Simple nuclear $C^*$-algebras of tracial topological rank one

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Abstract

We give a classification theorem for unital separable nuclear $C^*$-algebras with tracial rank no more than one. Let $A$ and $B$ be two unital separable simple nuclear $C^*$-algebras with $TR(A), TR(B) \leq 1$ which satisfy the universal coefficient theorem. We show that $A \cong B$ if and only if there is an order and unit preserving isomorphism

$$\gamma = (\gamma_0, \gamma_1, \gamma_2) : \left( K_0(A), K_0(A)_+[1_A], K_1(A), T(A) \right) \cong \left( K_0(B), K_0(B)_+[1_B], K_1(B), T(B) \right),$$

where $\gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x))$ for each $x \in K_0(A)$ and $\tau \in T(B)$.

Keywords: TAI $C^*$-algebras; Classification of nuclear $C^*$-algebras; Tracial rank one

1. Introduction

This paper is a part of the program to classify separable nuclear $C^*$-algebras initiated by George A. Elliott (see [14] and [16]). By a classification theorem for a class of nuclear $C^*$-algebras, one means the following: two $C^*$-algebras in the class with the same $K$-theoretical data are isomorphic (as $C^*$-algebras) and the range of the invariant can be described for the class so that given a set of $K$-theoretical data in the range there is a $C^*$-algebra in the class...
which possesses the given $K$-theoretical data. By the $K$-theoretical data, one usually means the Elliott invariant which contains the $K$-theory and traces, at least for the simple case. In this paper we are only interested in simple $C^*$-algebras with lower rank. By $C^*$-algebras of lower rank, one often means that the $C^*$-algebras have real rank zero, or stable rank one. Many important $C^*$-algebras which arise naturally are of real rank zero or stable rank one. Notably, all purely infinite simple $C^*$-algebras have real rank zero and many $C^*$-algebras arising from dynamical systems are of stable rank one. One of the classical results of this kind states that all irrational rotation $C^*$-algebras are simple nuclear $C^*$-algebras with real rank zero and stable rank one (see [17] and [46]).

One may view (simple) $C^*$-algebras of real rank zero and stable rank one as some kind of generalization of AF-algebras. A more suitable generalization of AF-algebras has been demonstrated to be $C^*$-algebras with tracial topological rank zero. Simple $C^*$-algebras with tracial topological rank zero have real rank zero, stable rank one, with weakly unperforated $K_0$ and are quasicentral. All simple AH-algebras with slow dimension growth and with real rank zero have tracial topological rank zero. This shows that simple $C^*$-algebras with zero tracial rank could have rich $K$-theory. Simple AH-algebras with slow dimension growth and with real rank zero have been classified in [18] (together with [9,21] and [22]). A classification theorem for unital separable simple $C^*$-algebras with tracial topological rank zero which satisfy the UCT was given in [38] (see also [32,36] and [10] for earlier references). Simple $C^*$-algebras with tracial topological rank zero are also called TAF (tracially AF) $C^*$-algebras.

This paper studies $C^*$-algebras of tracial rank one. A standard example of a $C^*$-algebra with stable rank one is of course $M_2(C([0, 1]))$ (which also has tracial rank one). A notion of tracially approximately interval $C^*$-algebras (TAI $C^*$-algebras) is introduced in this paper—see Definition 2.2 below. It turns out that simple TAI $C^*$-algebras are the same as simple $C^*$-algebras with tracial topological rank no more than one. Roughly speaking, TAI $C^*$-algebras are those $C^*$-algebras whose finite subsets can be approximated by $C^*$-subalgebras which are finite direct sums of finite-dimensional $C^*$-algebras and matrix algebras over $C([0, 1])$ in “measure” or rather in trace. It is proved here that simple TAI $C^*$-algebras have stable rank one. From a result of G. Gong [23] we observe that all simple AH-algebras with very slow dimension growth are in fact TAI $C^*$-algebras. It is also shown here that simple TAI $C^*$-algebras are quasicentral, their ordered $K_0$-groups are weakly unperforated and satisfy the Riesz interpolation property, and these $C^*$-algebras also satisfy the Fundamental Comparison Property of Blackadar.

