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LINEAR ALGEBRA AND ITS APPLICATIONS

# A fixed point theorem and a norm inequality for operator means

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**Abstract**

There is one to one correspondence between positive operator monotone functions on $(0,\infty)$ and operator connections. For a symmetric connection $\sigma$, it is proved that the map $X \to (A\sigma X)\sigma^{\perp}(B\sigma X)$ from positive operators on a Hilbert space to itself, has a unique fixed point. Here $\sigma^{\perp}$ denotes the dual of $\sigma$. It is also proved that $|||A\sigma B||| \leqslant |||A|||\,\sigma\,|||B|||$ for all unitarily invariant norms $|||\cdot|||$ and for all positive operators $A,B$. © 1999 Elsevier Science Inc. All rights reserved.

## 1. Introduction

Throughout $H$ denotes the complex Hilbert space $\mathbb{C}^n$, $n \in \mathbf{N}$. $\mathscr{L}(H)$ is the space of bounded linear operators on $H$, while $\mathscr{L}_+(H)$ is the cone of positive semidefinite operators on $H$ and $\mathscr{P}(H)$ is the cone of positive operators on $H$.

The study of operator means began with the work of Anderson and Duffin [1]. They first studied the arithmetic and harmonic means and proved the arithmetic–harmonic inequality. Ando [5] defined the geometric mean and proved the arithmetic–geometric inequality. The axiomatic theory for connections and means for pairs of positive operators has been developed by Nishio and Ando [17] and Kubo and Ando [13]. Let $A,B,C,\ldots,$ denote elements of $\mathscr{L}_+(H)$. An operator connection $\sigma$ is a binary operation on $\mathscr{L}_+(H)$ satisfying the following axioms:

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*Monotonicity*

$A \leqslant C, B \leqslant D$ imply $A\sigma B \leqslant C\sigma D$,

*Transformer inequality*

$C(A\sigma B)C \leqslant (CAC)\sigma(CBC)$,

*Upper continuity*

$A_n \downarrow A$ and $B_n \downarrow B$ imply$(A_n\sigma B_n) \downarrow (A\sigma B)$.

A mean is a connection with normalization condition

$I\sigma I = I$.

Kubo and Ando [13] showed the existence of an affine order isomorphism between the class of connections and the class of positive operator monotone functions on $\mathbb{R}_+$. This isomorphism $\sigma \leftrightarrow f$ is characterized by the relation

$$A\sigma B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2} \tag{1}$$

for $A, B \in \mathscr{P}(H)$. The operator monotone function $f$ is called the representing function of $\sigma$. The following inequality holds.

Let $A, B, C, D \in \mathscr{P}(H)$. Then for any connection $\sigma$,

$$(A + C)\sigma(B + D) \geqslant (A\sigma B) + (C\sigma D). \tag{2}$$

The transpose $\sigma'$ of a connection $\sigma$ is defined by

$A\sigma' B = B\sigma A$.

For a connection $\sigma$ its dual $\sigma^\perp$ is defined by

$A\sigma^\perp B = (B^{-1}\sigma A^{-1})^{-1}$,

$A, B \in \mathscr{P}(H)$. If $f$ is the representing function of $\sigma$ then $f'(t) = tf(t^{-1})$ is the representing function of $\sigma'$ and $f^\perp(t) = t(f(t))^{-1}$ is the representing function of $\sigma^\perp$. A connection $\sigma$ is called symmetric if $\sigma' = \sigma$ and is called selfdual if $\sigma^\perp = \sigma$.

The operator mean corresponding to the operator monotone function $t \to t^{1/2}$ is called the geometric mean and is denoted by $\#$. For $A, B \in \mathscr{P}(H)$, their parallel sum is defined by

$A : B = (A^{-1} + B^{-1})^{-1}$.

In his unpublished thesis F. Kubo considered the following problem.

Given a pair $\sigma, \tau$ of operator means and $A, B \in \mathscr{P}(H)$, define a map $\theta$ by

$\theta(X) = (A\sigma X)\tau(B\sigma X)$,

$X \in \mathscr{P}(H)$. Then the iterates converges to a fixed point of $\theta$

$$\lim_{n\to\infty} \theta^n\left(\frac{A+B}{2}\right) = \lim_{n\to\infty} \theta^n(2(A:B))$$

which is a unique fixed point among $X$'s such that $X \geqslant c(A:B)$ for some $c > 0$.

