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Non-Strong Uniqueness in Real and Complex Chebyshev Approximation

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1. INTRODUCTION

In this paper we review some basic facts on strong uniqueness (strong unicity) in real and complex Chebyshev approximation and draw a number of simple conclusions. For example, we note the relation to the local Kolmogorov criterion [4], and the connection between the linear and the Fréchet-differentiable nonlinear case [1, 5, 20]. However, we stress *the difference between real and complex Chebyshev approximation*. By counter-examples we show that a best approximation in a complex Haar subspace need not be strongly unique (contrary to a related theorem of Dunham [11]), that a critical point of a complex rational approximation problem need not be a local best approximation (contrary to a theorem by Ellacott and Williams [12]), and that Klotz [13] sufficient condition for strong uniqueness in complex polynomial approximation is incorrect. In contrast to the real case there are particular situations in complex approximation, where non-strong uniqueness is the normal case (cf. Theorem 1). Moreover, with respect to strong uniqueness, approximation problems with Hilbert space valued functions behave like complex problems too.

In the last section we investigate *the length of primitive extremal signatures*. If the best approximation is not strongly unique, this length is at most n in the real case and at most $2n$ in the complex case (as partly conjectured by Dunham in a private communication), and these bounds are the best possible. Finally, we improve a related theorem of Bartelt [2] and show that his bounds are also the best possible.

The possible lack of strong uniqueness has *an impact on numerical computations*. Finding a strong unique local best approximation is a well-conditioned problem. In a distinct case there holds even a uniform Lipschitz condition for the dependence of best approximations from f [3]. Many results on numerical algorithms are mainly based on strong uniqueness, e.g., the quadratic convergence of Newton's method [8], and, reportedly [2], the

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convergence of the Remes algorithm. Yet, an algorithm for complex Chebyshev approximation must be able to approach non-strongly unique best approximations. Also, if a local best approximation is not strongly unique, it becomes impossible to distinguish it by virtue of the local Kolmogorov criterion from a saddle point. Moreover, there are general results on nonlinear families which first of all depend on strong uniqueness [5, 20].

2. Linear Approximation

Suppose Z is a compact space, K stands for either $\mathbb{R}$ or $\mathbb{C}$, and $C(Z) := C(Z, K)$ denotes the Banach space of continuous functions $f: Z \to K$, endowed with the uniform norm. We assume at first that $V \subset C(Z)$ is an n-dimensional (linear) subspace. A given function $f \in C(Z)$ is to be approximated by elements from V. Then, a *best approximation* (BA) $v \in V$ is called *strongly unique* iff there is a $\gamma > 0$ (depending on f) such that

$$\|f - w\| \geqslant \|f - v\| + \gamma \|v - w\| \qquad \text{for } \forall w \in V. \tag{1}$$

Note that (1) holds with $\gamma = 1$ whenever $v = f \in V$. Hence, we will assume $f \notin V$ in the sequel.

Newman and Shapiro [17, Theorem 4] derived a fundamental result on strong uniqueness: *If V is a real Haar subspace, the best approximation is strongly unique.* In this respect real Chebyshev approximation is completely different from linear approximation in the mean (or in any other smooth space), where the BA is never strongly unique [20]. For the BA v in a complex Haar subspace V, Newman and Shapiro [17, Theorem 4′] established the existence of positive constants β_1, β_2 (depending on f) such that

$$\|w - v\| \leqslant \beta_1\{\|f - w\| - \|f - v\|\}^{1/2} + \beta_2\{\|f - w\| - \|f - v\|\} \qquad \text{for } \forall w \in V, \tag{2}$$

from which we get for every w in a neighborhood of v

$$\|f - w\| \geqslant \|f - v\| + \beta \|w - v\|^2 \tag{3}$$

(with $\beta > 0$). However, they did not give a counterexample showing that (1) does not hold for any $\gamma > 0$.