Elliott, Gong and Li in [20] (also [23]) give a complete classification (up to isomorphism) for simple AH-algebras with bounded dimension growth by their $K$-theoretical data (an important special case can be found in K. Thomsen’s work [55]). G. Gong also has a proof [24] that simple AH-algebras with very slow dimension growth can be rewritten as simple AH-algebras with bounded dimension growth (the proof of this article also implies that—see 10.6).

These $C^*$-algebras are nuclear separable simple $C^*$-algebras of stable rank one. Their work is a significant advance in classifying finite simple $C^*$-algebras after the remarkable result of [18] which classifies simple AH-algebras of real rank zero (with slow dimension growth). Therefore, it is the time to classify nuclear simple separable finite $C^*$-algebras with real rank other than zero without assuming that they are inductive limits (AH-algebras are inductive limits of finite direct sums of some standard homogeneous $C^*$-algebras) of certain special building blocks. The main purpose of this paper is to present such a result.

Sections 1–6, Section 8 and most of Section 9 were written in 1998. Together with later sections, the original preprint has two parts. A preliminary report on the results in the two-part preprint was reported in EU Conference on Operator Algebra in Copenhagen in August 1998.
Since then a great deal of progress on the subject has been made. The present paper absorbs both parts of the original preprint and reflects the new developments. But it is significantly shorter than the original preprint. More importantly, the main result of the paper has been greatly improved and a technical condition in original preprint has been removed. The main result of the paper is the following. Let $A$ and $B$ be two unital separable nuclear simple $C^*$-algebras with $\text{TR}(A) \leq 1$, $\text{TR}(B) \leq 1$ and satisfying the UCT. Then $A \cong B$ if and only if they have the same Elliott invariant—see Theorem 10.10.

Consider two $C^*$-algebras $A$ and $B$ as above. As in [20], we will construct the following approximately commutative diagram:

$$
\begin{array}{ccc}
B & \overset{\text{id}_B}{\longrightarrow} & B \\
\downarrow{L_1} & & \downarrow{L_2} \\
A & \overset{\text{id}_A}{\longrightarrow} & A \\
\end{array}
$$

We then apply an approximate intertwining argument of Elliott to obtain an isomorphism. It requires two types of results: an existence theorem and a uniqueness theorem. The existence theorem should state that if $A$ and $B$ have the same Elliott invariant, then there are unital $*$-homomorphisms from $A$ to $B$ and from $B$ to $A$ which induce the isomorphism at the level of the Elliott invariant. However, the existence theorem that we proved in the process only provides a sequence of completely positive maps from $A$ to $B$ (and from $B$ to $A$) which are eventually multiplicative. In order to get the approximately commutative diagram, one also needs the uniqueness theorem: two such maps which induce the same (partial) map at the level of the Elliott invariant are approximately unitarily equivalent. In other words, we also need a uniqueness theorem which works for maps that are not necessary homomorphisms.

An important fact is the following classification of monomorphisms from $\bigoplus_{k=1}^2 M_{r(k)}(C([0, 1]))$ to a simple TAI-algebra. For any unital $C^*$-algebra $C$, denote by $T(C)$ the tracial state space of $C$ (it could be an empty set). Let $A = \bigoplus_{k=1}^n M_{r(k)}(C([0, 1]))$ and $B$ be a unital simple TAI $C^*$-algebra. Suppose that $\phi_i : A \rightarrow B$ ($i = 1, 2$) are two unital monomorphisms which induce the same map at the level of $K_0$ and satisfy

$$
\tau \circ \phi_1(a) = \tau \circ \phi_2(a)
$$

for all $a \in A$ and all $\tau \in T(B)$. Then there exists a sequence of unitaries $u_n \in B$ such that

$$
\lim_{n \rightarrow \infty} u_n^* \phi_1(a) u_n = \phi_2(a) \quad \text{for all} \quad a \in A.
$$