In Ref. [8], Arlinskii proved that $A\#B$ is the unique fixed point of the map

$$X \to (A+X):(B+X).$$

The operator connections : and + are symmetric and are duals of each other. In Section 2, we shall prove a generalization of this result for all symmetric connections. Our Theorem 2.5 gives a characterization of symmetric connections. In Section 3, we shall prove that for all connections $\sigma$ and for all unitarily invariant norms $|||\cdot|||$,

$$|||A\sigma B||| \leqslant |||A|||\,\sigma\,|||B|||$$

for $A, B \in \mathscr{P}(H)$.

## 2. A fixed point theorem

**Theorem 2.1.** *Let $\sigma$ be a symmetric connection and let $A, B \in \mathscr{P}(H)$. Then $A\#B$ is the unique fixed point of the map $\Phi_{A,B} : \mathscr{P}(H) \to \mathscr{P}(H)$ defined by*

$$\Phi_{A,B}(X) = (A\sigma X)\sigma^{\perp}(B\sigma X),$$

$X \in \mathscr{P}(H)$.

**Proof.** Let $f$ be the representing function of $\sigma$. We shall prove that $A\#B$ is a fixed point of $\Phi_{A,B}$, that is, $\Phi_{A,B}(A\#B) = A\#B$. Indeed, using the representation (1), we have

$$\begin{aligned}
\Phi_{A,B}(A\#B) &= [A\sigma(A\#B)]\sigma^{\perp}[B\sigma(A\#B)] \\
&= [A^{1/2}f(A^{-1/2}(A\#B)A^{-1/2})A^{1/2}]\sigma^{\perp}[B^{1/2}f(B^{-1/2}(A\#B)B^{-1/2})B^{1/2}] \\
&= [A^{1/2}f(A^{-1/2}(A\#B)A^{-1/2})A^{1/2}]\sigma^{\perp}[B^{1/2}f(B^{-1/2}(B\#A)B^{-1/2})B^{1/2}] \\
&= [A^{1/2}f((A^{-1/2}BA^{-1/2})^{1/2})A^{1/2}]\sigma^{\perp}[B^{1/2}f((B^{-1/2}AB^{-1/2})^{1/2})B^{1/2}] \\
&= A^{1/2}[\{f((A^{-1/2}BA^{-1/2})^{1/2})\}\sigma^{\perp}\{A^{-1/2}B^{1/2}f((B^{-1/2}AB^{-1/2})^{1/2})B^{1/2}A^{-1/2}\}]A^{1/2}.
\end{aligned} \tag{$*$}$$

Now

$$\begin{aligned}
&\{f((A^{-1/2}BA^{-1/2})^{1/2})\}\sigma^{\perp}\{A^{-1/2}B^{1/2}f((B^{-1/2}AB^{-1/2})^{1/2})B^{1/2}A^{-1/2}\} \\
&= \{f((A^{-1/2}BA^{-1/2})^{1/2})\}\sigma^{\perp}\{A^{-1/2}B^{1/2}(I\sigma(B^{-1/2}AB^{-1/2})^{1/2})B^{1/2}A^{-1/2}\} \\
&= \{f((A^{-1/2}BA^{-1/2})^{1/2})\}\sigma^{\perp}\{(A^{-1/2}BA^{-1/2})\sigma(A^{-1/2}(B\#A)A^{-1/2})\} \\
&= \{f((A^{-1/2}BA^{-1/2})^{1/2})\}\sigma^{\perp}\{(A^{-1/2}BA^{-1/2})\sigma(A^{-1/2}(A\#B)A^{-1/2})\} \\
&= \{f((A^{-1/2}BA^{-1/2})^{1/2})\}\sigma^{\perp}\{(A^{-1/2}BA^{-1/2})\sigma(A^{-1/2}BA^{-1/2})^{1/2}\} \\
&= \{f((A^{-1/2}BA^{-1/2})^{1/2})\}\sigma^{\perp}\{(A^{-1/2}BA^{-1/2})^{1/2}[(A^{-1/2}BA^{-1/2})^{1/2})\sigma I]\} \\
&= \{f((A^{-1/2}BA^{-1/2})^{1/2})\}\sigma^{\perp}\{(A^{-1/2}BA^{-1/2})^{1/2}[I\sigma(A^{-1/2}BA^{-1/2})^{1/2}]\}
\end{aligned}$$

