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Note

On recursive bounds for the exceptional values in speed-up

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Abstract

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This note contains a proof that there is no recursive function of the initial index that gives a bound for the exceptional values in Blum speed-up, but that there is a recursive bounding function of the speed-up index. All the proofs given are constructive.

1. Introduction

The literature on the speed-up theorem contains several references to the nonexistence of a recursive bound for the exceptional values in the speed-up; but the only result on this topic seems to be the one in Schnorr [14, 15], which deals with a simultaneous recursive bound for both the speed-up index and the exceptional values. This note shows that, in general, there is no recursive function of the initial index that gives a bound for the exceptional values in speed-up; but that if the

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bounding function is taken as a function of the speed-up index, then it can be chosen to be recursive.

We shall assume familiarity with, or access to, Bridges [4], Calude [5], Machtey and Young [11], or Salomaa [13].

Let $\mathbb{N}=\{0,1,\ldots\}$ be the set of natural numbers, and let $(\varphi_i)_{i\in\mathbb{N}}$ be an acceptable gödelization of the set of unary partial recursive (p.r.) functions from $\mathbb{N}$ to $\mathbb{N}$. The domain of the p.r. function φ is denoted by domain(φ). The relation $i\in$domain(φ) is abbreviated by $\varphi(i)\downarrow$. If $f=\varphi_i$, then i is an *index* of f.

We shall make use of the following result.

Double Recursion Theorem (Smullyan [17]). *If i_0, i_1 are natural numbers, then there effectively exist two recursive functions $g_t: \mathbb{N}^2 \to \mathbb{N}$ such that*

$$\varphi_{g_t(i,z)}(x) = \varphi_{i_t}(g_0(i,z), g_1(i,z), x),$$

for all $i,z,x\in\mathbb{N}$ and for $t=0,1$.

A sequence $\Gamma=(\gamma_i)_{i\in\mathbb{N}}$ of p.r. functions is called a *complexity measure* (with respect to the acceptable gödelization $(\varphi_i)_{i\in\mathbb{N}}$) if the following two axioms—*Blum's axioms*—are satisfied:

- domain(γ_i)=domain(φ_i) for all i.
- the ternary predicate

$$\text{costs}(i,x,y)\begin{cases} =1 & \text{if } \gamma_i(x)\leqslant y,\\ =0 & \text{otherwise}\end{cases}$$

is recursive.

By a *speed-up factor* we mean a recursive function on $\mathbb{N}^2$ that is increasing in its second argument. A fundamental result in abstract complexity theory is the following.

Speed-up Theorem (Blum [1]). *If Γ is a complexity measure and F is a speed-up factor, then there exists a recursive function f with the following property: for each index i of f there exists an index j of f such that $F(n,\gamma_j(n))\leqslant\gamma_i(n)$ for all sufficiently large values of n.*

A recursive function f satisfying the conclusion of the Speed-up Theorem is called an *F-speedable function.* In the speed-up inequality

$$F(n,\gamma_j(n))\leqslant\gamma_i(n),$$

the index i will be called the *initial index*, and the index j the *speed-up index*. The finite set of natural numbers n for which the speed-up inequality fails to hold is called the set of *exceptional values* in the speed-up.

Presentations of the Speed-up Theorem are given by Blum [1], Bridges [4], Calude [5], Hartmanis and Hopcroft [8], Machtey and Young [11], van Emde Boas [18], Salomaa [13], Seiferas [16] and Young [19]. The topological analysis in Calude et al. [6] shows that speedable functions form a fairly large class of recursive functions, so the speed-up phenomenon is by no means esoteric or artificial.

For the rest of this paper we fix an acceptable gödelization $(\varphi_i)_{i\in\mathbb{N}}$ and a complexity measure $\Gamma=(\gamma_i)_{i\in\mathbb{N}}$.

2. The main results

Blum [2] has proved that in speed-up the better programs cannot be obtained effectively from the given ones: the speed-up index j of f cannot be computed as a recursive function of the initial index i. As the speed-up is iterated, the size of the increasingly better programs increases; it is sometimes [2], but not always [9, 12], possible to bound the size of the speeded-up program (the one corresponding to the index j) as a recursive function of the size of the initial program (the one corresponding to the index i).

Similarly, the number of exceptional values must increase as we iterate the speed-up: otherwise, we would obtain an infinitely descending sequence of complexities, which is impossible.

Schnorr [14] has proved that there is no simultaneous recursive bound for both the speed-up index and the exceptional values. In this section we address the question: *Can we compute a bound for the exceptional values in speed-up*? We shall show that the answer depends on whether we want the bound to be given by a recursive function of the initial index i or by a recursive function of the speed-up index j: in the former case the answer is "no", whereas in the latter it is "yes".