Combining this with the more general uniqueness theorem of [35], we are able to obtain a uniqueness theorem for nuclear separable simple TAI $C^*$-algebras. As in [20], our invariant includes not only Banach algebra $K$-theory, but also an additional datum, namely, the tracial state space $T(A)$ together with a pairing of $T(A)$ and $K_0(A)$. Since traces are part of the invariant as in [15, 26] and [20], we also use some earlier results of J. Cuntz and G.K. Pedersen. However, we are able to avoid some difficult topological techniques involving higher-dimensional CW-complexes in [20]. We also show that the set of Elliott invariants for unital separable simple $C^*$-algebras with $\text{TR}(A) \leq 1$ which satisfy the UCT is the same as that of simple AH-algebras with no dimension growth as described by J. Villadsen [57] (see 10.2 below). The uniqueness theorem also
has to be adjusted to deal with other complications caused by the fact that our $C^*$-algebras are no longer assumed to have real rank zero. A careful treatment on exponential length is needed. Our existence theorem also needs to be improved from that in [38]. The existence theorem should also control the exponential length. It turns out that when $C^*$-algebras are assumed to have only torsion $K_1$, the proof can be made much shorter. This is done without using de la Harpe and Skandalis determinants as in [20].

The paper is organized as follows. Section 2 gives the definition of TAI $C^*$-algebras. Section 3 gives some elementary properties of simple TAI $C^*$-algebras. In Section 4, we show that simple TAI $C^*$-algebras have stable rank one, weakly unperforated $K_0$, the fundamental comparison property, and are MF. Starting from Section 5, we will use term “simple $C^*$-algebras with $\text{TR}(A) \leq 1$” instead of “TAI $C^*$-algebra $A$.” Even though that the term “$\text{TR}(A) \leq 1$” has appeared in [33], the term “TAI” has been used and results in the first 4 sections have been quoted in a number of places including [33]. We feel that we can keep the literature consistent by keeping the term “TAI” in the first 4 sections here. In Section 5, we show that every simple (nonelementary) $C^*$-algebra $A$ with $\text{TR}(A) \leq 1$ is tracially approximately divisible. We also give a classification theorem for monomorphisms from $\mathcal{M}_r(C([0,1]))$ to a unital simple TAI $C^*$-algebra mentioned earlier. In Section 6, we study the unitary group of a simple $C^*$-algebra $A$ with $\text{TR}(A) \leq 1$. Exponential rank of a simple $C^*$-algebra $A$ with $\text{TR}(A) \leq 1$ is proved to be no more than $3 + \varepsilon$ in the sense of [44]. Let $\text{CU}(A)$ be the closure of commutator group of $U(A)$. We show that $U_0(A)/\text{CU}(A)$ is always divisible, and if $A$ is simple and $\text{TR}(A) \leq 1$ then $U_0(A)/\text{CU}(A)$ is torsion free. In Section 7, we present some results concerning homomorphisms from $U(C)/\text{CU}(C)$ to $U(B)/\text{CU}(B)$, where $B$ is a unital simple $C^*$-algebra with $\text{TR}(B) \leq 1$ and $C$ is a very special unital $C^*$-algebra. These results may be viewed as part of the existence theorem which controls the exponential length of unitaries under certain maps. In Section 8, we present a uniqueness theorem suitable to be used in the proof of 10.4 which is based on results in [35]. One immediate consequence of it is the following. Let $A$ be a unital separable simple nuclear $C^*$-algebra with $\text{TR}(A) \leq 1$ and with torsion $K_1(A)$. Then an automorphism $\alpha : A \to A$ is approximately inner if and only if $[\alpha] = [\text{id}_A]$ in $KL(A)$ and $\tau \circ \alpha(x) = \tau(x)$ for each self-adjoint $x \in A$ and $\tau \in T(A)$. In Section 9, we present several existence theorems. The purpose is to establish a map from $A$ to $B$ if $\text{TR}(A) \leq 1$, $\text{TR}(B) \leq 1$ and $(K_0(A), K_0(A)_+, \{1_A\}, K_1(A), T(A)) = (K_0(B), K_0(B)_+, \{1_B\}, K_1(B), T(B))$. Finally, in Section 10, we give the proof of the main theorem—Theorems 10.4 and 10.10.

The following terminology and notation will be used throughout this paper.

