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Some integral inequalities for functions with $(n-1)$st derivatives of bounded variation

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Abstract

In this paper, we generalize Cerone's results, and a unified treatment of error estimates for a general inequality satisfying $f^{(n-1)}$ being of bounded variation is presented. We derive the estimates for the remainder terms of the mid-point, trapezoid, and Simpson formulas. All constants of the errors are sharp. Applications in numerical integration are also given.

Keywords: Bounded variation; Appell type polynomial; Bernoulli polynomial; Ostrowski's inequality; Trapezoidal inequality

1. Introduction

In 2000, Cerone, Dragomir and Pearce [1] proved the following trapezoid type inequalities.

Theorem 1. *Let $f : [a, b] \to R$ be a function of bounded variation. Then we have the inequality*

$$\left|\int_a^b f(t)\,\mathrm{d}t - [(x-a)f(a) + (b-x)f(b)]\right| \le \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{1.1}$$

for all $x \in [a, b]$, where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$.

The inequality (1.1) is a perturbed generalization of the trapezoidal inequality for mapping of bounded variation. Using (1.1), Cerone et al. further obtained the following error estimate for the composite quadrature rule.

Theorem 2. *Let f be defined as in Theorem 1; then we have*

$$\int_a^b f(t)\,\mathrm{d}t = \sum_{i=0}^{n-1}[(\xi_i - x_i)f(x_i) + (x_{i+1} - \xi_i)f(x_{i+1})] + R(f). \tag{1.2}$$

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The remainder term $R(f)$ satisfies the estimate

$$|R(f)| \le \left[\frac{\nu(l)}{2} + \max_{i=0,1,\dots,n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b (f) \le \nu(l) \bigvee_a^b (f), \tag{1.3}$$

where $\nu(l) := \max\{l_i | i = 0, 1, \dots, n-1\}$, $l_i = x_{i+1} - x_i$ *and* $\xi_i \in [x_i, x_{i+1}]$.

In this paper, following the main ideas of Vinogradov [2], we give a unified treatment of error estimates for a general quadrature rule satisfying $f^{(n-1)}$ being of bounded variation. Using the perturbed inequality, we obtain the error bounds for the mid-point, trapezoid and Simpson quadrature formulas. We also generalize Euler trapezoid formulas [3].

2. The main results

A sequence of polynomials $\{u_k\}_0^\infty$ is called a sequence of Appell type polynomials if $u_0 = 1$, $u_k' = u_{k-1} (k \in Z_+)$.

Lemma 1. *Let $f : [a,b] \to R$ be such that $f^{(n-1)}$ is a function of bounded variation on $[a,b]$ for some $n \ge 1$, $n \in Z_+$. Moreover, if $n = 1$, $f(t)$ is continuous at x, $x \in [a,b]$. Suppose that $\{r_k\}$, $\{s_k\}$ are sequences of Appell type polynomials on $[a,x)$ and $\{u_k\}$, $\{v_k\}$ are sequences of Appell type polynomials on $(x,b]$. Let $m \in N$, $m \le n$,*

$$k_n(x,t) = \begin{cases} p_n(t) = r_{n-m}(t) s_m(t), & t \in [a,x); \\ q_n(t) = u_{n-m}(t) v_m(t), & t \in (x,b]. \end{cases}$$

Then we have the following equality:

$$\int_a^b f(t)\,\mathrm{d}t - \frac{(-1)^n}{C_n^m} \int_a^b k_n(x,t)\,\mathrm{d}f^{(n-1)}(t)$$

