\rangle$	$\mathcal{L}_m(J)\cup\langle x\rangle$	$\mathcal{L}_m(J)\cup\langle m-1\rangle$
$\mathcal{R}_m(J\cup\langle x\rangle)$	$\mathcal{R}_m(J)$	$\mathcal{R}_m(J)\cup\langle x\rangle$	$\mathcal{R}_m(J)\cup\langle x\rangle$
$\mathcal{L}^m(I\cup\langle x\rangle)$	$\mathcal{L}^m(I)\cup\langle x\rangle$	$\mathcal{L}^m(I)\cup\langle x\rangle$	$\mathcal{L}^m(I)$
$\mathcal{R}^m(I\cup\langle x\rangle)$	$\mathcal{R}^m(I)\cup\langle m-1\rangle$	$\mathcal{R}^m(I)\cup\langle x\rangle$	$\mathcal{R}^m(I)\cup\langle x\rangle$
$\mathcal{L}_m(J_{x-1\mapsto x})$	$\mathcal{L}_m(J)_{x-1\mapsto x}$	$\mathcal{L}_m(J)_{x-1\mapsto x}$	$\mathcal{L}_m(J)$
$\mathcal{R}_m(J_{x-1\mapsto x})$	$\mathcal{R}_m(J)$	$\mathcal{R}_m(J)\cup\langle x\rangle$	$\mathcal{R}_m(J)_{x-1\mapsto x}$
$\mathcal{L}^m(I_{x+1\mapsto x})$	$\mathcal{L}^m(I)_{x+1\mapsto x}$	$\mathcal{L}^m(I)\cup\langle x\rangle$	$\mathcal{L}^m(I)$
$\mathcal{R}^m(I_{x+1\mapsto x})$	$\mathcal{R}^m(I)$	$\mathcal{R}^m(I)_{x+1\mapsto x}$	$\mathcal{R}^m(I)_{x+1\mapsto x}$

2.3. Rings and their quotients

Let $u_{i+1},\dots,u_{n-1}$ be commutative variables. Throughout the paper, we assume that $u_i=0$. We consider the field of rational fractions $\mathbf{Q}':=\mathbf{Q}(u_{i+1},\dots,u_{n-1})$ and the Lie algebra $\mathfrak{gl}_{\mathbf{Q}'}(n)$ of all $n\times n$ matrices over $\mathbf{Q}'$ with respect to usual commutation. Let $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ be the universal enveloping algebra of $\mathfrak{gl}_{\mathbf{Q}'}(n)$.

Let $X_{s,t}$ denote the $n\times n$ matrix with 1 at the intersection of row s and column t and 0 elsewhere. We also denote $H_1:=X_{1,1},\dots,H_n:=X_{n,n}$. Thus we have $X_{s,t}\in\mathfrak{gl}_{\mathbf{Q}'}(n)\subset\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ and

$$X_{s,t}X_{k,l}-X_{k,l}X_{s,t}=\delta_{t=k}X_{s,l}-\delta_{l=s}X_{k,t},$$

for $1\leqslant s,t,k,l\leqslant n$. It follows from this formula that the elements $H_1,\dots,H_n$, $u_{i+1},\dots,u_{n-1}$ commute with each other. Let $\mathcal{U}^0$ denote the subring of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ generated by these elements. By the Poincare–Birkhoff–Witt theorem, $\mathcal{U}^0$ is generated freely by these variables as a $\mathbf{Z}$-algebra. In particular, $\mathcal{U}^0$ is a unique factorization domain. This fact will often be used in this paper.

We obtain a $\mathbf{Z}^n$-grading of the algebra $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ if we assume that $X_{s,t}$ has weight $\alpha(s,t)$ and the elements of $\mathbf{Q}'$ have weight zero. Weights and homogeneity of elements of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ will always be understood with respect to this grading.

Choosing the appropriate orderings of elements $X_{s,t}$ for the Poincare–Birkhoff–Witt theorem, we easily obtain that nonzero elements of $\mathcal{U}^0$ are not zero divisors of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$.

We claim that $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ satisfies the right Ore condition with respect to the denominator set $\mathcal{U}^0\setminus\{0\}$: for any $s\in\mathcal{U}^0\setminus\{0\}$ and $a\in\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$, there exist $t\in\mathcal{U}^0\setminus\{0\}$ and $b\in\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ such that $at=sb$. If a is homogeneous, then this fact is obvious and we can take $b=a$. In the general case, consider the representation $a=a_1+\dots+a_k$, where $a_1,\dots,a_k$ are homogeneous. Let $t_1,\dots,t_k$ be elements of $\mathcal{U}^0\setminus\{0\}$ such that $a_jt_j=sa_j$ for $j=1,\dots,k$. Then $a\cdot t_1\cdots t_k=s\cdot\sum_{j=1}^k(a_jt_1\cdots t_{j-1}t_{j+1}\cdots t_k)$ as required.

Thus there exists the right ring of quotients $\bar{U}$ of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ with respect to $\mathcal{U}^0\setminus\{0\}$. As far as we work over $\mathbf{Z}$, the ring $\bar{U}$ is our universal object in the sense that we construct all the rings we need as subrings of $\bar{U}$.

Note that $\bar{U}$ is determined uniquely up to isomorphisms identical on $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ and that $\bar{U}$ is also the left ring of quotients of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ with respect to $\mathcal{U}^0\setminus\{0\}$. The algebra $\bar{U}$ inherits the grading from the grading of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ described above.

2.4. Hyperalgebra over $\mathbf{Z}'$

Consider the polynomial algebra $\mathbf{Z}':=\mathbf{Z}[u_{i+1},\dots,u_{n-1}]$ and denote by U the $\mathbf{Z}'$-subalgebra of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ generated by

$$X_{s,t}^{(r)}:=\frac{(X_{s,t})^r}{r!}\quad\text{for integers } 1\leqslant s,\ t\leqslant n,\ s\neq t \text{ and } r\geqslant 0;$$

$$\binom{X_{s,s}}{r}:=\frac{X_{s,s}(X_{s,s}-1)\cdots(X_{s,s}-r+1)}{r!}\quad\text{for integers } 1\leqslant s\leqslant n \text{ and } r\geqslant 0.$$

In this definition, the empty product means the identity of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$. It is convenient to define the above elements as zero if r is a negative integer. We shall use the notation $E_{s,t}^{(r)} := X_{s,t}^{(r)}$ and $F_{s,t}^{(r)} := X_{t,s}^{(r)}$, where $1 \leqslant s < t \leqslant n$, and $H_s := X_{s,s}$, where $s = 1, \dots, n$. We also put $E_s^{(r)} := E_{s,s+1}^{(r)}$ for $s = 1, \dots, n-1$ and omit the superscript $^{(1)}$.

For any $N \in UT(n)$, we define

$$F^{(N)} := \prod_{1 \leqslant a < b \leqslant n} F_{a,b}^{(N_{a,b})}, \qquad E^{(N)} := \prod_{1 \leqslant a < b \leqslant n} E_{a,b}^{(N_{a,b})},$$

where $F_{a,b}^{(N_{a,b})}$ precedes $F_{c,d}^{(N_{c,d})}$ iff $b < d$ or $b = d$ and $a < c$ in the first product and $E_{a,b}^{(N_{a,b})}$ precedes $E_{c,d}^{(N_{c,d})}$ iff $a < c$ or $a = c$ and $b < d$ in the second product. Obviously, $F^{(N)} = E^{(N)} = 0$ if N contains a negative entry.

Proposition 2.1. *Elements $F^{(N)} \binom{H_1}{r_1} \cdots \binom{H_n}{r_n} E^{(M)}$, where $N, M \in UT^{\geqslant 0}(n)$ and $r_1, \dots, r_n$ are nonnegative integers form a $\mathbf{Q}'$-space basis of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$. These elements form a $\mathbf{Z}'$-module basis of U.*

Proof. This can be proved similarly to [St, Theorem 2]. See also [CL, 2.1]. $\square$

The similar result holds for $\bar{U}$. Let $\bar{\mathcal{U}}^0$ denote the subfield of $\bar{U}$ generated by $\mathcal{U}^0$.

Lemma 2.2. *Any element $x \in \bar{U}$ is uniquely represented in the form $x = \sum_{N,M \in UT^{\geqslant 0}(n)} F^{(N)} H_{N,M} E^{(M)}$, where $H_{N,M} \in \bar{\mathcal{U}}^0$.*

Proof. The possibility of such a representation follows directly from Proposition 2.1. For any element $a \in \mathcal{U}^0$ and matrix $M \in UT^{\geqslant 0}(n)$, there is a uniquely determined element $a_M \in \mathcal{U}^0$ such that $E^{(M)} a = a_M E^{(M)}$.

Now suppose that

$$\sum_{N,M \in UT^{\geqslant 0}(n)} F^{(N)} H_{N,M} E^{(M)} = \sum_{N,M \in UT^{\geqslant 0}(n)} F^{(N)} \hat{H}_{N,M} E^{(M)}, \tag{2.4}$$

where $H_{N,M}, \hat{H}_{N,M} \in \bar{\mathcal{U}}^0$ and only finitely many of them are nonzero. Obviously, we can choose $a \in \mathcal{U}^0 \setminus \{0\}$ so that $H_{N,M} a_M, \hat{H}_{N,M} a_M \in \mathcal{U}^0$ for all $N, M \in UT^{\geqslant 0}(n)$. Multiplying (2.4) by a on the right and applying Proposition 2.1, we get $H_{N,M} a_M = \hat{H}_{N,M} a_M$ for all $N, M \in UT^{\geqslant 0}(n)$. Noting that $a_M \neq 0$, we get $H_{N,M} = \hat{H}_{N,M}$. $\square$

Let I^+ and $\bar{I}^+$ denote the left ideals of U and $\bar{U}$ respectively generated by the elements $E_s^{(r)}$, where $r > 0$ and $s = 1, \dots, n-1$.

Lemma 2.3. $\bar{I}^+ \cap U = I^+$.

Proof. We only need to prove that the left-hand side is contained in the right-hand side.

Let $x \in \bar{I}^+ \cap U$. The definition of $\bar{I}^+$ and Lemma 2.2 yield

$$x = \sum_{1 \leqslant s < n,\, r > 0} \left(\sum_{N,M \in UT^{\geqslant 0}(n)} F^{(N)} H_{N,M}^{(s,r)} E^{(M)} \right) E_s^{(r)},$$

where $H^{(s,r)}_{N,M} \in \bar{\mathcal{U}}^0$. It is a standard calculation to check that each element $E^{(M)}E^{(r)}_s$ is an integral linear combination of elements $E^{(M')}$ of the same weight. In particular, it is an integral linear combination of $E^{(M')}$ with $M' \in UT^{\geqslant 0}(n) \setminus \{0\}$, since $r > 0$. Hence we get

$$x = \sum_{N \in UT^{\geqslant 0}(n),\ M' \in UT^{\geqslant 0}(n) \setminus \{0\}} F^{(N)} H_{N,M'} E^{(M')},$$

where $H_{N,M'} \in \bar{\mathcal{U}}^0$. However, $x \in U$ and Proposition 2.1 and Lemma 2.2 imply that each $H_{N,M'}$ belongs to the $\mathbf{Z}'$-submodule of U generated by various products $\binom{H_1}{r_1} \cdots \binom{H_n}{r_n}$. Since for $M' \in UT^{\geqslant 0}(n) \setminus \{0\}$ we have $E^{(M')} \in I^+$, we obtain $x \in I^+$. □

Let U^0, U^-, $\mathcal{U}^{-,0}$ denote the $\mathbf{Z}'$-subalgebras of U generated by the sets $\{\binom{H_s}{r} \mid s = 1, \dots, n,\ r \in \mathbf{Z}\}$, $\{F^{(r)}_{s,t} \mid 1 \leqslant s < t \leqslant n,\ r \in \mathbf{Z}\}$ and $\{F^{(r)}_{s,t} \mid 1 \leqslant s < t \leqslant n,\ r \in \mathbf{Z}\} \cup \{H_s \mid s = 1, \dots, n\}$, respectively. We also denote by $\bar{\mathcal{U}}^{-,0}$ the subring of $\bar{U}$ generated by U^- and $\bar{\mathcal{U}}^0$. If an element $x \in \mathcal{U}^{-,0}$ is represented in the form stated in Lemma 2.2, then $H_{N,0}$ for $N \in UT^{\geqslant 0}(n)$ is called the *$F^{(N)}$-coefficient of x*.

In what follows, we shall use the abbreviation $C(k,l) := l - k + H_k - H_l$.

2.5. Hyperalgebras over fields

Let K be an arbitrary field. Suppose that $\tau : \mathbf{Z}' \to K$ is a ring homomorphism. Let K_τ denote the field K considered as a left $\mathbf{Z}'$-module via the multiplication rule $f \cdot \alpha = \tau(f)\alpha$, where $f \in \mathbf{Z}'$ and $\alpha \in K$. For $x \in U$, we denote $x^\tau := x \otimes 1_K$, which is an element of $U \otimes_{\mathbf{Z}'} K_\tau$. It follows from Proposition 2.1 that the algebra $U \otimes_{\mathbf{Z}'} K_\tau$ has K-space basis

$$\left\{ \left(F^{(N)} \binom{H_1}{r_1} \cdots \binom{H_n}{r_n} E^{(M)} \right) \otimes 1_K \;\middle|\; N, M \in UT^{\geqslant 0}(n),\ r_1, \dots, r_n \text{ integers } \geqslant 0 \right\}.$$

Therefore, this algebra actually does not depend on τ. We denote it by $U_K(n)$. Note that $U_K(n)$ is naturally isomorphic to the algebra denoted by $U(n)$ in [BKS, §2.1]. The reason we constructed the hyperalgebra $U_K(n)$ in such a way is to obtain the projection $x \mapsto x^\tau$, which unlike $U_K(n)$ depends on τ, and to have at our disposal the variables $u_{i+1}, \dots, u_{n-1}$. These variables will be used first to construct elementary expressions in Section 3 and then to constructing lowering operators in Section 6. Assuming this projection is weight-preserving and that elements of K have weight zero, we obtain the $\mathbf{Z}^n$-grading of $U_K(n)$.

We denote $\mathcal{E}^{(r)}_s := E^{(r)}_s \otimes 1_K$ for $s = 1, \dots, n-1$, $\mathcal{F}^{(r)}_{s,t} := F^{(r)}_{s,t} \otimes 1_K$ for $1 \leqslant s < t \leqslant n$, $\mathcal{F}^{(N)} := F^{(N)} \otimes 1_K$ for $N \in UT^{\geqslant 0}(n)$ and $\mathcal{H}_s := H_s \otimes 1_K$, $\binom{\mathcal{H}_s}{r} := \binom{H_s}{r} \otimes 1_K$ for $s = 1, \dots, n$. Let $U^0_K(n)$ and $U^-_K(n)$ denote the K-subalgebras of $U_K(n)$ generated by the sets $\{\binom{\mathcal{H}_s}{r} \mid s = 1, \dots, n,\ r \in \mathbf{Z}\}$ and $\{\mathcal{F}^{(r)}_{s,t} \mid 1 \leqslant s < t \leqslant n,$ $r \in \mathbf{Z}\}$, respectively. As usual, the *$\mathcal{F}^{(N)}$-coefficient* of $\sum_{M \in UT^{\geqslant 0}(n)} \mathcal{F}^{(M)} h_M$, where $h_M \in U^0_K(n)$, is h_N.

3. Elementary expressions

Let $l = 1, \dots, n-1$ and $N \in UT(n)$. One can easily check that

$$\begin{aligned} \left[E_l, F^{(N)}\right] = {} & \sum_{1 \leqslant s < l} (N_{s,l} + 1) F^{(N - e_{s,l+1} + e_{s,l})} \\ & + F^{(N - e_{l,l+1})} \left(H_l - H_{l+1} + 1 - \sum_{l < b \leqslant n} N_{l,b} + \sum_{l+1 < b \leqslant n} N_{l+1,b} \right) \\ & - \sum_{l+1 < t \leqslant n} (N_{l+1,t} + 1) F^{(N - e_{l,t} + e_{l+1,t})}. \end{aligned} \tag{3.1}$$

Suppose that for each $N \in UT(n)$, there is given an element $H_N \in U^0$ so that only finitely many of these elements are nonzero. It follows directly from (3.1) and (2.1) that

$$E_l \sum_{N\in UT(n)} F^{(N)} H_N \equiv \sum_{M\in UT(n)} F^{(M)} \biggl(\sum_{1\leqslant s<l} M_{s,l} H_{M+e_{s,l+1}-e_{s,l}} + \bigl(H_l - H_{l+1} - M_l + M_{l+1} + M(l-1) - 2M(l) + M(l+1)\bigr) H_{M+e_{l,l+1}} - \sum_{l+1<t\leqslant n} M_{l+1,t} H_{M+e_{l,t}-e_{l+1,t}} \biggr) \pmod{U \cdot E_l}. \tag{3.2}$$

Notice also that

$$F^{(N)} \text{ has weight } -\sum_{t=0}^{n-1} N(t)\alpha_t \quad \text{for any } N \in UT^{\geqslant 0}(n). \tag{3.3}$$

Throughout this section we fix a subset $\mathcal{M} = \{m_1 < \cdots < m_k\}$ of $(i..n)$ and additionally assume $m_0 := i$, $m_{k+1} := n$ and $J_0 := \langle (i-1)^d \rangle$ (i.e. J_0 is the multiset of length d whose every entry is $i-1$). We consider a sequence of multisets $I_1, \dots, I_{k+1}, J_1, \dots, J_k$ such that

(M1) the entries of I_s belong to $[m_{s-1}-1..m_s)$, where $s = 2, \dots, k+1$, and the entries of I_1 belong to $[i..m_1)$;
(M2) the entries of J_s belong to $[m_s - 1..m_{s+1})$, where $s = 1, \dots, k$;
(M3) $|I_{s+1}|^{\{m_s-1\}} + |J_s| = |I_s| + |J_{s-1}|^{\{m_s-1\}}$ for any $s = 1, \dots, k$.

We define the weight

$$\lambda^{(d)}_{i,n,\mathcal{M}}(I_1, \dots, I_{k+1}, J_1, \dots, J_k) := \sum_{s=0}^{k} \sum_{t=m_s}^{m_{s+1}-1} \bigl(-d + |I_{s+1}|^{(-\infty..t]} + |J_s|^{[t..+\infty)}\bigr)\alpha_t. \tag{3.4}$$

Lemma 3.1. *For $s = 0, \dots, k$ and $t = m_s - 1, \dots, m_{s+1} - 1$, the α_t-coefficient of $\lambda^{(d)}_{i,n,\mathcal{M}}(I_1, \dots, I_{k+1}, J_1, \dots, J_k)$ is $-d + |I_{s+1}|^{(-\infty..t]} + |J_s|^{[t..+\infty)}$.*

Proof. We obviously need to consider only the case $t = m_s - 1$. Applying conditions (M1)–(M3), we obtain that for $s = 1, \dots, k$, the α_{m_s-1}-coefficient of $\lambda^{(d)}_{i,n,\mathcal{M}}(I_1, \dots, I_{k+1}, J_1, \dots, J_k)$ is

$$\begin{aligned} -d + |I_s|^{(-\infty..m_s-1]} + |J_{s-1}|^{[m_s-1..+\infty)} &= -d + |I_s| + |J_{s-1}|^{\{m_s-1\}} \\ &= -d + |I_{s+1}|^{\{m_s-1\}} + |J_s| \\ &= -d + |I_{s+1}|^{(-\infty..m_s-1]} + |J_s|^{[m_s-1..+\infty)} \end{aligned}$$

and the α_{i-1}-coefficient of $\lambda^{(d)}_{i,n,\mathcal{M}}(I_1, \dots, I_{k+1}, J_1, \dots, J_k)$ is $0 = -d + |I_1|^{(-\infty..i-1]} + |J_0|^{[i-1..+\infty)}$. □

Lemma 3.2. *Suppose that conditions* (M1)–(M3) *hold for a sequence $I_1, \dots, I_{k+1}, J_1, \dots, J_k$ of multisets and a set $\mathcal{M} = \{m_1 < \cdots < m_k\}$. Choose any well-defined sequence of the following list*:

(i) $I_1, \dots, I_{k+1}, J_1, \dots, J_{r-1}, (J_r)_{l-1\mapsto l}, J_{r+1}, \dots, J_k$;
$I_1, \dots, I_r, (I_{r+1})_{l+1\mapsto l}, I_{r+2}, \dots, I_{k+1}, J_1, \dots, J_k$,
where $m_r \leqslant l < m_{r+1} - 1$, $0 \leqslant r \leqslant k$ and $r > 0$ for the first sequence,

(ii) $I_1,\dots,I_{k+1},J_1,\dots,J_{r-2},(J_{r-1})_{m_r-2\mapsto m_r-1},J_r\cup\langle m_r-1\rangle,J_{r+1},\dots,J_k$;
$I_1,\dots,I_{r-1},I_r\cup\langle m_r-1\rangle,I_{r+1},\dots,I_{k+1},J_1,\dots,J_{r-1},J_r\cup\langle m_r-1\rangle,J_{r+1},\dots,J_k$;
$I_1,\dots,I_{r-1},I_r\cup\langle m_r-1\rangle,(I_{r+1})_{m_r\mapsto m_r-1},I_{r+2},\dots,I_{r+1},J_1,\dots,J_k$,
where $1\leqslant r\leqslant k$ *and* $r>1$ *for the first sequence,*

(iii) $I_1,\dots,I_{k+1},J_1,\dots,J_{k-1},(J_k)_{n-2\mapsto n-1}$;
$I_1,\dots,I_k,I_{k+1}\cup\langle n-1\rangle,J_1,\dots,J_k$,
where $k>0$ *for the first sequence,*

(iv) $I_1,\dots,I_r,\mathcal{L}^m(I_{r+1}),\mathcal{R}^m(I_{r+1}),I_{r+2},\dots,I_{k+1},J_1,\dots,J_{r-1},\mathcal{L}_m(J_r),\mathcal{R}_m(J_r),J_{r+1},\dots,J_k$,
where $m_r<m<m_{r+1}$ *and* $0\leqslant r\leqslant k$.

In cases (i)–(iii)*, conditions* (M1)–(M3) *hold for the chosen sequence and the set* $\mathcal{M}$; $\lambda^{(d)}_{i,n,\mathcal{M}}$ *with this sequence as its argument equals* $\lambda^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)+\alpha_l$*, where* $l=m_r-1$ *in case* (ii) *and* $l=n-1$ *in case* (iii)*. In case* (iv)*, conditions* (M1)–(M3) *hold for the chosen sequence and the set* $\mathcal{M}\cup\{m\}$; $\lambda^{(d)}_{i,n,\mathcal{M}\cup\{m\}}$ *with this sequence as its argument equals* $\lambda^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)$.

We introduce the following element of $\mathcal{U}^{-,0}$:

$$\begin{aligned}&S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\&:=\sum\Biggl\{F^{(N)}\prod_{s=0}^{k}\prod_{t=m_s}^{m_{s+1}-1}\bigl(N_t+|J_s|^{\{t-1\}}\bigr)!\bigl(C(m_s,t)+u_{m_s}\bigr)^{\underline{d-(N_t+|J_s|^{\{t-1\}}+|I_{s+1}|^{(-\infty..t)})}}\Bigm| N\in UT^{\geqslant 0}(n)\text{ and}\\&\qquad F^{(N)}\text{ has weight }\lambda^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\Biggr\},\end{aligned}\tag{3.5}$$

which we call an *elementary expression.*

In order to prove that we have actually obtained an element of $\mathcal{U}^{-,0}$, we must show that $d\geqslant N_t+|J_s|^{\{t-1\}}+|I_{s+1}|^{(-\infty..t)}$ for $m_s\leqslant t<m_{s+1}$ and $N\in UT^{\geqslant 0}(n)$. Since all the entries of N are nonnegative, by Lemma 3.1 we have

$$N_t\leqslant N(t-1)=d-|J_s|^{[t-1..+\infty)}-|I_{s+1}|^{(-\infty..t-1]}\leqslant d-|J_s|^{\{t-1\}}-|I_{s+1}|^{(-\infty..t)}.\tag{3.6}$$

Choose any $l=1,\dots,n-1$. The aim of the present section is to calculate $E_lS^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)$ modulo $U\cdot E_l$. It follows from (3.2) that

$$E_lS^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\equiv T\pmod{U\cdot E_l},$$

where $T\in\mathcal{U}^{-,0}$. Let $\mathcal{H}_M$ denote the $F^{(M)}$-coefficient of T, where $M\in UT^{\geqslant 0}(n)$. It follows from (3.2) and (3.5) that

$$T=\sum\bigl\{F^{(M)}\mathcal{H}_M\bigm| M\in UT^{\geqslant 0}(n)\text{ and }F^{(M)}\text{ has weight }\lambda^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)+\alpha_l\bigr\}.\tag{3.7}$$

Therefore, we take any matrix $M\in UT^{\geqslant 0}(n)$ such that $F^{(M)}$ has weight $\lambda^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)+\alpha_l\leqslant 0$ and calculate $\mathcal{H}_M$. By (3.3) and Lemma 3.1, we have

$$M(t)=d-|I_{s+1}|^{(-\infty..t]}-|J_s|^{[t..+\infty)}-\delta_{t=l}\tag{3.8}$$

for $s=0,\dots,k$ and $t=m_s-1,\dots,m_{s+1}-1$. Hence similarly to (3.6) we have

$$d-\big(M_t+|J_s|^{\{t-1\}}+|I_{s+1}|^{(-\infty..t)}+\delta_{t=l+1}\big)\geqslant 0 \tag{3.9}$$

for $s=0,\dots,k$ and $t=m_s,\dots,m_{s+1}-1$. By (3.2), we have $\mathcal{H}_M=S_1+S_2+S_3$ with S_1, S_2, S_3 calculated below. Inequalities (3.9) show that all exponents of the decreasing factorial powers we consider are nonnegative.

Case 0: $l<i$. We have $\mathcal{H}_M=S_1=S_2=S_3=0$.

Case 1: $m_r\leqslant l<m_{r+1}-1$ *for some* $0\leqslant r\leqslant k$. We put

$$\begin{aligned}\Phi:=\prod\big\{&\big(M_t+|J_s|^{\{t-1\}}\big)!\big(C(m_s,t)+u_{m_s}\big)^{\underline{d-(M_t+|J_s|^{\{t-1\}}+|I_{s+1}|^{(-\infty..t)})}}\;\big|\\ &0\leqslant s\leqslant k,\ m_s\leqslant t<m_{s+1},\ (s,t)\neq(r,l),(r,l+1)\big\}.\end{aligned}$$

By (3.2) we have

$$\begin{aligned}S_1&=\Phi M_l\big(M_l-1+|J_r|^{\{l-1\}}\big)!\big(C(m_r,l)+u_{m_r}\big)^{\underline{d-(M_l-1+|J_r|^{\{l-1\}}+|I_{r+1}|^{(-\infty..l)})}}\\ &\quad\times\big(M_{l+1}+1+|J_r|^{\{l\}}\big)!\big(C(m_r,l+1)+u_{m_r}\big)^{\underline{d-(M_{l+1}+1+|J_r|^{\{l\}}+|I_{r+1}|^{(-\infty..l+1)})}},\\ S_2&=\Phi\big(H_l-H_{l+1}-M_l+M_{l+1}+M(l-1)-2M(l)+M(l+1)\big)\\ &\quad\times\big(M_l+|J_r|^{\{l-1\}}\big)!\big(C(m_r,l)+u_{m_r}\big)^{\underline{d-(M_l+|J_r|^{\{l-1\}}+|I_{r+1}|^{(-\infty..l)})}}\\ &\quad\times\big(M_{l+1}+1+|J_r|^{\{l\}}\big)!\big(C(m_r,l+1)+u_{m_r}\big)^{\underline{d-(M_{l+1}+1+|J_r|^{\{l\}}+|I_{r+1}|^{(-\infty..l+1)})}},\\ S_3&=-\Phi M^{l+1}\big(M_l+|J_r|^{\{l-1\}}\big)!\big(C(m_r,l)+u_{m_r}\big)^{\underline{d-(M_l+|J_r|^{\{l-1\}}+|I_{r+1}|^{(-\infty..l)})}}\\ &\quad\times\big(M_{l+1}+|J_r|^{\{l\}}\big)!\big(C(m_r,l+1)+u_{m_r}\big)^{\underline{d-(M_{l+1}+|J_r|^{\{l\}}+|I_{r+1}|^{(-\infty..l+1)})}}.\end{aligned}$$

We put

$$\begin{aligned}X:=\Phi&\big(M_l+|J_r|^{\{l-1\}}-1\big)!\big(C(m_r,l)+u_{m_r}\big)^{\underline{d-(M_l+|J_r|^{\{l-1\}}+|I_{r+1}|^{(-\infty..l)})}}\\ &\times\big(M_{l+1}+|J_r|^{\{l\}}\big)!\big(C(m_r,l+1)+u_{m_r}\big)^{\underline{d-(M_{l+1}+|J_r|^{\{l\}}+|I_{r+1}|^{(-\infty..l+1)}+1)}}.\end{aligned}$$

Note that (2.1) and (3.8) imply

$$\begin{aligned}M_l-M^l&=M(l-1)-M(l)\\ &=-|I_{r+1}|^{(-\infty..l-1]}-|J_r|^{[l-1..+\infty)}+|I_{r+1}|^{(-\infty..l]}+|J_r|^{[l..+\infty)}+1\\ &=-|J_r|^{\{l-1\}}+|I_{r+1}|^{\{l\}}+1>-|J_r|^{\{l-1\}},\end{aligned}$$

whence $M_l+|J_r|^{\{l-1\}}>M^l\geqslant 0$. Moreover (2.1) and (3.8) also imply

$$M^{l+1}=M_{l+1}-M(l)+M(l+1)=M_{l+1}+|I_{r+1}|^{(-\infty..l]}+|J_r|^{\{l\}}+1-|I_{r+1}|^{(-\infty..l+1]}.$$

Summing S_1, S_2, S_3 and applying (3.8), we obtain

$$\begin{aligned}
\mathcal{H}_M = {} & X\big(M_l\big(C(m_r,l)+u_{m_r}-d+M_l+|J_r|^{\{l-1\}}+|I_{r+1}|^{(-\infty..l)}\big)\big(M_{l+1}+|J_r|^{\{l\}}+1\big)\\
& +\big(H_l-H_{l+1}-M_l+M_{l+1}-|I_{r+1}|^{(-\infty..l-1]}-|J_r|^{[l-1..+\infty)}\\
& +2|I_{r+1}|^{(-\infty..l]}+2|J_r|^{[l..+\infty)}+2-|I_{r+1}|^{(-\infty..l+1]}-|J_r|^{[l+1..+\infty)}\big)\\
& \times\big(M_l+|J_r|^{\{l-1\}}\big)\big(M_{l+1}+|J_r|^{\{l\}}+1\big)\\
& -\big(M_{l+1}+|I_{r+1}|^{(-\infty..l]}+|J_r|^{\{l\}}+1-|I_{r+1}|^{(-\infty..l+1]}\big)\big(M_l+|J_r|^{\{l-1\}}\big)\\
& \times\big(C(m_r,l+1)+u_{m_r}-d+M_{l+1}+|J_r|^{\{l\}}+|I_{r+1}|^{(-\infty..l+1)}+1\big)\big)\\
= {} & X\big(-|J_r|^{\{l-1\}}\big(C(m_r,l)+u_{m_r}-d+M_l+|J_r|^{\{l-1\}}+|I_{r+1}|^{(-\infty..l)}\big)\big(M_{l+1}+|J_r|^{\{l\}}+1\big)\\
& +|I_{r+1}|^{\{l+1\}}\big(M_l+|J_r|^{\{l-1\}}\big)\big(C(m_r,l+1)+u_{m_r}-d+|I_{r+1}|^{(-\infty..l+1)}\big)\big)\\
= {} & -|J_r|^{\{l-1\}}\Phi\big(M_l+|J_r|^{\{l-1\}}-1\big)!\big(C(m_r,l)+u_{m_r}\big)^{\underline{d-(M_l+|J_r|^{\{l-1\}}-1+|I_{r+1}|^{(-\infty..l)})}}\\
& \times\big(M_{l+1}+|J_r|^{\{l\}}+1\big)!\big(C(m_r,l+1)+u_{m_r}\big)^{\underline{d-(M_{l+1}+|J_r|^{\{l\}}+1+|I_{r+1}|^{(-\infty..l+1)})}}\\
& +|I_{r+1}|^{\{l+1\}}\Phi\big(M_l+|J_r|^{\{l-1\}}\big)!\big(C(m_r,l)+u_{m_r}\big)^{\underline{d-(M_l+|J_r|^{\{l-1\}}+|I_{r+1}|^{(-\infty..l)})}}\\
& \times\big(M_{l+1}+|J_r|^{\{l\}}\big)!\big(C(m_r,l+1)+u_{m_r}\big)^{\underline{d-(M_{l+1}+|J_r|^{\{l\}}+|I_{r+1}|^{(-\infty..l+1)}+1)}}\\
& \times\big(C(m_r,l+1)+u_{m_r}-d+|I_{r+1}|^{(-\infty..l]}\big).
\end{aligned}$$

Note that for $l=i$, we have $M_l=0$, $|J_0|^{\{l-1\}}=d$ and $|I_1|^{(-\infty..l)}=0$. Since $u_i=0$, the first summand in the right-hand side of the above formula equals zero. Taking into account (3.7), (2.2) and Lemma 3.2, we obtain

$$\begin{aligned}
& E_lS^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
& \quad\equiv -\delta_{l>i}|J_r|^{\{l-1\}}S^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_{k+1},J_1,\dots,J_{r-1},(J_r)_{l-1\mapsto l},J_{r+1},\dots,J_k\big)\\
& \qquad +|I_{r+1}|^{\{l+1\}}S^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_r,(I_{r+1})_{l+1\mapsto l},I_{r+2},\dots,I_{k+1},J_1,\dots,J_k\big)\\
& \qquad \times\big(C(m_r,l+1)+u_{m_r}-d+|I_{r+1}|^{(-\infty..l]}\big)\pmod{U\cdot E_l}.
\end{aligned}\tag{3.10}$$

Here and in what follows we assume the rule: a product of zero by anything not well defined (e.g. a function having an argument $J_{l-1\mapsto l}$, where $|J|^{\{l-1\}}=0$) is zero. Moreover, the restriction $\lambda^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)+\alpha_l\leqslant 0$ is unnecessary in (3.10) as well as in (3.11) and (3.12).