Let $A$ be a $C^*$-algebra.

(i) Two projections in $A$ are said to be equivalent if they are Murray–von Neumann equivalent. We write $p \approx q$ if $p$ is equivalent to a projection in $q A q$. We use $[p]$ for the equivalence class of projections equivalent to $p$. Let $a \in A_+$. We write $p \preceq a$ if $p \preceq q$ for some projection $q \in a \overline{A} a$.

(ii) An element in $A$ is said to be full if the (closed) ideal generated by the element is $A$ itself. Every nonzero element in a simple $C^*$-algebra is full.

(iii) Let $\varepsilon > 0$, $\mathcal{F}$ and $S$ be a subset of $A$. We write $x \in_{\varepsilon} S$ if there exists $y \in S$ such that $\|x - y\| < \varepsilon$, and write $\mathcal{F} \subseteq_{\varepsilon} S$, if $x \in_{\varepsilon} S$ for all $x \in \mathcal{F}$.

(iv) Let $A$ be a $C^*$-algebra. Denote by $A_{\text{sa}}$ the set of all self-adjoint elements of $A$ and denote by $A_+$ the set of all positive elements of $A$.

(v) Let $\mathcal{G} \subseteq A$ and $\delta > 0$. A contractive completely positive linear map $L : A \to B$ is said to be $\mathcal{G}$-$\delta$-multiplicative if

$$\left\| L(ab) - L(a)L(b) \right\| < \delta \quad \text{for all } a, b \in \mathcal{G}.$$
(vi) Let $X$ be a compact metric space and $h : PM_r(C(X))P \to A$, where $P \in M_r(C(X))$ is a projection, be a homomorphism. We say $h$ is homotopically trivial, if $h$ is homotopic to a point-evaluation. A contractive completely positive linear map $L : PM_r(C(X))P \to A$ is said to be homotopically trivial, if $L$ factors through a homotopically trivial homomorphism, i.e., $L = L' \circ h$, where $h$ is homotopically trivial.

2. Definition of tracially AI $C^*$-algebras

2.1. Definition. We denote by $I$ the class of all unital $C^*$-algebras with the form $\bigoplus_{i=1}^n B_i$, where each $B_i \cong M_{k(i)}$ for some integer $k(i)$ or $B_i \cong M_{k(i)}(C([0, 1]))$. Let $A \in I$. We have the following well-known facts.

(i) Every $C^*$-algebra in $I$ is of stable rank one.

(ii) Two projections $p$ and $q$ in a $C^*$-algebra $A \in I$ are equivalent if and only if $\tau(p) = \tau(q)$ for all $\tau \in T(A)$.

(iii) For any $\varepsilon > 0$ and any finite subset $F \subset A$, there exist $\delta > 0$ and a finite subset $G \subset A$ satisfying the following: if $L : A \to B$ is a $G$-$\delta$-multiplicative contractive completely positive linear map, where $B$ is a $C^*$-algebra, then there exists a homomorphism $h : A \to B$ such that

$$\|h(a) - L(a)\| < \varepsilon \quad \text{for all } a \in F.$$ 

These facts will be used throughout the paper without further notice.

2.2. Definition. A unital $C^*$-algebra $A$ is said to be tracially AI (TAI) if for any finite subset $F \subset A$ containing a nonzero element $b$, $\varepsilon > 0$, integer $n > 0$ and any full element $a \in A_+$, there exist a nonzero projection $p \in A$ and a $C^*$-subalgebra $I \subset A$ with $I \in I$ and $1_I = p$, such that:

1. $\|[x, p]\| < \varepsilon$ for all $x \in F$.
2. $pxp \in \varepsilon I$ for all $x \in F$ and $\|pbp\| \geq \|b\| - \varepsilon$.
3. $n[1 - p] \leq [p]$ and $1 - p \preceq a$.

A non-unital $C^*$-algebra $A$ is said to be TAI if $\tilde{A}$ is TAI.

In 4.10, we show that, if $A$ is simple, condition (3) can be replaced by

(3') $1 - p$ is unitarily equivalent to a projection in $e Ae$ for any previously given nonzero projection $e \in A$.