On the other hand, Dunham's Theorem 1 in [11] would imply strong uniqueness in the complex case. When applied to the linear case this theorem states: If V is a complex Haar subspace, $v \in V$ is a BA to $f \notin V$ if and only if

$$\mu(v, w) := \max_{z \in E(v)} \operatorname{Re}\{\overline{[f(z) - v(z)]}\, w(z)\} > 0 \qquad \text{for } \forall w \in V\backslash\{0\}, \tag{4}$$

where $E(v)$ denotes the set of extremal points of $f - v$. In contrast to Kolmogorov's criterion the equality sign has been excluded in (4). If S denotes the unit sphere in $C(Z)$, (4) is equivalent to

$$\gamma(v) := \frac{1}{\| f - v \|} \min_{w \in S \cap V} \mu(v, w) > 0, \tag{5}$$

while (1) is equivalent to $\gamma \leqslant \gamma(v)$ [4, Theorem 5]. Thus, the following three statements hold:

(S1) $\gamma(v) \geqslant 0 \Leftrightarrow v$ is a BA.

(S2) $\gamma(v) > 0 \Leftrightarrow v$ is the strongly unique BA.

(S3) If V is a real Haar subspace, then

$$\gamma(v) > 0 \Leftrightarrow \gamma(v) \geqslant 0.$$

The equivalence in (S3) is characteristic for real Haar subspaces [15].

Now we present a counterexample to strong uniqueness in complex Haar subspaces. It implies that Theorem 1 in [11] is incorrect, and that Theorem 4 in [2] no longer holds in the complex case.

EXAMPLE 1. Let $Z := \{-1, 1\}$, $f(z) := z$, $v_b(z) :\equiv b$ $(b \in \mathbb{C})$, $V := \{v_b : b \in \mathbb{C}\}$, and thus $n := 1$. Obviously the BA is v_0, i.e., $b = 0$, with $\| f - v_0 \| = 1$, $E(v_0) = Z$. But for every purely imaginary b we get $\mu(v_0, v_b) = 0$. Thus $\gamma(v_0) = 0$, and v_0 is not strongly unique. In fact, we get for $b \to 0$ on the imaginary axis

$$\| f - v_b \| = (1 + | b |^2)^{1/2} \leqslant \| f - v_0 \| + \tfrac{1}{2} \| v_b - v_0 \|^2.$$

3. Nonlinear Approximation

Now let V be a nonlinear family of functions. A *local best approximation* (LBA) $v \in V$ is called *strongly unique* iff there is a neighborhood $U \subset V$ of v such that v is the strongly unique best approximation in U [20].

For simplicity let us assume that $B \subset K^n$ is open, and $p : b \in B \mapsto v_b \in V \subset C(Z)$ is a continuously Fréchet-differentiable mapping. Let $T_b := \{ p_b' d \in C(Z) : d \in K^n\}$ denote the tangent space to V at v_b, and $\dim T_b$ its dimension. Then we may define $\gamma(v_b)$ again by (5) if we replace V by T_b there. We may also assume that the restriction of p to $\{b \in B : \dim T_b = n\}$ is one-to-one.

Supposing $K = \mathbb{R}$ and $\dim T_b = n$ Wulbert [20] has shown: *v_b is a strongly unique LBA to f iff* 0 *is the strongly unique BA to $f - v_b$ from T_b*. Generally, v_b

is called a *critical point* iff 0 is a BA to $f - v_b$ from T_b. Since Wulbert's result remains correct in the complex case, we have instead of (S1) through (S3):

(S1′) $\gamma(v_b) \geqslant 0 \Leftrightarrow v_b$ is a critical point.

(S2′) If $\dim T_b = n$, then

$$\gamma(v_b) > 0 \Leftrightarrow v_b \text{ is a strongly unique LBA.}$$

(S3′) If T_b is an n-dimensional real Haar subspace, then

$$\gamma(v_b) > 0 \Leftrightarrow \gamma(v_b) \geqslant 0 \Leftrightarrow v_b \text{ is LBA.}$$

For real functions (S3′) has essentially been proved by Barrar and Loeb [1, Theorem 3]. Another proof is due to Dunham [10, Theorem 1]. (S2′) and (S3′) have been generalized by Braess [5] and Cromme [9] to include the important cases of manifolds with boundary (e.g., exponential sums) and restricted range approximation, respectively. The local Kolmogorov criterion [16, Theorem 8] is only a necessary condition for a LBA:

(S4′) $\gamma(v_b) \geqslant 0 \Leftarrow v_b$ is LBA.