Case 2: *$l=m_r-1$ for some $1\leqslant r\leqslant k$.* We put

$$\begin{aligned}
\Phi := \prod\big\{&\big(M_t+|J_s|^{\{t-1\}}\big)!\big(C(m_s,t)+u_{m_s}\big)^{\underline{d-(M_t+|J_s|^{\{t-1\}}+|I_{s+1}|^{(-\infty..t)})}}\ \big|\\
& 0\leqslant s\leqslant k,\ m_s\leqslant t<m_{s+1},\ (s,t)\neq(r-1,l),(r,l+1)\big\}.
\end{aligned}$$

By (3.2) we have

$$\begin{aligned}
S_1 = {} & \Phi M_l\big(M_l-1+|J_{r-1}|^{\{l-1\}}\big)!\big(C(m_{r-1},l)+u_{m_{r-1}}\big)^{\underline{d-(M_l-1+|J_{r-1}|^{\{l-1\}}+|I_r|^{(-\infty..l)})}}\\
& \times\big(M_{l+1}+1+|J_r|^{\{l\}}\big)!\big(C(m_r,l+1)+u_{m_r}\big)^{\underline{d-(M_{l+1}+1+|J_r|^{\{l\}}+|I_{r+1}|^{(-\infty..l+1)})}},\\
S_2 = {} & \Phi\big(H_l-H_{l+1}-M_l+M_{l+1}+M(l-1)-2M(l)+M(l+1)\big)
\end{aligned}$$

$$\times \big(M_l + |J_{r-1}|^{\{l-1\}}\big)!\big(C(m_{r-1}, l) + u_{m_{r-1}}\big)^{\underline{d-(M_l+|J_{r-1}|^{\{l-1\}}+|I_r|^{(-\infty..l)})}}$$
$$\times \big(M_{l+1} + 1 + |J_r|^{\{l\}}\big)!\big(C(m_r, l+1) + u_{m_r}\big)^{\underline{d-(M_{l+1}+1+|J_r|^{\{l\}}+|I_{r+1}|^{(-\infty..l+1)})}},$$
$$S_3 = -\Phi M^{l+1}\big(M_l + |J_{r-1}|^{\{l-1\}}\big)!\big(C(m_{r-1}, l) + u_{m_{r-1}}\big)^{\underline{d-(M_l+|J_{r-1}|^{\{l-1\}}+|I_r|^{(-\infty..l)})}}$$
$$\times \big(M_{l+1} + |J_r|^{\{l\}}\big)!\big(C(m_r, l+1) + u_{m_r}\big)^{\underline{d-(M_{l+1}+|J_r|^{\{l\}}+|I_{r+1}|^{(-\infty..l+1)})}}.$$

We put

$$X := \Phi\big(M_l + |J_{r-1}|^{\{l-1\}} - 1\big)!\big(C(m_{r-1}, l) + u_{m_{r-1}}\big)^{\underline{d-(M_l+|J_{r-1}|^{\{l-1\}}+|I_r|^{(-\infty..l)})}}$$
$$\times \big(M_{l+1} + |J_r|^{\{l\}}\big)!\big(C(m_r, l+1) + u_{m_r}\big)^{\underline{d-(M_{l+1}+|J_r|^{\{l\}}+|I_{r+1}|^{(-\infty..l+1)}+1)}}.$$

Similarly to case 1 we obtain $M_l + |J_{r-1}|^{\{l-1\}} > M^l \geqslant 0$ from (2.1) and (3.8). Moreover, (2.1) and (3.8) imply

$$\begin{aligned} M^{l+1} &= M_{l+1} - M(l) + M(l+1) \\ &= M_{l+1} + |I_r|^{(-\infty..l]} + |J_{r-1}|^{[l..+\infty)} + 1 - |I_{r+1}|^{(-\infty..l+1]} - |J_r|^{[l+1..+\infty)}. \end{aligned}$$

Summing S_1, S_2, S_3 and applying (3.8), we obtain

$$\begin{aligned} \mathcal{H}_M &= X\big(M_l\big(C(m_{r-1}, l) + u_{m_{r-1}} - d + M_l + |J_{r-1}|^{\{l-1\}} + |I_r|^{(-\infty..l)}\big)\big(M_{l+1} + |J_r|^{\{l\}} + 1\big) \\ &\quad + \big(H_l - H_{l+1} - M_l + M_{l+1} - |I_r|^{(-\infty..l-1]} - |J_{r-1}|^{[l-1..+\infty)} \\ &\quad + 2|I_r|^{(-\infty..l]} + 2|J_{r-1}|^{[l..+\infty)} + 2 - |I_{r+1}|^{(-\infty..l+1]} - |J_r|^{[l+1..+\infty)}\big) \\ &\quad \times \big(M_l + |J_{r-1}|^{\{l-1\}}\big)\big(M_{l+1} + |J_r|^{\{l\}} + 1\big) \\ &\quad - \big(M_{l+1} + |I_r|^{(-\infty..l]} + |J_{r-1}|^{[l..+\infty)} + 1 - |I_{r+1}|^{(-\infty..l+1]} - |J_r|^{[l+1..+\infty)}\big)\big(M_l + |J_{r-1}|^{\{l-1\}}\big) \\ &\quad \times \big(C(m_r, l+1) + u_{m_r} - d + M_{l+1} + |J_r|^{\{l\}} + |I_{r+1}|^{(-\infty..l+1)} + 1\big)\big) \\ &= X\big(-|J_{r-1}|^{\{l-1\}}\big(C(m_{r-1}, l) + u_{m_{r-1}} - d + M_l + |J_{r-1}|^{\{l-1\}} + |I_r|^{(-\infty..l)}\big) \\ &\quad \times \big(M_{l+1} + |J_r|^{\{l\}} + 1\big) + \big(M_l + |J_{r-1}|^{\{l-1\}}\big)\big(M_{l+1} + |J_r|^{\{l\}} + 1\big) \\ &\quad \times \big(C(m_{r-1}, m_r) + u_{m_{r-1}} - u_{m_r} + |I_r| - |I_{r+1}|^{\{m_r-1\}}\big) \\ &\quad + \big(|I_{r+1}|^{(-\infty..m_r]} - |I_r| - |J_{r-1}|^{\{m_r-1\}} + |J_r|\big)\big(M_l + |J_{r-1}|^{\{l-1\}}\big)\big(u_{m_r} - d + |I_{r+1}|^{\{m_r-1\}}\big)\big). \end{aligned}$$

Applying condition (M3) for $s = r$, we obtain $-|I_r| - |J_{r-1}|^{\{m_r-1\}} + |J_r| = -|I_{r+1}|^{\{m_r-1\}}$, whence

$$\begin{aligned} \mathcal{H}_M &= X\big(-|J_{r-1}|^{\{m_r-2\}}\big(C(m_{r-1}, l) + u_{m_{r-1}} - d + M_l + |J_{r-1}|^{\{l-1\}} + |I_r|^{(-\infty..l)}\big) \\ &\quad \times \big(M_{l+1} + |J_r|^{\{l\}} + 1\big) + \big(M_l + |J_{r-1}|^{\{l-1\}}\big)\big(M_{l+1} + |J_r|^{\{l\}} + 1\big) \\ &\quad \times \big(C(m_{r-1}, m_r) + u_{m_{r-1}} - u_{m_r} + |I_r| - |I_{r+1}|^{\{m_r-1\}}\big) \\ &\quad + |I_{r+1}|^{\{m_r\}}\big(M_l + |J_{r-1}|^{\{l-1\}}\big)\big(u_{m_r} - d + |I_{r+1}|^{\{m_r-1\}}\big)\big) \\ &= -|J_{r-1}|^{\{m_r-2\}}\Phi\big(M_l + |J_{r-1}|^{\{l-1\}} - 1\big)!\big(C(m_{r-1}, l) + u_{m_{r-1}}\big)^{\underline{d-(M_l+|J_{r-1}|^{\{l-1\}}-1+|I_r|^{(-\infty..l)})}} \\ &\quad \times \big(M_{l+1} + |J_r|^{\{l\}} + 1\big)!\big(C(m_r, l+1) + u_{m_r}\big)^{\underline{d-(M_{l+1}+|J_r|^{\{l\}}+1+|I_{r+1}|^{(-\infty..l+1)})}} \end{aligned}$$

$$
\begin{aligned}
&+\Phi\big(M_l+|J_{r-1}|^{\{l-1\}}\big)!\big(C(m_{r-1},l)+u_{m_{r-1}}\big)^{\underline{d-(M_l+|J_{r-1}|^{\{l-1\}}+|I_r|^{(-\infty..l)})}}\\
&\times\big(M_{l+1}+|J_r|^{\{l\}}+1\big)!\big(C(m_r,l+1)+u_{m_r}\big)^{\underline{d-(M_{l+1}+|J_r|^{\{l\}}+1+|I_{r+1}|^{(-\infty..l+1)})}}\\
&\times\big(C(m_{r-1},m_r)+u_{m_{r-1}}-u_{m_r}+|I_r|-|I_{r+1}|^{\{m_r-1\}}\big)\\
&+|I_{r+1}|^{\{m_r\}}\Phi\big(M_l+|J_{r-1}|^{\{l-1\}}\big)!\big(C(m_{r-1},l)+u_{m_{r-1}}\big)^{\underline{d-(M_l+|J_{r-1}|^{\{l-1\}}+|I_r|^{(-\infty..l)})}}\\
&\times\big(M_{l+1}+|J_r|^{\{l\}}\big)!\big(C(m_r,l+1)+u_{m_r}\big)^{\underline{d-(M_{l+1}+|J_r|^{\{l\}}+|I_{r+1}|^{(-\infty..l+1)}+1)}}\big(u_{m_r}-d+|I_{r+1}|^{\{m_r-1\}}\big).
\end{aligned}
$$

If $l=i$ then similarly to case 1 the first summand of the right-hand side of the above formula equals zero. Taking into account (3.7), (2.2) and Lemma 3.2, we obtain

$$
\begin{aligned}
&E_{m_r-1}S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad\equiv-\delta_{m_r-1>i}|J_{r-1}|^{\{m_r-2\}}\\
&\qquad\times S^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_{k+1},J_1,\dots,J_{r-2},(J_{r-1})_{m_r-2\mapsto m_r-1},J_r\cup\langle m_r-1\rangle,J_{r+1},\dots,J_k\big)\\
&\qquad+S^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_{r-1},I_r\cup\langle m_r-1\rangle,I_{r+1},\dots,I_{k+1},J_1,\dots,J_{r-1},J_r\cup\langle m_r-1\rangle,J_{r+1},\dots,J_k\big)\\
&\qquad\times\big(C(m_{r-1},m_r)+u_{m_{r-1}}-u_{m_r}+|I_r|-|I_{r+1}|^{\{m_r-1\}}\big)\\
&\qquad+|I_{r+1}|^{\{m_r\}}S^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_{r-1},I_r\cup\langle m_r-1\rangle,(I_{r+1})_{m_r\mapsto m_r-1},I_{r+2},\dots,I_{r+1},J_1,\dots,J_k\big)\\
&\qquad\times\big(u_{m_r}-d+|I_{r+1}|^{\{m_r-1\}}\big)\pmod{U\cdot E_{m_r-1}}.
\end{aligned}\tag{3.11}
$$

Case 3: $l=n-1$. We put

$$
\begin{aligned}
\Phi:=\prod\big\{&\big(M_t+|J_s|^{\{t-1\}}\big)!\big(C(m_s,t)+u_{m_s}\big)^{\underline{d-(M_t+|J_s|^{\{t-1\}}+|I_{s+1}|^{(-\infty..t)})}}\ \big|\\
&0\leqslant s\leqslant k,\ m_s\leqslant t<m_{s+1},\ (s,t)\neq(k,n-1)\big\}.
\end{aligned}
$$

By (3.2) we have

$$
\begin{aligned}
S_1&=\Phi M_l\big(M_l-1+|J_k|^{\{l-1\}}\big)!\big(C(m_k,l)+u_{m_k}\big)^{\underline{d-(M_l-1+|J_k|^{\{l-1\}}+|I_{k+1}|^{(-\infty..l)})}},\\
S_2&=\Phi\big(H_l-H_{l+1}-M_l+M_{l+1}+M(l-1)-2M(l)+M(l+1)\big)\\
&\quad\times\big(M_l+|J_k|^{\{l-1\}}\big)!\big(C(m_k,l)+u_{m_k}\big)^{\underline{d-(M_l+|J_k|^{\{l-1\}}+|I_{k+1}|^{(-\infty..l)})}},\\
S_3&=-\Phi M^{l+1}\big(M_l+|J_k|^{\{l-1\}}\big)!\big(C(m_k,l)+u_{m_k}\big)^{\underline{d-(M_l+|J_k|^{\{l-1\}}+|I_{k+1}|^{(-\infty..l)})}}.
\end{aligned}
$$

We put

$$
X:=\Phi\big(M_l+|J_k|^{\{l-1\}}-1\big)!\big(C(m_k,l)+u_{m_k}\big)^{\underline{d-(M_l+|J_k|^{\{l-1\}}+|I_{k+1}|^{(-\infty..l)})}}.
$$

Similarly to the previous case we obtain $M_l+|J_k|^{\{l-1\}}>M^l\geqslant0$. Notice also that $M(l+1)=0$. Summing S_1, S_2, S_3 and applying (3.8), we obtain

$$
\begin{aligned}
\mathcal{H}_M=X\big(&M_l\big(C(m_k,l)+u_{m_k}-d+M_l+|J_k|^{\{l-1\}}+|I_{k+1}|^{(-\infty..l)}\big)\\
&+\big(H_l-H_{l+1}-M_l+M_{l+1}-|I_{k+1}|^{(-\infty..l-1]}-|J_k|^{[l-1..+\infty)}
\end{aligned}
$$

$$
\begin{aligned}
&+2|I_{k+1}|^{(-\infty..l]}+2|J_k|^{[l..+\infty)}+2-d\big)\big(M_l+|J_k|^{\{l-1\}}\big)\\
&-\big(M_{l+1}+|I_{k+1}|^{(-\infty..l]}+|J_k|^{[l..+\infty)}+1-d\big)\big(M_l+|J_k|^{\{l-1\}}\big)\big)\\
={}&X\big(-|J_k|^{\{l-1\}}\big(C(m_k,l)+u_{m_k}-d+M_l+|J_k|^{\{l-1\}}+|I_{k+1}|^{(-\infty..l)}\big)\\
&+\big(M_l+|J_k|^{\{l-1\}}\big)\big(C(m_k,l+1)+u_{m_k}-d+|I_{k+1}|^{(-\infty..l+1)}\big)\big)\\
={}&-|J_k|^{\{n-2\}}\Phi\big(M_l+|J_k|^{\{l-1\}}-1\big)!\big(C(m_k,l)+u_{m_k}\big)^{\underline{d-(M_l+|J_k|^{\{l-1\}}-1+|I_{k+1}|^{(-\infty..l)})}}\\
&+\Phi\big(M_l+|J_k|^{\{l-1\}}\big)!\big(C(m_k,l)+u_{m_k}\big)^{\underline{d-(M_l+|J_k|^{\{l-1\}}+|I_{k+1}|^{(-\infty..l)})}}\big(C(m_k,l+1)+u_{m_k}-d+|I_{k+1}|\big).
\end{aligned}
$$

If $l=i$ then similarly to case 1 the first summand of the right-hand side of the above formula equals zero. Taking into account (3.7), (2.2) and Lemma 3.2, we obtain

$$
\begin{aligned}
&E_{n-1}S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad\equiv-\delta_{n-1>i}|J_k|^{\{n-2\}}S^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_{k+1},J_1,\dots,J_{k-1},(J_k)_{n-2\mapsto n-1}\big)\\
&\qquad+S^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_k,I_{k+1}\cup\langle n-1\rangle,J_1,\dots,J_k\big)\\
&\qquad\times\big(C(m_k,n)+u_{m_k}-d+|I_{k+1}|\big)\pmod{U\cdot E_{n-1}}.
\end{aligned}
\tag{3.12}
$$

4. Coefficients

In this section, we fix a subset $\mathcal{M}=\{m_1<\cdots<m_k\}$ of $(i..n)$ and put $m_0:=i$, $m_{k+1}:=n$, $\Gamma:=\{(s,t)\mid 0\leqslant s\leqslant k,\ m_s<t\leqslant m_{s+1}\}$.

Let $I_1,\dots,I_{k+1},J_1,\dots,J_k$ be a sequence of multisets satisfying conditions (M1)–(M3) together with the set $\mathcal{M}$. Consider the representation

$$\lambda^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)=-a_1\alpha_1-\cdots-a_{n-1}\alpha_{n-1}.$$

The integers $a_1,\dots,a_{n-1}$ are determined by (3.4) or alternatively by Lemma 3.1. We also put $a_0:=0$. In this section, we find the polynomial $P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)$ of $\mathcal{U}^0$, which is uniquely determined, such that

$$E_1^{(a_1)}\cdots E_{n-1}^{(a_{n-1})}S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\equiv P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\pmod{I^+}.\tag{4.1}$$

Lemma 4.1. *If $a_1,\dots,a_{n-1}$ are nonnegative, then*

$$
\begin{aligned}
&P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad=\frac{1}{a_{n-1}!}\left(\prod_{s=0}^{k}u_{m_s}^{\underline{|J_s|^{[m_s..+\infty)}}}\right)\left(\prod_{s=0}^{k}\prod_{t=m_s+1}^{m_{s+1}}|J_s|^{\{t-2\}}!\right.\\
&\qquad\times\sum_{q=0}^{+\infty}\frac{(C(m_s,t)+u_{m_s})^{\underline{d+1-|I_{s+1}|^{(-\infty..t-1]}}}}{(C(m_s,t)+u_{m_s}-|J_s|^{[t..+\infty)})\cdots(C(m_s,t)+u_{m_s}-|J_s|^{[t-1..+\infty)}-q)}\\
&\qquad\left.\times\binom{|I_{s+1}|^{\{t-1\}}}{a_{t-2}-a_{t-1}+q}\binom{a_{t-1}}{q}(H_{t-1}-H_t)^{\underline{q}}\right).
\end{aligned}
\tag{4.2}
$$

Otherwise $P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)=0$.

Proof. If $a_t < 0$ for some t, then $S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)=0$. Therefore, further we consider only the case where $a_1,\dots,a_{n-1}$ are nonnegative. We apply induction on the weight $\lambda^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)$, assuming that for greater weights the assertion of the lemma holds.

Case 1: $\lambda^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)=0$. In (3.5), the summation parameter N can be only the zero matrix. Hence

$$S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k) = \left(\prod_{s=0}^{k}\prod_{t=m_s}^{m_{s+1}-1}|J_s|^{\{t-1\}}!\right)\left(\prod_{s=0}^{k}\prod_{t=m_s}^{m_{s+1}-1}\big(C(m_s,t)+u_{m_s}\big)^{\underline{d-(|J_s|^{\{t-1\}}+|I_{s+1}|^{(-\infty..t)})}}\right). \tag{4.3}$$

It follows from Lemma 3.1 that $d-|I_{s+1}|^{(-\infty..m_s-1]}-|J_s|^{\{m_s-1\}}=|J_s|^{[m_s..+\infty)}$. Therefore, the factor in the second pair of brackets of the right-hand side of (4.3) for $t=m_s$ equals $u_{m_s}^{\underline{|J_s|^{[m_s..+\infty)}}}$. Let us see what value the same factor takes for $t=m_{s+1}$. By Lemma 3.1, we have

$$d-\big(|J_s|^{\{m_{s+1}-1\}}+|I_{s+1}|^{(-\infty..m_{s+1})}\big)=d-\big(|J_s|^{[m_{s+1}-1..+\infty)}+|I_{s+1}|^{(-\infty..m_{s+1}-1]}\big)=0.$$

Hence for $t=m_{s+1}$ our factor equals 1. Therefore, (4.3) can be rearranged as

$$S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k) = \left(\prod_{s=0}^{k}u_{m_s}^{\underline{|J_s|^{[m_s..+\infty)}}}\right)\left(\prod_{s=0}^{k}\prod_{t=m_s+1}^{m_{s+1}}|J_s|^{\{t-2\}}!\big(C(m_s,t)+u_{m_s}\big)^{\underline{d-(|J_s|^{\{t-1\}}+|I_{s+1}|^{(-\infty..t-1]})}}\right).$$

If $m_s+1\leqslant t\leqslant m_{s+1}$ then applying Lemma 3.1, we easily obtain

$$\big(C(m_s,t)+u_{m_s}\big)^{\underline{d-(|J_s|^{\{t-1\}}+|I_{s+1}|^{(-\infty..t-1]})}} = \frac{(C(m_s,t)+u_{m_s})^{\underline{d+1-|I_{s+1}|^{(-\infty..t-1]}}}}{(C(m_s,t)+u_{m_s}-|J_s|^{[t..+\infty)})\cdots(C(m_s,t)+u_{m_s}-|J_s|^{[t-1..+\infty)})}.$$

It remains to notice that we may assume $q=0$ in (4.2).

Case 2: $\lambda^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)<0$. Let l be the greatest integer of $1,\dots,n-1$ such that $a_l>0$. We have $i\leqslant l$ and

$$E_1^{(a_1)}\cdots E_{n-1}^{(a_{n-1})}S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k) = \frac{1}{a_l}E_1^{(a_1)}\cdots E_{l-1}^{(a_{l-1})}E_l^{(a_l-1)}\cdot E_l S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k). \tag{4.4}$$

Case 2.1: $m_r\leqslant l<m_{r+1}-1$ *for some* $0\leqslant r\leqslant k$. Applying (3.10) to the last factor of the right-hand side of (4.4), we obtain

$$\begin{aligned}P^{(d)}_{i,n,\mathcal{M}}&(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\ &=\frac{1}{a_l}\big(-\delta_{l>i}|J_r|^{\{l-1\}}P^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_{k+1},J_1,\dots,J_{r-1},(J_r)_{l-1\mapsto l},J_{r+1},\dots,J_k\big)\\ &\quad+|I_{r+1}|^{\{l+1\}}P^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_r,(I_{r+1})_{l+1\mapsto l},I_{r+2},\dots,I_{k+1},J_1,\dots,J_k\big)\\ &\quad\times\big(C(m_r,l+1)+u_{m_r}-d+|I_{r+1}|^{(-\infty..l]}\big)\big).\end{aligned} \tag{4.5}$$

Since $a_{l+1}=0$, by Lemma 3.1 we have $0<a_l=a_l-a_{l+1}=|I_{r+1}|^{\{l+1\}}-|J_r|^{\{l\}}$. Hence $|I_{r+1}|^{\{l+1\}}>0$. We put

$$\Phi := \frac{1}{a_l}\frac{1}{a_{n-1}!}\prod\bigl\{|J_s|^{\{t-2\}}!\bigm|(s,t)\in\Gamma\setminus\bigl\{(r,l+1),(r,l+2)\bigr\}\bigr\}\left(\prod_{s=0}^{k}u_{m_s}{}^{\underline{|J_s|^{[m_s..+\infty)}}}\right)$$
$$\times\prod\left\{\sum_{q=0}^{+\infty}\frac{(C(m_s,t)+u_{m_s})^{\underline{d+1-|I_{s+1}|^{(-\infty..t-1]}}}}{(C(m_s,t)+u_{m_s}-|J_s|^{[t..+\infty)})\cdots(C(m_s,t)+u_{m_s}-|J_s|^{[t-1..+\infty)}-q)}\right.$$
$$\left.\times\binom{|I_{s+1}|^{\{t-1\}}}{a_{t-2}-a_{t-1}+q}\binom{a_{t-1}}{q}(H_{t-1}-H_t)^{\underline{q}}\biggm|(s,t)\in\Gamma\setminus\bigl\{(r,l+1),(r,l+2)\bigr\}\right\}.$$

Considering separately the cases $m_r<l$ and $m_r=l$, applying the induction hypothesis, the equality $a_{l+1}=0$ and (4.5), we obtain

$$P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)$$
$$=-\Phi|J_r|^{\{l-1\}}!\bigl(|J_r|^{\{l\}}+1\bigr)!\bigl(C(m_r,l)+u_{m_r}-|J_r|^{[l..+\infty)}\bigr)$$
$$\times\left(\sum_{q=0}^{+\infty}\frac{(C(m_r,l+1)+u_{m_r})^{\underline{d+1-|I_{r+1}|^{(-\infty..l]}}}}{(C(m_r,l+1)+u_{m_r}-|J_r|^{[l+1..+\infty)})\cdots(C(m_r,l+1)+u_{m_r}-|J_r|^{[l..+\infty)}-1-q)}\right.$$
$$\left.\times\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+1+q}\binom{a_l-1}{q}(H_l-H_{l+1})^{\underline{q}}\right)$$
$$\times\frac{(C(m_r,l+2)+u_{m_r})^{\underline{d+1-|I_{r+1}|^{(-\infty..l+1]}}}}{(C(m_r,l+2)+u_{m_r}-|J_r|^{[l+2..+\infty)})\cdots(C(m_r,l+2)+u_{m_r}-|J_r|^{[l+1..+\infty)})}\binom{|I_{r+1}|^{\{l+1\}}}{a_l-1}$$
$$+\Phi|J_r|^{\{l-1\}}!|J_r|^{\{l\}}!|I_{r+1}|^{\{l+1\}}$$
$$\times\left(\sum_{q=0}^{+\infty}\frac{(C(m_r,l+1)+u_{m_r})^{\underline{d+1-|I_{r+1}|^{(-\infty..l]}}}}{(C(m_r,l+1)+u_{m_r}-|J_r|^{[l+1..+\infty)})\cdots(C(m_r,l+1)+u_{m_r}-|J_r|^{[l..+\infty)}-q)}\right.$$
$$\left.\times\binom{|I_{r+1}|^{\{l\}}+1}{a_{l-1}-a_l+1+q}\binom{a_l-1}{q}(H_l-H_{l+1})^{\underline{q}}\right)$$
$$\times\frac{(C(m_r,l+2)+u_{m_r})^{\underline{d+1-|I_{r+1}|^{(-\infty..l+1]}}}}{(C(m_r,l+2)+u_{m_r}-|J_r|^{[l+2..+\infty)})\cdots(C(m_r,l+2)+u_{m_r}-|J_r|^{[l+1..+\infty)})}\binom{|I_{r+1}|^{\{l+1\}}-1}{a_l-1}.$$

To see that this formula holds in the cases $|J_r|^{\{l-1\}}=0$ and $l=i$, we notice that $a_{l-1}-a_l=|I_{r+1}|^{\{l\}}$ by Lemma 3.1 in the former case and $r=0$, $u_{m_r}=0$, $|J_r|^{[l..+\infty)}=0$ in the latter case. Therefore, the first summand of the right-hand side of the above formula equals zero in either of these cases. We put

$$X := \Phi|J_r|^{\{l-1\}}!|J_r|^{\{l\}}!\frac{(C(m_r,l+1)+u_{m_r})^{\underline{d+1-|I_{r+1}|^{(-\infty..l]}}}}{(C(m_r,l+1)+u_{m_r}-|J_r|^{[l+1..+\infty)})\cdots(C(m_r,l+1)+u_{m_r}-|J_r|^{[l..+\infty)})}$$
$$\times\frac{(C(m_r,l+2)+u_{m_r})^{\underline{d+1-|I_{r+1}|^{(-\infty..l+1]}}}}{(C(m_r,l+2)+u_{m_r}-|J_r|^{[l+2..+\infty)})\cdots(C(m_r,l+2)+u_{m_r}-|J_r|^{[l+1..+\infty)})}.$$

Then we have

$$\begin{aligned}
&P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad=-X\big(|J_r|^{\{l\}}+1\big)\left(\sum_{q=1}^{+\infty}\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q}\right.\\
&\qquad\times\left.\binom{a_l-1}{q-1}\frac{(C(m_r,l)+u_{m_r}-|J_r|^{[l..+\infty)})(H_l-H_{l+1})^{\underline{q-1}}}{(C(m_r,l+1)+u_{m_r}-|J_r|^{[l..+\infty)}-1)^{\underline{q}}}\right)\binom{|I_{r+1}|^{\{l+1\}}}{a_l-1}\\
&\qquad+X|I_{r+1}|^{\{l+1\}}\left(\sum_{q=0}^{+\infty}\binom{|I_{r+1}|^{\{l\}}+1}{a_{l-1}-a_l+1+q}\binom{a_l-1}{q}\right.\\
&\qquad\times\left.\frac{(H_l-H_{l+1})^{\underline{q}}}{(C(m_r,l+1)+u_{m_r}-|J_r|^{[l..+\infty)}-1)^{\underline{q}}}\right)\binom{|I_{r+1}|^{\{l+1\}}-1}{a_l-1}.
\end{aligned}\tag{4.6}$$

Noting that $C(m_r,l)+u_{m_r}-|J_r|^{[l..+\infty)}=(C(m_r,l+1)+u_{m_r}-|J_r|^{[l..+\infty)}-q)-((H_l-H_{l+1})-q+1)$, we obtain

$$\begin{aligned}
&\sum_{q=1}^{+\infty}\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q}\binom{a_l-1}{q-1}\frac{(C(m_r,l)+u_{m_r}-|J_r|^{[l..+\infty)})(H_l-H_{l+1})^{\underline{q-1}}}{(C(m_r,l+1)+u_{m_r}-|J_r|^{[l..+\infty)}-1)^{\underline{q}}}\\
&\quad=\sum_{q=1}^{+\infty}\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q}\binom{a_l-1}{q-1}\frac{(H_l-H_{l+1})^{\underline{q-1}}}{(C(m_r,l+1)+u_{m_r}-|J_r|^{[l..+\infty)}-1)^{\underline{q-1}}}\\
&\qquad-\sum_{q=0}^{+\infty}\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q}\binom{a_l-1}{q-1}\frac{(H_l-H_{l+1})^{\underline{q}}}{(C(m_r,l+1)+u_{m_r}-|J_r|^{[l..+\infty)}-1)^{\underline{q}}}\\
&\quad=\sum_{q=0}^{+\infty}\left(\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q+1}\binom{a_l-1}{q}-\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q}\binom{a_l-1}{q-1}\right)\\
&\qquad\times\frac{(H_l-H_{l+1})^{\underline{q}}}{(C(m_r,l+1)+u_{m_r}-|J_r|^{[l..+\infty)}-1)^{\underline{q}}}.
\end{aligned}$$

Substituting this back to (4.6) (the first pair of big brackets containing the sum equal to the left-hand side of the above formula), we obtain

$$\begin{aligned}
P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)=X\sum_{q=0}^{+\infty}\Bigg(\bigg[&-\big(|J_r|^{\{l\}}+1\big)\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q+1}\binom{a_l-1}{q}\binom{|I_{r+1}|^{\{l+1\}}}{a_l-1}\\
&+\big(|J_r|^{\{l\}}+1\big)\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q}\binom{a_l-1}{q-1}\binom{|I_{r+1}|^{\{l+1\}}}{a_l-1}\\
&+|I_{r+1}|^{\{l+1\}}\binom{|I_{r+1}|^{\{l\}}+1}{a_{l-1}-a_l+1+q}\binom{a_l-1}{q}\binom{|I_{r+1}|^{\{l+1\}}-1}{a_l-1}\bigg]\\
&\times\frac{(H_l-H_{l+1})^{\underline{q}}}{(C(m_r,l+1)+u_{m_r}-|J_r|^{[l..+\infty)}-1)^{\underline{q}}}\Bigg).
\end{aligned}$$

Since $|J_r|^{\{l\}}+1=|I_{r+1}|^{\{l+1\}}-a_l+1$, the expression in the square brackets is

$$-\big(|I_{r+1}|^{\{l+1\}}-a_l+1\big)\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q+1}\binom{a_l-1}{q}\binom{|I_{r+1}|^{\{l+1\}}}{a_l-1}$$
$$+\big(|I_{r+1}|^{\{l+1\}}-a_l+1\big)\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q}\binom{a_l-1}{q-1}\binom{|I_{r+1}|^{\{l+1\}}}{a_l-1}$$
$$+|I_{r+1}|^{\{l+1\}}\binom{|I_{r+1}|^{\{l\}}+1}{a_{l-1}-a_l+1+q}\binom{a_l-1}{q}\binom{|I_{r+1}|^{\{l+1\}}-1}{a_l-1}$$
$$=a_l\bigg(-\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q+1}\binom{a_l-1}{q}+\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q}\binom{a_l-1}{q-1}$$
$$+\binom{|I_{r+1}|^{\{l\}}+1}{a_{l-1}-a_l+1+q}\binom{a_l-1}{q}\bigg)\binom{|I_{r+1}|^{\{l+1\}}}{a_l}$$
$$=a_l\binom{|I_{r+1}|^{\{l\}}}{a_{l-1}-a_l+q}\binom{a_l}{q}\binom{|I_{r+1}|^{\{l+1\}}}{a_l-a_{l+1}},$$

which immediately yields the required expression.