$$= \begin{cases} \dfrac{1}{C_n^m} \displaystyle\sum_{k=0}^{n-1} (-1)^{n-1-k} \Big[q_n^{(k)}(b) f^{(n-1-k)}(b) \\ \quad - q_n^{(k)}(a+) f^{(n-1-k)}(a) \Big], & x = a; \\ \dfrac{1}{C_n^m} \displaystyle\sum_{k=0}^{n-1} (-1)^{n-1-k} \Big[(p_n^{(k)}(x-) - q_n^{(k)}(x+)) f^{(n-1-k)}(x) \\ \quad + q_n^{(k)}(b) f^{(n-1-k)}(b) - p_n^{(k)}(a) f^{(n-1-k)}(a) \Big], & x \in (a,b); \\ \dfrac{1}{C_n^m} \displaystyle\sum_{k=0}^{n-1} (-1)^{n-1-k} \Big[p_n^{(k)}(b-) f^{(n-1-k)}(b) \\ \quad - p_n^{(k)}(a) f^{(n-1-k)}(a) \Big], & x = b, \end{cases}$$

where $C_n^m = \frac{n!}{m!(n-m)!}$.

Proof. Integrating by parts in the sense of Riemann and Stieltjes, we can easily obtain Lemma 1. □

Remark 1. Actually, $f^{(n-1)}$ is continuous if it is of bounded variation when $n > 1$. If $k_1(x,t)$ is continuous at x, we can weaken the conditions of Lemma 1. In this case, it is not necessary that $f(t)$ is continuous at x.

Theorem 3. *Let f be defined as in Lemma 1. Suppose that $m \in N$, $n \in Z_+$, $m \le n$ and $\lambda \in [0,1]$. Then we have*

$$\Bigg| \int_a^b f(t)\,\mathrm{d}t - \frac{1}{C_n^m} \sum_{j=0}^{n-1} \left[\sum_{i=L}^{U} C_j^i C_{n-j}^{n-m-i} (1-\lambda)^{m-j+i} \right] \frac{(b-x)^{n-j} - (a-x)^{n-j}}{(n-j)!} f^{(n-1-j)}(x)$$

$$- \frac{1}{C_n^m} \sum_{j=n-m}^{n-1} C_j^{n-m} \lambda^{n-j} \frac{(x-a)^{n-j} f^{(n-1-j)}(a) - (x-b)^{n-j} f^{(n-1-j)}(b)}{(n-j)!} \Bigg|$$

$$\leq \frac{\sup\limits_{0<\tau<1} \tau^{n-m}|\tau-\lambda|^m}{n!} \bigvee_a^b (f^{(n-1)}) \max\{|x-a|^n, |x-b|^n\}, \tag{2.1}$$

where $U = \min\{j, n-m\}$, $L = \max\{0, j-m\}$.

Proof. Let

$$k_n(x,t) = \begin{cases} p_n(t) = \dfrac{(t-a)^{n-m}(t-\alpha)^m}{(n-m)!m!}, & t \in [a,x); \\ q_n(t) = \dfrac{(t-b)^{n-m}(t-\beta)^m}{(n-m)!m!}, & t \in (x,b], \end{cases}$$

and $\alpha = \lambda x + (1-\lambda)a$, $\beta = \lambda x + (1-\lambda)b$.

Thus, it follows from a straightforward calculation that

$$p_n^{(j)}(x+) = \sum_{i=L}^{U} C_j^i C_{n-j}^{n-m-i} \frac{(1-\lambda)^{m-j+i}(x-a)^{n-j}}{(n-j)!},$$

$$q_n^{(j)}(x-) = \sum_{i=L}^{U} C_j^i C_{n-j}^{n-m-i} \frac{(1-\lambda)^{m-j+i}(x-b)^{n-j}}{(n-j)!}.$$