Theorem 1. *There exists a total recursive function $B:\mathbb{N}\to\mathbb{N}$ with the following property: if F is a speed-up factor such that $F(n,0)>B(n)$ for each n, if f is a binary F-speedable function, and if $\varphi_s(i)\downarrow$ for each index i of f, then there exist an index k of f, and a natural number $m>\varphi_s(k)$, such that $\gamma_k(m)<F(m,\gamma_j(m))$ for each index j of f.*

Proof. Define the p.r. function $E:\mathbb{N}^5\to\mathbb{N}$ by

$$E(u,v,i,z,s)=1+\max\{i,z,s,\varphi_s(u),\gamma_s(u),\varphi_s(v),\gamma_s(v)\}.$$

Applying the Double Recursion Theorem to the p.r. functions

$$\varphi_{i_t}(u,v,i,z,s,x)\begin{cases}=t & \text{if } x=E(u,v,i,z,s),\\ =\varphi_i(x) & \text{otherwise,}\end{cases}$$

we obtain two recursive functions $g_t:\mathbb{N}^3\to\mathbb{N}$ satisfying the equations

$$\varphi_{g_t(i,z,s)}(x)\begin{cases}=t & \text{if } x=E(g_0(i,z,s),g_1(i,z,s),i,z,s),\\ =\varphi_i(x) & \text{otherwise}\end{cases}$$

for $t=0,1$ and for all $i,x,z\in\mathbb{N}$. Setting

$$e(i,z,s)\equiv E(g_0(i,z,s),g_1(i,z,s),i,z,s),$$

note that the predicate

$$e(i,z,s)=x$$

is recursive, that $e(i,z,s)>\max\{i,z,s\}$ and that $\varphi_{g_t(i,z,s)}(e(i,z,s))=t$. It readily follows that

$$B(x)=1+\max\{\gamma_{g_t(i,z,s)}(x)\colon t=0,1;\ e(i,z,s)=x;\ i,z,s\in\mathbb{N}\}$$

defines a total recursive function $B\colon\mathbb{N}\to\mathbb{N}$.

Given a speed-up factor F such that $F(x,0)\geqslant B(x)$ for all natural numbers x, now consider any F-speedable binary function f, and any p.r. function φ_s whose domain contains each index of f. Choosing an index i of f, set

$$k\equiv g_{f(m)}(i,i,s),\quad m\equiv e(i,i,s).$$

Then

- $\varphi_k=f$. For we have $\varphi_k(x)=\varphi_{g_{f(m)}(i,i,s)}(x)=\varphi_i(x)=f(x)$ whenever $x\neq m$, and $\varphi_k(m)=\varphi_{g_{f(m)}(i,i,s)}(e(i,i,s))=f(m)$.
- $\varphi_s(k)\downarrow$ and $m>\varphi_s(k)$. For k is an index of f; $\varphi_s(k)=\varphi_s(g_{f(m)}(i,i,s))$; and

$$\begin{aligned}m=e(i,i,s)=1+\max\{i,\ &\varphi_s(g_0(i,i,s)),\gamma_s(g_0(i,i,s)),\\&\varphi_s(g_1(i,i,s)),\gamma_s(g_1(i,i,s))\}>\varphi_s(k).\end{aligned}$$

- $\gamma_k(m)<B(m)$. For

$$\begin{aligned}\gamma_k(m)&=\gamma_{g_{f(m)}(i,i,s)}(e(i,i,s))\\&<1+\max\{\gamma_{g_t(j,z,w)}(x);\ t=0,1;\ e(i,i,s)=e(j,z,w);\ j,z,w\in\mathbb{N}\}\\&\leqslant B(m).\end{aligned}$$

It now follows that for each index j of f we have

$$\gamma_k(m)<B(m)\leqslant F(m,0)\leqslant F(m,\gamma_j(m))$$

and $m>\varphi_s(k)$. $\square$

Corollary 1. *Let the recursive function B be as in Theorem* 1, *and let F be a speed-up factor satisfying $F(n,0)\geqslant B(n)$ for all n. There is no partial recursive function θ such that if φ_i is any F-speedable recursive function, then $\theta(i)\downarrow$ and there exists a speed-up index j for φ_i such that $F(n,\gamma_j(n))\leqslant\gamma_i(n)$ whenever $n\geqslant\theta(i)$. (In fact, there is no partial recursive bound $\theta(i)$ for the exceptional values in the speed-up of F-speedable* binary *functions φ_i).*

Proof. Suppose such a partial recursive θ exists, let f be a binary F-speedable function, and apply Theorem 1 with $\varphi_s=\theta$ to obtain a contradiction. $\square$

The restriction on the size of the speed-up factor cannot be removed from the statement of Corollary 1, since for a sufficiently small speed-up we can get a recursive bound for the exceptional values; see [10].