Case 2.2: *$l=m_r-1$ for some $1\leqslant r\leqslant k$.* Applying (3.11) to the last factor of the right-hand side of (4.4), we obtain

$$\begin{aligned}
&P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad=\frac{1}{a_l}\big(-\delta_{m_r-1>i}|J_{r-1}|^{\{m_r-2\}}P^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_{k+1},\\
&\qquad J_1,\dots,J_{r-2},(J_{r-1})_{m_r-2\mapsto m_r-1},J_r\cup\langle m_r-1\rangle,J_{r+1},\dots,J_k\big)\\
&\qquad+P^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_{r-1},I_r\cup\langle m_r-1\rangle,I_{r+1},\dots,I_{k+1},J_1,\dots,J_{r-1},J_r\cup\langle m_r-1\rangle,J_{r+1},\dots,J_k\big)\\
&\qquad\times\big(C(m_{r-1},m_r)+u_{m_{r-1}}-u_{m_r}+|I_r|-|I_{r+1}|^{\{m_r-1\}}\big)\\
&\qquad+|I_{r+1}|^{\{m_r\}}P^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_{r-1},I_r\cup\langle m_r-1\rangle,(I_{r+1})_{m_r\mapsto m_r-1},I_{r+2},\dots,I_{r+1},J_1,\dots,J_k\big)\\
&\qquad\times\big(u_{m_r}-d+|I_{r+1}|^{\{m_r-1\}}\big)\big).
\end{aligned}\tag{4.7}$$

Since $a_{m_r}=0$, by Lemma 3.1, we have $|I_{r+1}|^{\{m_r\}}=a_{m_r-1}+|J_r|^{\{m_r-1\}}>0$. We put

$$\begin{aligned}
\varPhi:=&\frac{1}{a_{m_r-1}}\frac{1}{a_{n-1}!}\prod\big\{|J_s|^{\{t-2\}}!\;\big|\;(s,t)\in\varGamma\setminus\{(r-1,m_r),(r,m_r+1)\}\big\}\Bigg(\prod_{s=0}^{k}u_{m_s}\underline{{}^{|J_s|^{[m_s..+\infty)}}}\Bigg)\\
&\times\prod\Bigg\{\sum_{q=0}^{+\infty}\frac{(C(m_s,t)+u_{m_s})\underline{{}^{d+1-|I_{s+1}|^{(-\infty..t-1]}}}}{(C(m_s,t)+u_{m_s}-|J_s|^{[t..+\infty)})\cdots(C(m_s,t)+u_{m_s}-|J_s|^{[t-1..+\infty)}-q)}\\
&\times\binom{|I_{s+1}|^{\{t-1\}}}{a_{t-2}-a_{t-1}+q}\binom{a_{t-1}}{q}(H_{t-1}-H_t)\underline{^q}\;\Big|\;(s,t)\in\varGamma\setminus\{(r-1,m_r),(r,m_r+1)\}\Bigg\}.
\end{aligned}$$

Considering separately the cases $m_{r-1}<m_r-1$ and $m_{r-1}=m_r-1$, applying the induction hypothesis, the equality $a_{m_r}=0$ and (4.7), we obtain

$$\begin{aligned}
&P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad=-\varPhi|J_{r-1}|^{\{m_r-2\}}!\big(|J_r|^{\{m_r-1\}}+1\big)!\big(C(m_{r-1},m_r-1)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)}\big)
\end{aligned}$$

$$\times\left(\sum_{q=0}^{+\infty}\frac{(C(m_{r-1},m_r)+u_{m_{r-1}})^{\underline{d+1-|I_r|^{(-\infty..m_r-1]}}}}{(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r..+\infty)})\cdots(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)}-1-q)}\right.$$

$$\left.\times\binom{|I_r|^{\{m_r-1\}}}{a_{m_r-2}-a_{m_r-1}+1+q}\binom{a_{m_r-1}-1}{q}(H_{m_r-1}-H_{m_r})^{\underline{q}}\right)$$

$$\times\frac{(C(m_r,m_r+1)+u_{m_r})^{\underline{d+1-|I_{r+1}|^{(-\infty..m_r]}}}}{(C(m_r,m_r+1)+u_{m_r}-|J_r|^{[m_r+1..+\infty)})\cdots(C(m_r,m_r+1)+u_{m_r}-|J_r|^{[m_r..+\infty)})}\binom{|I_{r+1}|^{\{m_r\}}}{a_{m_r-1}-1}$$

$$+\Phi|J_{r-1}|^{\{m_r-2\}}!\big(|J_r|^{\{m_r-1\}}+1\big)!\big(C(m_{r-1},m_r)+u_{m_{r-1}}-u_{m_r}+|I_r|-|I_{r+1}|^{\{m_r-1\}}\big)$$

$$\times\left(\sum_{q=0}^{+\infty}\frac{(C(m_{r-1},m_r)+u_{m_{r-1}})^{\underline{d-|I_r|^{(-\infty..m_r-1]}}}}{(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r..+\infty)})\cdots(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)}-q)}\right.$$

$$\left.\times\binom{|I_r|^{\{m_r-1\}}+1}{a_{m_r-2}-a_{m_r-1}+1+q}\binom{a_{m_r-1}-1}{q}(H_{m_r-1}-H_{m_r})^{\underline{q}}\right)$$

$$\times\frac{(C(m_r,m_r+1)+u_{m_r})^{\underline{d+1-|I_{r+1}|^{(-\infty..m_r]}}}}{(C(m_r,m_r+1)+u_{m_r}-|J_r|^{[m_r+1..+\infty)})\cdots(C(m_r,m_r+1)+u_{m_r}-|J_r|^{[m_r..+\infty)})}\binom{|I_{r+1}|^{\{m_r\}}}{a_{m_r-1}-1}$$

$$+\Phi|I_{r+1}|^{\{m_r\}}|J_{r-1}|^{\{m_r-2\}}!|J_r|^{\{m_r-1\}}!\big(u_{m_r}-d+|I_{r+1}|^{\{m_r-1\}}\big)$$

$$\times\left(\sum_{q=0}^{+\infty}\frac{(C(m_{r-1},m_r)+u_{m_{r-1}})^{\underline{d-|I_r|^{(-\infty..m_r-1]}}}}{(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r..+\infty)})\cdots(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)}-q)}\right.$$

$$\left.\times\binom{|I_r|^{\{m_r-1\}}+1}{a_{m_r-2}-a_{m_r-1}+1+q}\binom{a_{m_r-1}-1}{q}(H_{m_r-1}-H_{m_r})^{\underline{q}}\right)$$

$$\times\frac{(C(m_r,m_r+1)+u_{m_r})^{\underline{d+1-|I_{r+1}|^{(-\infty..m_r]}}}}{(C(m_r,m_r+1)+u_{m_r}-|J_r|^{[m_r+1..+\infty)})\cdots(C(m_r,m_r+1)+u_{m_r}-|J_r|^{[m_r..+\infty)})}\binom{|I_{r+1}|^{\{m_r\}}-1}{a_{m_r-1}-1}.$$

To see that this formula holds in the cases $|J_{r-1}|^{\{m_r-2\}}=0$ and $m_r-1=i$, notice that $a_{m_r-2}-a_{m_r-1}=|I_r|^{\{m_r-1\}}$ by Lemma 3.1 in the former case and $r=1$, $u_{m_{r-1}}=0$, $|J_{r-1}|^{[m_r-1..+\infty)}=0$ in the latter case. Therefore, the first summand of the right-hand side of the above formula equals zero in either of these cases. We put

$$X:=\Phi|J_{r-1}|^{\{m_r-2\}}!|J_r|^{\{m_r-1\}}!$$
$$\times\frac{(C(m_{r-1},m_r)+u_{m_{r-1}})^{\underline{d+1-|I_r|^{(-\infty..m_r-1]}}}}{(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r..+\infty)})\cdots(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)})}$$
$$\times\frac{(C(m_r,m_r+1)+u_{m_r})^{\underline{d+1-|I_{r+1}|^{(-\infty..m_r]}}}}{(C(m_r,m_r+1)+u_{m_r}-|J_r|^{[m_r+1..+\infty)})\cdots(C(m_r,m_r+1)+u_{m_r}-|J_r|^{[m_r..+\infty)})}.$$

Noting that by Lemma 3.1

$$\big(|J_r|^{\{m_r-1\}}+1\big)\binom{|I_{r+1}|^{\{m_r\}}}{a_{m_r-1}-1}=|I_{r+1}|^{\{m_r\}}\binom{|I_{r+1}|^{\{m_r\}}-1}{a_{m_r-1}-1}=a_{m_r-1}\binom{|I_{r+1}|^{\{m_r\}}}{a_{m_r-1}},$$

we obtain

$$
\begin{aligned}
&P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad= -a_{m_r-1}X\Bigg(\sum_{q=1}^{+\infty}\binom{|I_r|^{\{m_r-1\}}}{a_{m_r-2}-a_{m_r-1}+q}\binom{a_{m_r-1}-1}{q-1}\\
&\qquad\times\frac{(C(m_{r-1},m_r-1)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)})(H_{m_r-1}-H_{m_r})^{\underline{q-1}}}{(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)}-1)^{\underline{q}}}\Bigg)\binom{|I_{r+1}|^{\{m_r\}}}{a_{m_r-1}}\\
&\qquad+a_{m_r-1}X\Bigg(\sum_{q=0}^{+\infty}\binom{|I_r|^{\{m_r-1\}}+1}{a_{m_r-2}-a_{m_r-1}+1+q}\binom{a_{m_r-1}-1}{q}\\
&\qquad\times\frac{(H_{m_r-1}-H_{m_r})^{\underline{q}}}{(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)}-1)^{\underline{q}}}\Bigg)\binom{|I_{r+1}|^{\{m_r\}}}{a_{m_r-1}}.
\end{aligned}\tag{4.8}
$$

Noting that $C(m_{r-1},m_r-1)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)}=-(H_{m_r-1}-H_{m_r}-q+1)+(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)}-q)$, we obtain similarly to case 2.1 that

$$
\begin{aligned}
&P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad=a_{m_r-1}X\binom{|I_{r+1}|^{\{m_r\}}}{a_{m_r-1}}\\
&\qquad\times\sum_{q=0}^{+\infty}\Bigg(\binom{|I_r|^{\{m_r-1\}}}{a_{m_r-2}-a_{m_r-1}+q}\binom{a_{m_r-1}-1}{q-1}-\binom{|I_r|^{\{m_r-1\}}}{a_{m_r-2}-a_{m_r-1}+1+q}\binom{a_{m_r-1}-1}{q}\\
&\qquad+\binom{|I_r|^{\{m_r-1\}}+1}{a_{m_r-2}-a_{m_r-1}+1+q}\binom{a_{m_r-1}-1}{q}\Bigg)\frac{(H_{m_r-1}-H_{m_r})^{\underline{q}}}{(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)}-1)^{\underline{q}}}\\
&\quad=a_{m_r-1}X\binom{|I_{r+1}|^{\{m_r\}}}{a_{m_r-1}}\\
&\qquad\times\sum_{q=0}^{+\infty}\binom{|I_r|^{\{m_r-1\}}}{a_{m_r-2}-a_{m_r-1}+q}\binom{a_{m_r-1}}{q}\frac{(H_{m_r-1}-H_{m_r})^{\underline{q}}}{(C(m_{r-1},m_r)+u_{m_{r-1}}-|J_{r-1}|^{[m_r-1..+\infty)}-1)^{\underline{q}}},
\end{aligned}
$$

which is what the lemma claims.

Case 2.3: $l=n-1$. Applying (3.12) to the last factor of the right-hand side of (4.4), we obtain

$$
\begin{aligned}
&P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad=\frac{1}{a_{n-1}}\big(-\delta_{n-1>i}|J_k|^{\{n-2\}}P^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_{k+1},J_1,\dots,J_{k-1},(J_k)_{n-2\mapsto n-1}\big)\\
&\qquad+P^{(d)}_{i,n,\mathcal{M}}\big(I_1,\dots,I_k,I_{k+1}\cup\langle n-1\rangle,J_1,\dots,J_k\big)\big(C(m_k,n)+u_{m_k}-d+|I_{k+1}|\big)\big).
\end{aligned}\tag{4.9}
$$

We put

$$
\begin{aligned}
\Phi:=&\frac{1}{a_{n-1}!}\Bigg(\prod_{s=0}^{k}\prod_{t=m_s+1}^{m_{s+1}}|J_s|^{\{t-2\}}!\Bigg)\Bigg(\prod_{s=0}^{k}u_{m_s}\underline{{}^{|J_s|^{[m_s..+\infty)}}}\Bigg)\\
&\times\prod\Bigg\{\sum_{q=0}^{+\infty}\frac{(C(m_s,t)+u_{m_s})^{\underline{d+1-|I_{s+1}|^{(-\infty..t-1]}}}}{(C(m_s,t)+u_{m_s}-|J_s|^{[t..+\infty)})\cdots(C(m_s,t)+u_{m_s}-|J_s|^{[t-1..+\infty)}-q)}
\end{aligned}
$$

$$\times \binom{|I_{s+1}|^{\{t-1\}}}{a_{t-2}-a_{t-1}+q}\binom{a_{t-1}}{q}(H_{t-1}-H_t)^{\underline{q}} \;\Big|\; (s,t)\in\Gamma\setminus\{(k,n)\}\Bigg\}.$$

Considering separately the cases $m_k < n-1$ and $m_k = n-1$, applying the induction hypothesis and (4.9), we obtain

$$\begin{aligned}
&P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad = -\Phi\big(C(m_k,n-1)+u_{m_k}-|J_k|^{[n-1..+\infty)}\big)\\
&\qquad \times\Bigg(\sum_{q=0}^{+\infty}\frac{(C(m_k,n)+u_{m_k})^{\underline{d+1-|I_{k+1}|^{(-\infty..n-1]}}}}{(C(m_k,n)+u_{m_k}-|J_k|^{[n..+\infty)})\cdots(C(m_k,n)+u_{m_k}-|J_k|^{[n-1..+\infty)}-1-q)}\\
&\qquad \times\binom{|I_{k+1}|^{\{n-1\}}}{a_{n-2}-a_{n-1}+1+q}\binom{a_{n-1}-1}{q}(H_{n-1}-H_n)^{\underline{q}}\Bigg)\\
&\qquad + \Phi\big(C(m_k,n)+u_{m_k}-d+|I_{k+1}|\big)\\
&\qquad \times\Bigg(\sum_{q=0}^{+\infty}\frac{(C(m_k,n)+u_{m_k})^{\underline{d-|I_{k+1}|^{(-\infty..n-1]}}}}{(C(m_k,n)+u_{m_k}-|J_k|^{[n..+\infty)})\cdots(C(m_k,n)+u_{m_k}-|J_k|^{[n-1..+\infty)}-q)}\\
&\qquad \times\binom{|I_{k+1}|^{\{n-1\}}+1}{a_{n-2}-a_{n-1}+1+q}\binom{a_{n-1}-1}{q}(H_{n-1}-H_n)^{\underline{q}}\Bigg).
\end{aligned}$$

We put

$$X := \Phi\frac{(C(m_k,n)+u_{m_k})^{\underline{d+1-|I_{k+1}|^{(-\infty..n-1]}}}}{(C(m_k,n)+u_{m_k}-|J_k|^{[n..+\infty)})\cdots(C(m_k,n)+u_{m_k}-|J_k|^{[n-1..+\infty)})}.$$

We have

$$\begin{aligned}
&P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad = -X\Bigg(\sum_{q=1}^{+\infty}\binom{|I_{k+1}|^{\{n-1\}}}{a_{n-2}-a_{n-1}+q}\binom{a_{n-1}-1}{q-1}\frac{(C(m_k,n-1)+u_{m_k}-|J_k|^{[n-1..+\infty)})(H_{n-1}-H_n)^{\underline{q-1}}}{(C(m_k,n)+u_{m_k}-|J_k|^{[n-1..+\infty)}-1)^{\underline{q}}}\Bigg)\\
&\qquad + X\Bigg(\sum_{q=0}^{+\infty}\binom{|I_{k+1}|^{\{n-1\}}+1}{a_{n-2}-a_{n-1}+1+q}\binom{a_{n-1}-1}{q}\frac{(H_{n-1}-H_n)^{\underline{q}}}{(C(m_k,n)+u_{m_k}-|J_k|^{[n-1..+\infty)}-1)^{\underline{q}}}\Bigg).
\end{aligned}$$

The right-hand side of this formula is exactly the right-hand side of (4.8) for $r=k+1$ without the factor $a_{m_r-1}\binom{|I_{r+1}|^{\{m_r\}}}{a_{m_r-1}}$. Hence similarly to case 2.2 we obtain

$$\begin{aligned}
&P^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\\
&\quad = X\sum_{q=0}^{+\infty}\binom{|I_{k+1}|^{\{n-1\}}}{a_{n-2}-a_{n-1}+q}\binom{a_{n-1}}{q}\frac{(H_{n-1}-H_n)^{\underline{q}}}{(C(m_k,n)+u_{m_k}-|J_k|^{[n-1..+\infty)}-1)^{\underline{q}}},
\end{aligned}$$

which is what the lemma claims. □

Remark. In case 2, we have $a_{n-1} \leqslant d$ and, therefore, $|J_0|^{\{i-1\}}!/a_{n-1}! = d!/a_{n-1}!$ is an integer. Moreover in (4.2) the summation actually goes over $q = 0, \ldots, a_{t-1} = d - |I_{s+1}|^{(-\infty..t-1]} - |J_s|^{[t-1..+\infty)}$. This shows that in (4.2) the denominator always divides the numerator in each nonzero summand. Hence $P^{(d)}_{i,n,\mathcal{M}}(I_1, \ldots, I_{k+1}, J_1, \ldots, J_k) \in \mathcal{U}^0$ as claimed.

5. Geometry of integer plane

5.1. Decreasing injections

Let X be a finite set with a nonstrict partial order $\preccurlyeq$. We put $\operatorname{cone}(x) := \{y \in X \mid y \preccurlyeq x\}$ for any $x \in X$ and $\operatorname{cone}(S) := \bigcup_{x \in S} \operatorname{cone}(x)$ for any $S \subset X$. A map $\psi : A \to B$, where $A, B \subset X$, is called *weakly decreasing* if $\psi(x) \preccurlyeq x$ for any $x \in A$.

Proposition 5.1. *Let $A, B \subset X$. The following conditions are equivalent*:

(i) *there exists a weakly decreasing injection from A to B*;
(ii) *for any $S \subset X$, there holds $|\operatorname{cone}(S) \cap A| \leqslant |\operatorname{cone}(S) \cap B|$*;
(iii) *for any $S \subset A$, there holds $|\operatorname{cone}(S) \cap A| \leqslant |\operatorname{cone}(S) \cap B|$.*

Proof. (i) $\Rightarrow$ (ii). Let $\psi : A \to B$ be a weakly decreasing injection. Take any $S \subset X$. Let $y \in \operatorname{cone}(S) \cap A$. By definition $\psi(y) \in B$. On the other hand, there exists $x \in S$ such that $y \preccurlyeq x$. However $\psi(y) \preccurlyeq y$. Hence $\psi(y) \preccurlyeq x$ and $\psi(y) \in \operatorname{cone}(S)$. We have proved $\psi(\operatorname{cone}(S) \cap A) \subset \operatorname{cone}(S) \cap B$. Since ψ is an injection, we obtain (ii).

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Let $A = \{a_1, \ldots, a_N\}$, where $a_1, \ldots, a_N$ are mutually distinct. Suppose that for some $k = 1, \ldots, N$, we have constructed a weakly decreasing injection $\psi_{k-1} : \{a_1, \ldots, a_{k-1}\} \to B$. We construct the sequence $S_{-1}, S_0, \ldots, S_m$ of subsets of $\{a_1, \ldots, a_k\}$ as follows. We put $S_{-1} := \emptyset$. Suppose that the subsets $S_{-1}, S_0, \ldots, S_q$ have already been constructed, where $q \geqslant -1$. If $\operatorname{cone}(S_q)$ contains an element of B distinct from $\psi_{k-1}(a_1), \ldots, \psi_{k-1}(a_{k-1})$, then we put $m := q$ and stop. Otherwise, we put

$$S_{q+1} := \{a_k\} \cup \bigl\{a_l \bigm| l = 1, \ldots, k-1 \text{ and } \psi_{k-1}(a_l) \in \operatorname{cone}(S_q)\bigr\}.$$

If $a \in S_q \setminus \{a_k\}$ then $\psi_{k-1}(a) \in \operatorname{cone}(a) \subset \operatorname{cone}(S_q)$, since ψ_{k-1} is weakly decreasing. Hence $a \in S_{q+1}$. Therefore, $S_q \subset S_{q+1}$ ($a_k \in S_{q+1}$ as $q + 1 \geqslant 0$). Moreover, if $q \geqslant 0$ then by hypothesis we have

$$\bigl|S_q \setminus \{a_k\}\bigr| < |S_q| \leqslant \bigl|\operatorname{cone}(S_q) \cap A\bigr| \leqslant \bigl|\operatorname{cone}(S_q) \cap B\bigr|.$$

Thus $\psi_{k-1}(S_q \setminus \{a_k\}) \subsetneq \operatorname{cone}(S_q) \cap B$. Therefore, there exists some $b \in (\operatorname{cone}(S_q) \cap B) \setminus \psi_{k-1}(S_q \setminus \{a_k\})$. Since we consider the case $q < m$, we have $b = \psi_{k-1}(a)$, where $a \in \{a_1, \ldots, a_{k-1}\}$ and $a \notin S_q$. We see that $a \in S_{q+1} \setminus S_q$. Therefore, the process of constructing the sets $S_0, S_1, \ldots$ will eventually terminate. Note that $m \geqslant 0$ and $S_0 = \{a_k\}$.

Let b_m be an element of $\operatorname{cone}(S_m)$ distinct from $\psi_{k-1}(a_1), \ldots, \psi_{k-1}(a_{k-1})$. We are going to construct inductively the elements $b_m, \ldots, b_0$ of B and the elements $a'_m, \ldots, a'_0$ of $\{a_1, \ldots, a_k\}$ so that

(a) $b_l \in \operatorname{cone}(a'_l) \setminus \operatorname{cone}(S_{l-1})$ for any $l = 0, \ldots, m$;
(b) $a'_l \in S_l \setminus S_{l-1}$ for any $l = 0, \ldots, m$;
(c) $\psi_{k-1}(a'_l) = b_{l-1}$ for any $l = 1, \ldots, m$.

If b_m belonged to $\operatorname{cone}(S_{m-1})$, then this element would coincide with one of the elements $\psi_{k-1}(a_1), \ldots, \psi_{k-1}(a_{k-1})$. Therefore $b_m \in \operatorname{cone}(S_m) \setminus \operatorname{cone}(S_{m-1})$. We take for a'_m an arbitrary element of $S_m \setminus S_{m-1}$ such that $b_m \in \operatorname{cone}(a'_m)$.

Now suppose that the elements $b_m, \ldots, b_q$ and $a'_m, \ldots, a'_q$, where $m \geqslant q > 0$, have already been chosen so that conditions (a) and (b) hold for $l = q, \ldots, m$ and condition (c) holds for $l = q+1, \ldots, m$.

To conform with condition (c), we put $b_{q-1} := \psi_{k-1}(a'_q)$. Note that $a'_q \in S_q \setminus S_{q-1} \subset S_q \setminus \{a_k\} \subset \{a_1, \dots, a_{k-1}\}$. We obtain $b_{q-1} \in \operatorname{cone}(S_{q-1}) \setminus \operatorname{cone}(S_{q-2})$. Now we take for a'_{q-1} an arbitrary element of S_{q-1} such that $b_{q-1} \in \operatorname{cone}(a'_{q-1})$. If a'_{q-1} belonged to S_{q-2}, then b_{q-1} would belong to $\operatorname{cone}(S_{q-2})$, which is wrong.

Note that $a'_0 = a_k$. Now a weakly decreasing injection $\psi_k : \{a_1, \dots, a_k\} \to B$ is given by

$$\psi_k(a) := \begin{cases} \psi_{k-1}(a) & \text{if } a \notin \{a'_0, \dots, a'_m\}; \\ b_l & \text{if } a = a'_l \text{ and } l = 0, \dots, m. \end{cases}$$

The injection we need is ψ_N. □

5.2. Decreasing injections for $\mathbf{Z}^2$

Recall that in Section 1, we introduced the strict order $\dot{<}$ and defined strictly decreasing maps. However, a better theory exists for nonstrict versions of these concepts. See, for example, Proposition 5.1.

We introduce the nonstrict partial order $\dot{\leqslant}$ on $\mathbf{Z}^2$ as follows: $(a, b) \dot{\leqslant} (x, y)$ holds if and only if $a \leqslant x$ and $b \leqslant y$. Let $A, B \subset \mathbf{Z}^2$ and $\varphi : A \to B$ be a map. We call φ *weakly decreasing* if $\varphi(\alpha) \dot{\leqslant} \alpha$ for any $\alpha \in A$. For any point $\alpha \in \mathbf{Z}^2$, we put $\operatorname{cone}(\alpha) := \{\beta \in \mathbf{Z}^2 \mid \beta \dot{\leqslant} \alpha\}$. For any subset $\Gamma \subset \mathbf{Z}^2$, we put $\operatorname{cone}(\Gamma) := \bigcup_{\alpha \in \Gamma} \operatorname{cone}(\alpha)$. We also endow $\mathbf{Z}^2$ with the componentwise arithmetic operations. For a point $\alpha \in \mathbf{Z}^2$ and a subset $\Gamma \subset \mathbf{Z}^2$, we put $\Gamma \pm \alpha := \{\gamma \pm \alpha \mid \gamma \in \Gamma\}$.

Proposition 5.2. *Let A and B be finite subsets of $\mathbf{Z}^2$. If there is a strictly decreasing injection from A to B, then $|\operatorname{cone}(\Gamma) \cap A| \leqslant |\operatorname{cone}(\Gamma - (1, 1)) \cap B|$ for any $\Gamma \subset \mathbf{Z}^2$. Conversely, if $|\operatorname{cone}(\Gamma) \cap A| \leqslant |\operatorname{cone}(\Gamma - (1, 1)) \cap B|$ for any $\Gamma \subset A$, then there exists a strictly decreasing injection from A to B.*

Proof. Clearly, there is a strictly decreasing injection from A to B if and only if there is a weakly decreasing injection from A to $B + (1, 1)$. Now the result follows from Proposition 5.1 and the obvious formula

$$\operatorname{cone}(\Gamma) \cap \big(B + (1, 1)\big) = \big(\operatorname{cone}\big(\Gamma - (1, 1)\big) \cap B\big) + (1, 1),$$

where Γ is an arbitrary subset of $\mathbf{Z}^2$. □

5.3. Snakes

Let $A \subset \mathbf{Z}^2$. A point $a \in A$ is called an *interior point* of A if there exists $b \in A$ such that $a \dot{<} b$. Otherwise a is called a *boundary point* of A. The sets of all interior points of A and of all boundary points of A are called the *interior* of A and the *boundary* of A, respectively. For $\Gamma \subset \mathbf{Z}^2$, we define $\operatorname{snake}(\Gamma)$ to be the boundary of $\operatorname{cone}(\Gamma)$. Clearly, $\operatorname{snake}(\Gamma) = \operatorname{cone}(\Gamma) \setminus \operatorname{cone}(\Gamma - (1, 1))$.

Proposition 5.3. *Any two points of a snake are incomparable with respect to $\dot{<}$. Conversely, if Γ is a subset of $\mathbf{Z}^2$ whose points are incomparable with respect to $\dot{<}$, then $\Gamma \subset \operatorname{snake}(\Gamma)$.*

Lemma 5.4. *Let $R = [a..b] \times [c..d]$ be a nonempty rectangle and $X \subset R$. We put $R_0 := [a..b] \times \{c\}$, $R_1 := [a..b] \times (c..d]$ and $Y := X \cap R_1$. The following conditions are equivalent*:

(i) *there exists a strictly decreasing injection $\psi : Y \to X$*;
(ii) *for any subset Δ of Y whose points are incomparable with respect to $\dot{<}$, there is a strictly decreasing injection from Δ to $X \cap R_0$.*

Proof. (i) $\Rightarrow$ (ii). We endow X with the following nonstrict partial order: $x \preccurlyeq y$ if and only if there are elements $z_0, \dots, z_m$, where $m \geqslant 0$, of X such that $y = z_0$, $x = z_m$ and $z_s = \psi(z_{s-1})$ for any $s = 1, \dots, m$. For any $y \in X$, let $\hat{\psi}(y)$ denote the smallest (w.r.t. $\preccurlyeq$) element of $\{x \in X \mid x \preccurlyeq y\}$. Obviously, $\hat{\psi}(y) \in X \cap R_0$ and $\hat{\psi}(y) = \hat{\psi}(y')$ if and only if y and y' comparable with respect to $\preccurlyeq$.

Let Δ be a subset of Y whose points are incomparable with respect to $\dot{<}$. Then distinct points of Δ are incomparable with respect to $\preccurlyeq$, since ψ is strictly decreasing. Hence $\hat{\psi}|_\Delta$ is a required injection.

(ii) $\Rightarrow$ (i). We shall use Proposition 5.2. Let $\Gamma \subset Y$. We must prove the inequality

$$\bigl|\operatorname{cone}(\Gamma) \cap Y\bigr| \leqslant \bigl|\operatorname{cone}\bigl(\Gamma - (1,1)\bigr) \cap X\bigr|. \tag{5.1}$$

We have

$$\operatorname{cone}\bigl(\Gamma - (1,1)\bigr) \cap X = \bigl(\operatorname{cone}\bigl(\Gamma - (1,1)\bigr) \cap Y\bigr) \cup \bigl(\operatorname{cone}\bigl(\Gamma - (1,1)\bigr) \cap (X \cap R_0)\bigr).$$

Since the intersection of $\operatorname{cone}(\Gamma) \cap Y$ and $\operatorname{cone}(\Gamma - (1,1)) \cap X$ is $\operatorname{cone}(\Gamma - (1,1)) \cap Y$, inequality (5.1) is equivalent to

$$\bigl|\bigl(\operatorname{cone}(\Gamma) \setminus \operatorname{cone}\bigl(\Gamma - (1,1)\bigr)\bigr) \cap Y\bigr| \leqslant \bigl|\operatorname{cone}\bigl(\Gamma - (1,1)\bigr) \cap (X \cap R_0)\bigr|. \tag{5.2}$$

Consider the snake $S := \operatorname{snake}(\Gamma)$. We have

$$\operatorname{cone}(\Gamma) \cap (Y \cap S) = \bigl(\operatorname{cone}(\Gamma) \setminus \operatorname{cone}\bigl(\Gamma - (1,1)\bigr)\bigr) \cap Y.$$

The last set is exactly the set of the left-hand side of (5.2). Therefore (5.1) is equivalent to

$$\bigl|\operatorname{cone}(\Gamma) \cap (Y \cap S)\bigr| \leqslant \bigl|\operatorname{cone}\bigl(\Gamma - (1,1)\bigr) \cap (X \cap R_0)\bigr|,$$

which holds in view of Proposition 5.2 and the condition of part (ii). Here we should take $\Delta := Y \cap S$ and apply Proposition 5.3. $\square$

Also, we call a map $\varphi : A \to B$, where $A, B \subset \mathbf{Z}^2$, *strictly increasing* if $\varphi(\alpha) \dot{>} \alpha$ for any $\alpha \in A$.

5.4. Set S

Consider a nonempty subset S of $\mathbf{Z}^2$ having the form

$$S = \bigl(\{a\} \times [f_a..l_a]\bigr) \cup \bigl(\{a+1\} \times [f_{a+1}..l_{a+1}]\bigr) \cup \dots \cup \bigl(\{b\} \times [f_b..l_b]\bigr), \tag{5.3}$$

where $a \leqslant b$, $f_{s+1} \leqslant f_s \leqslant l_{s+1} \leqslant l_s$ for $s = a, \dots, b-1$ and $f_s \geqslant l_{s+2}$ for any $s = a, \dots, b-2$. We say that a point $x = (x_1, x_2)$ of the strip $[a..b] \times \mathbf{Z}$ lies below S or above S if $x_2 < f_{x_1}$ or $x_2 > l_{x_1}$, respectively.

Lemma 5.5. *Let $M \subset X \subset [a..b] \times \mathbf{Z}$ and $\varphi : M \to X$ be a strictly increasing injection. Suppose that $X \cap S$ does not contain points comparable with respect to $\dot{<}$. Then there is an injection $\varphi_S : M \to X \setminus S$ such that for any $x \in M$ we have either $x \dot{<} \varphi_S(x)$ or $\varphi_S(x) = x$ and x lies below S.*

Proof. We denote by B and T the set of all points of $[a..b] \times \mathbf{Z}$ lying below S and above S, respectively. For $x \in M$, we put $\varphi_S(x) := x$ if $x \in B \cap M$ and $\varphi_S(x) := \varphi(x)$ otherwise. It remains to show that φ_S is an injection from M to $X \setminus S$.

Take a point $x = (x_1, x_2) \in M$ and denote $(y_1, y_2) := \varphi(x)$. We have $y_1 > x_1$ and $y_2 > x_2$. If $x \in B$ then $\varphi_S(x) = x \in B \cap M \subset B \cap X \subset X \setminus S$.

Now suppose $x \in S$. If we had $\varphi(x) \in S$, then x and $\varphi(x)$ would be points of $X \cap S$ comparable with respect to $\dot{<}$. Hence $\varphi_S(x) = \varphi(x) \notin S$. Suppose that $\varphi(x) \in B$. Then $y_2 < f_{y_1} \leqslant f_{x_1} \leqslant x_2$, which is a contradiction. Therefore $\varphi_S(x) = \varphi(x) \in T \cap X \subset X \setminus S$.

Finally, let $x \in T$. We have $y_2 > x_2 > l_{x_1} \geqslant l_{y_1}$. Hence $\varphi_S(x) = \varphi(x) \in T \cap X \subset X \setminus S$. $\square$

Now let us consider the case where S contains points comparable with respect to $\dot{<}$. We fix some $\lambda \in \mathbf{Z}^n$ and suppose that $1 \leqslant a \leqslant b \leqslant n$.

Lemma 5.6. *Let $x, y \in S$ and $x \dot{<} y$. Then x and y belong to adjacent columns and*

$$(\lambda_t - \lambda_{t+1})^{\underline{l_{t+1} - f_t}} \equiv 0 \pmod{\operatorname{dist}_\lambda(x, y)},$$

where x belongs to column t.

Proof. We denote $x = (t, u)$ and $y = (s, v)$. Since $x \dot{<} y$, we have $t < s$ and $u < v$. If x and y did not belong to adjacent columns, we would have $t + 2 \leqslant s$. By the conditions imposed on S, we get $v > u \geqslant f_t \geqslant l_{t+2} \geqslant l_s$ and therefore $y \notin S$, which is a contradiction.

Thus $x = (t, u)$, $y = (t + 1, v)$, $f_t \leqslant u \leqslant l_t$ and $f_{t+1} \leqslant v \leqslant l_{t+1}$. Hence we get

$$0 < v - u \leqslant l_{t+1} - f_t. \tag{5.4}$$

By definition, we have $\lambda_t - \lambda_{t+1} \equiv v - u - 1 \pmod{\operatorname{dist}_\lambda(x, y)}$. This equivalence together with (5.4), yield

$$(\lambda_t - \lambda_{t+1})^{\underline{l_{t+1} - f_t}} \equiv (v - u - 1)^{\underline{l_{t+1} - f_t}} = 0 \pmod{\operatorname{dist}_\lambda(x, y)}. \qquad \square$$

5.5. Diagrams

Let $k < j$ and I be a multiset with integer entries of length no greater than d. We put

$$\Sigma^{(d)}_{k,j}(I) := \{(t, h) \in \mathbf{Z}^2 \mid k < t \leqslant j \ \& \ 0 \leqslant h \leqslant d - |I|^{(-\infty..t-1]}\},$$
$$\Omega^{(d)}_{k,j}(I) := \{(t, h) \in \mathbf{Z}^2 \mid k < t < j \ \& \ 0 \leqslant h < d - |I|^{(-\infty..t]}\}.$$

One can easily see that $\Omega^{(d)}_{k,j}(I)$ is the interior of $\Sigma^{(d)}_{k,j}(I)$.

