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Uniqueness of Inverse Scattering Problem for a Penetrable Obstacle with Rigid Core

P. Y. H. PANG AND G. YAN*
Department of Mathematics, National University of Singapore
2 Science Drive 2, Republic of Singapore 117543
matpyh@nus.edu.sg yanguoz@ccnu.edu.cn

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Abstract—In this paper, we discuss the inverse scattering problem for a penetrable obstacle with an impenetrable rigid core. Using a generalization of Schiffer's method to nonsmooth domains due to Ramm, we prove that the rigid core is uniquely determined by the far field patterns for a range of interior wavenumbers.

Keywords—Inverse obstacle scattering, Multilayered obstacle, Neumann boundary condition, Helmholtz equation, Green's theorem for nonsmooth domains.

1. INTRODUCTION

This paper is concerned with the uniqueness of the multilayered obstacle inverse scattering problem for acoustic waves for a penetrable scatterer with an impenetrable core. In particular, we consider the case where the core is rigid, or sound-hard, i.e., the Neumann boundary condition holds. For discussions on other types of boundary conditions, such as Dirichlet (soft) and Robin (impedance) conditions, and related inverse scattering problems, we refer to [1–5].

For the existence and uniqueness of the direct acoustic scattering problem by multilayered obstacles, we refer to [1,4,6–8]. Dassios developed the low-frequency theory for acoustic scattering by a soft body [9] and by a penetrable body with either a soft or a rigid core [10]. Twersky in [11] proved reciprocity and scattering theorems for both soft and hard obstacles, and using low-frequency expansions, he obtained the leading term approximation of the real part of the scattering amplitude by direct application of the scattering theorems.

However, for the inverse problem for multilayered obstacles, few results have been found. In this paper, we study whether and when the core can be uniquely determined by the far field pattern of the scattered wave.

2. FORMULATION OF THE PROBLEM

Consider a finite body (the scatterer) in R^3 with C^2 boundary S_0. Let S_1 be the C^2 boundary of another such body (the core) Ω lying entirely within the scatterer S_0 ($S_0 \cap S_1 = \emptyset$). Denote by Ω^+ the region exterior to S_0, and Ω^- the region between S_0 and S_1.

*Current address: Department of Mathematics, Central China Normal University, Wuhan, P.R. China

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The time-independent scattering problem of a plane acoustic wave by the penetrable body S_0 with the impenetrable rigid core S_1 may be formulated as follows:

$$\Delta u(x) + k^2 u(x) = 0, \qquad x \in \Omega^+, \tag{1}$$
$$\Delta v(x) + k_0^2 v(x) = 0, \qquad x \in \Omega^-, \tag{2}$$
$$u(x) = v(x), \qquad x \in S_0, \tag{3}$$
$$\frac{\partial u(x)}{\partial \nu} = \lambda \frac{\partial v(x)}{\partial \nu}, \qquad x \in S_0, \tag{4}$$
$$\frac{\partial v(x)}{\partial \nu} = 0, \qquad x \in S_1, \tag{5}$$

where $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative. The constants k and k_0 are called the exterior and interior wavenumbers, respectively, and the constant λ, given by the ratio of the exterior (Ω^+) and interior (Ω^-) densities, is known as the jump parameter.

The total field u in the region Ω^+ can be decomposed as

$$u(x) = u^i(x) + u^s(x),$$

where $u^i(x) = e^{ikx\cdot d}$ is the incident plane wave with incidence direction d, and u^s is the scattered wave. We assume that u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \int_{S_r} \left| \frac{\partial u^s}{\partial \nu} - iku^s \right|^2 ds = 0,$$

where S_r is the sphere of radius r centered at 0. It follows that the scattered wave has the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \qquad |x| \to \infty,$$

uniformly in all directions $\hat{x} = x/|x|$. The function u_∞ defined on the unit sphere S^2 is known as the *far field pattern* or *scattering amplitude*. To emphasize the dependence of u_∞ on the incidence direction d and interior wavenumber k_0, we also denote it by $u_\infty(\cdot, d, k_0)$.

The well-posedness of the boundary value problem (1)–(5) using integral equation methods is discussed in [1,4]. Our result is the following.