$$= \{f((A^{-1/2}BA^{-1/2})^{1/2})\}\sigma^{\perp}\{(A^{-1/2}BA^{-1/2})^{1/2}f((A^{-1/2}BA^{-1/2})^{1/2})\}$$
$$= f((A^{-1/2}BA^{-1/2})^{1/2})[I\sigma^{\perp}\{(A^{-1/2}BA^{-1/2})^{1/2}]$$
$$= f((A^{-1/2}BA^{-1/2})^{1/2})f^{\perp}((A^{-1/2}BA^{-1/2})^{1/2})$$
$$= f((A^{-1/2}BA^{-1/2})^{1/2})(A^{-1/2}BA^{-1/2})^{1/2}[f((A^{-1/2}BA^{-1/2})^{1/2})]^{-1}$$
$$= (A^{-1/2}BA^{-1/2})^{1/2}.$$

Thus it follows from (*) that

$$\Phi_{A,B}(A\#B) = A\#B.$$

To prove the uniqueness, we need the following lemma.

**Lemma 2.2.** *Let $A, B \in \mathscr{P}(H)$ and $\sigma$ be a symmetric connection. Then*

$$(A\sigma I)\sigma^{\perp}(B\sigma I) = I \tag{3}$$

*if and only if $B = A^{-1}$.*

**Proof.** Observe that if $B = A^{-1}$ then $A\#B = I$ and hence (3) follows from the proof given in above theorem. Conversely, suppose (3) holds. Let $f$ be the representing function of $\sigma$. Then

$$f(A)\sigma^{\perp}f(B) = I,$$

which implies

$$(f(A))^{-1}\sigma(f(B))^{-1} = I.$$

Thus

$$f((f(A))^{1/2}(f(B))^{-1}(f(A))^{1/2}) = f(A).$$

Since a non-constant operator monotone function is strictly increasing and hence is one to one. Thus for a non-constant operator monotone function $g$, $g(C) = g(D)$, implies $C = D$. The function $f$ is non-constant operator monotone, we have from the above equality

$$(f(A))^{1/2}(f(B))^{-1}(f(A))^{1/2} = A,$$

which implies

$$f(B) = A^{-1}f(A). \tag{4}$$

Now using that $f(t) = tf(t^{-1})$, we have

$$A^{-1}f(A) = f(A^{-1}).$$

Thus from Eq. (4), we get

$$f(B) = f(A^{-1}),$$

which further implies

$$B = A^{-1}. \qquad \square$$

**Proof of uniqueness.** Suppose $\Phi_{A,B}(X) = X$, $X \in \mathscr{P}(H)$. We shall prove that $X = A\#B$. We have

$$(A\sigma X)\sigma^{\perp}(B\sigma X) = X,$$

which implies

$$X^{1/2}[\{(X^{-1/2}AX^{-1/2})\sigma I\}\sigma^{\perp}\{(X^{-1/2}BX^{-1/2})\sigma I\}]X^{1/2} = X,$$

so

$$[(X^{-1/2}AX^{-1/2})\sigma I]\sigma^{\perp}[(X^{-1/2}BX^{-1/2})\sigma I] = I.$$

Therefore, by Lemma 2.2,

$$X^{-1/2}BX^{-1/2} = [X^{-1/2}AX^{-1/2}]^{-1}.$$

Consequently,

$$(X^{-1/2}AX^{-1/2})\#(X^{-1/2}BX^{-1/2}) = I.$$

Hence

$$X = A\#B,$$

which completes the proof. $\square$

**Corollary 2.3** ([8], Theorem 2). *Let $A, B \in \mathscr{P}(H)$. Then the maps*

$$\Psi^{(1)}_{A,B}, \Psi^{(2)}_{A,B} : \mathscr{P}(H) \to \mathscr{P}(H)$$

*defined by*

$$\Psi^{(1)}_{A,B}(X) = (A + X) : (B + X)$$

*and*

$$\Psi^{(2)}_{A,B}(X) = (A : X) + (B : X)$$

*have the unique fixed point $A\#B$.*

**Remark 2.4.** Theorem 2.1 need not be true if $\sigma$ is not symmetric. Indeed, if $\sigma$ is the operator mean corresponding to the operator monotone function $f(t) = 1$, then

$$X \to (A\sigma X)\sigma^{\perp}(B\sigma X) = B$$

is a constant map. If $f(t) = t$ then

$$X \to (A\sigma X)\sigma^{\perp}(B\sigma X) = X,$$

so every point of this map is a fixed point. Infact our next result gives a characterization of the symmetric connections.