Lemma 5.7. *Let Γ be a subset of $(i..n] \times [0..d]$ whose points are incomparable with respect to $\dot{<}$. Then there exists a multiset I of length no greater than d with entries in the interval $[i..n)$ such that Γ is contained in the boundary of $\Sigma^{(d)}_{i,n}(I)$.*

Proof. We denote by Γ' the set of all points of Γ not belonging to column n. Let $t_1 < t_2 < \cdots < t_k$ be the numbers of columns that contain at least one point of Γ'. We denote $h_j := \max\{h \mid (t_j, h) \in \Gamma'\}$ for $j = 1, \ldots, k$. Let $h_{k+1} := \max\{h \mid (n, h) \in \Gamma\} \cup \{0\}$. Since the points of Γ are incomparable with respect to $\dot{<}$, we have $h_1 \geqslant h_2 \geqslant \cdots \geqslant h_k \geqslant h_{k+1}$. Now we put

$$I := \langle i^{d-h_1}, t_1^{h_1-h_2}, t_2^{h_2-h_3}, \ldots, t_{k-1}^{h_{k-1}-h_k}, t_k^{h_k-h_{k+1}} \rangle.$$

This multiset has length $d - h_{k+1} \leqslant d$. We put $t_{k+1} := n$.

Take a point $\gamma=(t_j,h)\in\Gamma$. By definition, we have $h\leqslant h_j$. Since $|I|^{(-\infty..t_j-1]}=d-h_j$, we have $\gamma\in\Sigma^{(d)}_{i,n}(I)$.

It remains to show that $\gamma\notin\Omega^{(d)}_{i,n}(I)$. This is true if γ belongs to column n. Therefore assume that $\gamma\in\Gamma'$. If $h<h_{j+1}$, then $\gamma\mathrel{\dot{<}}(t_{j+1},h_{j+1})\in\Gamma$ and we get a contradiction with incomparability of points of Γ. Hence $h_{j+1}\leqslant h$. Since $|I|^{(-\infty..t_j]}=d-h_{j+1}$, we get $\gamma\notin\Omega^{(d)}_{i,n}(I)$. □

6. Formal factorization of elementary expressions

6.1. Cutting operators

Let $i\leqslant l<m<n$. For $t=1,\dots,n$, we put

$$\sigma_{l,m}(H_t)=\begin{cases}H_t & \text{if } t<m;\\ H_t+C(l,m)-u_m+u_l & \text{if } t\geqslant m.\end{cases}$$

We extend $\sigma_{l,m}$ to a ring endomorphism of $\mathcal{U}^0$, assuming that $\sigma_{l,m}$ acts identically on $u_{i+1},\dots,u_{n-1}$. One can check directly that $\sigma_{l,m}$ is an idempotent operator on $\mathcal{U}^0$ and that for $1\leqslant q\leqslant t\leqslant n$, there holds

$$\sigma_{l,m}\big(C(q,t)+u_l\big)=\begin{cases}C(q,t)+u_l & \text{if } q\leqslant t<m \text{ or } m\leqslant q\leqslant t;\\ C(m,t)+C(q,l)+u_m & \text{if } q<m\leqslant t.\end{cases}\tag{6.1}$$

We extend $\sigma_{l,m}$ to a $\mathbf{Z}'$-module endomorphism of $\mathcal{U}^{-,0}$ by the rule $\sigma_{l,m}(FH)=F\sigma_{l,m}(H)$, where $F\in U^-$ and $H\in\mathcal{U}^0$.

Lemma 6.1. *An element $x\in\mathcal{U}^{-,0}$ is representable in the form $x=y(C(l,m)-u_m+u_l)$ for some $y\in\mathcal{U}^{-,0}$ if and only if $\sigma_{l,m}(x)=0$.*

Proof. We put for brevity $\sigma:=\sigma_{l,m}$. Clearly, if $x=y(C(l,m)-u_m+u_l)$ for some $y\in\mathcal{U}^{-,0}$ then $\sigma(x)=0$, since $\sigma(C(l,m)-u_m+u_l)=0$ by (6.1).

Conversely, suppose that $\sigma(x)=0$. Take any matrix $M\in UT^{\geqslant 0}(n)$ and let $\mathcal{H}_M$ be the $F^{(M)}$-coefficient of x. By our assumption and Lemma 2.2, $\sigma(\mathcal{H}_M)=0$. Let R be a subring of $\mathcal{U}^0$ generated by $H_1,\dots,H_{m-1},H_{m+1},\dots,H_n$ and $u_{i+1},\dots,u_{n-1}$. We have the representation

$$\mathcal{H}_M=h_kH_m^k+h_{k-1}H_m^{k-1}+\dots+h_1H_m+h_0,$$

where $h_0,\dots,h_k\in R$. Applying σ to this representation, we obtain

$$\sigma(h_k)\sigma(H_m)^k+\sigma(h_{k-1})\sigma(H_m)^{k-1}+\dots+\sigma(h_1)\sigma(H_m)+\sigma(h_0)=0.\tag{6.2}$$

Consider the ring endomorphism τ of $\mathcal{U}^0$ such that $\tau(H_m)=m-l+H_l-u_m+u_l$ and τ acts identically on the generators $H_1,\dots,H_{m-1},H_{m+1},\dots,H_n$, $u_{i+1},\dots,u_{n-1}$. We claim that $(\tau\circ\sigma)(H_t)=H_t$ for any $t\in\{1,\dots,n\}\setminus\{m\}$. If $t<m$ then $(\tau\circ\sigma)(H_t)=\tau(H_t)=H_t$. Now let $t>m$. We have

$$\begin{aligned}(\tau\circ\sigma)(H_t)&=\tau(H_t+m-l+H_l-H_m-u_m+u_l)\\ &=H_t+m-l+H_l-\tau(H_m)-u_m+u_l=H_t.\end{aligned}$$

Thus, we have $(\tau\circ\sigma)(h_0)=h_0,\dots,(\tau\circ\sigma)(h_k)=h_k$. Moreover, $\sigma(H_m)=m-l+H_l-u_m+u_l$ and it does not depend on H_m. Therefore $(\tau\circ\sigma)(H_m)=m-l+H_l-u_m+u_l$. Thus applying τ to (6.2), we obtain

$$h_k(m-l+H_l-u_m+u_l)^k+h_{k-1}(m-l+H_l-u_m+u_l)^{k-1}+\cdots+h_1(m-l+H_l-u_m+u_l)+h_0=0.$$

We have proved that $m-l+H_l-u_m+u_l$ is a root of the polynomial $h_kX^k+h_{k-1}X^{k-1}+\cdots+h_1X+h_0\in R[X]$ belonging to the ground ring R. By the Bezout theorem, this polynomial is divisible by $X-m+l-H_l+u_m-u_l$, that is

$$h_kX^k+h_{k-1}X^{k-1}+\cdots+h_1X+h_0=P(X)\cdot(X-m+l-H_l+u_m-u_l)$$

for some $P\in R[X]$. Substituting H_m for X, we obtain

$$\mathcal{H}_M=P(H_m)\cdot(H_m-m+l-H_l+u_m-u_l)=-P(H_m)\cdot\big(C(l,m)-u_m+u_l\big).$$

It remains to notice that $P(H_m)\in\mathcal{U}^0$. □

Remark. We shall often use the following simple fact immediately following from (6.1): for $l<m$ and $q<t$, the equalities $\sigma_{l,m}(C(q,t)-u_t+u_q)=0$ and $(l,m)=(q,t)$ are equivalent.

Lemma 6.2. *Let $\mathcal{M}=\{m_1<\cdots<m_k\}$ be a subset of $(i..n)$, $m_0:=i$, $m_{k+1}:=n$ and $I_1,\dots,I_{k+1},J_1,\dots,J_k$ be a sequence of multisets that with $\mathcal{M}$ satisfies conditions* (M1)–(M3)*. Suppose that $0\leqslant r\leqslant k$, $m_r<m\leqslant m_{r+1}$ and $m<n$. Then we have*

$$\begin{aligned}&\sigma_{m_r,m}\big(S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\big)\\&\quad=S^{(d)}_{i,n,\mathcal{M}\cup\{m\}}\big(I_1,\dots,I_r,\mathcal{L}^m(I_{r+1}),\mathcal{R}^m(I_{r+1}),I_{r+2},\dots,I_{k+1},\\&\qquad J_1,\dots,J_{r-1},\mathcal{L}_m(J_r),\mathcal{R}_m(J_r),J_{r+1},\dots,J_k\big)\end{aligned}$$

if $m<m_{r+1}$ and

$$\sigma_{m_r,m}\big(S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)\big)=S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)$$

if $m=m_{r+1}$.

Proof. First consider the case $m_r<m<m_{r+1}$. Take an arbitrary matrix $M\in UT^{\geqslant 0}(n)$ such that $F^{(M)}$ has weight $\lambda^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)$ and denote by $\mathcal{H}_M$ the $F^{(M)}$-coefficient of $S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)$. By (3.5) and (6.1), we have

$$\begin{aligned}\sigma_{m_r,m}(\mathcal{H}_M)&=\left(\prod_{s\in[0..k]\setminus\{r\}}\prod_{t=m_s}^{m_{s+1}-1}\big(M_t+|J_s|^{\{t-1\}}\big)!\big(C(m_s,t)+u_{m_s}\big)^{\underline{d-(M_t+|J_s|^{\{t-1\}}+|I_{s+1}|^{(-\infty..t)})}}\right)\\&\quad\times\sigma_{m_r,m}\left(\prod_{t=m_r}^{m_{r+1}-1}\big(M_t+|J_r|^{\{t-1\}}\big)!\big(C(m_r,t)+u_{m_r}\big)^{\underline{d-(M_t+|J_r|^{\{t-1\}}+|I_{r+1}|^{(-\infty..t)})}}\right)\\&=\left(\prod_{s\in[0..k]\setminus\{r\}}\prod_{t=m_s}^{m_{s+1}-1}\big(M_t+|J_s|^{\{t-1\}}\big)!\big(C(m_s,t)+u_{m_s}\big)^{\underline{d-(M_t+|J_s|^{\{t-1\}}+|I_{s+1}|^{(-\infty..t)})}}\right)\end{aligned}$$

$$\times \prod_{t=m_r}^{m-1} \left(M_t + |J_r|^{\{t-1\}}\right)! \left(C(m_r,t) + u_{m_r}\right)^{\underline{d-(M_t+|J_r|^{\{t-1\}}+|I_{r+1}|^{(-\infty..t)})}}$$
$$\times \prod_{t=m}^{m_{r+1}-1} \left(M_t + |J_r|^{\{t-1\}}\right)! \left(C(m,t) + u_m\right)^{\underline{d-(M_t+|J_r|^{\{t-1\}}+|I_{r+1}|^{(-\infty..t)})}}.$$

To complete the proof in this case, it remains to notice that

$$|J_r|^{\{t-1\}} = \left|\mathcal{L}_m(J_r)\right|^{\{t-1\}} \quad \text{and} \quad |I_{r+1}|^{(-\infty..t)} = \left|\mathcal{L}^m(I_{r+1})\right|^{(-\infty..t)} \quad \text{for } t < m;$$
$$|J_r|^{\{t-1\}} = \left|\mathcal{R}_m(J_r)\right|^{\{t-1\}} \quad \text{and} \quad |I_{r+1}|^{(-\infty..t)} = \left|\mathcal{R}^m(I_{r+1})\right|^{(-\infty..t)} \quad \text{for } t \geqslant m,$$

and apply case (iv) of Lemma 3.2 (the sequence of this case is always well defined).

In the case $m = m_{r+1}$, it follows from (3.5) and (6.1) that $\sigma_{m_r,m}$ acts identically on any $F^{(M)}$-coefficient of $S^{(d)}_{i,n,\mathcal{M}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)$. □

6.2. Factorization

We consider the formal commutative variables $\mathcal{S}^{(d)}_{m,m'}(I,J)$, where $i \leqslant m < m' \leqslant n$ and

(F1) I is a multiset with entries in $[m-1..m')$ if $m > i$ and with entries in $[i..m')$ if $m = i$;
(F2) J is a multiset with entries in $[m-1..m')$ if $m > i$ and $J = \langle (i-1)^d \rangle$ if $m = i$.

These variables are called *formal operators*. Let $F^{(d)}_{i,n}$ denote the free commutative algebra (polynomial algebra) over the field $\bar{\mathcal{U}}^0$ generated by all formal operators. This algebra is called the *algebra of formal operators*.

A monomial $Q \cdot f/g$, where Q is a product of formal operators, $f, g \in \mathcal{U}^0$ and $g \neq 0$, is called *regular* if the following conditions hold for Q and g.

(F3) If $\mathcal{S}^{(d)}_{\hat{m},m}(\bar{I},\bar{J})$ and $\mathcal{S}^{(d)}_{m,m'}(I,J)$ occur in Q, then $|I|^{\{m-1\}} + |J| = |\bar{I}| + |\bar{J}|^{\{m-1\}}$.
(F4) For any $t \in \mathbf{Z}$, there is at most one factor of Q having the form $\mathcal{S}^{(d)}_{m,m'}(I,J)$, where $t \in [m..m')$.
(F5) g is a product of polynomials $C(m,m') - u_{m'} + u_m$, where $i \leqslant m < m' < n$, each in degree at most 1. Moreover, if this polynomial occurs in g, then some formal operator $\mathcal{S}^{(d)}_{m,m''}(I,J)$, where $m' \leqslant m''$, occurs in Q.

A polynomial of $F^{(d)}_{i,n}$ is called *regular* if it is a sum of regular monomials.

We are going to introduce the cutting operators for $F^{(d)}_{i,n}$ similar to the cutting operators of Section 6.1, for which we shall use the same notation $\sigma_{m,m'}$, where $i \leqslant m < m' < n$.

At first, we define the action of $\sigma_{m,m'}$ on the formal operator $\mathcal{S}^{(d)}_{\hat{m},m''}(I,J)$. If $\hat{m} = m$ and $m' < m''$, then we put

$$\sigma_{m,m'}\left(\mathcal{S}^{(d)}_{m,m''}(I,J)\right) := \mathcal{S}^{(d)}_{m,m'}\left(\mathcal{L}^{m'}(I), \mathcal{L}_{m'}(J)\right)\mathcal{S}^{(d)}_{m',m''}\left(\mathcal{R}^{m'}(I), \mathcal{R}_{m'}(J)\right).$$

If the intervals $[m..m')$ and $[\hat{m}..m'')$ either do not intersect or coincide, then we assume that $\sigma_{m,m'}$ acts on $\mathcal{S}^{(d)}_{\hat{m},m''}(I,J)$ identically. In all other cases, we define the result of this action to be 0.

Further, we assume that $\sigma_{m,m'}$ acts on $\mathcal{U}^0$ as defined in Section 6.1. We define the action of $\sigma_{m,m'}$ on an element $P \in F^{(d)}_{i,n}$ as follows. Suppose that P is representable as

$$P=\sum_{j=1}^{l} S_1^{(j)}\cdots S_{k_j}^{(j)}\cdot f_j/g_j,$$

where $S_q^{(j)}$ are formal operators, $f_j, g_j \in \mathcal{U}^0$ and $\sigma_{m,m'}(g_j)\neq 0$ for any $j=1,\dots,l$. Then we define

$$\sigma_{m,m'}(P)=\sum_{j=1}^{l}\sigma_{m,m'}\bigl(S_1^{(j)}\bigr)\cdots\sigma_{m,m'}\bigl(S_{k_j}^{(j)}\bigr)\cdot\sigma_{m,m'}(f_j)/\sigma_{m,m'}(g_j).$$

It is easy to understand that $\sigma_{m,m'}(P)$ thus defined does not depend on the choice of the representation of P.

Remark. The operator $\sigma_{m,m'}$ is not applicable to elements of $F_{i,n}^{(d)}$ not representable in the above form (see the example below). However if $\sigma_{m,m'}$ is applicable to P and P' of $F_{i,n}^{(d)}$, then $\sigma_{m,m'}(P\pm P')=\sigma_{m,m'}(P)\pm\sigma_{m,m'}(P')$ and $\sigma_{m,m'}(PP')=\sigma_{m,m'}(P)\sigma_{m,m'}(P')$.

A polynomial $P\in F_{i,n}^{(d)}$ is called *integral* if it is regular and for any $i\leqslant m<m'<n$, there holds

$$\sigma_{m,m'}\bigl(P\cdot\bigl(C(m,m')-u_{m'}+u_m\bigr)\bigr)=0.$$

In this formula, $\sigma_{m,m'}$ is applicable to $P\cdot(C(m,m')-u_{m'}+u_m)$, since the denominator of the last polynomial can contain only factors of the form $C(m_0,m_0')-u_{m_0'}+u_{m_0}$, where $(m_0,m_0')\neq(m,m')$. By (6.1), the operator $\sigma_{m,m'}$ does not take such polynomials to zero.

Example. Let $i<m<n$, I be a multiset with entries in $[i..n)$, $J_0=\langle(i-1)^d\rangle$ and

$$P:=\frac{\mathcal{S}_{i,n}^{(d)}(I,J_0)-\mathcal{S}_{i,m}^{(d)}(\mathcal{L}^m(I),J_0)\mathcal{S}_{m,n}^{(d)}(\mathcal{R}^m(I),\emptyset)}{C(i,m)-u_m}.$$

This polynomial is integral. Note that $\sigma_{i,m}$ is not applicable to P. Indeed, suppose that there exists a representation $P=\sum_{j=1}^{l}Q_j\cdot f_j/g_j$, where Q_j are products of formal operators, $f_j,g_j\in\mathcal{U}^0$ and $\sigma_{i,m}(g_j)\neq 0$. Then $g_1,\dots,g_l$ are not divisible by $C(i,m)-u_m$ in $\mathcal{U}^0$. We have

$$\begin{aligned}&g_1\cdots g_l\bigl(\mathcal{S}_{i,n}^{(d)}(I,J_0)-\mathcal{S}_{i,m}^{(d)}\bigl(\mathcal{L}^m(I),J_0\bigr)\mathcal{S}_{m,n}^{(d)}\bigl(\mathcal{R}^m(I),\emptyset\bigr)\bigr)\\&\quad=\bigl(C(i,m)-u_m\bigr)\sum_{j=1}^{l}Q_j\cdot f_jg_1\cdots g_{j-1}g_{j+1}\cdots g_l.\end{aligned}$$

The coefficient of $\mathcal{S}_{i,n}^{(d)}(I,J_0)$ in the left-hand side equals $g_1\cdots g_l$ and is not divisible by $C(i,m)-u_m$ unlike the entire right-hand side. This is a contradiction.

Finally, we introduce the notion of weight for elements of $F_{i,n}^{(d)}$. We say that

- $\mathcal{S}_{m,m'}^{(d)}(I,J)$ has weight $\sum_{t=m}^{m'-1}(-d+|I|^{(-\infty..t]}+|J|^{[t..+\infty)})\alpha_t$;
- elements of $\bar{\mathcal{U}}^0$ have weight 0;
- if P has weight λ and P' has weight μ, then PP' has weight $\lambda+\mu$;
- a sum of elements having fixed weight λ has weight λ.
- a sum of elements having weight $\leqslant 0$ is called an *element of weight* $\leqslant 0$.

6.3. Raising operators

Following the multiplication rules of Section 3, we introduce the operators $\rho_l^{(1)}$, $\rho_l^{(2,L)}$, $\rho_l^{(2,R)}$, $\rho_l^{(3)}$, where $i \leqslant l < n$, on the algebra of formal operators $F_{i,n}^{(d)}$.

We assume that all these operators act identically on $\bar{\mathcal{U}}^0$. Take a formal operator $\mathcal{S}_{m,m'}^{(d)}(I, J)$ and an integer l such that $i \leqslant l < n$. We put

$$\rho_l^{(1)}\big(\mathcal{S}_{m,m'}^{(d)}(I, J)\big) := \begin{cases} -\delta_{l>i}|J|^{\{l-1\}}\mathcal{S}_{m,m'}^{(d)}(I, J_{l-1\mapsto l}) & \text{if } m \leqslant l < m'; \\ \mathcal{S}_{m,m'}^{(d)}(I, J \cup \langle m-1\rangle) & \text{if } l = m-1; \\ \mathcal{S}_{m,m'}^{(d)}(I, J) & \text{otherwise,} \end{cases}$$

$$\rho_l^{(2,L)}\big(\mathcal{S}_{m,m'}^{(d)}(I, J)\big) := \begin{cases} 0 & \text{if } m \leqslant l < m'-1; \\ \mathcal{S}_{m,m'}^{(d)}(I \cup \langle m'-1\rangle, J)(C(m,m') + u_m - d + |I|) & \text{if } l = m'-1; \\ \mathcal{S}_{m,m'}^{(d)}(I, J \cup \langle m-1\rangle) & \text{if } l = m-1; \\ \mathcal{S}_{m,m'}^{(d)}(I, J) & \text{otherwise,} \end{cases}$$

$$\rho_l^{(2,R)}\big(\mathcal{S}_{m,m'}^{(d)}(I, J)\big) := \begin{cases} 0 & \text{if } m \leqslant l < m'-1; \\ \mathcal{S}_{m,m'}^{(d)}(I \cup \langle m'-1\rangle, J) & \text{if } l = m'-1; \\ \mathcal{S}_{m,m'}^{(d)}(I, J \cup \langle m-1\rangle)(u_m - d + |I|^{\{m-1\}}) & \text{if } l = m-1; \\ \mathcal{S}_{m,m'}^{(d)}(I, J) & \text{otherwise,} \end{cases}$$

$$\rho_l^{(3)}\big(\mathcal{S}_{m,m'}^{(d)}(I, J)\big)$$
$$:= \begin{cases} |I|^{\{l+1\}}\mathcal{S}_{m,m'}^{(d)}(I_{l+1\mapsto l}, J)(C(m,l+1) + u_m - d + |I|^{(-\infty..l]}) & \text{if } m-1 \leqslant l < m'-1; \\ \mathcal{S}_{m,m'}^{(d)}(I \cup \langle m'-1\rangle, J) & \text{if } l = m'-1; \\ \mathcal{S}_{m,m'}^{(d)}(I, J) & \text{otherwise.} \end{cases}$$

We extend the operators $\rho_l^{(1)}$, $\rho_l^{(2,L)}$, $\rho_l^{(2,R)}$, $\rho_l^{(3)}$ to ring endomorphisms of $F_{i,n}^{(d)}$. Finally we put $\rho_l^{(2)} := \rho_l^{(2,L)} - \rho_l^{(2,R)}$.

Remark. Clearly, each $\rho_l^{(1)}$, $\rho_l^{(3)}$, $\rho_l^{(2,L)}$, $\rho_l^{(2,R)}$ acts on a formal operator $\mathcal{S}_{m,m'}^{(d)}(I, J)$ identically unless $m-1 \leqslant l \leqslant m'-1$. The reader should be careful with $\rho_l^{(2)}$, keeping in mind that it is not a ring endomorphism. However, if both $\rho_l^{(2,L)}$ and $\rho_l^{(2,R)}$ act identically on $x \in F_{i,n}^{(d)}$, then $\rho_l^{(2)}(xy) = x\rho_l^{(2)}(y)$ for any $y \in F_{i,n}^{(d)}$.

6.4. Operators $\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I, J)$

We are going to define the principal object of our study, the elements $\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I, J)$ of $F_{i,n}^{(d)}$, where $i \leqslant k < j \leqslant n$; $\mathcal{M} \subset (k..j)$; I and J are multisets with entries in $[k-1..j)$ if $k > i$; I is a multiset with entries in $[i..j)$ and $J = \langle (i-1)^d\rangle$ if $k = i$.

Definition 6.3. We put $\mathcal{T}_{k,j,\emptyset}^{(d)}(I, J) := \mathcal{S}_{k,j}^{(d)}(I, J)$. For $\mathcal{M} \neq \emptyset$, we put

$$\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I, J) = \big(\mathcal{T}_{k,j,\mathcal{M}\setminus\{m\}}^{(d)}(I, J) - \mathcal{S}_{k,m}^{(d)}\big(\mathcal{L}^m(I), \mathcal{L}_m(J)\big)$$

$$\times \mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I),\mathcal{R}_m(J)\big)\big)/\big(C(k,m)-u_m+u_k\big),$$

where $m=\min\mathcal{M}$.

Lemma 6.4.

(i) $\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)$ *is regular.*

(ii) $\sigma_{k,m_0}(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J))=\mathcal{S}^{(d)}_{k,m_0}(\mathcal{L}^{m_0}(I),\mathcal{L}_{m_0}(J))\mathcal{T}^{(d)}_{m_0,j,\mathcal{M}}(\mathcal{R}^{m_0}(I),\mathcal{R}_{m_0}(J))$, *where* $k<m_0<\min\mathcal{M}\cup\{j\}$.

(iii) $\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)$ *is integral.*

Proof. (i) The result follows from the following observation, which can easily be proved by induction on $|\mathcal{M}|$: $\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)$ is an integral linear combination of monomials Q/g such that

- $Q=\prod_{s=0}^{c}\mathcal{S}^{(d)}_{m_s,m_{s+1}}(\mathcal{L}^{m_{s+1}}(\mathcal{R}^{m_s}(I)),\mathcal{L}_{m_{s+1}}(\mathcal{R}_{m_s}(J)))$;
- g is a product of polynomials of the form $C(m_s,m')-u_{m'}+u_{m_s}$, where $s=0,\dots,c$, $m_s<m'\leqslant m_{s+1}$ and $m'\in\mathcal{M}$, each occurring in g in degree at most one,

where $c\geqslant 0$, $k=m_0<m_1<\cdots<m_c<m_{c+1}=j$ and $m_1,\dots,m_c\in\mathcal{M}$. To see this, it suffices to apply Definition 6.3 and (2.3).

(ii) Induction on $|\mathcal{M}|$. The case $\mathcal{M}=\emptyset$ is obvious.

Now let $\mathcal{M}\neq\emptyset$ and suppose that the result is true for sets of smaller cardinality. We put $m:=\min\mathcal{M}$. Applying Definition 6.3, (6.1) and the inductive hypothesis, we obtain

$$\begin{aligned}
\sigma_{k,m_0}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big) &= \big[\sigma_{k,m_0}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I,J)\big)-\sigma_{k,m_0}\big(\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)\big)\big)\\
&\quad\times \mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I),\mathcal{R}_m(J)\big)\big]/\big(C(m_0,m)-u_m+u_{m_0}\big)\\
&= \big[\mathcal{S}^{(d)}_{k,m_0}\big(\mathcal{L}^{m_0}(I),\mathcal{L}_{m_0}(J)\big)\mathcal{T}^{(d)}_{m_0,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^{m_0}(I),\mathcal{R}_{m_0}(J)\big)\\
&\quad-\mathcal{S}^{(d)}_{k,m_0}\big(\mathcal{L}^{m_0}\big(\mathcal{L}^m(I)\big),\mathcal{L}_{m_0}\big(\mathcal{L}_m(J)\big)\big)\mathcal{S}^{(d)}_{m_0,m}\big(\mathcal{R}^{m_0}\big(\mathcal{L}^m(I)\big),\mathcal{R}_{m_0}\big(\mathcal{L}_m(J)\big)\big)\\
&\quad\times \mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I),\mathcal{R}_m(J)\big)\big]/\big(C(m_0,m)-u_m+u_{m_0}\big)\\
&= \mathcal{S}^{(d)}_{k,m_0}\big(\mathcal{L}^{m_0}(I),\mathcal{L}_{m_0}(J)\big)\big[\mathcal{T}^{(d)}_{m_0,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^{m_0}(I),\mathcal{R}_{m_0}(J)\big)\\
&\quad-\mathcal{S}^{(d)}_{m_0,m}\big(\mathcal{R}^{m_0}\big(\mathcal{L}^m(I)\big),\mathcal{R}_{m_0}\big(\mathcal{L}_m(J)\big)\big)\\
&\quad\times \mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I),\mathcal{R}_m(J)\big)\big]/\big(C(m_0,m)-u_m+u_{m_0}\big)\\
&= \mathcal{S}^{(d)}_{k,m_0}\big(\mathcal{L}^{m_0}(I),\mathcal{L}_{m_0}(J)\big)\big[\mathcal{T}^{(d)}_{m_0,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^{m_0}(I),\mathcal{R}_{m_0}(J)\big)\\
&\quad-\mathcal{S}^{(d)}_{m_0,m}\big(\mathcal{L}^{m}\big(\mathcal{R}^{m_0}(I)\big),\mathcal{L}_{m}\big(\mathcal{R}_{m_0}(J)\big)\big)\\
&\quad\times \mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m\big(\mathcal{R}^{m_0}(I)\big),\mathcal{R}_m\big(\mathcal{R}_{m_0}(J)\big)\big)\big]/\big(C(m_0,m)-u_m+u_{m_0}\big)\\
&= \mathcal{S}^{(d)}_{k,m_0}\big(\mathcal{L}^{m_0}(I),\mathcal{L}_{m_0}(J)\big)\mathcal{T}^{(d)}_{m_0,j,\mathcal{M}}\big(\mathcal{R}^{m_0}(I),\mathcal{R}_{m_0}(J)\big).
\end{aligned}$$

(iii) Induction on $|\mathcal{M}|$. The case $\mathcal{M}=\emptyset$ is obvious. Now let $\mathcal{M}\neq\emptyset$ and suppose that the result is true for sets of smaller cardinality. We put $m:=\min\mathcal{M}$. Applying the inductive hypothesis and the remark of Section 6.2, we obtain

$$\begin{aligned}
&\sigma_{m_0,m_0'}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\cdot\big(C(m_0,m_0')-u_{m_0'}+u_{m_0}\big)\big)\\
&\quad=\big[\sigma_{m_0,m_0'}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I,J)\cdot\big(C(m_0,m_0')-u_{m_0'}+u_{m_0}\big)\big)\\
&\qquad-\sigma_{m_0,m_0'}\big(\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)\big)\big)\\
&\qquad\times\sigma_{m_0,m_0'}\big(\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I),\mathcal{R}_m(J)\big)\cdot\big(C\big(m_0,m_0'\big)-u_{m_0'}+u_{m_0}\big)\big)\big]/\sigma_{m_0,m_0'}\big(C(k,m)-u_m+u_k\big)\\
&\quad=0
\end{aligned}$$

for any m_0, m_0' such that $i\leqslant m_0<m_0'<n$ and $(m_0,m_0')\neq(k,m)$. The only remaining equality

$$\sigma_{k,m}\big(\mathcal{T}^{],(d)}_{k,j,\mathcal{M}}(I,J)\cdot\big(C(k,m)-u_m+u_k\big)\big)=0$$

follows immediately from part (ii) of the current lemma. □

Lemma 6.5. *Let $i\leqslant k<j\leqslant n$ and $k\leqslant l<j$. Then we have*

$$\rho^{(1)}_l\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big)=-\delta_{l>i}|J|^{\{l-1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J_{l-1\mapsto l}).$$

If $i<k$ then $\rho^{(1)}_{k-1}(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J))=\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J\cup\langle k-1\rangle)$.

Proof. We apply induction on $|\mathcal{M}|$. In the case $\mathcal{M}=\emptyset$, the required formulas immediately follow from the definition. Therefore we assume that $\mathcal{M}\neq\emptyset$ and put $m:=\min\mathcal{M}$.