Meinardus and Schwedt [16], Brosowski [7], and many other authors have specified nonlinear families of approximants every critical point of which is a BA; see the references in [6]. In particular, real rational functions defined on an interval are such a family, but complex rationals are not.

For *complex rational functions* Ellacott and Williams [12, Theorems 2.1 and 2.2] state that v_b is a LBA if and only if $\gamma(v_b) \geqslant 0$, but their proof is not complete in the case $\gamma(v_b) = 0$. In fact, even under the additional assumption $\dim T_b = n$, their statement is incorrect as the following counterexample shows:

EXAMPLE 2. Let $Z := \{-1, 1, 2\}$, $f(z) := 1/z + e(z)$, $e(-1) := e(2) := 4$, $e(1) := -4$,

$$V := R_{01}^{C} := \left\{\frac{b_0}{b_1 + b_2 z} : b_i \in \mathbb{C},\ b_1 + b_2 z \neq 0\right\}.$$

In a neighborhood of $v_{00}(z) := 1/z$ every function in V is of the form $v_{ab}(z) := (1 + a)/(z + b)$, with $a, b \in \mathbb{C}$. We get $E(v_{00}) = Z$, and for $a, b \to 0$

$$v_{ab}(z) - v_{00}(z) = \frac{a}{b} - \frac{b}{z^2} - \frac{ab}{z^2} + \frac{b^2}{z^3} + O(ab^2) + O(b^3).$$

Hence, the tangent space at v_{00} is

$$T_{00} = \{w_{ab}(z) \colon a, b \in \mathbb{C}\}, \qquad w_{ab}(z) := \frac{a}{z} - \frac{b}{z^2}.$$

It is easy to verify that $\mu(v_{00}, w_{ab}) \geqslant 0$ for every $w_{ab} \in T_{00}$, and $\mu(v_{00}, w_{ab}) = 0$ iff $\operatorname{Re} a = \operatorname{Re} b = 0$. Thus v_{00} is a critical point. If a and b were restricted to real values, v_{00} would be a LBA. It is even the BA among those functions in V having a pole in $(-1, 1)$ and real coefficients because this family is regular (on Z) [7, p. 91]; but it is not the BA among all real functions in V since, e.g., $v(z) \equiv 0.75$ is a better one. However, here a and b may be complex, and if we choose $a = 2\delta i$, $b = 3\delta i$, straightforward calculations yield

$$\| f - v_{ab} \|^2 = \| f - v_{00} \|^2 - \tfrac{47}{16}\, \delta^2 + O(\delta^3) \qquad \text{as } \delta \to 0.$$

This proves that v_{00} is not a LBA but a saddle point.

More generally, let R^C_{lm} denote the family of complex rational functions with nominator degree (at most) $l\,(\geqslant 0)$ and denominator degree (at most) $m\ (\geqslant 0)$, and let R^R_{lm} be the subset of functions having real coefficients. Then we get

THEOREM 1. *Let $Z \subset \mathbb{R}, f \in C(Z)$, with f real valued. Then no $v_b \in R^R_{lm}$ $(l, m \geqslant 0)$ is a strongly unique LBA to f with respect to R^C_{lm}.*

Proof. Assume v_b is a LBA with respect to R^C_{lm}. Then the local Kolmogorov criterion $\mu(v_b, p_b'd) \geqslant 0$ is satisfied for every $d \in \mathbb{C}^n$ $(n := l + m + 1)$, and $\mu(v_b, p_b'd) = 0$ for any $d = (d_1, \ldots, d_n)^T$ with $\operatorname{Re} d_k = 0$ and *arbitrary* $\operatorname{Im} d_k$ $(k = 1, \ldots, n)$. Thus $\gamma(v_b) = 0$, and v_b is not a strongly unique LBA. □

So, critical points that are not strongly unique LBA's are very common in complex polynomial and complex rational approximation problems. Saff and Varga [19] have shown that even in the case where Z is an interval and the function v_b in Theorem 1 is thus the BA from R^R_{lm} this function v_b may not be the BA from R^C_{lm}. Then the BA from R^C_{lm} is obviously not unique. One must expect that in this case v_b is usually a saddle point.