**Theorem 2.5.** *Let $A, B \in \mathscr{P}(H)$ and let $\sigma$ be a connection. Then $\sigma$ is symmetric if and only if $A\#B$ is the unique fixed point of the map $\Phi_{A,B} : \mathscr{P}(H) \to \mathscr{P}(H)$ defined by*

$$\Phi_{A,B}(X) = (A\sigma X)\sigma^{\perp}(B\sigma X),$$

$X \in \mathscr{P}(H)$.

**Proof.** Suppose $A\#B$ is the unique fixed point of $\Phi_{A,B}$. Therefore, for $X, Y \in \mathscr{P}(H)$

$$(X\sigma I)\sigma^{\perp}(Y\sigma I) = I \quad \text{implies} \quad Y = X^{-1}.$$

Let $g$ be the representing function of $\sigma^{\perp}$. Note that $g$ is non-constant operator monotone function. Let $t > 0$. Then

$$\begin{aligned}
&(t\sigma 1)\sigma^{\perp}(t^{-1}\sigma 1) = 1 \\
&\quad \Rightarrow \ (g(t^{-1}))^{-1}\sigma^{\perp}(g(t))^{-1} = 1 \\
&\quad \Rightarrow \ g((g(t))^{-1}g(t^{-1})) = g(t^{-1}) \\
&\quad \Rightarrow \ (g(t))^{-1}g(t^{-1}) = t^{-1} \\
&\quad \Rightarrow \ g(t) = tg(t^{-1}).
\end{aligned}$$

Thus $\sigma^{\perp}$ is symmetric and hence $\sigma$ is symmetric. The other part of the theorem is Theorem 2.1. □

## 3. A norm inequality

For an operator $A \in \mathscr{L}(H)$, its singular values are denoted by $s_j(A)$ and its eigen values are denoted by $\lambda_j(A)$. The Schatten $p$-norms on $\mathscr{L}(H)$ are defined as

$$\|A\|_p = \left[\sum (s_j(A))^p\right]^{1/p}, \quad 1 \leqslant p < \infty,$$

$$\|A\|_\infty = \|A\| = s_1(A),$$

$A \in \mathscr{L}(H)$.

A norm $|||\cdot|||$ on $\mathscr{L}(H)$ is called symmetric or unitarily invariant, if $|||A||| = |||UAV|||$ for all $A \in \mathscr{L}(H)$ and for all unitary operators $U, V \in \mathscr{L}(H)$. The operator norm $\|\cdot\|$ and the Schatten $p$-norms are such norms. A basic property of the unitarily invariant norms is that they are symmetric gauge functions of the singular values of the operator. For a positive operator $T$ its singular values are the same as its eigen values. Let $T \in \mathscr{L}(H)$ and let its singular values be enumerated as

$$s_1(T) \geqslant s_2(T) \geqslant \cdots \geqslant s_n(T).$$

The *generalized spectral norms* have been introduced in Ref. [14].

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n_+$ be such that $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$. Then

$$|||T|||_\alpha = \sum_{j=1}^{n} \alpha_j s_j(T)$$

is a unitarily invariant norm and is called the generalized spectral norm.

In Ref. [8], Arlinskii proved that for all connections $\sigma$

$$\|A\sigma B\|_p \leqslant \|A\|_p \, \sigma \, \|B\|_p$$

for all $p$-norms. We shall extend this inequality for all unitarily invariant norms. We need some lemmas.

**Lemma 3.1.** *Let $A, B \in \mathscr{P}(H)$ and let $\sigma$ be a connection. Then*

$$\langle (A\sigma B)x, x\rangle \leqslant \langle Ax, x\rangle \sigma \langle Bx, x\rangle$$

*for all $x \in H$.*

**Proof.** Let $A, B \in \mathscr{P}(H)$ and $x \in H$. Using the fact that an operator monotone function $f$ on $(0, \infty)$ can be represented as an integral

$$f(s) = a + bs + \int_0^\infty (1+t)\frac{s}{t+s}\, \mathrm{d}\mu(t),$$

where $a, b \geqslant 0$ and $\mu$ is a finite positive measure, one can show

$$A\sigma B = aA + bB + \int_0^\infty \frac{1+t}{t}\{(tA) : B\}\, \mathrm{d}\mu(t).$$