Case $k\leqslant l<m-1$. Applying the inductive hypothesis, we obtain

$$\begin{aligned}
\rho^{(1)}_l\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big)&=\big(-\delta_{l>i}|J|^{\{l-1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I,J_{l-1\mapsto l})\\
&\quad+\delta_{l>i}\big|\mathcal{L}_m(J)\big|^{\{l-1\}}\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)_{l-1\mapsto l}\big)\\
&\quad\times\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I),\mathcal{R}_m(J)\big)\big)/\big(C(k,m)-u_m+u_k\big)\\
&=-\delta_{l>i}|J|^{\{l-1\}}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I,J_{l-1\mapsto l})-\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J_{l-1\mapsto l})\big)\\
&\quad\times\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I),\mathcal{R}_m(J_{l-1\mapsto l})\big)\big)/\big(C(k,m)-u_m+u_k\big)\\
&=-\delta_{l>i}|J|^{\{l-1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J_{l-1\mapsto l}).
\end{aligned}$$

Case $l=m-1$. Similarly to the previous case, applying the inductive hypothesis, we obtain

$$\begin{aligned}
\rho^{(1)}_l\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big)&=\big(-\delta_{m-1>i}|J|^{\{m-2\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I,J_{m-2\mapsto m-1})\\
&\quad+\delta_{m-1>i}\big|\mathcal{L}_m(J)\big|^{\{m-2\}}\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)_{m-2\mapsto m-1}\big)\\
&\quad\times\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I),\mathcal{R}_m(J)\cup\langle m-1\rangle\big)\big)/\big(C(k,m)-u_m+u_k\big)\\
&=-\delta_{m-1>i}|J|^{\{m-2\}}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I,J_{m-2\mapsto m-1})\\
&\quad-\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J_{m-2\mapsto m-1})\big)\\
&\quad\times\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I),\mathcal{R}_m(J_{m-2\mapsto m-1})\big)\big)/\big(C(k,m)-u_m+u_k\big)\\
&=-\delta_{m-1>i}|J|^{\{m-2\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J_{m-2\mapsto m-1}).
\end{aligned}$$

Case $m \leqslant l$. Applying the inductive hypothesis, we obtain

$$\begin{aligned}\rho_l^{(1)}\big(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)\big) &= \big(-|J|^{\{l-1\}}\mathcal{T}_{k,j,\mathcal{M}\setminus\{m\}}^{(d)}(I,J_{l-1\mapsto l}) + \big|\mathcal{R}_m(J)\big|^{\{l-1\}}\mathcal{S}_{k,m}^{(d)}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)\big)\\ &\quad\times \mathcal{T}_{m,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(\mathcal{R}^m(I),\mathcal{R}_m(J)_{l-1\mapsto l}\big)\big)/\big(C(k,m)-u_m+u_k\big)\\ &= -|J|^{\{l-1\}}\big(\mathcal{T}_{k,j,\mathcal{M}\setminus\{m\}}^{(d)}(I,J_{l-1\mapsto l}) - \mathcal{S}_{k,m}^{(d)}\big(\mathcal{L}^m(I),\mathcal{L}_m(J_{l-1\mapsto l})\big)\\ &\quad\times \mathcal{T}_{m,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(\mathcal{R}^m(I),\mathcal{R}_m(J_{l-1\mapsto l})\big)\big)/\big(C(k,m)-u_m+u_k\big)\\ &= -|J|^{\{l-1\}}\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J_{l-1\mapsto l}).\end{aligned}$$

It remains to prove the last formula. Suppose that $i<k$. Then applying the inductive hypothesis, we obtain

$$\begin{aligned}\rho_{k-1}^{(1)}\big(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)\big) &= \big(\mathcal{T}_{k,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(I,J\cup\langle k-1\rangle\big) - \mathcal{S}_{k,m}^{(d)}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)\cup\langle k-1\rangle\big)\\ &\quad\times \mathcal{T}_{m,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(\mathcal{R}^m(I),\mathcal{R}_m(J)\big)\big)/\big(C(k,m)-u_m+u_k\big)\\ &= \big(\mathcal{T}_{k,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(I,J\cup\langle k-1\rangle\big) - \mathcal{S}_{k,m}^{(d)}\big(\mathcal{L}^m(I),\mathcal{L}_m\big(J\cup\langle k-1\rangle\big)\big)\\ &\quad\times \mathcal{T}_{m,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(\mathcal{R}^m(I),\mathcal{R}_m\big(J\cup\langle k-1\rangle\big)\big)\big)/\big(C(k,m)-u_m+u_k\big)\\ &= \mathcal{T}_{k,j,\mathcal{M}}^{(d)}\big(I,J\cup\langle k-1\rangle\big). \qquad\square\end{aligned}$$

Lemma 6.6. *Let $i \leqslant k < j \leqslant n$ and $k \leqslant l < j-1$. Then $\rho_l^{(2)}(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)) = 0$ if $l+1 \notin \mathcal{M}$ and*

$$\begin{aligned}\rho_l^{(2)}\big(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)\big) &= -\mathcal{T}_{k,l+1,\mathcal{M}\cap(k..l+1)}^{(d)}\big(\mathcal{L}^{l+1}(I)\cup\langle l\rangle,\mathcal{L}_{l+1}(J)\big)\\ &\quad\times \mathcal{T}_{l+1,j,\mathcal{M}\cap(l+1..j)}^{(d)}\big(\mathcal{R}^{l+1}(I),\mathcal{R}_{l+1}(J)\cup\langle l\rangle\big)\end{aligned}$$

if $l+1\in\mathcal{M}$. If $i<k$ then

$$\begin{aligned}\rho_{k-1}^{(2,L)}\big(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)\big) &= \mathcal{T}_{k,j,\mathcal{M}}^{(d)}\big(I,J\cup\langle k-1\rangle\big),\\ \rho_{k-1}^{(2,R)}\big(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)\big) &= \mathcal{T}_{k,j,\mathcal{M}}^{(d)}\big(I,J\cup\langle k-1\rangle\big)\big(u_k-d+|I|^{\{k-1\}}\big).\end{aligned}$$

Finally, we have

$$\begin{aligned}\rho_{j-1}^{(2,L)}\big(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)\big) &= \mathcal{T}_{k,j,\mathcal{M}}^{(d)}\big(I\cup\langle j-1\rangle,J\big)\big(C(k,j)+u_k-d+|I|\big)\\ &\quad+\sum_{q\in\mathcal{M}}\mathcal{T}_{k,q,\mathcal{M}\cap(k..q)}^{(d)}\big(\mathcal{L}^q(I),L_q(J)\big)\mathcal{T}_{q,j,\mathcal{M}\cap(q..j)}^{(d)}\big(\mathcal{R}^q(I)\cup\langle j-1\rangle,\mathcal{R}_q(J)\big),\\ \rho_{j-1}^{(2,R)}\big(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)\big) &= \mathcal{T}_{k,j,\mathcal{M}}^{(d)}\big(I\cup\langle j-1\rangle,J\big).\end{aligned}$$

Proof. We apply induction on $|\mathcal{M}|$. We restrict ourselves to the case $l+1\in\mathcal{M}$, since otherwise both $\rho_l^{(2,L)}$ and $\rho_l^{(2,R)}$ take all summands of $\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)$ to zero.

In the case $\mathcal{M}=\emptyset$, the required formulas follow directly from the definition. Therefore, we assume $\mathcal{M}\neq\emptyset$ and put $m := \min\mathcal{M}$.

Suppose that $i < k$. Then applying the inductive hypothesis, we obtain

$$\begin{aligned}
\rho_{k-1}^{(2,R)}\big(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)\big) &= \big(\mathcal{T}_{k,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(I, J\cup\langle k-1\rangle\big)\big(u_k - d + |I|^{\{k-1\}}\big) \\
&\quad - \mathcal{S}_{k,m}^{(d)}\big(\mathcal{L}^m(I), \mathcal{L}_m(J)\cup\langle k-1\rangle\big)\big(u_k - d + \big|\mathcal{L}^m(I)\big|^{\{k-1\}}\big) \\
&\quad \times \mathcal{T}_{m,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(\mathcal{R}^m(I), \mathcal{R}_m(J)\big)\big)/\big(C(k,m) - u_m + u_k\big) \\
&= \big(u_k - d + |I|^{\{k-1\}}\big)\big(\mathcal{T}_{k,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(I, J\cup\langle k-1\rangle\big) \\
&\quad - \mathcal{S}_{k,m}^{(d)}\big(\mathcal{L}^m(I), \mathcal{L}_m\big(J\cup\langle k-1\rangle\big)\big) \\
&\quad \times \mathcal{T}_{m,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(\mathcal{R}^m(I), \mathcal{R}_m\big(J\cup\langle k-1\rangle\big)\big)\big)/\big(C(k,m) - u_m + u_k\big) \\
&= \big(u_k - d + |I|^{\{k-1\}}\big)\mathcal{T}_{k,j,\mathcal{M}}^{(d)}\big(I, J\cup\langle k-1\rangle\big).
\end{aligned}$$

Similar (but simpler) argument gives the formula for $\rho_{k-1}^{(2,L)}(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J))$.

Case $l = m-1$. By the inductive hypothesis, we have

$$\begin{aligned}
\rho_l^{(2,L)}\big(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)\big) &= -\frac{\mathcal{S}_{k,m}^{(d)}(\mathcal{L}^m(I)\cup\langle m-1\rangle, \mathcal{L}_m(J))(C(k,m) + u_k - d + |\mathcal{L}^m(I)|)}{C(k,m) - u_m + u_k} \\
&\quad \times \mathcal{T}_{m,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(\mathcal{R}^m(I), \mathcal{R}_m(J)\cup\langle m-1\rangle\big), \\
\rho_l^{(2,R)}\big(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)\big) &= -\frac{\mathcal{S}_{k,m}^{(d)}(\mathcal{L}^m(I)\cup\langle m-1\rangle, \mathcal{L}_m(J))(u_m - d + |\mathcal{R}^m(I)|^{\{m-1\}})}{C(k,m) - u_m + u_k} \\
&\quad \times \mathcal{T}_{m,j,\mathcal{M}\setminus\{m\}}^{(d)}\big(\mathcal{R}^m(I), \mathcal{R}_m(J)\cup\langle m-1\rangle\big),
\end{aligned}$$

It remains to notice that $|\mathcal{L}^m(I)| = |\mathcal{R}^m(I)|^{\{m-1\}}$.

Case $m \leqslant l$ and $l+1 \in \mathcal{M}$. Since in this case $\rho_l^{(2,L)}$ and $\rho_l^{(2,R)}$ act on $\mathcal{S}_{k,m}^{(d)}(\mathcal{L}^m(I), \mathcal{L}_m(J))$ identically, the inductive hypothesis yields

$$\begin{aligned}
\rho_l^{(2)}\big(\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I,J)\big) &= \big(-\mathcal{T}_{k,l+1,(\mathcal{M}\setminus\{m\})\cap(k..l+1)}^{(d)}\big(\mathcal{L}^{l+1}(I)\cup\langle l\rangle, \mathcal{L}_{l+1}(J)\big) \\
&\quad \times \mathcal{T}_{l+1,j,\mathcal{M}\cap(l+1..j)}^{(d)}\big(\mathcal{R}^{l+1}(I), \mathcal{R}_{l+1}(J)\cup\langle l\rangle\big) + \mathcal{S}_{k,m}^{(d)}\big(\mathcal{L}^m(I), \mathcal{L}_m(J)\big) \\
&\quad \times \mathcal{T}_{m,l+1,\mathcal{M}\cap(m..l+1)}^{(d)}\big(\mathcal{L}^{l+1}\big(\mathcal{R}^m(I)\big)\cup\langle l\rangle, \mathcal{L}_{l+1}\big(\mathcal{R}_m(J)\big)\big) \\
&\quad \times \mathcal{T}_{l+1,j,\mathcal{M}\cap(l+1..j)}^{(d)}\big(\mathcal{R}^{l+1}\big(\mathcal{R}^m(I)\big), \mathcal{R}_{l+1}\big(\mathcal{R}_m(J)\big)\cup\langle l\rangle\big)\big)/\big(C(k,m) - u_m + u_k\big) \\
&= \big(-\mathcal{T}_{k,l+1,\mathcal{M}\cap(m..l+1)}^{(d)}\big(\mathcal{L}^{l+1}(I)\cup\langle l\rangle, \mathcal{L}_{l+1}(J)\big) \\
&\quad + \mathcal{S}_{k,m}^{(d)}\big(\mathcal{L}^m\big(\mathcal{L}^{l+1}(I)\cup\langle l\rangle\big), \mathcal{L}_m\big(\mathcal{L}_{l+1}(J)\big)\big) \\
&\quad \times \mathcal{T}_{m,l+1,\mathcal{M}\cap(m..l+1)}^{(d)}\big(\mathcal{R}^m\big(\mathcal{L}^{l+1}(I)\cup\langle l\rangle\big), \mathcal{R}_m\big(\mathcal{L}_{l+1}(J)\big)\big)\big) \\
&\quad \times \mathcal{T}_{l+1,j,\mathcal{M}\cap(l+1..j)}^{(d)}\big(\mathcal{R}^{l+1}(I), \mathcal{R}_{l+1}(J)\cup\langle l\rangle\big)/\big(C(k,m) - u_m + u_k\big) \\
&= -\mathcal{T}_{k,l+1,\mathcal{M}\cap(k..l+1)}^{(d)}\big(\mathcal{L}^{l+1}(I)\cup\langle l\rangle, \mathcal{L}_{l+1}(J)\big) \\
&\quad \times \mathcal{T}_{l+1,j,\mathcal{M}\cap(l+1..j)}^{(d)}\big(\mathcal{R}^{l+1}(I), \mathcal{R}_{l+1}(J)\cup\langle l\rangle\big).
\end{aligned}$$

It remains to prove the last two formulas. Applying the inductive hypothesis, we obtain

$$\begin{aligned}
\rho^{(2,L)}_{j-1}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big) &= \Bigg(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}\big(I\cup\langle j-1\rangle, J\big)\big(C(k,j)+u_k-d+|I|\big) \\
&\quad + \Bigg[\sum_{q\in\mathcal{M}\setminus\{m\}} \mathcal{T}^{(d)}_{k,q,(\mathcal{M}\setminus\{m\})\cap(k..q)}\big(\mathcal{L}^q(I),\mathcal{L}_q(J)\big) \\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)\cup\langle j-1\rangle,\mathcal{R}_q(J)\big)\Bigg] \\
&\quad - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I)\cup\langle j-1\rangle,\mathcal{R}_m(J)\big) \\
&\quad \times \big(C(m,j)+u_m-d+\big|\mathcal{R}^m(I)\big|\big) - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)\big) \\
&\quad \times \Bigg[\sum_{q\in\mathcal{M}\setminus\{m\}} \mathcal{T}^{(d)}_{m,q,\mathcal{M}\cap(m..q)}\big(\mathcal{L}^q\big(\mathcal{R}^m(I)\big),\mathcal{L}_q\big(\mathcal{R}_m(J)\big)\big) \\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q\big(\mathcal{R}^m(I)\big)\cup\langle j-1\rangle,\mathcal{R}_q\big(\mathcal{R}_m(J)\big)\big)\Bigg]\Bigg)\Big/\big(C(k,m)-u_m+u_k\big) \\
&= \big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}\big(I\cup\langle j-1\rangle, J\big) - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m\big(I\cup\langle j-1\rangle\big),\mathcal{L}_m(J)\big) \\
&\quad \times \mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m\big(I\cup\langle j-1\rangle\big),\mathcal{R}_m(J)\big)\big) \\
&\quad \times \big(C(k,j)+u_k-d+|I|\big)\big/\big(C(k,m)-u_m+u_k\big) \\
&\quad + \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\cap(m..j)}\big(\mathcal{R}^m(I)\cup\langle j-1\rangle,\mathcal{R}_m(J)\big) \\
&\quad + \Bigg[\sum_{q\in\mathcal{M}\setminus\{m\}} \big(\mathcal{T}^{(d)}_{k,q,\mathcal{M}\cap(m..q)}\big(\mathcal{L}^q(I),\mathcal{L}_q(J)\big) - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m\big(\mathcal{L}^q(I)\big),\mathcal{L}_m\big(\mathcal{L}_q(J)\big)\big) \\
&\quad \times \mathcal{T}^{(d)}_{m,q,\mathcal{M}\cap(m..q)}\big(\mathcal{R}^m\big(\mathcal{L}^q(I)\big),\mathcal{R}_m\big(\mathcal{L}_q(J)\big)\big)\big) \\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)\cup\langle j-1\rangle,\mathcal{R}_q(J)\big)\Bigg]\Big/\big(C(k,m)-u_m+u_k\big) \\
&= \mathcal{T}^{(d)}_{k,j,\mathcal{M}}\big(I\cup\langle j-1\rangle, J\big)\big(C(k,j)+u_k-d+|I|\big) \\
&\quad + \sum_{q\in\mathcal{M}} \mathcal{T}^{(d)}_{k,q,\mathcal{M}\cap(k..q)}\big(\mathcal{L}^q(I),\mathcal{L}_q(J)\big)\mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)\cup\langle j-1\rangle,\mathcal{R}_q(J)\big).
\end{aligned}$$

Similar (but simpler) argument gives the formula for $\rho^{(2,R)}_{j-1}(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J))$. $\square$

Lemma 6.7. *Let $i\leqslant k<j\leqslant n$ and $\max\{k-1,i\}\leqslant l<j-1$. Take an integer $m'\in(\mathcal{M}\cup\{k\})\cap(-\infty..l+1]$, which we call the origin in this lemma. If $m'=k$ then*

$$\begin{aligned}
\rho^{(3)}_l\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big) &= |I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I_{l+1\mapsto l}, J)\big(C(k,l+1)+u_k-d+|I|^{(-\infty..l]}\big) \\
&\quad + |I|^{\{l+1\}}\sum_{q\in(k..l+1]\cap\mathcal{M}} \mathcal{T}^{(d)}_{k,q,\mathcal{M}\cap(k..q)}\big(\mathcal{L}^q(I_{l+1\mapsto l}),\mathcal{L}_q(J)\big) \\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)_{l+1\mapsto l},\mathcal{R}_q(J)\big).
\end{aligned}$$

If $m' > k$ then

$$\rho_l^{(3)}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big) = |I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I_{l+1\mapsto l},J)\big(C(m',l+1)+u_{m'}-d+|I|^{(-\infty..l]}\big)$$
$$+|I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m'\}}(I_{l+1\mapsto l},J)$$
$$+|I|^{\{l+1\}}\sum_{q\in(m'..l+1]\cap\mathcal{M}}\mathcal{T}^{(d)}_{k,q,\mathcal{M}\cap(k..q)}\big(\mathcal{L}^q(I_{l+1\mapsto l}),\mathcal{L}_q(J)\big)$$
$$\times\mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)_{l+1\mapsto l},\mathcal{R}_q(J)\big).$$

Finally, we have $\rho_{j-1}^{(3)}(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)) = \mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I\cup\langle j-1\rangle, J)$.

Proof. We apply induction on $|\mathcal{M}|$. In the case $\mathcal{M}=\emptyset$, the required formulas follow immediately from the definition. Now let $\mathcal{M}\neq\emptyset$. We put $m := \min\mathcal{M}$.

Case $l < m-1$. We have $m' = k$. Applying the inductive hypothesis with origin at k, we obtain

$$\rho_l^{(3)}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big) = \big(|I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{l+1\mapsto l},J)\big(C(k,l+1)+u_k-d+|I|^{(-\infty..l]}\big)$$
$$-\big|\mathcal{L}^m(I)\big|^{\{l+1\}}\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I)_{l+1\mapsto l},\mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I),\mathcal{R}_m(J)\big)$$
$$\times\big(C(k,l+1)+u_k-d+\big|\mathcal{L}^m(I)\big|^{(-\infty..l]}\big)\big)/\big(C(k,m)-u_m+u_k\big)$$
$$=|I|^{\{l+1\}}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{l+1\mapsto l},J)$$
$$-\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I_{l+1\mapsto l}),\mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I_{l+1\mapsto l}),\mathcal{R}_m(J)\big)\big)$$
$$\times\big(C(k,l+1)+u_k-d+|I|^{(-\infty..l]}\big)/\big(C(k,m)-u_m+u_k\big)$$
$$=|I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I_{l+1\mapsto l},J)\big(C(k,l+1)+u_k-d+|I|^{(-\infty..l]}\big).$$

Case $l = m-1$ and $m' = m$. Applying the inductive hypothesis with origins at k and m, we obtain

$$\rho_l^{(3)}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big) = \big(|I|^{\{m\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{m\mapsto m-1},J)\big(C(k,m)+u_k-d+|I|^{(-\infty..m-1]}\big)$$
$$-\big|\mathcal{R}^m(I)\big|^{\{m\}}\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I)\cup\langle m-1\rangle,\mathcal{L}_m(J)\big)$$
$$\times\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I)_{m\mapsto m-1},\mathcal{R}_m(J)\big)$$
$$\times\big(u_m-d+\big|\mathcal{R}^m(I)\big|^{(-\infty..m-1]}\big)\big)/\big(C(k,m)-u_m+u_k\big)$$
$$=|I|^{\{m\}}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{m\mapsto m-1},J)\big(C(k,m)+u_k-d+|I|^{(-\infty..m-1]}\big)$$
$$-\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I_{m\mapsto m-1}),\mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I_{m\mapsto m-1}),\mathcal{R}_m(J)\big)$$
$$\times\big(u_m-d+|I|^{(-\infty..m-1]}\big)\big)/\big(C(k,m)-u_m+u_k\big)$$
$$=|I|^{\{m\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{m\mapsto m-1},J)+|I|^{\{m\}}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{m\mapsto m-1},J)$$
$$-\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I_{m\mapsto m-1}),\mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I_{m\mapsto m-1}),\mathcal{R}_m(J)\big)\big)$$
$$\times\big(u_m-d+|I|^{(-\infty..m-1]}\big)/\big(C(k,m)-u_m+u_k\big)$$
$$=|I|^{\{m\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{m\mapsto m-1},J)+|I|^{\{m\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I_{m\mapsto m-1},J)\big(u_m-d+|I|^{(-\infty..m-1]}\big).$$

In this case, the summation is empty.

Case $l = m-1$ and $m' = k$. Beginning as in the previous case, we obtain

$$\begin{aligned}
\rho_l^{(3)}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big) &= |I|^{\{m\}}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{m\mapsto m-1},J)\\
&\quad - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I_{m\mapsto m-1}),\mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I_{m\mapsto m-1}),\mathcal{R}_m(J)\big)\big)\\
&\quad \times \big(C(k,m)+u_k-d+|I|^{(-\infty..m-1]}\big)/\big(C(k,m)-u_m+u_k\big)\\
&\quad + |I|^{\{m\}}\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I_{m\mapsto m-1}),\mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I_{m\mapsto m-1}),\mathcal{R}_m(J)\big)\\
&= |I|^{\{m\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I_{m\mapsto m-1},J)\big(C(k,m)+u_k-d+|I|^{(-\infty..m-1]}\big)\\
&\quad + |I|^{\{m\}}\mathcal{T}^{(d)}_{k,m,\emptyset}\big(\mathcal{L}^m(I_{m\mapsto m-1}),\mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I)_{m\mapsto m-1},\mathcal{R}_m(J)\big).
\end{aligned}$$

The last summand corresponds to $q = m$, the only possible value of the summation parameter.

Case $l \geqslant m$ and $m' = m$. Applying the inductive hypothesis with origins at k and m, we obtain

$$\begin{aligned}
\rho_l^{(3)}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big) &= \Big(|I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{l+1\mapsto l},J)\big(C(k,l+1)+u_k-d+|I|^{(-\infty..l]}\big)\\
&\quad + |I|^{\{l+1\}}\Big[\sum_{q\in(k..l+1]\cap(\mathcal{M}\setminus\{m\})}\mathcal{T}^{(d)}_{k,q,(\mathcal{M}\setminus\{m\})\cap(k..q)}\big(\mathcal{L}^q(I_{l+1\mapsto l}),\mathcal{L}_q(J)\big)\\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)_{l+1\mapsto l},\mathcal{R}_q(J)\big)\Big] - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)\big)\\
&\quad \times \big|\mathcal{R}^m(I)\big|^{\{l+1\}}\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I)_{l+1\mapsto l},\mathcal{R}_m(J)\big)\\
&\quad \times \big(C(m,l+1)+u_m-d+\big|\mathcal{R}^m(I)\big|^{(-\infty..l]}\big)\\
&\quad - \big|\mathcal{R}^m(I)\big|^{\{l+1\}}\Big[\sum_{q\in(m..l+1]\cap\mathcal{M}}\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)\big)\\
&\quad \times \mathcal{T}^{(d)}_{m,q,\mathcal{M}\cap(m..q)}\big(\mathcal{L}^q\big(\mathcal{R}^m(I)_{l+1\mapsto l}\big),\mathcal{L}_q\big(\mathcal{R}_m(J)\big)\big)\\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q\big(\mathcal{R}^m(I)\big)_{l+1\mapsto l},\mathcal{R}_q\big(\mathcal{R}_m(J)\big)\big)\Big]\Big)/\big(C(k,m)-u_m+u_k\big)\\
&= |I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{l+1\mapsto l},J)\\
&\quad + |I|^{\{l+1\}}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{l+1\mapsto l},J) - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I_{l+1\mapsto l}),\mathcal{L}_m(J)\big)\\
&\quad \times \mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I_{l+1\mapsto l}),\mathcal{R}_m(J)\big)\big)\\
&\quad \times \big(C(m,l+1)+u_m-d+|I|^{(-\infty..l]}\big)/\big(C(k,m)-u_m+u_k\big)\\
&\quad + |I|^{\{l+1\}}\Big[\sum_{q\in(m..l+1]\cap\mathcal{M}}\big(\mathcal{T}^{(d)}_{k,q,\mathcal{M}\cap(m..q)}\big(\mathcal{L}^q(I_{l+1\mapsto l}),\mathcal{L}_q(J)\big)\\
&\quad - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m\big(\mathcal{L}^q(I_{l+1\mapsto l})\big),\mathcal{L}_m\big(\mathcal{L}_q(J)\big)\big)\\
&\quad \times \mathcal{T}^{(d)}_{m,q,\mathcal{M}\cap(m..q)}\big(\mathcal{R}^m\big(\mathcal{L}^q(I_{l+1\mapsto l})\big),\mathcal{R}_m\big(\mathcal{L}_q(J)\big)\big)\big)\\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)_{l+1\mapsto l},\mathcal{R}^q(J)\big)\Big]/\big(C(k,m)-u_m+u_k\big)
\end{aligned}$$

$$
\begin{aligned}
&= |I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{l+1\mapsto l}, J)\\
&\quad + |I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I_{l+1\mapsto l}, J)\big(C(m,l+1)+u_m-d+|I|^{(-\infty..l]}\big)\\
&\quad + |I|^{\{l+1\}}\sum_{q\in(m..l+1]\cap\mathcal{M}}\mathcal{T}^{(d)}_{k,q,\mathcal{M}\cap(k..q)}\big(\mathcal{L}^q(I_{l+1\mapsto l}), \mathcal{L}_q(J)\big)\\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)_{l+1\mapsto l}, \mathcal{R}^q(J)\big).
\end{aligned}
$$

Case $l \geqslant m$ and $m' = k$. Handling the sums in the square brackets as in the previous case, we obtain

$$
\begin{aligned}
\rho^{(3)}_l\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big) &= |I|^{\{l+1\}}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{l+1\mapsto l}, J)\big(C(k,l+1)+u_k-d+|I|^{(-\infty..l]}\big)\\
&\quad - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I), \mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I)_{l+1\mapsto l}, \mathcal{R}_m(J)\big)\\
&\quad \times \big(C(m,l+1)+u_m-d+|I|^{(-\infty..l]}\big)\big)/\big(C(k,m)-u_m+u_k\big)\\
&\quad + |I|^{\{l+1\}}\sum_{q\in(m..l+1]\cap\mathcal{M}}\mathcal{T}^{(d)}_{k,q,\mathcal{M}\cap(k..q)}\big(\mathcal{L}^q(I_{l+1\mapsto l}), \mathcal{L}_q(J)\big)\\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)_{l+1\mapsto l}, \mathcal{R}_q(J)\big)\\
&= |I|^{\{l+1\}}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{l+1\mapsto l}, J)\\
&\quad - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I_{l+1\mapsto l}), \mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I_{l+1\mapsto l}), \mathcal{R}_m(J)\big)\big)\\
&\quad \times \big(C(k,l+1)+u_k-d+|I|^{(-\infty..l]}\big)/\big(C(k,m)-u_m+u_k\big)\\
&\quad + |I|^{\{l+1\}}\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I_{l+1\mapsto l}), \mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I)_{l+1\mapsto l}, \mathcal{R}_m(J)\big)\\
&\quad + |I|^{\{l+1\}}\sum_{q\in(m..l+1]\cap\mathcal{M}}\mathcal{T}^{(d)}_{k,q,\mathcal{M}\cap(k..q)}\big(\mathcal{L}^q(I_{l+1\mapsto l}), \mathcal{L}_q(J)\big)\\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)_{l+1\mapsto l}, \mathcal{R}_q(J)\big)\\
&= |I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I_{l+1\mapsto l}, J)\big(C(k,l+1)+u_k-d+|I|^{(-\infty..l]}\big)\\
&\quad + |I|^{\{l+1\}}\sum_{q\in(k..l+1]\cap\mathcal{M}}\mathcal{T}^{(d)}_{k,q,\mathcal{M}\cap(k..q)}\big(\mathcal{L}^q(I_{l+1\mapsto l}), \mathcal{L}_q(J)\big)\\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)_{l+1\mapsto l}, \mathcal{R}_q(J)\big).
\end{aligned}
$$

Case $l \geqslant m$ and $m' > m$. Applying the inductive hypothesis with origin at m', we obtain

$$
\begin{aligned}
\rho^{(3)}_l\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big) &= \Big(|I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I_{l+1\mapsto l}, J)\big(C(m',l+1)+u_{m'}-d+|I|^{(-\infty..l]}\big)\\
&\quad + |I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m,m'\}}(I_{l+1\mapsto l}, J)\\
&\quad + |I|^{\{l+1\}}\Bigg[\sum_{q\in(m'..l+1]\cap\mathcal{M}}\mathcal{T}^{(d)}_{k,q,(\mathcal{M}\setminus\{m\})\cap(k..q)}\big(\mathcal{L}^q(I_{l+1\mapsto l}), \mathcal{L}_q(J)\big)\\
&\quad \times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)_{l+1\mapsto l}, \mathcal{R}_q(J)\big)\Bigg] - \big|\mathcal{R}^m(I)\big|^{\{l+1\}}S^{(d)}_{k,m}\big(\mathcal{L}^m(I), \mathcal{L}_m(J)\big)\\
&\quad \times \mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I)_{l+1\mapsto l}, \mathcal{R}_m(J)\big)\big(C(m',l+1)+u_{m'}-d+\big|\mathcal{R}^m(I)\big|^{(-\infty..l]}\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \left|\mathcal{R}^m(I)\right|^{\{l+1\}} S^{(d)}_{k,m}\big(\mathcal{L}^m(I), \mathcal{L}_m(J)\big)\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m,m'\}}\big(\mathcal{R}^m(I)_{l+1\mapsto l}, \mathcal{R}_m(J)\big)\\
&- \left|\mathcal{R}^m(I)\right|^{\{l+1\}} S^{(d)}_{k,m}\big(\mathcal{L}^m(I), \mathcal{L}_m(J)\big)\\
&\times \Bigg[\sum_{q\in(m'..l+1]\cap\mathcal{M}} \mathcal{T}^{(d)}_{m,q,\mathcal{M}\cap(m..q)}\big(\mathcal{L}^q\big(\mathcal{R}^m(I)_{l+1\mapsto l}\big), \mathcal{L}_q\big(\mathcal{R}_m(J)\big)\big)\\
&\times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q\big(\mathcal{R}^m(I)\big)_{l+1\mapsto l}, \mathcal{R}_q\big(\mathcal{R}_m(J)\big)\big)\Bigg]\Bigg)/\big(C(k,m)-u_m+u_k\big)\\
= {}&|I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I_{l+1\mapsto l}, J)\big(C(m', l+1)+u_{m'}-d+|I|^{(-\infty..l]}\big)\\
&+ |I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m'\}}(I_{l+1\mapsto l}, J)\\
&+ |I|^{\{l+1\}}\Bigg[\sum_{q\in(m'..l+1]\cap\mathcal{M}} \big(\mathcal{T}^{(d)}_{k,q,\mathcal{M}\cap(m..q)}\big(\mathcal{L}^q(I_{l+1\mapsto l}), \mathcal{L}_q(J)\big)\\
&- S^{(d)}_{k,m}\big(\mathcal{L}^m\big(\mathcal{L}^q(I_{l+1\mapsto l})\big), \mathcal{L}_m\big(\mathcal{L}_q(J)\big)\big)\\
&\times \mathcal{T}^{(d)}_{m,q,\mathcal{M}\cap(m..q)}\big(\mathcal{R}^m\big(\mathcal{L}^q(I_{l+1\mapsto l})\big), \mathcal{R}_m\big(\mathcal{L}_q(J)\big)\big)\big)\\
&\times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)_{l+1\mapsto l}, \mathcal{R}_q(J)\big)\Bigg]/\big(C(k,m)-u_m+u_k\big)\\
= {}&|I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I_{l+1\mapsto l}, J)\big(C(m', l+1)+u_{m'}-d+|I|^{(-\infty..l]}\big)\\
&+ |I|^{\{l+1\}}\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m'\}}(I_{l+1\mapsto l}, J)\\
&+ |I|^{\{l+1\}}\sum_{q\in(m'..l+1]\cap\mathcal{M}} \mathcal{T}^{(d)}_{k,q,\mathcal{M}\cap(k..q)}\big(\mathcal{L}^q(I_{l+1\mapsto l}), \mathcal{L}_q(J)\big)\\
&\times \mathcal{T}^{(d)}_{q,j,\mathcal{M}\cap(q..j)}\big(\mathcal{R}^q(I)_{l+1\mapsto l}, \mathcal{R}_q(J)\big).
\end{aligned}
$$

To prove the last formula, we apply the inductive hypothesis and obtain

$$
\begin{aligned}
\rho^{(3)}_{j-1}\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I, J)\big) &= \big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}\big(I\cup\langle j-1\rangle, J\big) - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I), \mathcal{L}_m(J)\big)\\
&\quad\times \mathcal{T}^{(d)}_{m,j,\mathcal{M}}\big(\mathcal{R}^m(I)\cup\langle j-1\rangle, \mathcal{R}_m(J)\big)\big)/\big(C(k,m)-u_m+u_k\big)\\
&= \big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}\big(I\cup\langle j-1\rangle, J\big) - \mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m\big(I\cup\langle j-1\rangle\big), \mathcal{L}_m(J)\big)\\
&\quad\times \mathcal{T}^{(d)}_{m,j,\mathcal{M}}\big(\mathcal{R}^m\big(I\cup\langle j-1\rangle\big), \mathcal{R}_m(J)\big)\big)/\big(C(k,m)-u_m+u_k\big)\\
&= \mathcal{T}^{(d)}_{k,j,\mathcal{M}}\big(I\cup\langle j-1\rangle, J\big). \qquad \Box
\end{aligned}
$$

It would be natural to ask whether the operators $\rho^{(1)}_l$, $\rho^{(2)}_l$, $\rho^{(3)}_l$ take integral polynomials to integral. Although the answer is actually affirmative, we prefer to work only with integral polynomials of the following special form.

Definition 6.8. A polynomial of $F^{(d)}_{i,n}$ of the form

$$
\mathcal{T}^{(d)}_{m_0,m_1,\mathcal{M}_1}(I_1, J_0)\mathcal{T}^{(d)}_{m_1,m_2,\mathcal{M}_2}(I_2, J_1)\cdots\mathcal{T}^{(d)}_{m_k,m_{k+1},\mathcal{M}_{k+1}}(I_{k+1}, J_k)f, \tag{6.3}
$$

where $i=m_0<m_1<\cdots<m_k<m_{k+1}=n$, $\mathcal{M}_1\subset(m_0..m_1)$, $\mathcal{M}_2\subset(m_1..m_2),\dots,\mathcal{M}_{k+1}\subset(m_k..m_{k+1})$, $f\in\mathcal{U}^0$, $J_0=\langle(i-1)^d\rangle$ and conditions (M1)–(M3) hold for the sequence of multisets $I_1,\dots,I_{k+1}$, $J_1,\dots,J_k$ and the set $\{m_1,\dots,m_k\}$ is called a $\mathcal{T}$-monomial and $\mathcal{T}^{(d)}_{m_k,m_{k+1},\mathcal{M}_{k+1}}(I_{k+1},J_k)$ is called its tail.

Clearly, (6.3) has weight $\lambda^{(d)}_{i,n,\{m_1,\dots,m_k\}}(I_1,\dots,I_{k+1},J_1,\dots,J_k)$.

Corollary 6.9. *Any $\mathcal{T}$-monomial is an integral polynomial. Each operator $\rho^{(1)}_l$, $\rho^{(2)}_l$, $\rho^{(3)}_l$ takes a $\mathcal{T}$-monomial of weight λ to a sum of $\mathcal{T}$-monomials of weight $\lambda+\alpha_l$.*

Proof. The first statement follows from Lemma 6.4(iii), representation given in the proof of Lemma 6.4(i) and the remark of Section 6.2. The second statement follows from Lemmas 6.5–6.7 and 3.2. □

6.5. Raising coefficients

Based on Lemma 4.1, we define the ring homomorphism $\mathrm{cf}_\varkappa: F^{(d)}_{i,n}\to\bar{\mathcal{U}}^0$ that acts identically on $\bar{\mathcal{U}}^0$ and acts on the formal operator $\mathcal{S}^{(d)}_{m,m'}(I,J)$ as follows

$$\begin{aligned}\mathrm{cf}_\varkappa\big(\mathcal{S}^{(d)}_{m,m'}(I,J)\big)&:=u_m\underline{{}^{|J|^{[m..+\infty)}}}\\&\times\prod_{t=m+1}^{m'}|J|^{\{t-2\}}!\frac{(C(m,t)+u_m)\underline{{}^{d+1-|I|^{(-\infty..t-1]}}}}{(C(m,t)+u_m-|J|^{[t..+\infty)})\cdots(C(m,t)+u_m-|J|^{[t-1..+\infty)}-q_t)}\\&\times\binom{|I|^{\{t-1\}}}{|I|^{\{t-1\}}-|J|^{\{t-2\}}+q_t}\binom{d-|I|^{(-\infty..t-1]}-|J|^{[t-1..+\infty)}}{q_t}(H_{t-1}-H_t)\underline{{}^{q_t}},\end{aligned}\tag{6.4}$$

where $\varkappa=(q_{i+1},\dots,q_n)$ is a sequence of nonnegative integers.