On the other hand, we have

$$\begin{aligned} \left|\int_a^b k_n(x,t)\,\mathrm{d}f^{(n-1)}(t)\right| &= \left|\int_a^x \frac{(t-a)^{n-m}(t-\alpha)^m}{m!(n-m)!}\,\mathrm{d}f^{(n-1)}(t) + \int_x^b \frac{(t-b)^{n-m}(t-\beta)^m}{m!(n-m)!}\,\mathrm{d}f^{(n-1)}(t)\right| \\ &\leq \frac{\sup\limits_{a<t<x} |(t-a)^{n-m}(t-\alpha)^m|}{m!(n-m)!} \int_a^x |\,\mathrm{d}f^{(n-1)}(t)| \\ &\quad + \frac{\sup\limits_{x<t<b} |(t-b)^{n-m}(t-\beta)^m|}{m!(n-m)!} \int_x^b |\,\mathrm{d}f^{(n-1)}(t)| \\ &= \frac{\sup\limits_{0<\tau<1} \tau^{n-m}|\tau-\lambda|^m}{m!(n-m)!} \left[\bigvee_a^x (f^{(n-1)})|x-a|^n + \bigvee_x^b (f^{(n-1)})|x-b|^n\right] \\ &\leq \frac{\sup\limits_{0<\tau<1} \tau^{n-m}|\tau-\lambda|^m}{(n-m)!m!} \bigvee_a^b (f^{(n-1)}) \max\{|x-a|^n, |x-b|^n\}. \end{aligned}$$

According to Lemma 1, we derive (2.1) and the proof is completed. □

Remark 2. Theorem 3 contains many classical formulas. The advantage of this theorem is that we have three parameters λ, x and m to choose.

Corollary 1. *Let* f *be defined as in* Theorem 3. *Suppose that* $n \in Z_+$, $\lambda \in [0,1]$. *Then we have*

$$\begin{aligned} &\left|\int_a^b f(t)\,\mathrm{d}t - \frac{1}{n}\left\{\lambda(x-a)f(a) + \lambda(b-x)f(b) + \sum_{j=0}^{n-1}\left(\sum_{i=max\{0,j-1\}}^{j} C_j^i C_{n-j}^{n-1-i}(1-\lambda)^{i+1-j}\right)\right.\right. \\ &\left.\left.\cdot \frac{(b-x)^{n-j}-(a-x)^{n-j}}{(n-j)!} f^{(n-1-j)}(x)\right\}\right| \leq C_n \bigvee_a^b (f^{(n-1)}) \max\{|x-a|^n, |x-b|^n\} \end{aligned} \tag{2.2}$$

where

$$C_n = \begin{cases} \max\{\lambda, 1-\lambda\}, & n = 1; \\ \dfrac{1}{n!}\max\left\{\dfrac{n-1}{n^n}\lambda^n, 1-\lambda\right\}, & n > 1. \end{cases}$$

Proof. Let $g(\tau) = \tau^{n-1}(\tau - \lambda)$, $\tau \in [0, 1]$. Hence

$$g'(\tau) = \begin{cases} 1, & n = 1; \\ \tau^{n-2}(n\tau - \lambda), & n > 1. \end{cases}$$

Clearly, we can obtain the following inequality:

$$|g(\tau)| \le \begin{cases} \max\{|g(0)|, |g(1)|\} = \max\{\lambda, 1-\lambda\}, & n = 1; \\ \max\left\{\left|g\left(\frac{\lambda}{n}\right)\right|, |g(1)|\right\} = \max\left\{\frac{n-1}{n^n}\lambda^n, 1-\lambda\right\}, & n > 1. \end{cases}$$

Therefore, setting $m = 1$ in Theorem 3 we have (2.2) and the proof is completed. □

Corollary 2. *Let f be defined as in Theorem 3. Suppose that $n \in Z_+$ and $0^0 = 1$. Then we have*

$$\left|\int_a^b f(t)\,\mathrm{d}t - \frac{1}{n}\left\{(b-a)f(x) + \sum_{j=1}^{n-1} \frac{n-j}{j!}[(x-b)^j f^{(j-1)}(b) - (x-a)^j f^{(j-1)}(a)]\right\}\right|$$
$$\le \frac{(n-1)^{n-1}}{n^n n!} \bigvee_a^b (f^{(n-1)}) \max\{|x-a|^n, |x-b|^n\}. \tag{2.3}$$