THEOREM 2.1. *For the boundary value problem (1)–(5), let the scatterer S_0 and the exterior wavenumber k be fixed. Suppose that there are two rigid cores Ω_1 and Ω_2 which lead to the same far field pattern $u^i(\cdot, d, k_0)$ for a given incident wave for any interior wavenumber $k_0 \in [a, b]$, $0 < a < b$. Suppose also that $u^i(\cdot, d, k_0) \neq u^i(\cdot, d, \tilde{k}_0)$ for $k_0 \neq \tilde{k}_0$. Then the two cores Ω_1 and Ω_2 coincide, i.e.,*

$$\Omega_1 \equiv \Omega_2.$$

3. PROOF

Our proof is inspired by a paper by Ramm [12] which uses a generalization of Schiffer's method. A main ingredient is a nonsmooth version of Green's theorem. First, we define a notion of (weak) solutions to (1)–(5) for nonsmooth domains.

DEFINITION 3.1. *A function $u \in H^2_{\mathrm{loc}}(\Omega') \cap H^1(\Omega'_R)$ is said to be a (weak) solution of (1)–(5) if it satisfies (5) and*

$$\int_{\Omega'} \left(\mu \nabla u \nabla \varphi - \kappa^2 u \varphi\right) dx = 0, \qquad \forall \varphi \in H^2_{\mathrm{loc}} \cap H^1_0(\Omega'),$$

where

$$\mu(x) = \begin{cases} 1, & x \in \Omega^+, \\ \lambda, & x \in \Omega^-, \end{cases} \qquad \kappa^2(x) = \begin{cases} k^2, & x \in \Omega^+, \\ \lambda k_0^2, & x \in \Omega^-, \end{cases}$$

$\Omega' = R^3 \setminus \Omega$, H^1, H^2_{loc}, *and* H^1_0 *denote the usual Sobolev spaces, and* $H^1(\Omega'_R) \equiv H^1(\Omega' \cap B_R)$, *where* B_R *is a ball centered at the origin of sufficiently large radius* R. *Note that this definition does not require any smoothness on the boundary.*

We also use the following notation:

$\Omega_{12} = \Omega_1 \cup \Omega_2$, $\Omega^{12} = \Omega_1 \cap \Omega_2$;
$\Gamma_1 = \partial\Omega_1$, $\Gamma_2 = \partial\Omega_2$, $\Gamma_{12} = \partial\Omega_{12}$, $\Gamma^{12} = \partial\Omega^{12}$;
$\Gamma'_1 = \Gamma_1 \setminus \Omega_2$, $\Gamma'_2 = \Gamma_2 \setminus \Omega_1$;
$\tilde{\Omega}_1$ is a connected component of $\Omega_1 \setminus \Omega^{12}$, and $\Omega_3 = \Omega_{12} \setminus \Omega^{12}$.

Now, suppose (u_j, v_j), $j = 1, 2$, are solutions of (1)–(5) corresponding to the cores Ω_j and have the same far field pattern, i.e., $u_{1\infty} = u_{2\infty}$. Then, by Rellich's lemma [6,8], $u_1 \equiv u_2$ in Ω^+. By the boundary condition (3), it follows from Holmgren's uniqueness theorem [13] that

$$v_1(x) \equiv v_2(x) := V(x), \qquad x \in \Omega^- \setminus \Omega_{12}. \tag{6}$$

The function V can be continued analytically, as a solution to (2), to the domains Ω_3 and $\Omega^- \setminus \Omega^{12}$, because either v_1 or v_2 is defined in these domains and solve (2) there. Except possibly on $\Gamma_1 \cap \Gamma_2$, V satisfies the Neumann boundary condition (5) on the boundary Γ_3 of Ω_3. However, we note that Ω_3 is in general not smooth, and indeed, not even Lipschitz. Thus, to complete the proof, we need the generalization of Green's formula to domains with nonsmooth boundary [12,14–17]. Before we state this result, we recall some relevant concepts.

DEFINITION 3.2. *The space* $BV(\mathcal{D})$ *of functions of bounded variation on* $\mathcal{D} \subset R^n$ *consists of locally integrable functions on* $\mathcal{D}$ *whose first-order partial derivatives, in the sense of distributions, are (signed) measures with finite total variation.*

DEFINITION 3.3. *A set* $\mathcal{D} \subset R^n$ *is said to have finite perimeter if the characteristic function* $\chi(\mathcal{D})$ *belongs to* $BV(R^n)$. *The perimeter is then defined to be the total variation of* $\nabla\chi(\mathcal{D})$.