It follows from (6.1) that no cutting operator takes the denominator of $\mathrm{cf}_\varkappa(\mathcal{S}^{(d)}_{m,m'}(I,J))$ to zero.

Lemma 6.10. *Let P be a regular polynomial of $F^{(d)}_{i,n}$ and $i\leqslant m<m'<n$. Then we have*

$$(\sigma_{m,m'}\circ\mathrm{cf}_\varkappa\circ\sigma_{m,m'})\big(P\cdot\big(C(m,m')-u_{m'}+u_m\big)\big)=(\sigma_{m,m'}\circ\mathrm{cf}_\varkappa)\big(P\cdot\big(C(m,m')-u_{m'}+u_m\big)\big).$$

Proof. Since $\sigma_{m,m'}$ and $\mathrm{cf}_\varkappa$ are $\mathbf{Z}$-linear, it suffices to assume that P is a regular monomial. Let $P=Q\cdot f/g$, where $f,g\in\mathcal{U}^0$, $g\neq0$ and conditions (F3)–(F5) hold for Q and g. If g is not divisible by $C(m,m')-u_{m'}+u_m$, then $\sigma_{m,m'}$ is applicable to both P and $\mathrm{cf}_\varkappa(P)$. Hence by virtue of the remark of Section 6.2 it follows that both sides of the required equality equal zero.

Now consider the case where g is divisible by $C(m,m')-u_{m'}+u_m$ and denote by $\bar{g}$ their quotient. By conditions (F4) and (F5), we obtain $Q=Q_0\mathcal{S}^{(d)}_{m,m''}(I,J)$, where $m'\leqslant m''$ and Q_0 is a product of formal operators of the form $\mathcal{S}^{(d)}_{\hat{m},\hat{m}'}(\hat{I},\hat{J})$ with $[\hat{m}..\hat{m}')\cap[m..m'')=\emptyset$. Thus we get that $\sigma_{m,m'}$ acts identically on Q_0. Moreover, if $m'=m''$, then $\sigma_{m,m'}$ acts identically on the whole of Q and the required equality immediately follows, using the fact that $\sigma_{m,m'}$ is an idempotent operator on $\mathcal{U}^0$.

Therefore, we consider the case $m'<m''$. We have

$$\begin{aligned}&(\sigma_{m,m'}\circ\mathrm{cf}_\varkappa\circ\sigma_{m,m'})\big(P\cdot\big(C(m,m')-u_{m'}+u_m\big)\big)\\&\quad=(\sigma_{m,m'}\circ\mathrm{cf}_\varkappa)\left(Q_0\cdot\frac{\sigma_{m,m'}(f)}{\sigma_{m,m'}(\bar{g})}\cdot\sigma_{m,m'}\big(\mathcal{S}^{(d)}_{m,m''}(I,J)\big)\right)\end{aligned}$$

$$= \sigma_{m,m'}\left(\mathrm{cf}_\varkappa(Q_0)\cdot\frac{\sigma_{m,m'}(f)}{\sigma_{m,m'}(\bar g)}\cdot(\mathrm{cf}_\varkappa\circ\sigma_{m,m'})\big(\mathcal{S}^{(d)}_{m,m''}(I,J)\big)\right)$$
$$= (\sigma_{m,m'}\circ\mathrm{cf}_\varkappa)(Q_0)\cdot\frac{\sigma_{m,m'}(f)}{\sigma_{m,m'}(\bar g)}\cdot(\sigma_{m,m'}\circ\mathrm{cf}_\varkappa\circ\sigma_{m,m'})\big(S^{(d)}_{m,m''}(I,J)\big),$$
$$(\sigma_{m,m'}\circ\mathrm{cf}_\varkappa)\big(P\cdot\big(C(m,m')-u_{m'}+u_m\big)\big)$$
$$= \sigma_{m,m'}\left(\mathrm{cf}_\varkappa(Q_0)\cdot\frac{f}{\bar g}\cdot\mathrm{cf}_\varkappa\big(\mathcal{S}^{(d)}_{m,m''}(I,J)\big)\right)$$
$$= (\sigma_{m,m'}\circ\mathrm{cf}_\varkappa)(Q_0)\cdot\frac{\sigma_{m,m'}(f)}{\sigma_{m,m'}(\bar g)}\cdot(\sigma_{m,m'}\circ\mathrm{cf}_\varkappa)\big(\mathcal{S}^{(d)}_{m,m''}(I,J)\big).$$

It remains to prove that

$$(\sigma_{m,m'}\circ\mathrm{cf}_\varkappa\circ\sigma_{m,m'})\big(\mathcal{S}^{(d)}_{m,m''}(I,J)\big)=(\sigma_{m,m'}\circ\mathrm{cf}_\varkappa)\big(\mathcal{S}^{(d)}_{m,m''}(I,J)\big). \tag{6.5}$$

We put

$$\Phi := \prod_{t=m+1}^{m'-1}\left(|J|^{\{t-2\}}!\frac{(C(m,t)+u_m)^{\underline{d+1-|I|^{(-\infty..t-1]}}}}{(C(m,t)+u_m-|J|^{[t..+\infty)})\cdots(C(m,t)+u_m-|J|^{[t-1..+\infty)}-q_t)}\right.$$
$$\left.\times\binom{|I|^{\{t-1\}}}{|I|^{\{t-1\}}-|J|^{\{t-2\}}+q_t}\binom{d-|I|^{(-\infty..t-1]}-|J|^{[t-1..+\infty)}}{q_t}(H_{t-1}-H_t)^{\underline{q_t}}\right)$$
$$\times\prod_{t=m'+1}^{m''}\left(|J|^{\{t-2\}}!\frac{(C(m',t)+u_{m'})^{\underline{d+1-|I|^{(-\infty..t-1]}}}}{(C(m',t)+u_{m'}-|J|^{[t..+\infty)})\cdots(C(m',t)+u_{m'}-|J|^{[t-1..+\infty)}-q_t)}\right.$$
$$\left.\times\binom{|I|^{\{t-1\}}}{|I|^{\{t-1\}}-|J|^{\{t-2\}}+q_t}\binom{d-|I|^{(-\infty..t-1]}-|J|^{[t-1..+\infty)}}{q_t}(H_{t-1}-H_t)^{\underline{q_t}}\right).$$

We clearly have

$$(\sigma_{m,m'}\circ\mathrm{cf}_\varkappa)\big(\mathcal{S}^{(d)}_{m,m''}(I,J)\big)=\Phi u_m^{\underline{|J|^{[m..+\infty)}}}|J|^{\{m'-2\}}!\frac{u_{m'}^{\underline{d+1-|I|^{(-\infty..m'-1]}}}}{(u_{m'}-|J|^{[m'..+\infty)})\cdots(u_{m'}-|J|^{[m'-1..+\infty)}-q_{m'})}$$
$$\times\binom{|I|^{\{m'-1\}}}{|I|^{\{m'-1\}}-|J|^{\{m'-2\}}+q_{m'}}\binom{d-|I|^{(-\infty..m'-1]}-|J|^{[m'-1..+\infty)}}{q_{m'}}$$
$$\times\sigma_{m,m'}(H_{m'-1}-H_{m'})^{\underline{q_{m'}}}. \tag{6.6}$$

We can rewrite

$$\Phi := \prod_{t=m+1}^{m'-1}\left(\big|\mathcal{L}_{m'}(J)\big|^{\{t-2\}}!\frac{(C(m,t)+u_m)^{\underline{d+1-|\mathcal{L}^{m'}(I)|^{(-\infty..t-1]}}}}{(C(m,t)+u_m-|\mathcal{L}_{m'}(J)|^{[t..+\infty)})\cdots(C(m,t)+u_m-|\mathcal{L}_{m'}(J)|^{[t-1..+\infty)}-q_t)}\right.$$
$$\times\binom{|\mathcal{L}^{m'}(I)|^{\{t-1\}}}{|\mathcal{L}^{m'}(I)|^{\{t-1\}}-|\mathcal{L}_{m'}(J)|^{\{t-2\}}+q_t}\binom{d-|\mathcal{L}^{m'}(I)|^{(-\infty..t-1]}-|\mathcal{L}_{m'}(J)|^{[t-1..+\infty)}}{q_t}$$
$$\left.\times(H_{t-1}-H_t)^{\underline{q_t}}\right)\prod_{t=m'+1}^{m''}\left(\big|\mathcal{R}_{m'}(J)\big|^{\{t-2\}}!\right.$$

$$\times \frac{(C(m',t)+u_{m'})\underline{{}^{d+1-|\mathcal{R}^{m'}(I)|^{(-\infty..t-1]}}}}{(C(m',t)+u_{m'}-|\mathcal{R}_{m'}(J)|^{[t..+\infty)})\cdots(C(m',t)+u_{m'}-|\mathcal{R}_{m'}(J)|^{[t-1..+\infty)}-q_t)}$$
$$\times \binom{|\mathcal{R}^{m'}(I)|^{\{t-1\}}}{|\mathcal{R}^{m'}(I)|^{\{t-1\}}-|\mathcal{R}_{m'}(J)|^{\{t-2\}}+q_t}\binom{d-|\mathcal{R}^{m'}(I)|^{(-\infty..t-1]}-|\mathcal{R}_{m'}(J)|^{[t-1..+\infty)}}{q_t}$$
$$\times (H_{t-1}-H_t)\underline{{}^{q_t}}\Big),$$

whence we obtain

$$\begin{aligned}
&(\mathrm{cf}_\varkappa\circ\sigma_{m,m'})\big(\mathcal{S}^{(d)}_{m,m''}(I,J)\big)\\
&= \mathrm{cf}_\varkappa\big(\mathcal{S}^{(d)}_{m,m'}\big(\mathcal{L}^{m'}(I),\mathcal{L}_{m'}(J)\big)\big)\cdot \mathrm{cf}_\varkappa\big(\mathcal{S}^{(d)}_{m',m''}\big(\mathcal{R}^{m'}(I),\mathcal{R}_{m'}(J)\big)\big)\\
&= \Phi u_m\underline{{}^{|\mathcal{L}_{m'}(J)|^{[m..+\infty)}}}u_{m'}\underline{{}^{|\mathcal{R}_{m'}(J)|^{[m'..+\infty)}}}\big|\mathcal{L}_{m'}(J)\big|^{\{m'-2\}}!\\
&\quad\times \frac{(C(m,m')+u_m)\underline{{}^{d+1-|\mathcal{L}^{m'}(I)|^{(-\infty..m'-1]}}}}{(C(m,m')+u_m)\cdots(C(m,m')+u_m-|\mathcal{L}_{m'}(J)|^{[m'-1..+\infty)}-q_{m'})}\\
&\quad\times\binom{|\mathcal{L}^{m'}(I)|^{\{m'-1\}}}{|\mathcal{L}^{m'}(I)|^{\{m'-1\}}-|\mathcal{L}_{m'}(J)|^{\{m'-2\}}+q_{m'}}\binom{d-|\mathcal{L}^{m'}(I)|^{(-\infty..m'-1]}-|\mathcal{L}_{m'}(J)|^{[m'-1..+\infty)}}{q_{m'}}\\
&\quad\times (H_{m'-1}-H_{m'})\underline{{}^{q_{m'}}}.
\end{aligned}$$

Noting that $\sigma_{m,m'}$ acts identically on Φ, we obtain

$$\begin{aligned}
(\sigma_{m,m'}\circ\mathrm{cf}_\varkappa\circ\sigma_{m,m'})\big(\mathcal{S}^{(d)}_{m,m''}(I,J)\big) &= \Phi u_m\underline{{}^{|J|^{[m..+\infty)}}}u_{m'}\underline{{}^{|J|^{[m'..+\infty)}}}\\
&\quad\times |J|^{\{m'-2\}}!\frac{u_{m'}\underline{{}^{d+1-|I|^{(-\infty..m'-1]}}}}{u_{m'}\cdots(u_{m'}-|J|^{[m'-1..+\infty)}-q_{m'})}\\
&\quad\times\binom{|I|^{\{m'-1\}}}{|I|^{\{m'-1\}}-|J|^{\{m'-2\}}+q_{m'}}\binom{d-|I|^{(-\infty..m'-1]}-|J|^{[m'-1..+\infty)}}{q_{m'}}\\
&\quad\times \sigma_{m,m'}(H_{m'-1}-H_{m'})\underline{{}^{q_{m'}}}.
\end{aligned}$$

Comparing this with (6.6) gives (6.5). □

Note that applying $\mathrm{cf}_\varkappa$ to a formal operator does not always give an element of $\mathcal{U}^0$. However, this is true in all cases of interest.

Lemma 6.11. *If we apply* $\mathrm{cf}_\varkappa$ *to a formal operator of weight* $\leqslant 0$*, then we obtain an element of* $\mathcal{U}^0$*.*

Proof. Let $\mathcal{S}^{(d)}_{m,m'}(I,J)$ be a formal operator of weight $\leqslant 0$. For any $t=m+1,\dots,m'$, we have $d+1-|I|^{(-\infty..t-1]}>d-|I|^{(-\infty..t-1]}-|J|^{[t-1..+\infty)}\geqslant 0$. Therefore $\mathrm{cf}_\varkappa(\mathcal{S}^{(d)}_{m,m'}(I,J))=0$ except the case where $q_t\leqslant d-|I|^{(-\infty..t-1]}-|J|^{[t-1..+\infty)}$ for all $t=m+1,\dots,m'$, in which the denominator divides the numerator in each fraction of (6.4) and thus $\mathrm{cf}_\varkappa(\mathcal{S}^{(d)}_{m,m'}(I,J))\in\mathcal{U}^0$. □

Corollary 6.12. *Let* $P\in F^{(d)}_{i,n}$ *be an integral polynomial of weight* $\leqslant 0$*. Then* $\mathrm{cf}_\varkappa(P)\in\mathcal{U}^0$*.*

Proof. Consider the representation $\mathrm{cf}_\varkappa(P) = A/B$, where $A, B \in \mathcal{U}^0$, $B \neq 0$ and the fraction A/B is irreducible. By property (F4) of the definition of a regular polynomial, all formal operators occurring in P have weights $\leqslant 0$. Therefore, by Lemma 6.11 and property (F5), we obtain that B is a product of polynomials of the form $C(m, m') - u_{m'} + u_m$, where $i \leqslant m < m' < n$, each occurring in degree at most one.

Lemma 6.10 together with the definition of an integral polynomial yields

$$\sigma_{m,m'}\left(\frac{A}{B}\big(C(m,m') - u_{m'} + u_m\big)\right) = (\sigma_{m,m'} \circ \mathrm{cf}_\varkappa)\big(P \cdot \big(C(m,m') - u_{m'} + u_m\big)\big)$$
$$= (\sigma_{m,m'} \circ \mathrm{cf}_\varkappa \circ \sigma_{m,m'})\big(P \cdot \big(C(m,m') - u_{m'} + u_m\big)\big) = 0.$$

Suppose that B is divisible by $C(m, m') - u_{m'} + u_m$, where $i \leqslant m < m' < n$, and denote their quotient by $\bar{B}$. We have $\sigma_{m,m'}(\bar{B}) \neq 0$ and

$$0 = \sigma_{m,m'}(A/\bar{B}) = \sigma_{m,m'}(A)/\sigma_{m,m'}(\bar{B}).$$

Hence $\sigma_{m,m'}(A) = 0$, whence by Lemma 6.1, we obtain that A is divisible by $C(m, m') - u_{m'} + u_m$. This contradicts the irreducibility of A/B. Thus we have proved that $B = 1$. □

In the sequel, we shall use the following notation

$$\mathrm{cf}^{(d)}_{k,j,\varkappa}(I, J) := \prod_{t=k+1}^{j} |J|^{\{t-2\}}! \binom{|I|^{\{t-1\}}}{|I|^{\{t-1\}} - |J|^{\{t-2\}} + q_t} \binom{d - |I|^{(-\infty..t-1]} - |J|^{[t-1..+\infty)}}{q_t} (H_{t-1} - H_t)^{\underline{q_t}}.$$

For $k < m < j$, we have

$$\mathrm{cf}^{(d)}_{k,j,\varkappa}(I, J) = \mathrm{cf}^{(d)}_{k,m,\varkappa}\big(\mathcal{L}^m(I), \mathcal{L}_m(J)\big)\mathrm{cf}^{(d)}_{m,j,\varkappa}\big(\mathcal{R}^m(I), \mathcal{R}_m(J)\big). \tag{6.7}$$

6.6. *Back to the hyperalgebra*

A monomial of $F^{(d)}_{i,n}$ of the form

$$\mathcal{S}^{(d)}_{m_0,m_1}(I_1, J_0)\mathcal{S}^{(d)}_{m_1,m_2}(I_2, J_1) \cdots \mathcal{S}^{(d)}_{m_k,m_{k+1}}(I_{k+1}, J_k) f/g, \tag{6.8}$$

where $i = m_0 < m_1 < \cdots < m_k < m_{k+1} = n$, $f, g \in \mathcal{U}^0$, $g \neq 0$, $J_0 = \langle (i-1)^d \rangle$ and conditions (M1)–(M3) hold for the sequence of multisets $I_1, \ldots, I_{k+1}, J_1, \ldots, J_k$ and the set $\{m_1, \ldots, m_k\}$ is called *full* (cf. Definition 6.8). A polynomial of $F^{(d)}_{i,n}$ is called *full* if it is a sum of full monomials. It follows from Corollary 6.9 that any $\mathcal{T}$-monomial is an integral full polynomial.

We define the function ev with values in $\bar{\mathcal{U}}^{-,0}$ as follows. Let the value of ev on (6.8) be $S^{(d)}_{i,n,\{m_1,\ldots,m_k\}}(I_1, \ldots, I_{k+1}, J_1, \ldots, J_k) f/g$. We extend ev to an arbitrary full polynomial by linearity.

Lemma 6.13. *Let P be a regular full polynomial and $i \leqslant m < m' < n$. Then we have*

$$(\sigma_{m,m'} \circ \mathrm{ev})\big(P \cdot \big(C(m,m') - u_{m'} + u_m\big)\big) = (\mathrm{ev} \circ \sigma_{m,m'})\big(P \cdot \big(C(m,m') - u_{m'} + u_m\big)\big).$$

Proof. It suffices to consider the case where P equals (6.8) and (F3)–(F5) hold for

$$Q := \mathcal{S}^{(d)}_{m_0,m_1}(I_1, J_0)\mathcal{S}^{(d)}_{m_1,m_2}(I_2, J_1) \cdots \mathcal{S}^{(d)}_{m_k,m_{k+1}}(I_{k+1}, J_k)$$

and g. It was proved in Section 3 that $\mathrm{ev}(Qf) \in \mathcal{U}^{-,0}$. Therefore, if g is not divisible by $C(m,m') - u_{m'} + u_m$, then $\sigma_{m,m'}$ is applicable to both P and $\mathrm{ev}(P)$. Hence both sides of the required equality equal zero by the remark of Section 6.2.

Now suppose that g is divisible by $C(m,m') - u_{m'} + u_m$. Condition (F5) then yields that there exists some $r = 0,\dots,k$ such that $m = m_r$ and $m' \leqslant m_{r+1}$. Now it remains to apply Lemma 6.2. □

Corollary 6.14. *Let P be an integral full polynomial. Then* $\mathrm{ev}(P) \in \mathcal{U}^{-,0}$.

Proof. This result can be proved similarly to Corollary 6.12. Indeed, consider a representation $\mathrm{ev}(P) = A \cdot B^{-1}$, where $A \in \mathcal{U}^{-,0}$, $B \in \mathcal{U}^0$, $B \neq 0$ and B has minimal possible degree. Since applying ev to a full monomial being a product of formal operators gives an element of $\mathcal{U}^{-,0}$, condition (F5) of the definition of a regular polynomial yields that B is a product of polynomials of the form $C(m,m') - u_{m'} + u_m$, where $i \leqslant m < m' < n$, each occurring in degree at most one.

Suppose that B is divisible by some polynomial $C(m,m') - u_{m'} + u_m$, where $i \leqslant m < m' < n$. Let $\bar{B}$ denote their quotient. Lemma 6.13 yields

$$\begin{aligned} 0 &= (\mathrm{ev} \circ \sigma_{m,m'})\big(P \cdot \big(C(m,m') - u_{m'} + u_m\big)\big) \\ &= \sigma_{m,m'}\big(\mathrm{ev}(P) \cdot \big(C(m,m') - u_{m'} + u_m\big)\big) \\ &= \sigma_{m,m'}(A \cdot \bar{B}^{-1}) = \sigma_{m,m'}(A) \cdot \sigma_{m,m'}(\bar{B})^{-1}. \end{aligned}$$

Hence $\sigma_{m,m'}(A) = 0$, whence by Lemma 6.1, we obtain $A = A'(C(m,m') - u_{m'} + u_m)$ for some $A' \in \mathcal{U}^{-,0}$. This fact contradicts the minimality of degree of B. Therefore $B = 1$. □

To formulate the next result, we introduce the function cf on any full polynomial P as follows

$$\mathrm{cf}(P) := \sum \big\{ \mathrm{cf}_\varkappa(P) \;\big|\; \varkappa \text{ a sequence of nonnegative integers of length } n-i \big\}.$$

Since P is full, only finitely many $\mathrm{cf}_\varkappa(P)$ are nonzero.

Lemma 6.15. *Let P be a full polynomial of weight $-a_1\alpha_1 - \dots - a_{n-1}\alpha_{n-1}$, where $a_1,\dots,a_{n-1} \geqslant 0$. Then*

$$a_{n-1}! \cdot E_1^{(a_1)} \cdots E_{n-1}^{(a_{n-1})} \mathrm{ev}(P) \equiv \mathrm{cf}(P) \pmod{\bar{I}^+}.$$

Proof. The result follows directly from Lemma 4.1. □

We put $\rho_l := \rho_l^{(1)} + \rho_l^{(2)} + \rho_l^{(3)}$. The next lemma explains the role of this operator.

Lemma 6.16. *Let P be a full polynomial and $i \leqslant l < n$. Then we have*

$$E_l\, \mathrm{ev}(P) \equiv (\mathrm{ev} \circ \rho_l)(P) \pmod{\bar{I}^+}.$$

Proof. The result follows directly from (3.10)–(3.12) and the definition of $\rho_l^{(1)}$, $\rho_l^{(2)}$, $\rho_l^{(3)}$. □

To prove Lemmas 6.15 and 6.16, it is useful to notice that ev is linear in the following sense: $\mathrm{ev}(P + P') = \mathrm{ev}(P) + \mathrm{ev}(P')$ and $\mathrm{ev}(Px) = \mathrm{ev}(P)x$, where P and P' are full polynomials and $x \in \bar{\mathcal{U}}^0$.

Lemma 6.17. *Let I be a multiset with entries in $[i..n)$, $J_0 = \langle (i-1)^d \rangle$ and $\mathcal{M} \subset (i..n)$. Suppose that $\mathcal{M} \neq \emptyset$ and denote $m := \min \mathcal{M}$. Then we have*

$$\mathrm{ev}\big(\mathcal{T}^{(d)}_{i,n,\mathcal{M}}(I, J_0)\big) = \big(\mathrm{ev}\big(\mathcal{T}^{(d)}_{i,n,\mathcal{M}\setminus\{m\}}(I, J_0)\big) - \sigma_{i,m}\big(\mathrm{ev}\big(\mathcal{T}^{(d)}_{i,n,\mathcal{M}\setminus\{m\}}(I, J_0)\big)\big)\big)\big(C(i,m) - u_m\big)^{-1}.$$

Proof. Applying Lemma 6.13 with $P = \mathcal{T}^{(d)}_{i,n,\mathcal{M}}(I, J_0)$, Definition 6.3 and Lemma 6.4(iii), we obtain

$$(\sigma_{i,m} \circ \mathrm{ev})\big(\mathcal{T}^{(d)}_{i,n,\mathcal{M}\setminus\{m\}}(I, J_0) - \mathcal{S}^{(d)}_{i,m}\big(\mathcal{L}^m(I), J_0\big)\mathcal{T}^{(d)}_{m,n,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I), \emptyset\big)\big) = 0.$$

Thus Lemma 6.2 (the second case) implies

$$\mathrm{ev}\big(\mathcal{S}^{(d)}_{i,m}\big(\mathcal{L}^m(I), J_0\big)\mathcal{T}^{(d)}_{m,n,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I), \emptyset\big)\big) = (\sigma_{i,m} \circ \mathrm{ev})\big(\mathcal{T}^{(d)}_{i,n,\mathcal{M}\setminus\{m\}}(I, J_0)\big).$$

Now it suffices to apply ev to $\mathcal{T}^{(d)}_{i,n,\mathcal{M}}(I, J_0)$, use Definition 6.3 and the remark on linearity preceding this lemma. □

6.7. Polynomials $\mathrm{cf}_\varkappa(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I, J))$

Throughout this section, we fix a sequence $\varkappa = (q_{i+1}, \dots, q_n)$ of nonnegative integers. For any integers k and j such that $i-1 \leqslant k < j \leqslant n$ and a multiset J with integer entries, we put

$$\Delta^\varkappa_{k,j}(J) := \big\{(t,h) \in \mathbf{Z}^2 \;\big|\; k < t \leqslant j \;\&\; |J|^{[t..+\infty)} \leqslant h \leqslant |J|^{[t-1..+\infty)} + q_t\big\},$$

where $q_i = 0$. We obviously have

$$\Sigma^{(d)}_{k,j}\big(\mathcal{L}^j(I)\big) = \Sigma^{(d)}_{k,j}(I), \qquad \Delta^\varkappa_{k,j}\big(\mathcal{L}_j(J)\big) = \Delta^\varkappa_{k,j}(J) \sqcup \big(\{j\}\times\big[0..|J|^{[j..+\infty)}\big)\big),$$
$$\Sigma^{(d)}_{k,j}\big(\mathcal{R}^k(I)\big) = \Sigma^{(d)}_{k,j}(I), \qquad \Delta^\varkappa_{k,j}\big(\mathcal{R}_k(J)\big) = \Delta^\varkappa_{k,j}(J). \tag{6.9}$$

Proposition 6.18. *Let $1 \leqslant m_1 < \dots < m_l \leqslant N$ be integers and $f_1, \dots, f_l$ be polynomials of $\mathbf{Z}[x_1, \dots, x_N]$ having the form $f_s = x_{m_s} - g_s$, where g_s is an integral linear combination of the unit and of the variables x_t for $t > m_s$. Let $\mathcal{I}$ be the ideal of $\mathbf{Z}[x_1, \dots, x_N]$ generated by $f_1, \dots, f_l$. Then*

(i) *$\mathcal{I}$ is a prime ideal*;
(ii) *an integral linear combination of the unit and the variables x_t belongs to $\mathcal{I}$ if and only if it is an integral linear combination of $f_1, \dots, f_l$.*

Proof. Applying nondegenerate linear transformations, we can without loss of generality assume that the polynomials $g_1, \dots, g_l$ do not depend on the variables $x_{m_1}, \dots, x_{m_l}$.

Let φ be the ring endomorphism of $\mathbf{Z}[x_1, \dots, x_N]$ that takes x_{m_s} to g_s for any $s = 1, \dots, l$ and acts on the remaining variables identically. We claim $\mathcal{I} = \ker\varphi$. Indeed $\varphi(f_s) = \varphi(x_{m_s}) - g_s = 0$. Hence $\varphi(\mathcal{I}) = 0$ and $\mathcal{I} \subset \ker\varphi$. Now suppose on the contrary that $\varphi(f) = 0$ for some $f \in \mathbf{Z}[x_1, \dots, x_N]$. Consider an arbitrary monomial v of f represented in the form

$$v = x^{a_1}_{m_1} \cdots x^{a_l}_{m_l} \cdot u,$$

where u is a product of powers of the variables x_t with $t \in \{1, \dots, N\} \setminus \{m_1, \dots, m_l\}$ and an integer. We have $\varphi(v) = g^{a_1}_1 \cdots g^{a_l}_l \cdot u$. Considering the representation

$$v = (f_1 + g_1)^{a_1} \cdots (f_l + g_l)^{a_l} \cdot u$$

and applying to the first l factors the binomial theorem, we get $v \equiv \varphi(v) \bmod \mathcal{I}$. This extends to $f \equiv \varphi(f) = 0 \bmod \mathcal{I}$. Hence $f \in \mathcal{I}$ as required.

(i) Let $fg \in \mathcal{I}$. Then we have $0 = \varphi(fg) = \varphi(f)\varphi(g)$. Hence $\varphi(f) = 0$ or $\varphi(g) = 0$, since $\mathbf{Z}[x_1, \dots, x_N]$ does not contain zero divisors. Therefore $f \in \mathcal{I}$ or $g \in \mathcal{I}$.

(ii) Let $f \in \mathcal{I}$ be an integral linear combination of the unit and the variables x_t. We write it as

$$f = b_1 x_{m_1} + \dots + b_l x_{m_l} - w, \tag{6.10}$$

where w is an integral linear combination of the unit and the variables x_t with $t \in \{1, \dots, N\} \setminus \{m_1, \dots, m_l\}$. We have

$$0 = \varphi(f) = b_1 g_1 + \dots + b_l g_l - w.$$

Hence $w = b_1 g_1 + \dots + b_l g_l$. Substituting this back to (6.10), we obtain

$$f = b_1(x_{m_1} - g_1) + \dots + b_l(x_{m_l} - g_l) = b_1 f_1 + \dots + b_l f_l. \qquad \square$$

Lemma 6.19. *Let k, j, I, J and $\mathcal{M}$ be as in Definition 6.3. Suppose that $\Delta^{\varkappa}_{k,j}(J) \subset \Sigma^{(d)}_{k,j}(I)$ and there is an injection $\iota : \mathcal{M} \to \Sigma^{(d)}_{k,j}(I) \setminus \Delta^{\varkappa}_{k,j}(J)$ such that $\iota_1(t) \geqslant t$ for any $t \in \mathcal{M}$ and $\iota_1(t) = t$ implies $\iota_2(t) < |J|^{[t..+\infty)}$, where $\iota(t) = (\iota_1(t), \iota_2(t))$. Then modulo the ideal $\mathcal{I}$ of $\mathcal{U}^0$ generated by $C(t, \iota_1(t)) + u_t - \iota_2(t)$ for $t \in \mathcal{M}$, we have*

$$\mathrm{cf}_\varkappa\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I, J)\big) \equiv u_k \underline{{}^{|J|^{[k..+\infty)}}} \mathrm{cf}^{(d)}_{k,j,\varkappa}(I, J) \prod \big\{C(k,t) + u_k - h \bigm| (t,h) \in \big(\Sigma^{(d)}_{k,j}(I) \setminus \Delta^{\varkappa}_{k,j}(J)\big) \setminus \mathrm{Im}\,\iota\big\}.$$

Proof. Notice that $\mathcal{I}$ has the form described in Proposition 6.18. To see this, one can put $x_1 := u_{i+1}, \dots, x_{n-1-i} := u_{n-1}$, $x_{n-i} := H_1, \dots, x_{2n-i-1} := H_n$ and take $m_1 := 1, \dots, m_{n-1-i} := n-1-i$ ($l = n-1-i$, $N = 2n-i-1$).

We note that the condition $\Delta^{\varkappa}_{k,j}(J) \subset \Sigma^{(d)}_{k,j}(I)$ is equivalent to $q_t \leqslant d - |I|^{(-\infty..t-1]} - |J|^{[t-1..+\infty)}$ for any $t = k+1, \dots, j$. Thus $\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I, J)$ is an integral element of weight $\leqslant 0$, whence by Corollary 6.12, we get $\mathrm{cf}_\varkappa(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I, J)) \in \mathcal{U}^0$. Therefore, both sides of the equivalence we must prove belong to $\mathcal{U}^0$.

We apply induction on $|\mathcal{M}|$. In the case $\mathcal{M} = \emptyset$, the result follows from (6.4). Now suppose that $\mathcal{M} \neq \emptyset$. We put $m := \min \mathcal{M}$ and $\varepsilon := C(k,m) - u_m + u_k$. We shall use the equality $C(k,t) + u_k = \varepsilon + C(m,t) + u_m$. Consider the restriction $\psi := \iota|_{\mathcal{M}\setminus\{m\}}$. It follows from (6.9) that ψ is an injection from $\mathcal{M} \setminus \{m\}$ to $\Sigma^{(d)}_{m,j}(\mathcal{R}^m(I)) \setminus \Delta^{\varkappa}_{m,j}(\mathcal{R}_m(J))$.