Proof. We consider the case $m = n - 1, \alpha = \beta = x$ in Theorem 3. In this case, we can get $\lambda = 1$. Let $g(\tau) = \tau(\tau - 1)^{n-1}$, $\tau \in [0, 1]$. Hence

$$g'(\tau) = \begin{cases} 1, & n = 1; \\ (\tau-1)^{n-2}(n\tau - 1), & n > 1. \end{cases}$$

We can obtain the following inequality:

$$|g(\tau)| \le \begin{cases} 1, & n = 1; \\ \left|g\left(\frac{1}{n}\right)\right| = \frac{(n-1)^{n-1}}{n^n}, & n > 1. \end{cases}$$

Therefore, by (2.1) the corollary is proved. □

Corollary 3. *Let f be defined as in Theorem 3, $n \in Z_+$, $\lambda \in [0, 1]$. Then*

$$\left|\int_a^b f(t)\,\mathrm{d}t - \sum_{j=0}^{n-1} \frac{(1-\lambda)^{n-j}[(b-x)^{n-j} - (a-x)^{n-j}]}{(n-j)!} f^{(n-1-j)}(x)\right.$$
$$\left. - \sum_{j=0}^{n-1} \frac{\lambda^{n-j}}{(n-j)!}\left[(x-a)^{n-j} f^{(n-1-j)}(a) - (x-b)^{n-j} f^{(n-1-j)}(b)\right]\right|$$
$$\le \frac{1}{n!} \max\{\lambda^n, (1-\lambda)^n\} \bigvee_a^b (f^{(n-1)}) \max\{|x-a|^n, |x-b|^n\}. \tag{2.4}$$

Proof. We take $m = n$ in Theorem 3, and the corollary is proved. □

Remark 3. For $n = 1$, we have

$$\left|\int_a^b f(t)\,\mathrm{d}t - (1-\lambda)(b-a)f(x) - \lambda(x-a)f(a) - \lambda(b-x)f(b)\right|$$
$$\le \max\{\lambda, 1-\lambda\} \bigvee_a^b (f)\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]. \tag{2.5}$$

Choosing $\lambda = 1$, we can obtain (1.1). Furthermore, when $x = (a+b)/2$, for $\lambda = 0$, $\lambda = 1$ and $\lambda = \frac{1}{3}$ we obtain the estimates for the errors of the mid-point rule, trapezoid rule and Simpson rule respectively.

Theorem 4. *Suppose that* $X := \{x_i \mid i = 0, 1, \ldots, k-1, k \in Z_+\}$ *is a set of* k *points satisfying* $a \le x_0 < x_1 < \cdots < x_{k-1} \le b$. *Let* $p_i \ge 0$, $\sum_{i=0}^{k-1} p_i = 1$, *and* $f^{(n-1)}$ *be a function of bounded variation. Moreover, when* $n = 1$, $f(t)$ *is continuous at* x_i, $i = 0, 1, \ldots, k-1$, *then we have*

$$\left|\int_a^b f(t)\,\mathrm{d}t - (b-a)\sum_{i=0}^{k-1} p_i f(x_i) + \sum_{j=1}^{n-1} \frac{(b-a)^j [f^{(j-1)}(b) - f^{(j-1)}(a)]}{j!} \sum_{i=0}^{k-1} p_i B_j\left(\frac{x_i - a}{b-a}\right)\right| \le K_n (b-a)^n \bigvee_a^b (f^{(n-1)}), \tag{2.6}$$

where

$$K_n = \frac{1}{n!} \sup_{a<t<b} \left|\sum_{i=0}^{k-1} p_i \left[B_n^*\left(\frac{x_i - t}{b-a}\right) - B_n\left(\frac{x_i - a}{b-a}\right)\right]\right|,$$

and B_n^* *is a 1-periodic function that coincides with the Bernoulli polynomial* B_n *on* $[0, 1)$.