Next, we need the notion of the normal to a nonsmooth boundary. For fixed $x, \nu \in R^n$, $\nu \neq 0$, we denote

$$A^{\pm} := \{y : \pm(y - x) \cdot \nu > 0\}, \qquad A^0 := \{y : (y - x) \cdot \nu = 0\}.$$

DEFINITION 3.4. *A unit vector* ν *is a normal to* $\partial\mathcal{D}$ *at the point* $x \in \partial\mathcal{D}$ *in the sense of Federer if*

$$\lim_{\rho \to 0} \rho^{-n} \ell_n \left(\mathcal{D} \cap B_\rho(x) \cap A^+\right) = 0,$$
$$\lim_{\rho \to 0} \rho^{-n} \ell_n (\mathcal{D}' \cap B_\rho(x) \cap A^-) = 0,$$

where ℓ_n *is the Lebesgue measure on* R^n, $B_\rho(x)$ *is the ball centered at* x *with radius* ρ, *and* $\mathcal{D}' = R^n \setminus \mathcal{D}$.

Using this, we can define the reduced boundary to a nonsmooth domain $\mathcal{D}$.

DEFINITION 3.5. *The set of points* $x \in \partial\mathcal{D}$ *for which the normal in the sense of Federer exists is called the reduced boundary of* $\mathcal{D}$ *and is denoted by* $\partial^*\mathcal{D}$.

We recall the following result.

LEMMA 3.1. *(See [18].) If a set* $\mathcal{D}$ *has finite perimeter and the boundary* $\partial\mathcal{D}$ *has full* $(n-1)$*-dimensional Hausdorff measure, then the normal in the sense of Federer is defined almost everywhere on* $\partial\mathcal{D}$ *with respect to the* $(n-1)$*-dimensional Hausdorff measure.*

THEOREM 3.1. GREEN'S THEOREM FOR NONSMOOTH DOMAIN. *(See [12].) Let Ω_3 be a domain with finite perimeter and let ψ be a function defined on Ω_3 whose first derivatives are in the space BV such that their rough traces are summable on the reduced boundary Γ_3^* of Ω_3 with respect to the $(n-1)$-dimensional Hausdorff measure. Then we have*

$$\int_{\Omega_3} \nabla \cdot \psi(x)\, dx = \int_{\Gamma_3^*} \psi(x)\nu(x)\, ds(x), \tag{7}$$

where $\nu(x)$ is the normal on Γ_3^*.

To complete the proof, consider the function

$$\psi := V_1 \nabla \bar{V}_2 - V_2 \nabla \bar{V}_1, \tag{8}$$

where V_j is the solution of (2) corresponding to the interior wavenumber k_{0j} extended to the domain Ω_3 (see (6)). It is a standard result that Ω_3 has finite perimeter. Also, as solutions to the Helmholtz equation in domains with smooth boundaries, $V_j \in H^1(\Omega_3)$ and

$$\nabla \cdot \psi = V_1 \Delta \bar{V}_2 - V_2 \Delta \bar{V}_1 = \left(k_{01}^2 - k_{02}^2\right) V_1 \bar{V}_2. \tag{9}$$

It follows that $V_1 \bar{V}_2 \in L^1(\Omega_3)$ and $\nabla \cdot \psi$ is a signed measure on Ω_3. We refer the reader to [12] to check that ψ has a summable rough trace on Γ_3^*.

Hence, by Theorem 3.1,

$$\int_{\Omega_3} \left(V_1 \Delta \bar{V}_2 - \bar{V}_2 \Delta V_1\right) dx = \left(k_{01}^2 - k_{02}^2\right) \int_{\Omega_3} V_1 \bar{V}_2\, dx = 0, \tag{10}$$

as V_j satisfies the Neumann boundary condition on the reduced boundary Γ_3^*. This implies that the functions V_j corresponding to different interior wavenumbers k_{0j} are orthogonal in the Hilbert space $L^2(\Omega_3)$. Now, by Rellich's lemma, as $u^i(\cdot, d, k_{0j})$ are distinct, so are V_j. This contradicts the separability of $L^2(\Omega_3)$. We conclude that

$$\Omega_1 \equiv \Omega_2.$$

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