Applying the inductive hypothesis and (6.7), we obtain modulo $\mathcal{I}$ the equivalences

$$\begin{aligned}
\varepsilon \cdot \mathrm{cf}_\varkappa\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I, J)\big) &= \mathrm{cf}_\varkappa\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I, J)\big) \\
&\quad - \mathrm{cf}_\varkappa\big(\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I), \mathcal{L}_m(J)\big)\big)\mathrm{cf}_\varkappa\big(\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I), \mathcal{R}_m(J)\big)\big) \\
&\equiv u_k \underline{{}^{|J|^{[k..+\infty)}}} \mathrm{cf}^{(d)}_{k,j,\varkappa}(I, J) \prod \big\{C(k,t) + u_k - h \bigm| (t,h) \in \big(\Sigma^{(d)}_{k,j}(I) \setminus \Delta^{\varkappa}_{k,j}(J)\big) \setminus \mathrm{Im}\,\psi\big\} \\
&\quad - u_k \underline{{}^{|\mathcal{L}_m(J)|^{[k..+\infty)}}} \mathrm{cf}^{(d)}_{k,m,\varkappa}\big(\mathcal{L}^m(I), \mathcal{L}_m(J)\big) \\
&\quad \times \prod \big\{C(k,t) + u_k - h \bigm| (t,h) \in \Sigma^{(d)}_{k,m}\big(\mathcal{L}^m(I)\big) \setminus \Delta^{\varkappa}_{k,m}\big(\mathcal{L}_m(J)\big)\big\} \\
&\quad \times u_m \underline{{}^{|\mathcal{R}_m(J)|^{[m..+\infty)}}} \mathrm{cf}^{(d)}_{m,j,\varkappa}\big(\mathcal{R}^m(I), \mathcal{R}_m(J)\big) \\
&\quad \times \prod \big\{C(m,t) + u_m - h \bigm| (t,h) \in \big(\Sigma^{(d)}_{m,j}\big(\mathcal{R}^m(I)\big) \setminus \Delta^{\varkappa}_{m,j}\big(\mathcal{R}_m(J)\big)\big) \setminus \mathrm{Im}\,\psi\big\}
\end{aligned}$$

$$\begin{aligned}&= u_k^{\underline{|J|^{[k..+\infty)}}}\mathrm{cf}^{(d)}_{k,j,\varkappa}(I,J)\\&\quad\times\prod\bigl\{C(k,t)+u_k-h\bigm|(t,h)\in\Sigma^{(d)}_{k,m}\bigl(\mathcal{L}^m(I)\bigr)\setminus\Delta^{\varkappa}_{k,m}\bigl(\mathcal{L}_m(J)\bigr)\bigr\}\\&\quad\times\Bigl[(\varepsilon+u_m)^{\underline{|J|^{[m..+\infty)}}}\\&\quad\times\prod\bigl\{\varepsilon+C(m,t)+u_m-h\bigm|(t,h)\in\bigl(\Sigma^{(d)}_{m,j}(I)\setminus\Delta^{\varkappa}_{m,j}(J)\bigr)\setminus\operatorname{Im}\psi\bigr\}\\&\quad-u_m^{\underline{|J|^{[m..+\infty)}}}\prod\bigl\{C(m,t)+u_m-h\bigm|(t,h)\in\bigl(\Sigma^{(d)}_{m,j}(I)\setminus\Delta^{\varkappa}_{m,j}(J)\bigr)\setminus\operatorname{Im}\psi\bigr\}\Bigr].\end{aligned}$$

Let X denote the polynomial in the square brackets.

Case 1: $\iota_1(m)>m$. We have $\iota(m)\in(\Sigma^{(d)}_{m,j}(I)\setminus\Delta^{\varkappa}_{m,j}(J))\setminus\operatorname{Im}\psi$, whence

$$\begin{aligned}X&=\varepsilon(\varepsilon+u_m)^{\underline{|J|^{[m..+\infty)}}}\prod\bigl\{\varepsilon+C(m,t)+u_m-h\bigm|(t,h)\in\bigl(\Sigma^{(d)}_{m,j}(I)\setminus\Delta^{\varkappa}_{m,j}(J)\bigr)\setminus\operatorname{Im}\iota\bigr\}\\&\quad+\bigl(C\bigl(m,\iota_1(m)\bigr)+u_m-\iota_2(m)\bigr)Y.\end{aligned}$$

In this sum, the last summand belongs to $\mathcal{I}$ and

$$\begin{aligned}Y&=(\varepsilon+u_m)^{\underline{|J|^{[m..+\infty)}}}\prod\bigl\{\varepsilon+C(m,t)+u_m-h\bigm|(t,h)\in\bigl(\Sigma^{(d)}_{m,j}(I)\setminus\Delta^{\varkappa}_{m,j}(J)\bigr)\setminus\operatorname{Im}\iota\bigr\}\\&\quad-u_m^{\underline{|J|^{[m..+\infty)}}}\prod\bigl\{C(m,t)+u_m-h\bigm|(t,h)\in\bigl(\Sigma^{(d)}_{m,j}(I)\setminus\Delta^{\varkappa}_{m,j}(J)\bigr)\setminus\operatorname{Im}\iota\bigr\}.\end{aligned}$$

Substituting X written in this form back, we get

$$\begin{aligned}&\varepsilon\cdot\Bigl(\mathrm{cf}_{\varkappa}\bigl(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\bigr)-u_k^{\underline{|J|^{[k..+\infty)}}}\mathrm{cf}^{(d)}_{k,j,\varkappa}(I,J)\\&\quad\times\prod\bigl\{C(k,t)+u_k-h\bigm|(t,h)\in\bigl(\Sigma^{(d)}_{k,j}(I)\setminus\Delta^{\varkappa}_{k,j}(J)\bigr)\setminus\operatorname{Im}\iota\bigr\}\Bigr)\in\mathcal{I}.\end{aligned}\tag{6.11}$$

Since ε depends on H_k and no polynomial $C(t,\iota_1(t))+u_t-\iota_2(t)$, where $t\in\mathcal{M}$, does, we obtain by Proposition 6.18(ii) that $\varepsilon\notin\mathcal{I}$. Applying Proposition 6.18(i), we obtain the required equivalence.

Case 2: $\iota_1(m)=m$. By hypothesis, $\iota_2(m)<|J|^{[m..+\infty)}$. Therefore

$$\begin{aligned}X&=\varepsilon\prod\bigl\{\varepsilon+u_m-h\bigm|h\in\bigl[0..|J|^{[m..+\infty)}\bigr)\setminus\bigl\{\iota_2(m)\bigr\}\bigr\}\\&\quad\times\prod\bigl\{\varepsilon+C(m,t)+u_m-h\bigm|(t,h)\in\bigl(\Sigma^{(d)}_{m,j}(I)\setminus\Delta^{\varkappa}_{m,j}(J)\bigr)\setminus\operatorname{Im}\psi\bigr\}\\&\quad+\bigl(u_m-\iota_2(m)\bigr)Y.\end{aligned}$$

In this sum, the last summand belongs to $\mathcal{I}$ and

$$\begin{aligned}Y&=\prod\bigl\{\varepsilon+u_m-h\bigm|h\in\bigl[0..|J|^{[m..+\infty)}\bigr)\setminus\bigl\{\iota_2(m)\bigr\}\bigr\}\\&\quad\times\prod\bigl\{\varepsilon+C(m,t)+u_m-h\bigm|(t,h)\in\bigl(\Sigma^{(d)}_{m,j}(I)\setminus\Delta^{\varkappa}_{m,j}(J)\bigr)\setminus\operatorname{Im}\psi\bigr\}\\&\quad-\prod\bigl\{u_m-h\bigm|h\in\bigl[0..|J|^{[m..+\infty)}\bigr)\setminus\bigl\{\iota_2(m)\bigr\}\bigr\}\\&\quad\times\prod\bigl\{C(m,t)+u_m-h\bigm|(t,h)\in\bigl(\Sigma^{(d)}_{m,j}(I)\setminus\Delta^{\varkappa}_{m,j}(J)\bigr)\setminus\operatorname{Im}\psi\bigr\}.\end{aligned}$$

Hence (6.11) follows similarly to case 1 and the argument of that case completes the proof. □

Lemma 6.20. *Let k, j, I, J and $\mathcal{M}$ be as in Definition 6.3. Suppose additionally that $\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)$ has weight $\leqslant 0$. Then $\mathrm{cf}_\varkappa(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J))$ is divisible by $\mathrm{cf}^{(d)}_{k,j,\varkappa}(I,J)$ in $\mathcal{U}^0$.*

Proof. Throughout this proof, we should keep in mind Corollary 6.12.

We apply induction on $|\mathcal{M}|$. Consider the case $\mathcal{M}=\emptyset$. If $q_t \leqslant d-|I|^{(-\infty..t-1]}-|J|^{[t-1..+\infty)}$ for any $t=k+1,\dots,j$, then the denominator divides the numerator in each fraction of (6.4), where $m=k$ and $m'=j$, and the result follows. On the other hand, if this condition is violated, then $\mathrm{cf}_\varkappa(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)) = \mathrm{cf}^{(d)}_{k,j,\varkappa}(I,J)=0$.

Now suppose that $\mathcal{M}\neq\emptyset$ and let $m:=\min\mathcal{M}$. We have

$$\begin{aligned}&\mathrm{cf}_\varkappa\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)\big)\big(C(k,m)-u_m+u_k\big)\\&\quad=\mathrm{cf}_\varkappa\big(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I,J)\big)-\mathrm{cf}_\varkappa\big(\mathcal{S}^{(d)}_{k,m}\big(\mathcal{L}^m(I),\mathcal{L}_m(J)\big)\big)\mathrm{cf}_\varkappa\big(\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}\big(\mathcal{R}^m(I),\mathcal{R}_m(J)\big)\big).\end{aligned}\tag{6.12}$$

By the inductive hypothesis, $\mathrm{cf}_\varkappa(\mathcal{S}^{(d)}_{k,m}(\mathcal{L}^m(I),\mathcal{L}_m(J)))$ is divisible by $\mathrm{cf}^{(d)}_{k,m,\varkappa}(\mathcal{L}^m(I),\mathcal{L}_m(J))$ and $\mathrm{cf}_\varkappa(\mathcal{T}^{(d)}_{m,j,\mathcal{M}\setminus\{m\}}(\mathcal{R}^m(I),\mathcal{R}_m(J)))$ is divisible by $\mathrm{cf}^{(d)}_{m,j,\varkappa}(\mathcal{R}^m(I),\mathcal{R}_m(J))$. Hence their product is divisible by $\mathrm{cf}^{(d)}_{k,m,\varkappa}(\mathcal{L}^m(I),\mathcal{L}_m(J))\mathrm{cf}^{(d)}_{m,j,\varkappa}(\mathcal{R}^m(I),\mathcal{R}_m(J))$, which by (6.7) equals $\mathrm{cf}^{(d)}_{k,j,\varkappa}(I,J)$. By the inductive hypothesis the last polynomial also divides $\mathrm{cf}_\varkappa(\mathcal{T}^{(d)}_{k,j,\mathcal{M}\setminus\{m\}}(I,J))$. Therefore the left-hand side of (6.12) is divisible by $\mathrm{cf}^{(d)}_{k,j,\varkappa}(I,J)$. Since $C(k,m)-u_m+u_k$ is not a prime factor of $\mathrm{cf}^{(d)}_{k,j,\varkappa}(I,J)$, we obtain that $\mathrm{cf}_\varkappa(\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J))$ is divisible by $\mathrm{cf}^{(d)}_{k,j,\varkappa}(I,J)$ as required. $\square$

Remark. Suppose that $\mathcal{T}^{(d)}_{k,j,\mathcal{M}}(I,J)$ has weight $\leqslant 0$. Then $\mathrm{cf}^{(d)}_{k,j,\varkappa}(I,J)=0$ unless $q_t\leqslant d-|I|^{(-\infty..t-1]}-|J|^{[t-1..+\infty)}$ and $|J|^{\{t-2\}}-|I|^{\{t-1\}}\leqslant q_t\leqslant|J|^{\{t-2\}}$ for any $t=k+1,\dots,j$. In particular, $\mathrm{cf}^{(d)}_{k,j,\varkappa}(I,J)=0$ unless $\Delta^\varkappa_{k,j}(J)\subset\Sigma^{(d)}_{k,j}(I)$, $|I|\leqslant d$ and $|J|\leqslant d$.

7. Proof of the main result

7.1. Operators $\mathfrak{T}^{(d)}_{i,n,M}(I)$

In this section, we fix an algebraically closed field K of characteristic $p>0$ and denote by $\bar{m}$ the sum $1_K+\cdots+1_K$ with m summands. Using the notation introduced in Section 2.5, we have $m^\tau=\bar{m}$ for any homomorphism $\tau:\mathbf{Z}[u_{i+1},\dots,u_{n-1}]\to K$.

We suppose that $d<p$. Any rational $\mathrm{GL}_n(K)$-module V can be considered as a $U_K(n)$-module (see [J, I.7.11]), since $U_K(n)$ is naturally isomorphic to the algebra of distributions $\mathrm{Dist}(\mathrm{GL}_n(K))$ (see [J, II.1.12]).

In particular, in the sense of Section 1, a vector $v\in V$ has weight $\lambda\in\mathbf{Z}^n$ iff $\binom{H_s}{r}v=\binom{\lambda_s}{r}v$ for any $s=1,\dots,n$ and $r\in\mathbf{Z}$; a vector $v\in V$ is a $\mathrm{GL}_n(K)$-high weight vector ($\mathrm{GL}_{n-1}(K)$-high weight vector) iff $\mathcal{E}^{(r)}_s v=0$ for any $r>0$ and $s=1,\dots,n-1$ (respectively $s=1,\dots,n-2$).

Throughout this section we use the notation $J_0:=\langle(i-1)^d\rangle$.

For a subset $M\subset(i..n)\times\mathbf{Z}$ having at most one point in each column and a multiset I of length no greater than d with entries in $[i..n)$, we define the following element of the hyperalgebra $U_K(n)$:

$$\mathfrak{T}^{(d)}_{i,n,M}(I):=\mathrm{ev}\big(\mathcal{T}^{(d)}_{i,n,\pi_1(M)}(I,J_0)\big)^\tau,$$

where τ is any ring homomorphism from $\mathbf{Z}[u_{i+1},\dots,u_{n-1}]$ to K such that

$$\tau(u_t)=\bar{h}\quad\text{if }(t,h)\in M.\tag{7.1}$$

Since $\mathrm{ev}(\mathcal{T}^{(d)}_{i,n,\pi_1(M)}(I, J_0))$ depends only on the variables u_t with $t \in \pi_1(M)$, the element $\mathfrak{T}^{(d)}_{i,n,M}(I)$ does not depend on the choice of particular τ satisfying (7.1).

7.2. Subspace $\mathcal{V}_\lambda$

For any $\lambda \in X^+(n)$, we consider the subspace $\mathcal{V}_\lambda$ of $L_n(\lambda)$ spanned by all vectors $\mathrm{ev}(u)^\tau v^+_\lambda$, where u is a $\mathcal{T}$-monomial of weight $\leqslant 0$, which we write as (6.3), such that for some subset M of $\Omega^{(d)}_{m_k-1,n}(I_{k+1})$ having at most one point in each column there hold

(a) $\pi_1(M) = \{m_k\} \cup \mathcal{M}_{k+1}$;
(b) $\mathrm{dist}_\lambda(x, y) \equiv 0 \pmod p$ for any $x, y \in M$;
(c) $\tau(u_t) = \bar{h}$ if $(t, h) \in M$;
(d) there exists a strictly increasing injection $\varphi : M \to \Sigma^{(d)}_{m_k,n}(I_{k+1})$ such that $\mathrm{dist}_\lambda(x, \varphi(x)) \equiv 0 \pmod p$ for any $x \in M$.

We claim the following properties of the vector $\mathrm{ev}(u)^\tau v^+_\lambda$ just mentioned:

(i) $\mathcal{E}^{(a_1)}_1 \cdots \mathcal{E}^{(a_{n-1})}_{n-1}\mathrm{ev}(u)^\tau v^+_\lambda = 0$, where u (and, therefore, $\mathrm{ev}(u)^\tau$) has weight $-a_1\alpha_1 - \cdots - a_{n-1}\alpha_{n-1}$;
(ii) $\mathcal{E}^{(r)}_l \mathrm{ev}(u)^\tau v^+_\lambda \in \mathcal{V}_\lambda$ for any $l = 1, \ldots, n-2$ and $r > 0$.

To prove (i), we note that by Corollaries 6.9 and 6.14, we have $\mathrm{ev}(u) \in \mathcal{U}^{-,0}$. Corollary 6.12 implies that $\mathrm{cf}(u) \in \mathcal{U}^0$. Hence it follows from Lemmas 6.15 and 2.3 that $a_{n-1}! \cdot E^{(a_1)}_1 \cdots E^{(a_{n-1})}_{n-1}\mathrm{ev}(u) \equiv \mathrm{cf}(u) \pmod{I^+}$. Therefore, since $(I^+)^\tau v^+_\lambda = 0$, we obtain

$$\overline{a_{n-1}!} \cdot \mathcal{E}^{(a_1)}_1 \cdots \mathcal{E}^{(a_{n-1})}_{n-1}\mathrm{ev}(u)^\tau v^+_\lambda = \big(a_{n-1}! \cdot E^{(a_1)}_1 \cdots E^{(a_{n-1})}_{n-1}\mathrm{ev}(u)\big)^\tau v^+_\lambda = \mathrm{cf}(u)^\tau v^+_\lambda. \tag{7.2}$$

We are going to prove that $\mathrm{cf}(u)^\tau v^+_\lambda = 0$, from which property (i) will follow, since $a_{n-1} \leqslant d < p$. Choose any sequence $\varkappa = (q_{i+1}, \ldots, q_n)$ of nonnegative integers. Each factor in product (6.3) is an integral polynomial of weight $\leqslant 0$. Thus if we apply $\mathrm{cf}_\varkappa$ to any of these factors, then by Corollary 6.12 we shall get an element of $\mathcal{U}^0$. Hence it remains to prove that

$$\mathrm{cf}_\varkappa\big(\mathcal{T}^{(d)}_{m_k,n,\mathcal{M}_{k+1}}(I_{k+1}, J_k)\big)^\tau v^+_\lambda = 0 \tag{7.3}$$

(recall that $\mathrm{cf}_\varkappa$ is a ring homomorphism).

By Lemma 6.20 and the remark following it, (7.3) is satisfied unless

$$\begin{aligned} q_t &\leqslant d - |I_{k+1}|^{(-\infty..t-1]} - |J_k|^{[t-1..+\infty)}, \\ |J_k|^{\{t-2\}} - |I_{k+1}|^{\{t-1\}} &\leqslant q_t \leqslant |J_k|^{\{t-2\}} \end{aligned} \tag{7.4}$$

for any $t = m_k + 1, \ldots, n$. Now assume that inequalities (7.4) hold. Therefore, $\Delta^\varkappa_{m_k,n}(J_k) \subset \Sigma^{(d)}_{m_k,n}(I_{k+1})$, $|I_{k+1}| \leqslant d$ and $|J_k| \leqslant d$.

We shall use, the theory developed in Section 5.4, for $a = m_k$, $b = n$ and $S = \Delta^\varkappa_{m_k-1,n}(J_k)$. The conditions imposed on S in Section 5.4 follow directly from (7.4). Let $X := M \cup \operatorname{Im}\varphi$. Note that any point of X not belonging to column m_k belongs to $\Sigma^{(d)}_{m_k,n}(I_{k+1})$ and that $\mathrm{dist}_\lambda(x, y) \equiv 0 \pmod p$ for any $x, y \in X$. The last assertion follows from properties (b) and (d).

If $X \cap S$ contains two points comparable with respect to $\dot{<}$, then by Lemma 5.6 we have $(\lambda_t - \lambda_{t+1})\underline{q_{t+1}} \equiv 0 \pmod p$ for some $t = m_k, \ldots, n-1$. Therefore $(\mathrm{cf}^{(d)}_{m_k,n,\varkappa}(I_{k+1}, J_k))^\tau v^+_\lambda = 0$, whence (7.3) follows by Lemma 6.20.

Now suppose that $X \cap S$ does not contain points comparable with respect to $\dot{<}$. In this case, we can apply Lemma 5.5 and obtain an injection $\varphi_S : M \to X \setminus S$ satisfying conditions described in that lemma. One can easily observe the following property of this injection:

(d′) $\operatorname{dist}_\lambda(x, \varphi_S(x)) \equiv 0 \pmod p$ for any $x \in M$.

We put $\iota(t) := \varphi_S((t,h))$ for any $t \in \mathcal{M}_{k+1}$, where $(t,h) \in M$, and apply Lemma 6.19 for this injection. Applying property (c), we get

$$\bigl(C\bigl(t, \iota_1(t)\bigr) + u_t - \iota_2(t)\bigr)^\tau v_\lambda^+ = \overline{\operatorname{dist}_\lambda\bigl((t,h), \varphi_S((t,h))\bigr)} v_\lambda^+ = 0$$

by property (d′). Here and in what follows $\iota(t) = (\iota_1(t), \iota_2(t))$. This implies $\mathcal{I}^\tau v_\lambda^+ = 0$, where $\mathcal{I}$ is the ideal of $\mathcal{U}^0$ generated by all $C(t, \iota_1(t)) + u_t - \iota_2(t)$ for $t \in \mathcal{M}_{k+1}$. Thus Lemma 6.19 implies

$$\begin{aligned}\operatorname{cf}_\varkappa\bigl(\mathcal{T}^{(d)}_{m_k,n,\mathcal{M}_{k+1}}(I_{k+1}, J_k)\bigr)^\tau v_\lambda^+ = \bar{\mathfrak{h}}^{\underline{|J_k|^{[m_k..+\infty)}}} \operatorname{cf}^{(d)}_{m_k,n,\varkappa}(I_{k+1}, J_k)^\tau \prod\bigl\{\bar{t} - \bar{m}_k + \bar{\lambda}_{m_k} - \bar{\lambda}_t + \bar{\mathfrak{h}} - \bar{h} \bigm| \\ (t,h) \in \bigl(\Sigma^{(d)}_{m_k,n}(I_{k+1}) \setminus \Delta^\varkappa_{m_k,n}(J_k)\bigr) \setminus \operatorname{Im}\iota\bigr\} v_\lambda^+,\end{aligned} \tag{7.5}$$

where $(m_k, \mathfrak{h}) \in M$. Since $0 \leqslant \mathfrak{h} < p$, property (c) implies $\mathfrak{h} = 0$ if $m_k = i$.

We put $(q_1, q_2) := \varphi_S((m_k, \mathfrak{h}))$. First consider the case $(m_k, \mathfrak{h}) \dot{<} (q_1, q_2)$. In that case (q_1, q_2) is a point of X not belonging to column m_k, which implies $(q_1, q_2) \in \Sigma^{(d)}_{m_k,n}(I_{k+1})$. On the other hand, $(q_1, q_2) \notin S$, whence $(q_1, q_2) \notin \Delta^\varkappa_{m_k,n}(J_k)$. Since φ_S is an injection, we have $(q_1, q_2) \notin \operatorname{Im}\iota = \varphi_S(M \setminus \{(m_k, \mathfrak{h})\})$. We have proved that $(q_1, q_2) \in (\Sigma^{(d)}_{m_k,n}(I_{k+1}) \setminus \Delta^\varkappa_{m_k,n}(J_k)) \setminus \operatorname{Im}\iota$. To prove (7.3) it remains to notice that

$$\bar{q}_1 - \bar{m}_k + \bar{\lambda}_{m_k} - \bar{\lambda}_{q_1} + \bar{\mathfrak{h}} - \bar{q}_2 = \overline{\operatorname{dist}_\lambda\bigl((m_k, \mathfrak{h}), (q_1, q_2)\bigr)} = 0$$

by property (d′) and substitute this to (7.5).

By Lemma 5.5, it remains to consider the case where $(m_k, \mathfrak{h}) = (q_1, q_2)$ and this point lies below S, that is $\mathfrak{h} < |J_k|^{[m_k..+\infty)}$. Hence $\mathfrak{h}^{\underline{|J_k|^{[m_k..+\infty)}}} = 0$. Substituting this back to (7.5), we get (7.3).

Now let us prove property (ii). We can obviously restrict ourselves to the case $i \leqslant l$. Since $a_l \leqslant d < p$ and $\mathcal{E}_l^{(r)} \operatorname{ev}(u)^\tau v_\lambda^+ = 0$ if $r > a_l$, it suffices to consider the case $r = 1$. By Corollaries 6.14 and 6.9, we get $\operatorname{ev}(u), (\operatorname{ev} \circ \rho_l)(u) \in \mathcal{U}^{-,0}$. Therefore, it follows from Lemmas 6.16 and 2.3 that $E_l \operatorname{ev}(u) \equiv (\operatorname{ev} \circ \rho_l)(u) \pmod{I^+}$. Since $(I^+)^\tau v_\lambda^+ = 0$, we obtain

$$\mathcal{E}_l \operatorname{ev}(u)^\tau v_\lambda^+ = \bigl(E_l \operatorname{ev}(u)\bigr)^\tau v_\lambda^+ = (\operatorname{ev} \circ \rho_l)(u)^\tau v_\lambda^+. \tag{7.6}$$

Now let us take $c = 1, 2, 3$ and use Lemma 6.5, Lemma 6.6 or Lemma 6.7 respectively to represent $\rho_l^{(c)}(u)$ as a linear combination $\mathcal{T}$-monomials. Let w be any of the $\mathcal{T}$-monomials occurring in this linear combination. We must prove that $\operatorname{ev}(w)^\tau v_\lambda^+ \in \mathcal{V}_\lambda$. For this, we clearly can assume that w has weight $\leqslant 0$, since otherwise $\operatorname{ev}(w) = 0$.

If $l < m_k - 1$ then the tail of w is the same as tail of u (see the remark of Section 6.3) and thus satisfies conditions (a)–(d). Thus we shall consider the case $m_k - 1 \leqslant l$ and see for what sets and injections the tail of w satisfies conditions (a)–(d).

Case I: $c = 1$. By Lemma 6.5, we obtain that the tail of w is either $\mathcal{T}^{(d)}_{m_k,n,\mathcal{M}_{k+1}}(I_{k+1}, (J_k)_{l-1\mapsto l})$ if $m_k \leqslant l$ or $\mathcal{T}^{(d)}_{m_k,n,\mathcal{M}_{k+1}}(I_{k+1}, J_k \cup \langle m_k - 1\rangle)$ if $l = m_k - 1$. In both cases, conditions (a)–(d) are satisfied for the same set and injection.

Case II: $c = 2$. By Lemma 6.6, we obtain that the tail of w is either $\mathcal{T}^{(d)}_{l+1,n,\mathcal{M}_{k+1}\cap(l+1..n)}(\mathcal{R}^{l+1}(I_{k+1}), \mathcal{R}_{l+1}(J_k) \cup \langle l\rangle)$ if $l + 1 \in \mathcal{M}_{k+1}$ or $\mathcal{T}^{(d)}_{m_k,n,\mathcal{M}_{k+1}}(I_{k+1}, J_k \cup \langle m_k - 1\rangle)$ if $l = m_k - 1$.

In the first case, conditions (a)–(d) are satisfied for the set $M \cap ([l+1..n) \times \mathbf{Z})$ and the injection φ' obtained by restricting φ to this set. Note that $M \cap ([l+1..n) \times \mathbf{Z})$ is a subset of $\Omega^{(d)}_{l,n}(I_{k+1}) = \Omega^{(d)}_{l,n}(\mathcal{R}^{l+1}(I_{k+1}))$ and $\operatorname{Im}\varphi'$ is a subset of $\Sigma^{(d)}_{l+1,n}(I_{k+1}) = \Sigma^{(d)}_{l+1,n}(\mathcal{R}^{l+1}(I_{k+1}))$ as required.

In the second case, conditions (a)–(d) are satisfied for the same set and injection.

Case III: $c = 3$ *and* $\operatorname{Im}\varphi \subset \Sigma^{(d)}_{m_k,n}((I_{k+1})_{l+1\mapsto l})$. Since $\operatorname{Im}\varphi$ is a subset of $\Sigma^{(d)}_{m_k-1,n}((I_{k+1})_{l+1\mapsto l})$ and φ is strictly increasing, M is a subset of the interior (see Section 5) of $\Sigma^{(d)}_{m_k-1,n}((I_{k+1})_{l+1\mapsto l})$. That is $M \subset \Omega^{(d)}_{m_k-1,n}((I_{k+1})_{l+1\mapsto l})$.

Applying Lemma 6.7 for $m' = m_k$, we obtain that the tail of w is either $\mathcal{T}^{(d)}_{m_k,n,\mathcal{M}_{k+1}}((I_{k+1})_{l+1\mapsto l}, J_k)$ or $\mathcal{T}^{(d)}_{q,n,\mathcal{M}_{k+1}\cap(q..n)}(\mathcal{R}^q(I_{k+1})_{l+1\mapsto l}, \mathcal{R}_q(J_k))$, where $q \in (m_k..l+1] \cap \mathcal{M}_{k+1}$.

In the first case, conditions (a)–(d) are satisfied for the same set and injection. In the second case, conditions (a)–(d) are satisfied for the set $M \cap ([q..n) \times \mathbf{Z})$ and the injection φ' obtained by restricting φ to this set. Note that $M \cap ([q..n) \times \mathbf{Z})$ is a subset of $\Omega^{(d)}_{q-1,n}((I_{k+1})_{l+1\mapsto l}) = \Omega^{(d)}_{q-1,n}(\mathcal{R}^q(I_{k+1})_{l+1\mapsto l})$ and $\operatorname{Im}\varphi'$ is a subset of $\Sigma^{(d)}_{q,n}((I_{k+1})_{l+1\mapsto l}) = \Sigma^{(d)}_{q,n}(\mathcal{R}^q(I_{k+1})_{l+1\mapsto l})$ as required.

Case IV: $c = 3$ *and* $\operatorname{Im}\varphi \not\subset \Sigma^{(d)}_{m_k,n}((I_{k+1})_{l+1\mapsto l})$. Since $\Sigma^{(d)}_{m_k,n}((I_{k+1})_{l+1\mapsto l}) = \Sigma^{(d)}_{m_k,n}(I_{k+1}) \setminus \{(l+1, d - |I_{k+1}|^{(-\infty..l]})\}$, we have $(l+1, d - |I_{k+1}|^{(-\infty..l]}) \in \operatorname{Im}\varphi$. Therefore $(l+1, d - |I_{k+1}|^{(-\infty..l]}) = \varphi((m', h'))$ for some point $(m', h') \in M$. Consider the set $\hat{M} := M \setminus \{(m', h')\}$ and the injection $\hat{\varphi} := \varphi|_{\hat{M}}$. We have $\operatorname{Im}\hat{\varphi} \subset \Sigma^{(d)}_{m_k,n}((I_{k+1})_{l+1\mapsto l})$ and hence $\hat{M} \subset \Omega_{m_k-1,n}((I_{k+1})_{l+1\mapsto l})$, since $\hat{\varphi}$ is strictly increasing.

Applying property (c), we get

$$\left(C(m', l+1) + u_{m'} - d + |I|^{(-\infty..l]}\right)^{\tau} = \overline{\operatorname{dist}_{\lambda}\left((m', h'), \varphi((m', h'))\right)} = 0$$

by property (d). Therefore, applying Lemma 6.7 (with origin at m'), we can assume that the tail of w is either $\mathcal{T}^{(d)}_{m_k,n,\mathcal{M}_{k+1}\setminus\{m'\}}((I_{k+1})_{l+1\mapsto l}, J_k)$ if $m_k < m'$ or $\mathcal{T}^{(d)}_{q,n,\mathcal{M}_{k+1}\cap(q..n)}(\mathcal{R}^q(I_{k+1})_{l+1\mapsto l}, \mathcal{R}_q(J_k))$, where $q \in (m'..l+1] \cap \mathcal{M}_{k+1}$.

In the first case, conditions (a)–(d) are satisfied for the set $\hat{M}$ and the injection $\hat{\varphi}$. In the second case, conditions (a)–(d) are satisfied for the set $\hat{M} \cap ([q..n) \times \mathbf{Z})$ and the injection obtained by restricting $\hat{\varphi}$ to this set. One can see this similarly to case III considering $\hat{M}$ instead of M and $\hat{\varphi}$ instead of φ. Note that $\hat{M} \cap ([q..n) \times \mathbf{Z}) = M \cap ([q..n) \times \mathbf{Z})$, whence condition (a) follows.

After considering all possible cases, we have prove that $\operatorname{ev}(w)^{\tau} v_{\lambda}^{+} \in \mathcal{V}_{\lambda}$. This implies property (ii). Also notice that $\mathcal{V}_{\lambda}$ is homogeneous, that is the sum of its weight spaces.

Lemma 7.1. $\mathcal{V}_{\lambda} = 0$.

Proof. Suppose that $\mathcal{V}_{\lambda}$ is nonzero. Let v be a nonzero element of $\mathcal{V}_{\lambda}$ of maximal weight. We can obviously assume that $v = \operatorname{ev}(u)^{\tau} v_{\lambda}^{+}$ for some $\mathcal{T}$-monomial u satisfying conditions (a)–(d).

Property (ii) implies then that v is a $GL_{n-1}(K)$-high weight vector. The irreducible module $L_n(\lambda)$ can be realized as the socle of the module $\nabla_n(\lambda)$ contravariantly dual to the Weyl module with highest weight λ. By [BKS, Corollary 3.3], we have $v = c \cdot f_{\mu,\lambda}$ for some $\mu \in X^{+}(n-1)$ and $c \in K$. Applying [BKS, Lemma 2.6(ii)] and property (i), we get

$$cf_{\lambda} = \mathcal{E}_1^{(a_1)} \cdots \mathcal{E}_{n-1}^{(a_{n-1})} \operatorname{ev}(u)^{\tau} v_{\lambda}^{+} = 0,$$

where u has weight $-a_1\alpha_1 - \cdots - a_{n-1}\alpha_{n-1}$. Hence $c = 0$ and $v = 0$ contrary to assumption. □

Remark. In this proof, we used the vectors $f_{\mu,\lambda}$ and f_{λ} introduced in [BKS, Section 2], where only the polynomial case (that is $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$) was considered. However, one can pass to the general case, multiplying $\nabla_n(\lambda)$ by a power of the determinant representation.

7.3. "If-part" of Theorem 1.1

We are going to apply Lemma 5.4 to the parameters $a := i+1$, $b := n$, $c := 0$, $d := d$ and $X := \mathcal{X}_d^\lambda(i,n)$. Then under the notation of Lemma 5.4, we have $Y := \mathfrak{Y}_d^\lambda(i,n)$ and $\mathfrak{C}^\lambda(i,n) \times \{0\} \subset X \cap R_0$. Therefore, condition (ii) of Lemma 5.4 is satisfied, whence there exists a strictly decreasing injection $\psi : Y \to X$. We put $M := \operatorname{Im}\psi$ and denote by φ the injection from M to Y inverting ψ.

We claim that $\mathfrak{T}_{i,n,M}^{(d)}(\emptyset)v_\lambda^+$ is a nonzero $\mathrm{GL}_{n-1}(K)$-high weight vector of $L_n(\lambda)$. Consider the $\mathcal{T}$-monomial $u = \mathcal{T}_{i,n,\pi_1(M)}^{(d)}(\emptyset, J_0)$ and any homomorphism τ satisfying (7.1).