Corollary 1 can be deduced from Theorem 2 of [7], since every F-speedable function is $F(n, 0)$-complex. The proof of Fulk's theorem is more complicated than ours and at one stage uses a nonconstructive argument, whereas *our proofs are fully constructive.*

The conclusion of Corollary 1 holds if the range of φ_i is finite. But, as can be seen by inspecting the proof of the Speed-up Theorem in Machtey and Young [11], there is a recursive bound $\theta(i)$ for the exceptional values in the speed-up of a function φ_i with infinite range[1]; see also Proposition 1 below.

Let $\mathbb{P}$ denote the set of all partial recursive functions from $\mathbb{N}$ to $\mathbb{N}$. A partial function $\mathscr{F}:\mathbb{P}\to\mathbb{P}$ is called an *effective operator* on $\mathbb{P}$ if there exists a p.r. function $\psi:\mathbb{N}\to\mathbb{N}$ such that for each $\varphi_i \in \text{domain}(\mathscr{F})$,

(i) $i \in \text{domain}(\psi)$ and

(ii) $\mathscr{F}(\varphi_i)(n) = \varphi_{\psi(i)}(n)$ for all $n \in \mathbb{N}$.

The effective operator is said to be *total* if for each total recursive function φ_i, $\varphi_i \in \text{domain}(\mathscr{F})$ and $\mathscr{F}(\varphi_i)$ is total.

The following theorem is due to Meyer and Fischer.

Operator Speed-up Theorem (Meyer and Fischer [12]). *Let $\mathscr{F}$ be a total effective operator on $\mathbb{P}$. There exists a recursive function $f:\mathbb{N}\to\mathbb{N}$ with the following property: For each index i of f there exists an index j of f such that $\mathscr{F}(\gamma_j)(n) < \gamma_i(n)$ for all sufficiently large n.*

The recursive function f in the conclusion of the Operator Speed-up Theorem is said to be *$\mathscr{F}$-speedable*, and the index j is called a *speed-up index* of f. The proof of the following result on bounds for the exceptional values in operator speed-up is similar to that of Corollary 1 and is left to the reader.

Corollary 2. *Let the recursive function B be as in Theorem* 1, *and let $\mathscr{F}$ be a total effective operator on $\mathbb{P}$ such that $\mathscr{F}(\varphi_i)(n) > B(n)$ for all total recursive functions φ_i and for all n. There is no partial recursive function θ such that if φ_i is any $\mathscr{F}$-speedable recursive function, then $\theta(i)\downarrow$ and there exists a speed-up index j for φ_i such that $\mathscr{F}(\gamma_j)(n) \leqslant \gamma_i(n)$ whenever $n \geqslant \theta(i)$.*

The following result was proved by Schnorr [14; 15, Satz 9.35] for a general recursive function f. Our version of it shows that the nonexistence of a simultaneous bound for the speed-up index and the exceptional values occurs even at the level of binary recursive functions. Moreover, our proof, unlike Schnorr's, is constructive throughout.

[1] Of course, there is no algorithm for deciding, for a given recursive function f, whether the range of f is finite or infinite.

Corollary 3. *Let the recursive function B be as in Theorem* 1, *let F be a speed-up factor satisfying $F(n,0)>B(n)$ for all n, and let f be a* binary *F-speedable function. There is no partial recursive function $\theta:\mathbb{N}\to\mathbb{N}$ with the following property: for each index i of f, $\theta(i)$ is defined, and there exists an index j of f such that* (i) $j\leqslant\theta(i)$ *and* (ii) $F(n,\gamma_j(n))\leqslant\gamma_i(n)$ *for all $n\geqslant\theta(i)$.*

Proof. Suppose such a partial recursive function θ exists. Taking s as an index of θ in Theorem 1, we obtain an index i of f, and a natural number $m>\theta(i)$, such that $\gamma_i(m)<F(m,\gamma_j(m))$ for each index j of f. Choosing an index j of f such that (i) and (ii) hold, we immediately obtain a contradiction. □

Having disposed of the negative aspect of our problem, we now turn to the positive one. The following lemma prepares the way for the proof of the existence of a recursive bound for the exceptional values in the speed-up of certain recursive functions with infinite range.