Proof. To prove this theorem, we set $m = 0$ in Lemma 1 and take

$$k_n(x, t) = (-1)^n \frac{(b-a)^n}{n!} B_n^*\left(\frac{x-t}{b-a}\right).$$

Hence we have

$$\int_a^b f(t)\,\mathrm{d}t = f(x)(b-a) - \sum_{j=1}^{n-1} \frac{(b-a)^j}{j!} B_j\left(\frac{x-a}{b-a}\right)\left[f^{(j-1)}(b) - f^{(j-1)}(a)\right] + \frac{(b-a)^n}{n!} \int_a^b \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] df^{(n-1)}(t).$$

Making the change of variables $x = x_i$, $i = 0, 1, \ldots, k-1$, and using $\sum_{i=0}^{k-1} p_i = 1$, we obtain

$$\int_a^b f(t)\,\mathrm{d}t = (b-a)\sum_{i=0}^{k-1} p_i f(x_i) + \sum_{j=1}^{n-1} \frac{(b-a)^j [f^{(j-1)}(b) - f^{(j-1)}(a)]}{j!} \sum_{i=0}^{k-1} p_i B_j\left(\frac{x_i - a}{b-a}\right) + \frac{(b-a)^n}{n!} \int_a^b \sum_{i=0}^{k-1} p_i \left[B_n^*\left(\frac{x_i - t}{b-a}\right) - B_n\left(\frac{x_i - a}{b-a}\right)\right] \mathrm{d}f^{(n-1)}(t).$$

Since

$$\int_a^b \sum_{i=0}^{k-1} p_i \left[B_n^*\left(\frac{x_i - t}{b-a}\right) - B_n\left(\frac{x_i - a}{b-a}\right)\right] \mathrm{d}f^{(n-1)}(t) \le \sup_{a<t<b} \left|\sum_{i=0}^{k-1} p_i \left[B_n^*\left(\frac{x_i - t}{b-a}\right) - B_n\left(\frac{x_i - a}{b-a}\right)\right]\right| \int_a^b |\,\mathrm{d}f^{(n-1)}(t)| = \sup_{a<t<b} \left|\sum_{i=0}^{k-1} p_i \left[B_n^*\left(\frac{x_i - t}{b-a}\right) - B_n\left(\frac{x_i - a}{b-a}\right)\right]\right| \bigvee_a^b (f^{(n-1)}),$$

We can easily derive (2.6) and the proof is completed. □

We define $h = (b-a)/k$. Setting $p_i = 1/k$, $x_i = a + (i + x)h$, $i = 0, 1, \ldots, k-1$, in Theorem 4, we obtain the Euler–Maclaurin formula;

$$\left|\int_a^b f(t)\,\mathrm{d}t - h\sum_{i=0}^{k-1} f(a+(i+x)h) + \sum_{j=1}^{n-1} h^j \frac{f^{(j-1)}(b)-f^{(j-1)}(a)}{j!} B_j(x)\right|$$
$$\leq \frac{(b-a)^n}{n!k^n} \sup_{0<t<1} |B_n(t)-B_n(x)| \bigvee_a^b (f^{(n-1)}).$$

As regards applications of the Euler–Maclaurin formula, one can see [4]. Now, we consider the general quadrature

$$\int_a^b f(t)\,\mathrm{d}t = (b-a)\sum_{i=0}^{k-1} p_i f(x_i) + R_n(f) \tag{2.7}$$

and obtain the following corollary.

Corollary 4. *Let $x_i \in [a,b]$ and $p_i \geq 0$ be such that*

$$\sum_{i=0}^{k-1} p_i x_i^j = \frac{b^{j+1}-a^{j+1}}{(j+1)(b-a)}, \quad j \in \{0,1,\ldots,n-1\}, \tag{2.8}$$

i.e. (2.7) is exact for any polynomial of degree less than n; then we have

$$\left|\int_a^b f(t)\,\mathrm{d}t - (b-a)\sum_{i=0}^{k-1} p_i f(x_i)\right| \leq K_n(b-a)^n \bigvee_a^b (f^{(n-1)}), \tag{2.9}$$

where $K_n = \frac{1}{n!}\sup_{a<t<b}\sum_{i=0}^{k-1}\left|B_n^*\left(\frac{x_i-t}{b-a}\right) - B_n\left(\frac{x_i-a}{b-a}\right)\right|$.