Take any $l = i, \dots, n-2$. Clearly, (7.6) holds in the present case. Lemmas 6.5–6.7 imply that either $\rho_l(u) = 0$ or

$$\rho_l(u) = \rho_l^{(2)}(u) = -\mathcal{T}_{i,l+1,\pi_1(M)\cap(i..l+1)}^{(d)}\big(\langle l\rangle, J_0\big)\mathcal{T}_{l+1,n,\pi_1(M)\cap(l+1..n)}^{(d)}\big(\emptyset, \langle l\rangle\big),$$

where $l+1 \in \pi_1(M)$. Note that $\rho_l(u)$ has weight $\leqslant 0$, since $d > 0$. The tail of $\rho_l(u)$ satisfies conditions (a)–(d) for the set $M \cap ([l+1..n) \times \mathbf{Z})$ and the injection φ' obtained by restricting φ to this set. Note that $M \subset (i..n) \times [0..d)$, whence $M \cap ([l+1..n) \times \mathbf{Z}) \subset (l..n) \times [0..d) = \Omega_{l,n}^{(d)}(\emptyset)$. Moreover, $\operatorname{Im}\varphi' \subset (l+1..n] \times (0..d] \subset \Sigma_{l+1,n}^{(d)}(\emptyset)$ and $\mathrm{dist}_\lambda(x,y) \equiv 0 \pmod p$ for any $x, y \in X$. Thus applying (7.6), we obtain

$$\mathcal{E}_l\mathfrak{T}_{i,n,M}^{(d)}(\emptyset)v_\lambda^+ = (\mathrm{ev}\circ\rho_l)(u)^\tau v_\lambda^+ \in \mathcal{V}_\lambda.$$

By Lemma 7.1, we have $\mathcal{E}_l\mathfrak{T}_{i,n,M}^{(d)}(\emptyset)v_\lambda^+ = 0$. Since $d < p$, we have $\mathcal{E}_l^{(r)}\mathfrak{T}_{i,n,M}^{(d)}(\emptyset)v_\lambda^+ = 0$ for any $r > 0$ and $l = 1, \dots, n-2$.

In the present case, (7.2) holds and takes the form

$$\overline{d!}\cdot\mathcal{E}_i^{(d)}\cdots\mathcal{E}_{n-1}^{(d)}\mathfrak{T}_{i,n,M}^{(d)}(\emptyset)v_\lambda^+ = \mathrm{cf}(u)^\tau v_\lambda^+ = \mathrm{cf}_\varkappa(u)^\tau v_\lambda^+, \tag{7.7}$$

where $\varkappa = (d, 0^{n-i-1})$ by virtue of (7.4), where $I_{k+1} = \emptyset$ and $J_k = \langle (i-1)^d\rangle$. We put $\iota(t) := \varphi((t,h))$ for any $(t,h) \in M$ and apply Lemma 6.19 for this injection. Note that $\Sigma_{i,n}^{(d)}(\emptyset) = (i..n] \times [0..d]$ and $\Delta_{i,n}^\varkappa(J_0) = (\{i+1\} \times [0..d]) \cup ((i+1..n] \times \{0\})$. Therefore, $\operatorname{Im}\iota = Y \subset (i+1..n] \times (0..d] = \Sigma_{i,n}^{(d)}(\emptyset) \setminus \Delta_{i,n}^\varkappa(J_0)$ as the hypothesis of Lemma 6.19 requires. Again, we have (7.5), which in the present case takes the form

$$\begin{aligned}\mathrm{cf}_\varkappa(u)^\tau v_\lambda^+ &= \bar d!(\bar\lambda_i - \bar\lambda_{i+1})^{\underline{d}}\prod\big\{\bar t - \bar\imath + \bar\lambda_i - \bar\lambda_t - \bar h \,\big|\, (t,h) \in (i+1..n] \times (0..d] \setminus \operatorname{Im}\iota\big\}v_\lambda^+ \\ &= \bar d!\prod\big\{\overline{\mathrm{dist}_\lambda\big((i,0),x\big)} \,\big|\, x \in (i..n] \times (0..d] \setminus Y\big\}v_\lambda^+ \neq 0\end{aligned}$$

by the definition of Y. Hence $\mathfrak{T}_{i,n,M}^{(d)}(\emptyset)v_\lambda^+ \neq 0$ by (7.7).

7.4. Some basis coefficients

Let I be a multiset of length no more than d with entries in $[i..n)$. We put

$$F_{i,n,I}^{(d)} := \left(\prod_{t=i+1}^{n-1} F_{i,t}^{(|I|^{\{t\}})}\right)F_{i,n}^{(d-|I|)}.$$

This element has weight $\lambda_{i,n,\emptyset}^{(d)}(I, J_0)$.

Lemma 7.2. *Suppose $M \subset \Omega^{(d)}_{i,n}(I)$. Then modulo the ideal $\mathcal{I}$ of $\mathcal{U}^0$ generated by the polynomials $u_t - h$, where $(t,h) \in M$, the $F^{(d)}_{i,n,I}$-coefficient of $\mathrm{ev}(\mathcal{T}^{(d)}_{i,n,\pi_1(M)}(I, J_0))$ is equivalent to*

$$d!\left(\prod_{t=i+1}^{n-1} |I|^{\{t\}}!\right)\prod\bigl\{C(i,t)-h \bigm| (t,h)\in \Omega^{(d)}_{i,n}(I)\setminus M\bigr\}.$$

Proof. Induction on $|M|$. For $M = \emptyset$, the result follows from (3.5).

Now suppose $M \neq \emptyset$. Let $x = (m, \mathfrak{h})$ be the element of M with smallest first coordinate. By the inductive hypothesis, modulo $\mathcal{I}$, the $F^{(d)}_{i,n,I}$-coefficient of $\mathrm{ev}(\mathcal{T}^{(d)}_{i,n,\pi_1(M\setminus\{x\})}(I, J_0))$ is equivalent to

$$d!\left(\prod_{t=i+1}^{n-1} |I|^{\{t\}}!\right)\prod\bigl\{C(i,t)-h \bigm| (t,h)\in \Omega^{(d)}_{i,n}(I)\setminus \bigl(M\setminus\{x\}\bigr)\bigr\}. \tag{7.8}$$

Since $\mathcal{I}$ is stable under $\sigma_{i,m}$, it follows from this equivalence that the $F^{(d)}_{i,n,I}$-coefficient of $\sigma_{i,m}(\mathrm{ev}(\mathcal{T}^{(d)}_{i,n,\pi_1(M\setminus\{x\})}(I, J_0)))$ is equivalent modulo $\mathcal{I}$ to

$$d!\left(\prod_{t=i+1}^{n-1} |I|^{\{t\}}!\right)\left(\prod_{t=i+1}^{m-1}\prod_{h=0}^{d-|I|^{(-\infty..t]}-1} C(i,t)-h\right)u_m\underline{{}^{d-|I|^{(-\infty..m]}}}$$
$$\times\prod\bigl\{C(m,t)-h \bigm| (t,h)\in \Omega^{(d)}_{m,n}(I)\setminus \bigl(M\setminus\{x\}\bigr)\bigr\},$$

which belongs to $\mathcal{I}$. More exactly, $u_m\underline{{}^{d-|I|^{(-\infty..m]}}} \in \mathcal{I}$, since $0 \leqslant \mathfrak{h} < d - |I|^{(-\infty..m]}$. Hence by Lemma 6.17, the $F^{(d)}_{i,n,I}$-coefficient of $\mathrm{ev}(\mathcal{T}^{(d)}_{i,n,\pi_1(M)}(I, J_0))(C(i,m) - u_m)$ is equivalent modulo $\mathcal{I}$ to (7.8), which in turn is equivalent to

$$d!\left(\prod_{t=i+1}^{n-1} |I|^{\{t\}}!\right)\prod\bigl\{C(i,t)-h \bigm| (t,h)\in \Omega^{(d)}_{i,n}(I)\setminus M\bigr\}\bigl(C(i,m)-u_m\bigr).$$

From Proposition 6.18(ii), it follows that $C(i,m) - u_m \notin \mathcal{I}$. Now the required result follows from Proposition 6.18(i). □

Proposition 7.3. *Let $N \in UT(n)$ and $1 \leqslant j < n$. Then we have*

$$F_{j,n}F^{(N)} = (N_{j,n}+1)F^{(N+e_{j,n})} + \sum_{k=1}^{j-1}(N_{k,n}+1)F^{(N-e_{k,j}+e_{k,n})}.$$

Let V_i denote the $\mathbf{Z}'$-submodule of U^- spanned by all $F^{(N)}$, where N belongs to $UT^{\geqslant 0}(n)$ and has 0 in all rows except row i and W_i denote the $\mathbf{Z}'$-submodule of U^- spanned by all other $F^{(N)}$. We have $U^- = V_i \oplus W_i$. We denote by proj the projection of U^- to the first summand.

Remark. Proposition 7.3 implies that W_i is closed under right multiplication by $F_{j,n}$.

Lemma 7.4. *Let $1 \leqslant j < n$, I be a multiset with entries in $[i..n)$ of length no greater than d and j occur in I. Then we have*

$$\mathrm{proj}\bigl(F_{j,n}F^{(d)}_{i,n,I}\bigr) = \bigl(d-|I|+1\bigr)F^{(d)}_{i,n,I\setminus\langle j\rangle}.$$

Proof. Let N be the matrix of $UT^{\geqslant 0}(n)$ such that $F^{(d)}_{i,n,I} = F^{(N)}$. Since N has 0 in all rows distinct from row i, we can assume that the summation parameter k in Proposition 7.3 takes the only value i.

Consider the case $j = i$. Then only the first summand remains, whence $\mathrm{proj}(F_{j,n}F^{(d)}) = (N_{i,n} + 1)F^{(N+e_{i,n})}$. Now consider the case $j > i$. Then only the second summand contributes to V_i and we obtain $\mathrm{proj}(F_{j,n}F^{(d)}) = (N_{i,n} + 1)F^{(N-e_{i,j}+e_{i,n})}$. □

For a multiset I with entries in $[i..n)$ of length no greater than d, we define

$$\hat{F}^{(d)}_{i,n,I} := \left(\prod_{t=i}^{n-1} F^{(|I|^{\{t\}})}_{t,n}\right),$$

which may be viewed as a complimentary element for $F^{(d)}_{i,n,I}$

Corollary 7.5. *Let p be a prime greater than d and I be a multiset with entries in $[i..n)$ of length no greater than d. Then the $F^{(d)}_{i,n}$-coefficient of $\hat{F}^{(d)}_{i,n,I}F^{(d)}_{i,n,I}$ is an integer not divisible by p.*

Lemma 7.6. *$F^{(d)}_{i,n,I}$ is the only element $F^{(N)}$, where $N \in UT^{\geqslant 0}(n)$, of given weight belonging to V_i.*

Proof. The sums of elements in columns $i+1, \dots, n$ can be expressed via the weight of $F^{(d)}_{i,n,I}$ by (2.1) and (3.3). However, each of this columns has 0 everywhere but in row i. □

We denote by $\mathcal{F}^{(d)}_{i,n,I}$ and $\hat{\mathcal{F}}^{(d)}_{i,n,I}$ the images of $F^{(d)}_{i,n,I}$ and $\hat{F}^{(d)}_{i,n,I}$ in $U_K(n)$, respectively.

7.5. "Only if-part" of Theorem 1.1

Without loss of generality we may suppose that $\lambda_1 \geqslant \dots \geqslant \lambda_n \geqslant 0$. We need this stipulation to apply the theory developed in [BKS].

We apply induction on $n - i$. Suppose first that $n - i = 1$. Then $\mathcal{F}^{(d)}_{n-1,n}v^+_\lambda \neq 0$. Hence

$$0 \neq \mathcal{E}^{(d)}_{n-1}\mathcal{F}^{(d)}_{n-1,n}v^+_\lambda = \binom{\lambda_{n-1} - \lambda_n}{d} v^+_\lambda.$$

Since $d < p$, it follows from the above formula that $\lambda_{n-1} - \lambda_n - q \not\equiv 0 \pmod p$ for any $q = 0, \dots, d-1$. Hence $\mathfrak{Y}^\lambda_d(i,n) = \emptyset$, as required.

Now suppose that $n - i > 1$. Then there exists some $F \in U^-_K(n)$ of weight $-d\alpha(i,n)$ such that Fv^+_λ is a nonzero $\mathrm{GL}_{n-1}(K)$-high weight vector. Recall that $L_n(\lambda)$ can be realized as the socle of the module $\nabla_n(\lambda)$. Therefore $\mathcal{E}^{(d)}_i \cdots \mathcal{E}^{(d)}_{n-1}Fv^+_\lambda \neq 0$ by [BKS, Lemma 2.6(ii)]. In particular, $\mathcal{E}^{(d)}_{n-1}Fv^+_\lambda \neq 0$. Using the commutator formulas for $U_K(n)$, we obtain $\mathcal{E}^{(d)}_{n-1}F = F'H + E$, where F' is an element of $U^-_K(n-1)$ having weight $-d\alpha(i, n-1)$, $H \in U^0_K(n)$ and E belongs to the left ideal of $U_K(n)$ generated by the elements $\mathcal{E}^{(r)}_{n-1}$ for $r = 1, \dots, d$. Hence $F'Hv^+_\lambda = \mathcal{E}^{(d)}_{n-1}Fv^+_\lambda \neq 0$. We have $Hv^+_\lambda = c \cdot v^+_\lambda$ for some $c \in K$. Therefore $F'v^+_\lambda \neq 0$ and $c \neq 0$. Finally, take any $s = 1, \dots, n-3$ and $r > 0$. We have

$$\mathcal{E}^{(r)}_s F'v^+_\lambda = c^{-1}\mathcal{E}^{(r)}_s F'Hv^+_\lambda = c^{-1}\mathcal{E}^{(r)}_s\mathcal{E}^{(d)}_{n-1}Fv^+_\lambda = c^{-1}\mathcal{E}^{(d)}_{n-1}\mathcal{E}^{(r)}_s Fv^+_\lambda = 0.$$

We have proved that $F'v^+_\lambda$ is a nonzero $\mathrm{GL}_{n-2}(K)$-high weight vector of weight $\lambda - d\alpha(i, n-1)$ belonging to the module $U^-_K(n-1)v^+_\lambda$, which is isomorphic to $L_{n-1}(\lambda_1, \dots, \lambda_{n-1})$ by [Sm] (this can be seen directly).

Applying the inductive hypothesis, we obtain that for each subset Δ of $\mathfrak{Y}^\lambda_d(i, n-1)$ whose points are incomparable with respect to $\dot{<}$, there exists a strictly decreasing injection from Δ to $\mathfrak{C}^\lambda(i, n-1) \times$

$\{0\}$. Applying Lemma 5.4 similarly to how we did it in Section 7.3, we obtain that there exists a strictly decreasing injection $\psi : \mathfrak{Y}_d^\lambda(i,n-1) \to \mathcal{X}_d^\lambda(i,n-1)$. We endow the set $\mathcal{X}_d^\lambda(i,n-1)$ with the nonstrict partial order $\preccurlyeq$ as in Lemma 5.4. That is $x \preccurlyeq y$ if and only if there are elements $z_0,\dots,z_m$, where $m \geqslant 0$, of $\mathcal{X}_d^\lambda(i,n-1)$ such that $y = z_0$, $x = z_m$ and $z_s = \psi(z_{s-1})$ for any $s = 1,\dots,m$. Following the proof of Lemma 5.4, we put $\hat{\psi}(y)$ to be the smallest (w.r.t. $\preccurlyeq$) element of $\{x \in X \mid x \preccurlyeq y\}$. Thus $\hat{\psi}$ maps $\mathcal{X}_d^\lambda(i,n-1)$ to $\mathcal{X}_d^\lambda(i,n-1) \cap ((i..n-1] \times \{0\})$, which equals $\mathfrak{C}^\lambda(i,n) \times \{0\}$. Moreover, $\hat{\psi}(x) = \hat{\psi}(x')$ if and only if x and x' are comparable with respect to $\preccurlyeq$.

Suppose that there exists a subset Γ of $\mathfrak{Y}_d^\lambda(i,n)$ whose points are incomparable with respect to $\dot{<}$ such that there is no strictly decreasing injection from Γ to $\mathfrak{C}^\lambda(i,n) \times \{0\}$. By Lemma 5.7, there exists a multiset I of length no greater than d with entries in the interval $[i..n)$ such that Γ is contained in the boundary of $\Sigma_{i,n}^{(d)}(I)$. Note that Γ contains a point z' in column n, since otherwise $\Gamma \subset \mathfrak{Y}_d^\lambda(i,n-1)$.

Consider the set $M := \Omega_{i,n}^{(d)}(I) \cap \mathcal{X}_d^\lambda(i,n-1)$. Take a point $x \in M$. We claim that there exists a point $y \in \Sigma_{i,n}^{(d)}(I) \cap \mathcal{X}_d^\lambda(i,n-1)$ such that $x = \psi(y)$. Indeed, consider the map $\xi : \Gamma \to \mathfrak{C}^\lambda(i,n) \times \{0\}$ defined by

$$\xi(z) = \begin{cases} \hat{\psi}(z) & \text{if } z \neq z'; \\ \hat{\psi}(x) & \text{if } z = z', \end{cases}$$

where $z \in \Gamma$. Observing that $\hat{\psi}(x) \dot{<} z'$, we obtain that ξ is strictly decreasing. Hence ξ is no injection. This is only possible if $\hat{\psi}(x) = \hat{\psi}(x')$ for some $x' \in \Gamma$ not belonging to column n. In that case, the points x and x' are comparable with respect to $\preccurlyeq$. Note that x' is a boundary point of $\Sigma_{i,n}^{(d)}(I)$ and x is an interior point of this set. Therefore by definition there are points $z_0,\dots,z_m$ of $\mathcal{X}_d^\lambda(i,n-1)$ such that $m > 0$, $x' = z_0$, $x = z_m$ and $z_s = \psi(z_{s-1})$ for any $s = 1,\dots,m$. It is easy to notice that $z_0,\dots,z_m$ belong to $\Sigma_{i,n-1}^{(d)}(I)$. Thus we can take $y := z_{m-1}$.

Let $\varphi : \{(i,0)\} \cup M \to \Sigma_{i,n}^{(d)}(I) \cap \mathcal{X}_d^\lambda(i,n)$ be the map such that $\varphi((i,0)) = z'$ and $\varphi(x) = y$ if $x \in M$ and $x = \psi(y)$. Obviously, φ is a strictly increasing injection. Also note that $\{(i,0)\} \cup M \subset \Sigma_{i-1,n}^{(d)}(I)$ and thus $\{(i,0)\} \cup M \subset \Omega_{i-1,n}^{(d)}(I)$, since φ is strictly increasing. Hence $\mathcal{T}_{i,n,\pi_1(M)}^{(d)}(I, J_0)$ satisfies conditions (a)–(d) of Section 7.2 for the set $\{(i,0)\} \cup M$ and the injection φ, where τ defined by (7.1). Therefore $\mathfrak{T}_{i,n,M}^{(d)}(I)v_\lambda^+ \in \mathcal{V}_\lambda$. Applying Lemma 7.1, we obtain $\mathfrak{T}_{i,n,M}^{(d)}(I)v_\lambda^+ = 0$.

Lemma 7.2 implies that the $\mathcal{F}_{i,n,I}^{(d)}$-coefficient of $\mathfrak{T}_{i,n,M}^{(d)}(I)$ is

$$\bar{d}!\left(\prod_{t=i+1}^{n-1} \overline{|I|^{\{t\}}!}\right) \prod\left\{\bar{t} - \bar{\imath} + \mathcal{H}_i - \mathcal{H}_t - \bar{h} \;\middle|\; (t,h) \in \Omega_{i,n}^{(d)}(I) \setminus M\right\}.$$

Applying Corollary 7.5 and Lemma 7.6 along with the remark of Section 7.4, we obtain that $\mathcal{F}_{i,n}^{(d)}$-coefficient of $\hat{\mathcal{F}}_{i,n,I}^{(d)}\mathfrak{T}_{i,n,M}^{(d)}(I)$ is

$$c \cdot \prod\left\{\bar{t} - \bar{\imath} + \mathcal{H}_i - \mathcal{H}_t - \bar{h} \;\middle|\; (t,h) \in \Omega_{i,n}^{(d)}(I) \setminus M\right\}, \tag{7.9}$$

where c is a nonzero element of K.

Denote by S the element of $U_K^-(n)$ obtained from $\hat{\mathcal{F}}_{i,n,I}^{(d)}\mathfrak{T}_{i,n,M}^{(d)}(I)$ written in the basis of Section 2.5 by the specialization $\mathcal{H}_t \mapsto \bar{\lambda}_t$ for $t = 1,\dots,n$. Our choice of M and (7.9) ensure that the $\mathcal{F}_{i,n}^{(d)}$-coefficient of S is nonzero. However,

$$Sv_\lambda^+ = \hat{\mathcal{F}}_{i,n,I}^{(d)}\mathfrak{T}_{i,n,M}^{(d)}(I)v_\lambda^+ = 0.$$

Applying [BKS, Lemma 3.6] for $\mu = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - d, \lambda_{i+1}, \dots, \lambda_{n-1})$, we obtain that there is no nonzero $\mathrm{GL}_{n-1}(K)$-high weight vector of weight μ in $L_n(\lambda)$ (μ is not normal for λ). This is a contradiction.

Remark. The condition $d < p$ is necessary in Theorem 1.1. Indeed, if we take, for example, $d = p$, $i = n-1$ and $\lambda \in X^+(n)$ such that $\lambda_{n-1} - \lambda_n = 2p - 1$, then we obtain that the vector $\mathcal{F}_{i,n}^{(d)} v_\lambda^+$ is a nonzero $\mathrm{GL}_{n-1}(K)$-high weight vector, since

$$\mathcal{E}_{n-1}^{(d)} \mathcal{F}_{n-1,n}^{(d)} v_\lambda^+ = \binom{\lambda_{n-1} - \lambda_n}{d} v_\lambda^+ = \binom{2p-1}{p} v_\lambda^+ = v_\lambda^+.$$

However, $\mathfrak{C}^\lambda(i,n) = \emptyset$ and $(n,p) \in \mathfrak{Y}_d^\lambda(i,n)$.

Appendix A. List of notations

K	algebraically closed field of characteristic $p > 0$;
$X^+(n)$	$\{\lambda \in \mathbf{Z}^n\colon\ \lambda_1 \geqslant \cdots \geqslant \lambda_n\}$, set of dominant weights;
$L_n(\lambda)$	irreducible $\mathrm{GL}_n(K)$-module with highest weight λ;
v_λ^+	fixed nonzero vector of $L_n(\lambda)$ of weight λ;
$[s..t]$, $[s..t)$, etc.	$\{x \in \mathbf{Z} \mid s \leqslant x \leqslant t\}$, $\{x \in \mathbf{Z} \mid s \leqslant x < t\}$, etc., p. 28;
$(a,b) \dot{<} (x,y)$	$a < x$ and $b < y$;
$\mathfrak{Y}_d^\lambda(i,n)$	$\{(t,h) \in (i..n] \times [1..d] \mid t - i + \lambda_i - \lambda_t - h \equiv 0 \pmod p\}$;
$\mathfrak{C}^\lambda(i,n)$	$\{s \in (i..n) \mid s - i + \lambda_i - \lambda_s \equiv 0 \pmod p\}$;
$\alpha(s,t)$	$(0,\dots,0,1,0,\dots,0,-1,\dots,0)$ of length n with 1 at position s and -1 at position t;
π_1	map from $\mathbf{Z}^2$ to $\mathbf{Z}$ taking (s,t) to s;
α_0	$(-1,0,\dots,0)$ of length n;
α_t	$\alpha(t,t+1)$;
$x^{\underline{n}}$	$x \cdots (x-n+1)$ if $n \geqslant 0$ and $1/(x+1)\cdots(x-n)$ if $n < 0$;
a^m	sequence of length m whose every entry is a;
$A \sqcup B = C$	$A \cup B = C$ and $A \cap B = \emptyset$;
$\|S\|$	cardinality of set S;
$\delta_{\mathcal{P}}$	1 or 0 if $\mathcal{P}$ is true or false, respectively;
$UT(n)$	set of integer $n \times n$ matrices N such that $N_{a,b} = 0$ unless $a < b$;
$UT^{\geqslant 0}(n)$	subset of $UT(n)$ consisting of matrices with nonnegative entries;
$e_{i,j}$, $X_{i,j}$	$n \times n$ matrix with 1 at the ijth position and 0 elsewhere;
N_t	$\sum_{a=1}^n N_{a,t}$ sum of elements in column t of N, p. 30;
N^s	$\sum_{b=1}^n N_{s,b}$ sum of elements in row s of N, p. 30;
$N(k)$	$\sum_{1 \leqslant a \leqslant k < b \leqslant n} N_{a,b}$, p. 30;
$\mathrm{dist}_\lambda(x,y)$	$y_1 - x_1 + \lambda_{x_1} - \lambda_{y_1} + x_2 - y_2$ if $x = (x_1,x_2)$ and $y = (y_1,y_2)$, p. 30;
$\mathcal{X}_d^\lambda(i,n)$	$\{x \in (i..n] \times [0..d] \mid \mathrm{dist}_\lambda((i,0),x) \equiv 0 \pmod p\}$;
$\|I\|^S$	number of elements of multiset I belonging to set S;
$\|I\|$	total number of elements of multiset I;
$I_{x \mapsto y}$	multiset obtained from I by replacing one x with y;
$\mathcal{L}_m(J)$	$\langle \min\{j_1, m-1\}, \dots, \min\{j_k, m-1\}\rangle$ if $J = \langle j_1, \dots, j_k\rangle$;
$\mathcal{R}_m(J)$	$\langle j_s \mid s = 1,\dots,k,\ j_s \geqslant m-1\rangle$ if $J = \langle j_1,\dots,j_k\rangle$;
$\mathcal{L}^m(I)$	$\langle i_s \mid s = 1,\dots,l,\ i_s \leqslant m-1\rangle$ if $I = \langle i_1,\dots,i_l\rangle$;

$\mathcal{R}^m(I)$	$\langle \max\{i_1, m-1\}, \dots, \max\{i_l, m-1\}\rangle$ if $I = \langle i_1, \dots, i_l\rangle$;
u_i	0;
$\mathbf{Q}'$	$\mathbf{Q}(u_{i+1}, \dots, u_{n-1})$ field of rational fractions, p. 32;
H_s	$X_{s,s}$;
$\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$	universal enveloping algebra of $\mathfrak{gl}_{\mathbf{Q}'}(n)$, p. 32;
$\mathcal{U}^0$	subring of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ generated by $H_1, \dots, H_n$, $u_{i+1}, \dots, u_{n-1}$, p. 32;
$\bar{U}$	right (left) ring of quotients of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ with respect to $\mathcal{U}^0 \setminus \{0\}$, p. 32;
$\mathbf{Z}'$	$\mathbf{Z}[u_{i+1}, \dots, u_{n-1}]$ polynomial algebra over $\mathbf{Z}$;
$X_{s,t}^{(r)}$	$(X_{s,t})^r/r!$, where $s \neq t$, $1 \leqslant s, t \leqslant n$ and $r \geqslant 0$, p. 32;
$\binom{X_{s,s}}{r}$	$X_{s,s} \cdots (X_{s,s} - r + 1)/r!$, where $1 \leqslant s \leqslant n$ and $r \geqslant 0$, p. 32;
U	$\mathbf{Z}'$-subalgebra of $\mathfrak{U}(\mathfrak{gl}_{\mathbf{Q}'}(n))$ generated by $X_{s,t}^{(r)}$ and $\binom{X_{s,s}}{r}$;
$E_{s,t}^{(r)}, F_{s,t}^{(r)}$	$X_{s,t}^{(r)}, X_{t,s}^{(r)}$ respectively, where $1 \leqslant s < t \leqslant n$;
$E_s^{(r)}$	$E_{s,s+1}^{(r)}$;
$F^{(N)}$	$\prod_{1 \leqslant a < b \leqslant n} F_{a,b}^{(N_{a,b})}$, p. 33;
$E^{(N)}$	$\prod_{1 \leqslant a < b \leqslant n} E_{a,b}^{(N_{a,b})}$, p. 33;
$\bar{\mathcal{U}}^0$	subfield of $\bar{U}$ generated by $\mathcal{U}^0$;
$I^+, \bar{I}^+$	left ideals of U and $\bar{U}$ respectively generated by $E_s^{(r)}$ with $r > 0$;
U^0	$\mathbf{Z}'$-subalgebra of U generated by all $\binom{H_s}{r}$;
U^-	$\mathbf{Z}'$-subalgebra of U generated by all $F_{s,t}^{(r)}$;
$\mathcal{U}^{-,0}$	$\mathbf{Z}'$-subalgebras of U generated by all $F_{s,t}^{(r)}$ and H_s;
$\bar{\mathcal{U}}^{-,0}$	subring of $\bar{U}$ generated by U^- and $\bar{\mathcal{U}}^0$;
$C(k,l)$	$l - k + H_k - H_l$;
K_τ	field K considered as a left $\mathbf{Z}'$-module via the multiplication rule $f \cdot \alpha = \tau(f)\alpha$, where $f \in \mathbf{Z}'$ and $\alpha \in K$, p. 34;
$U_K(n)$	$U \otimes_{\mathbf{Z}'} K_\tau$, hyperalgebra over filed K, p. 34;
x^τ	$x \otimes 1_K$, image of $x \in U$ in $U_K(n) = U \otimes_{\mathbf{Z}'} K_\tau$ under τ, p. 34;
$\mathcal{E}_s^{(r)}, \mathcal{F}_{s,t}^{(r)}$	images of $E_s^{(r)}$ and $F_{s,t}^{(r)}$ respectively in $U_K(n)$, p. 34;
$\mathcal{F}^{(N)}, \mathcal{H}_s$	images of $F^{(N)}$ and H_s respectively in $U_K(n)$, p. 34;
$U_K^0(n)$	K-subalgebra of $U_K(n)$ generated by all $\binom{\mathcal{H}_s}{r}$;
$U_K^-(n)$	K-subalgebra of $U_K(n)$ generated by all $\mathcal{F}_{s,t}^{(r)}$;
$\lambda_{i,n,\mathcal{M}}^{(d)}$	Eq. (3.4), p. 35;
$S_{i,n,\mathcal{M}}^{(d)}$	Eq. (3.5), p. 36;
$P_{i,n,\mathcal{M}}^{(d)}$	Eqs. (4.1) and (4.2), p. 41;
cone(x)	set of elements less than or equal to x, pp. 49, 50;
cone(S)	$\bigcup_{x \in S}$ cone(x);
$(a,b) \mathrel{\dot{\leqslant}} (x,y)$	$a \leqslant x$ and $b \leqslant y$;
snake(Γ)	boundary of cone(Γ), p. 50;
$\Sigma_{k,j}^{(d)}(I)$	$\{(t,h) \in \mathbf{Z}^2 \mid k < t \leqslant j \;\&\; 0 \leqslant h \leqslant d - \lvert I\rvert^{(-\infty..t-1]}\}$, p. 52;
$\Omega_{k,j}^{(d)}(I)$	$\{(t,h) \in \mathbf{Z}^2 \mid k < t < j \;\&\; 0 \leqslant h < d - \lvert I\rvert^{(-\infty..t]}\}$, p. 52;
$\sigma_{l,m}$	p. 53;
$\mathcal{S}_{m,m'}^{(d)}(I,J)$	formal operator, p. 55;
$F_{i,n}^{(d)}$	algebra of formal operators, p. 55;
$\rho_l^{(1)}$	p. 57

$\rho_l^{(2,L)}, \rho_l^{(2,R)}, \rho_l^{(3)}$	p. 57;
$\rho_l^{(2)}$	$\rho_l^{(2,L)} - \rho_l^{(2,R)}$;
$\mathcal{T}_{k,j,\mathcal{M}}^{(d)}(I, J)$	Definition 6.3, p. 57;
$\mathrm{cf}_\varkappa$	Eq. (6.4), p. 67;
$\mathrm{cf}_{k,j,\varkappa}^{(d)}(I, J)$	p. 70;
ev	evaluation function, p. 70;
$\mathrm{cf}(P)$	sum of $\mathrm{cf}_\varkappa(P)$ over all sequences of nonnegative integers $\varkappa$ of length $n-i$, p. 71;
ρ_l	$\rho_l^{(1)} + \rho_l^{(2)} + \rho_l^{(3)}$;
$\Delta_{k,j}^{\varkappa}(J)$	$\{(t,h)\in\mathbf{Z}^2 \mid k < t \leqslant j \ \& \ \|J\|^{[t..+\infty)} \leqslant h \leqslant \|J\|^{[t-1..+\infty)} + q_t\}$, p. 72;
$\bar{m}$	$1_K + \cdots + 1_K$ with m summands;
$\mathfrak{T}_{i,n,M}^{(d)}(I)$	$\mathrm{ev}(\mathcal{T}_{i,n,\pi_1(M)}^{(d)}(I, J_0))^\tau$, where $J_0 = \langle (i-1)^d \rangle$, p. 75;
$\mathcal{V}_\lambda$	p. 76;
$F_{i,n,I}^{(d)}$	$(\prod_{t=i+1}^{n-1} F_{i,t}^{(\|I\|^{\{t\}})}) F_{i,n}^{(d-\|I\|)}$, p. 79;
proj	projection of U^- to V_i along W_i, p. 80;
$\hat{F}_{i,n,I}^{(d)}$	$(\prod_{t=i}^{n-1} F_{t,n}^{(\|I\|^{\{t\}})})$, p. 81;
$\mathfrak{F}_{i,n,I}^{(d)}, \hat{\mathfrak{F}}_{i,n,I}^{(d)}$	images of $F_{i,n,I}^{(d)}$ and $\hat{F}_{i,n,I}^{(d)}$ in $U_K(n)$ respectively, p. 81.

References

[B] J. Brundan, Modular branching rules and the Mullineux map for Hecke algebras of type A, Proc. Lond. Math. Soc. 77 (3) (1998) 551–581.

[BKS] J. Brundan, A. Kleshchev, I. Suprunenko, Semisimple restrictions from GL(n) to GL($n-1$), J. Reine Angew. Math. 500 (1998) 83–112.

[CL] R.W. Carter, G. Lusztig, On the modular representations of the general linear and symmetric groups, Math. Z. 136 (1974) 193–242.

[GKP] R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics, second ed., Addison–Wesley, 1998.

[J] J.C. Jantzen, Representations of Algebraic Groups, Math. Surveys Monogr., vol. 107, Amer. Math. Soc., Providence, RI, 2003.

[K] A.S. Kleshchev, Branching rules for modular representations of symmetric groups. II, J. Reine Angew. Math. 459 (1995) 163–212.

[Sh] V.V. Shchigolev, Generalization of modular lowering operators for GL_n, Comm. Algebra 36 (4) (2008) 1250–1288.

[Sm] S.D. Smith, Irreducible modules and parabolic subgroups, J. Algebra 75 (1982) 286–289.

[St] R. Steinberg, Lectures on Chevalley Groups, Yale University, New Haven, 1968.