The *computation of* $\gamma(v_b)$ is rather complicated if Z is an infinite set. But assuming $\dim T_b = n$ we may use in T_b a different norm, for which the unit sphere instead of $S \cap T_b$ is

$$\hat{S}_b := \{ p_b'd \in T_b : d = (d_1, \ldots, d_n) \in K^n,\ \max_k \max\{ |\operatorname{Re} d_k|, |\operatorname{Im} d_k| \} = 1 \}.$$

We get a function $\hat{\gamma}(v_b)$ that is equivalent to $\gamma(v_b)$ in the sense that $\operatorname{sign} \gamma(v_b) = \operatorname{sign} \hat{\gamma}(v_b)$. It is easily verified that $\hat{\gamma}(v_b)$ is the maximum value of the object function of a linear optimization problem containing only a finite number of restrictions if $E(v_b)$ is a finite set. Hence, $\hat{\gamma}(v_b)$ is easy to compute in this case. In the case $\dim T_b < n$ we always get $\hat{\gamma}(v_b) \leqslant 0$, however.

4. The Length of Primitive Extremal Signatures

Owing to the connections mentioned in Section 3 we may restrict the discussion to the linear case again. We assume that V is a (real or complex) subspace of $C(Z)$ and $v \in V$ is a BA to $f \notin V$. We define a *primitive extremal point set* to be any subset $A \subset E(v)$ with the property that v is a BA to f on A but no more a BA to f on any proper subset of A. The corresponding set $\{(z, [f(z) - v(z)]/\|f - v\| : z \in A\} \subset Z \times \mathbb{C}$ is called the *primitive extremal signature* [7, 18]. The *length* $|A|$ of A is the number of elements of A. As is well known [18], $|A| \leqslant n + 1$ in the real case, and $|A| \leqslant 2n + 1$ in the complex case. Moreover, $|A| \geqslant n + 1$ if V is a Haar subspace. Dunham has conjectured that $|A| < 2n + 1$ if v is not strongly unique and $K = \mathbb{C}$. In fact, even without requiring Haar's condition, we get

Theorem 2. *If $v \in V$ is a BA but not a strongly unique one, then the length of a primitive extremal point set A is at most $2n$ if $K = \mathbb{C}$ and at most n if $K = \mathbb{R}$.*

As we have seen above, $|A| = n + 1$ if V is a real Haar subspace; the theorem thus implies statement (S3).

Proof. Since v is a BA but not a strongly unique BA to f on Z, so it is on A. Indeed, for $\gamma_A(v)$ [defined by (5) and (4) if $E(v)$ is replaced by A there] we conclude: $\gamma_A(v) \geqslant 0$ since A is primitive, and $\gamma_A(v) \leqslant \gamma = 0$ since $A \subset E(v)$ and v is not strongly unique; hence $\gamma_A(v) = 0$. Now, let $\{\phi_1, ..., \phi_n\}$ be a basis of V, and define $h : A \to K^n$ by

$$h(z) := [f(z) - v(z)](\overline{\phi_1(z)}, ..., \overline{\phi_n(z)})^T, \qquad z \in A. \tag{6}$$

Because v is a BA, the origin of K^n is in the convex hull C of these $|A|$ vectors $h(z)$, $z \in A$ [17, Theorem 1]; but since v is not strongly unique, $0 \in \partial C$ [4, Theorem 6]. In fact, if $w = \sum d_k \phi_k \neq 0$ satisfies $0 = \gamma_A(v) = \mu_A(v, w)$ [defined by (4) with $E(v)$ replaced by A], and if $d := (d_1, ..., d_n)^T$, then

$$0 = \mu_A(v, w) = \max_{z \in A} \operatorname{Re}\{\overline{[f(z) - v(z)]}\, w(z)\} = \max_{z \in A} \operatorname{Re}(h(z), d), \tag{7}$$

where (.,.) denotes the inner product in K^n. Thus, if we identify $\mathbb{C}$ with $\mathbb{R}^2$,

$$H := \{c \in K^n : \operatorname{Re}(c, d) = 0\}$$

is in $\mathbb{R}^{2n}$ (or $\mathbb{R}^n$ if $K = \mathbb{R}$) a supporting hyperplane of C at the origin.