Lemma 1. *There exists a speed-up factor F_0 with the following property: for all natural numbers i, k there exists $j>k$ such that* (i) $\varphi_j=\varphi_i$ *and (ii) $\gamma_j(n)\leqslant F_0(n,\gamma_i(n))$ whenever $\varphi_i(n)$ is defined and $n\geqslant\max\{i,j\}$.*

Proof. Let $\gamma_i^*(n)$ be the number of cells read by the read/write head of Turing machine number i in the computation of $\varphi_i(n)$. By the Recursive Relatedness Theorem [4, (6.4)], there exists a speed-up factor $G:\mathbb{N}^2\to\mathbb{N}$ such that for all i, and for all $n\geqslant i$ for which $\varphi_i(n)$ is defined, we have both $\gamma_i(n)\leqslant G(n,\gamma_i^*(n))$ and $\gamma_i^*(n)\leqslant G(n,\gamma_i(n))$. Set

$$F_0(n,k)\equiv G(n,G(n,k)).$$

Given i and k, choose $j>k$ such that $\varphi_j=\varphi_i$ and $\gamma_i^*=\gamma_j^*$. If $n\geqslant\max\{i,j\}$, and $\varphi_i(n)$ is defined, then

$$\gamma_j(n)\leqslant G(n,\gamma_j^*(n))=G(n,\gamma_i^*(n))\leqslant G(n,G(n,\gamma_i(n)))=F_0(n,\gamma_i(n)). \qquad \square$$

Proposition 1. *For each speed-up factor F there exists an F-speedable function f with the following property: for each index i of f there exists an index j of f such that $F(n,\gamma_j(n))\leqslant\gamma_i(n)$ for all $n\geqslant j$.*

Proof. Let F_0 be as in Lemma 1, and let F be a speed-up factor. Let f be speedable relative to the function $(n,k)\mapsto F(n,F_0(n,k))$, and let $\varphi_i=f$. Choose l and k such that $\varphi_l=f$ and $F(n,F_0(n,\gamma_l(n)))\leqslant\gamma_i(n)$ for all $n\geqslant k$. By our choice of F_0, there exists $j>\max\{l,k\}$ such that $\varphi_j=f$ and such that for all $n\geqslant j$,

$$F(n,\gamma_j(n))\leqslant F(n,F_0(n,\gamma_l(n)))\leqslant\gamma_i(n). \qquad \square$$

Theorem 2. *Let F be a speed-up factor. For each F-speedable function f, and each index i of f, there exists an index j of f such that $\gamma_i(n)\geqslant F(n,\gamma_j(n))$ whenever $n>j$.*

Proof. Start with an acceptable gödelization $\Phi = \varphi_0, \varphi_1, \ldots$ and a complexity measure $\Gamma = \gamma_0, \gamma_1, \ldots$, and introduce a new acceptable gödelization $\Phi^* \equiv \varphi_0^*, \varphi_1^*, \ldots$ and a corresponding complexity measure $\Gamma^* \equiv \gamma_0^*, \gamma_1^*, \ldots$ as follows. Φ^* is the sequence

$$\varphi_0, \varphi_1, \varphi_0, \varphi_0, \varphi_1, \varphi_1, \varphi_2, \varphi_0, \varphi_0, \varphi_0, \varphi_1, \varphi_1, \varphi_1, \varphi_2, \varphi_2, \varphi_3, \ldots$$

and Γ^* is the corresponding sequence

$$\gamma_0, \gamma_1, \gamma_0, \gamma_0, \gamma_1, \gamma_1, \gamma_2, \gamma_0, \gamma_0, \gamma_0, \gamma_1, \gamma_1, \gamma_1, \gamma_2, \gamma_2, \gamma_3, \ldots$$

Clearly, there is a recursive function $g: \mathbb{N} \to \mathbb{N}$ such that for each n,

$$\varphi_n^* = \varphi_{g(n)} \quad \text{and} \quad \gamma_n^* = \gamma_{g(n)}.$$

Given any F-speedable function f, and any index i of f, choose k, N such that $F(n, \gamma_k(n)) \leqslant \gamma_i(n)$ for all $n \geqslant N$. Then choose an index m such that $g(m) \geqslant N$, $\varphi_m^* = \varphi_k$, and $\gamma_m^* = \gamma_k$. Setting $j \equiv g(m)$, for all $n \geqslant j$ we have

$$F(n, \gamma_j(n)) = F(n, \gamma_m^*(n)) = F(n, \gamma_k(n)) \leqslant \gamma_i(n).$$

This completes the proof. $\square$

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