(see Ref. [13], Theorem 3.4 for details). Therefore

$$\begin{aligned} \langle (A\sigma B)x, x\rangle &= a\langle Ax, x\rangle + b\langle Bx, x\rangle + \int_0^\infty \frac{1+t}{t}\langle (((tA) : B)x, x\rangle\, \mathrm{d}\mu(t) \\ &\leqslant a\langle Ax, x\rangle + b\langle Bx, x\rangle + \int_0^\infty \frac{1+t}{t}\{t\langle Ax, x\rangle : \langle Bx, x\rangle\}\, \mathrm{d}\mu(t) \\ &= \langle Ax, x\rangle \sigma \langle Bx, x\rangle, \end{aligned}$$

using that $\langle (C : D)x, x\rangle \leqslant \langle Cx, x\rangle : \langle Dx, x\rangle$ for $C, D \in \mathscr{P}(H), x \in H$, [3]. $\square$

The following lemma is known as *Fan Maximum Principle* and can be found in Ref. [10].

**Lemma 3.2.** *Let $A \in \mathscr{L}_+(H)$. Then for $k = 1, 2, \ldots, n$,*

$$\sum_{j=1}^{k} \lambda_j(A) = \max \sum_{j=1}^{k} |\langle Ax_j, y_j \rangle|,$$

*$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, where the maximum is taken over all choices of the orthonormal vectors $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_k\}$.*

**Lemma 3.3.** *Let $h : \mathbb{R}_+^2 \to \mathbb{R}_+$ be non-decreasing in each component, and let $A, B, C \in \mathscr{L}(H)$. If*

$$|||A|||_\alpha \leqslant h(|||B|||_\alpha, |||C|||_\alpha)$$

*for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}_+^n$; $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$, then*

$$|||A||| \leqslant h(|||B|||, |||C|||)$$

*for all unitarily invariant norms $|||\cdot|||$.*

For a proof of the above lemma the reader may refer to ([12], Corollary 3.5.11).

**Theorem 3.4.** *Let $A, B \in \mathscr{P}(H)$ and let $\sigma$ be a connection. Then*

$$|||A\sigma B||| \leqslant |||A|||\,\sigma\,|||B|||$$

*for all unitarily invariant norms $|||\cdot|||$.*

**Proof.** Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}_+^n$; $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$. Choose orthonormal vectors $x_1, x_2, \ldots x_n$, such that, we have, for $k = 1, 2, \ldots, n$,

$$\begin{aligned}
\sum_{j=1}^{k} \alpha_j \lambda_j(A\sigma B) &= \sum_{j=1}^{k} \alpha_j \langle (A\sigma B)x_j, x_j \rangle \\
&\leqslant \sum_{j=1}^{k} \alpha_j \{\langle Ax_j, x_j \rangle \sigma \langle Bx_j, x_j \rangle\} \\
&\leqslant \left( \sum_{j=1}^{k} \alpha_j \langle Ax_j, x_j \rangle) \sigma \sum_{j=1}^{k} \alpha_j \langle Bx_j, x_j \rangle \right) \\
&\leqslant \left( \sum_{j=1}^{k} \alpha_j \lambda_j(A) \right) \sigma \left( \sum_{j=1}^{k} \alpha_j \lambda_j(B) \right),
\end{aligned}$$

using Lemma 3.1, the inequality (2) and Fan Maximum Principle, respectively. Thus

$$|||A\sigma B|||_\alpha \leqslant |||A|||_\alpha \, \sigma \, |||B|||_\alpha .$$

The function $h : \mathbb{R}_+^2 \to \mathbb{R}_+$ defined by $h(s,t) = s\sigma t$ is non-decreasing in each component. Consequently, by Lemma 3.3

$$|||A\sigma B||| \leqslant |||A||| \, \sigma \, |||B||| . \square$$

**Corollary 3.5.** *Let f be a positive operator monotone function on* $(0,\infty)$. *Then*

$$|||f(T)||| \leqslant |||I||| \; f((|||I|||)^{-1} \; |||T|||)$$

*for all unitarily invariant norms* $|||\cdot|||$ *and for all* $T \in \mathscr{P}(H)$.

**Proof.** Taking $A = I$, $B = T$ and $\sigma$ to be the connection corresponding to the operator monotone function $f$ and then using representation (1), we get the desired result. □

For related subjects see Refs. [2,4,6,7,9,11,15–16].

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