According to Caratheodory's theorem $0 \in K^n$ is a convex combination of $m \leqslant 2n + 1$ (or $n + 1$ if $K = \mathbb{R}$) vectors $h(z)$, say

$$0 = \sum_{j=1}^{m} \alpha_j h(z_j), \qquad \text{where} \quad z_j \in A, \quad \alpha_j > 0, \quad \sum_{j=1}^{m} \alpha_j = 1. \tag{8}$$

Here, $h(z_j) \in H$ for every j, because $z_l \notin H$ would imply

$$\sum_{j \neq l} \alpha_j \operatorname{Re}(h(z_j), d) = -\alpha_l \operatorname{Re}(h(z_l), d) > 0,$$

which requires Re $(h(z_j), d) > 0$ for at least one z_j, in contrast to (7). Finally, we note that H has the real dimension $2n - 1$ (or $n - 1$ if $K = \mathbb{R}$) and thus Caratheodory's theorem implies that we only need $m \leqslant 2n$ (if $K = \mathbb{C}$) or $m \leqslant n$ (if $K = \mathbb{R}$) points z_j in (8) and hence in A. Since a simplex of maximum dimension in H has $2n$ (or n, respectively) corners, *the bounds* $2n$ *and* n *are the best possible.* □

Conversely, if $v \in V$ is a strongly unique BA, we may consider subsets $A' \subset E(v)$ with the property that v is the strongly unique BA on A' but not on any proper subset of A'. One might call such a set A' a *primitive strongly extremal point set*. (However, note that "strongly extremal" has not the same meaning here as "strong extremal" in [17].) Since 0 lies then in the interior of the convex hull C' of the vectors $h(z)$, $z \in A'$, defined by (6), we need $|A'| \geqslant 2n + 1$ (or $n + 1$ if $K = \mathbb{R}$) [4, Remark 2]. This statement is in contrast to Klotz' Lemma 3.1 [13, p. 19] and Theorem 3.2 [13, p. 21], where a special extremal signature of length $2n - 1$ is claimed to imply strong uniqueness in $R^{\mathbb{C}}_{n-1,0}$. Our next example, which is a generalization of Example 1, shows that Klotz' assertion is indeed wrong. Moreover, it manifests that non-strong uniqueness exists in complex Haar subspaces of arbitrary finite dimension n, and that even there the bound $2n$ in Theorem 2 is attained.

EXAMPLE 3. Let Z be the unit circle,

$$Z' := \{z_k := \exp(i\pi k/n) \colon k = 1, \dots, 2n\},$$

$$f(z) := \tfrac{1}{2} z^n + \tfrac{1}{2} z^{3n} \qquad (z \in Z),$$

$$v_b(z) := \sum_{k=0}^{n-1} b_{n-k} z^k \qquad (z \in \mathbb{C},\ b := (b_1, \dots, b_n)^T \in \mathbb{C}^n),$$

and $V := \{v_b : b \in \mathbb{C}^n\}$ as usual. We assert that v_0 is again the unique BA, but is not strongly unique. First, $f(z_k) - v_0(z_k) = f(z_k) = (-1)^k$, $k = 1, \dots, 2n$. Thus $\|f - v_0\| = 1$ and $E(v_0) = Z'$. The optimality of v_0 follows from another theorem by Klotz [13, Theorem 2.2] or from its generalization [13, Theorem 9.2; 14, Theorem 3], but we will give here a much shorter proof, which can be modified easily to prove Klotz theorem, too. Suppose v_d is a better approximation than v_0. Then, according to Kolmogorov's theorem

$$(-1)^k \operatorname{Re} v_d(z_k) > 0, \qquad k = 1,..., 2n. \tag{9}$$

It follows that

$$\operatorname{Re} v_d(e^{it}) = \sum_{k=0}^{n-1} (\operatorname{Re} d_{n-k} \cos kt - \operatorname{Im} d_{n-k} \sin kt) \tag{10}$$

is a trigonometric polynomial of degree $n - 1$ that has at least $2n$ zeros in $(0, 2\pi]$. Thus it must be the zero function, i.e.,

$$\operatorname{Re} d_n = 0, \qquad d_1 = \cdots = d_{n-1} = 0. \tag{11}$$

But this contradicts (9); hence v_0 is the (unique) BA. However, v_0 is not strongly unique: If we choose d satisfying (11) and with arbitrary $\operatorname{Im} d_n$, then $(-1)^k \operatorname{Re} v_d(z_k) = 0$ for every k; and hence $\eta(v_0, v_d) = \gamma(v_0) = 0$.