Proof. We first note that $B_j((t-a)/(b-a))$ is a polynomial of degree j. By (2.8), we obtain

$$\sum_{i=0}^{k-1} p_i B_j\left(\frac{x_i-a}{b-a}\right) = \frac{1}{b-a}\int_a^b B_j\left(\frac{t-a}{b-a}\right)\mathrm{d}t = 0, \quad j \in \{1,\ 2,\ \ldots,\ n-1\}.$$

According to (2.6), we derive (2.9) and complete the proof. □

Remark 4. It is worth mentioning that the result was derived by Wang [5] in 1978. Further, if f is discontinuous at x_i when $n=1$, Corollary 4 also holds. We can generalize it to the functions of bounded p-variation [6]. Theorem 4 generalizes the classical Euler–Maclaurin formula as can be found in [7,8]. It is also a generalization of Euler trapezoid formulas [3]. In particular, we can evaluate the error constants for some quadrature formulas which are well known.

(1) Mid-point rule: $k=1$, $p_0=1$, $x_0=\frac{b+a}{2}$, $K_1=\frac{1}{2}$, $K_2=\frac{1}{8}$.
(2) Trapezoid rule: $k=2$, $p_0=p_1=\frac{1}{2}$, $x_0=a$, $x_1=b$, $K_1=\frac{1}{2}$, $K_2=\frac{1}{8}$.
(3) Simpson rule: $k=3$, $p_0=p_2=\frac{1}{6}$, $p_1=\frac{2}{3}$, $x_0=a$, $x_1=\frac{a+b}{2}$, $x_2=b$, $K_1=\frac{1}{3}$, $K_2=\frac{1}{24}$, $K_3=\frac{1}{324}$, $K_4=\frac{1}{1152}$.

All constants of the errors are sharp. It is obvious that $f(t)$ is of bounded variation if $|f'(t)|<\infty$ or $f(t)$ is Lipschitz continuous. For further investigation of these cases, one can refer to Ostrowski's inequality and its extensions ([9–13]).

We define $\widehat{h}=(b-a)/r$, $a_j=a+j\widehat{h}$, $(j=0,1,\ldots,r)$. We apply Corollary 4 on the interval $[a_j,a_{j+1}]$ and we have the following corollary.

Corollary 5 (*Cf. [5]*). *Let* $0 \leq t_0 < t_1 < \cdots < t_{k-1} \leq 1$. *Suppose that the following quadrature rule:*

$$\int_0^1 f(t)\,\mathrm{d}t = \sum_{i=0}^{k-1} p_i f(t_i)$$

is exact for any polynomial of degree less than n. Let $f : [a, b] \to R$ *be such that* $f^{(n-1)}$ *is a function of bounded variation. Then we have*

$$\int_a^b f(t)\,\mathrm{d}t = \widehat{h}\sum_{j=0}^{r-1}\sum_{i=0}^{k-1} p_i f(a_j + t_i\widehat{h}) + R(f), \tag{2.10}$$

where

$$R(f) = \widehat{h}^n \int_a^b G_n\left(r\frac{t-a}{b-a}\right)\mathrm{d}f^{(n-1)}(t)$$

and

$$G_n(t) = \frac{1}{n!}\sum_{i=0}^{k-1} p_i(B_n^*(t_i - t) - B_n(t_i)).$$

Moreover,

$$|R(f)| \le \frac{1}{n!}\left(\frac{b-a}{r}\right)^n \bigvee_a^b (f^{(n-1)}) \sup_{0<t<1}\left|\sum_{i=0}^{k-1} p_i(B_n^*(t_i - t) - B_n(t_i))\right|.$$

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