Finally note that if we delete any point of Z', say z_1, we can choose d such that (10) has a zero in each interval $(k\pi/n, (k+1)\pi/n)$, $k = 2,..., 2n - 1$, and is positive in z_1. Then (9) holds for every $k \neq 1$, which means that v_0 is not the BA on $Z' \backslash \{z_1\}$. We conclude that Z' is a primitive extremal point set.

The following theorem is a slightly improved version of Theorem 2 of Bartelt [2],

THEOREM 3. *Let $v \in V$ be a strongly unique BA. Then the length of a primitive strongly extremal point set A' is at most $4n$ if $K = \mathbb{C}$ and at most $2n$ if $K = \mathbb{R}$.*

Proof. According to Bartelt's proof [2] any set $A \subset E(v)$ on which v is a strongly unique BA contains a subset A' consisting of at most $4n$ (or $2n$, respectively) points such that

$$0 \in \operatorname{int} C' = \operatorname{int} \operatorname{conv}\{h(z) : z \in A'\} \tag{12}$$

and

$$\mu_{A'}(v, w) := \max_{z \in A'} \operatorname{Re}\{\overline{[f(z) - v(z)]}\, w(z)\} > 0 \qquad \text{for} \quad \forall w \in V,\ w \not\equiv 0 \text{ on } A'.$$

Now, assume $w \equiv 0$ on A' but $w \not\equiv 0$ on V. Then, using the same notation as in the proof of Theorem 2, we get $w = \sum d_k \phi_k$ (with $d \neq 0$) and

$$0 = \operatorname{Re}\{\overline{[f(z) - v(z)]}\, w(z)\} = \operatorname{Re}(h(z), d) \qquad \text{for} \quad \forall z \in A'.$$

Hence $h(z) \in H$ for $\forall z \in A'$, which implies $\operatorname{int} C' \subset \operatorname{int} H = \varnothing$ and thus contradicts (12). Therefore, $\mu_{A'}(v, w) > 0$ for $\forall w \in V \backslash \{0\}$, i.e., v is the strongly unique BA on A'. □

Bartelt [2] also posed the question: *Is* $4n(2n)$ *the best possible upper bound in Theorem* 3? Our next example proves that this is in fact true.

EXAMPLE 4. Let $K = \mathbb{R}$, $Z := \{-n,..., -1, 1,..., n\}$,

$$\phi_k(z) := \operatorname{sign}(z)\, \delta_{k,|z|} \qquad (z \in Z, k = 1,..., n) \tag{13}$$

(here δ_{kj} denotes Kronecker's symbol), $V := \{\phi_1, ..., \phi_n\}$. Obviously, the BA to $f \equiv 1$ is $v \equiv 0$, and $E(v) = Z$, $|Z| = 2n$. We assert that Z is even a primitive strongly extremal point set. In fact, for $j = 1,..., n$ the vector $h(j)$ is the jth standard basis vector in $\mathbb{R}^n$, and $h(-j) = -h(j)$. So, (12) is satisfied for $A' = Z$, but not for any proper subset of Z.

In the case $K = \mathbb{C}$ we let $Z := \{-n,..., -1; 1,..., n; -in,..., -i, i,..., in\}$, $f :\equiv 1$, define ϕ_k again by (13), and take advantage of the equivalence of $\mathbb{C}$ and $\mathbb{R}^2$. Then the set $\{h(z) : z \in Z\}$ consists of all standard basis vectors of $\mathbb{R}^{2n}$ and, in addition, of the corresponding negative vectors. So, the situation is the same as in the real case, but n is replaced by $2n$.

Note added in proof. Independently, Williams [21] has also constructed an example of a (non-normal) real rational function that is a saddle point of a complex approximation problem. Recently, important related results were established by Wulbert [22].

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