If $A$ has the Fundamental Comparability (see [2]), condition (3) can be replaced by

(3'') $\tau(1 - p) < \sigma$ for any prescribed $\sigma > 0$ and for all normalized quasi-traces of $A$.

From the definition, one sees that the part of $A$ which may not be approximated by $C^*$-algebras in $I$ has small “measure” or trace. Note in the above, if $I$ is replaced by finite-dimensional $C^*$-algebras, then it is precisely the definition of TAF $C^*$-algebras (see [31]).
2.3. Example. Every AF-algebra is TAI. Every TAF \( C^* \)-algebra introduced in [31] is a TAI \( C^* \)-algebra. However, in general, TAI \( C^* \)-algebras have real rank other than zero. In 4.5 we will show that every simple TAI \( C^* \)-algebra has stable rank one, which implies that simple TAI \( C^* \)-algebras have real rank one or zero. It is obvious that every direct limit of \( C^* \)-algebras in \( T \) is a TAI \( C^* \)-algebra. These \( C^* \)-algebras provide many examples of TAI \( C^* \)-algebras that have real rank one. However, TAI \( C^* \)-algebras may not be inductive limits of \( C^* \)-algebras in \( T \).

Let \( A = \lim_{n \to \infty} (A_n, \phi_n, a_n) \), where \( A_n = \bigoplus_{i=1}^{s(n)} P_{n,i} M_{n,i}(C(X_{n,i})) P_{n,i} \), \( X_{n,i} \) is a finite-dimensional compact metric space and \( P_{n,i} \in M_{n,i}(C(X_{n,i})) \) is a projection for all \( n \) and \( i \). Such a \( C^* \)-algebra is called an AH-algebra. Suppose that \( A \) is unital. Following [23], \( A \) is said to have very slow dimension growth if

\[
\lim_{n \to \infty} \min_i \frac{\text{rank}(P_{n,i})}{(\dim X_{n,i} + 1)^3} = \infty.
\]

\( A \) is said to have no dimension growth if there is an integer \( m > 0 \) such that \( \dim X_{n,i} \leq m \). Note these \( C^* \)-algebras may not be of real rank zero. Since these \( C^* \)-algebras could have non-trivial \( K_1 \)-groups (see 10.1), they are not inductive limits of \( C^* \)-algebras in \( T \). In [31], example of simple TAF \( C^* \)-algebras which are non-nuclear was given. In particular, there are simple TAI \( C^* \)-algebras that are not even nuclear.

2.4. Lemma. Let \( a \) be a positive element in a unital \( C^* \)-algebra \( A \) with \( \text{sp}(a) \subset [0, 1] \). Then for any \( \varepsilon > 0 \), there exists \( b \in A_+ \) such that \( \text{sp}(b) \) is a union of finitely many mutually disjoint closed intervals and finitely many points and

\[
\|a - b\| < \varepsilon.
\]

Proof. Fix \( \varepsilon > 0 \). Let \( I_1, I_2, \ldots, I_k \) be all disjoint closed intervals in \( \text{sp}(a) \) with length at least \( \varepsilon/8 \) such that if \( I \supset I_j \) is an interval, then \( I \not\subset \text{sp}(a) \). Let \( d' = \min\{\text{dist}(I_i, I_j), i \neq j\} \) and \( d = \min(d'/2, \varepsilon/16) \).

Choose \( J_i = \{ \xi \in [0, 1]: \text{dist}(\xi, I_j) < d_i \} \), \( i = 1, 2, \ldots, k \) with \( d_i \leq d \) and the endpoints of \( J_i \) are not in \( \text{sp}(a) \). Since the endpoints of \( J_i \) are not in \( \text{sp}(a) \), there are open intervals \( J'_i \subset J_i \) such that \( J'_i \subset J_i \) and \( I_i \subset J'_i \). Set \( Y = \text{sp}(a) \setminus (\bigcup_{i=1}^{k} J_i) \). Then \( Y = \text{sp}(a) \setminus (\bigcup_{i=1}^{k} J'_i) = \text{sp}(a) \setminus (\bigcup_{i=1}^{k} J_i) \). Since \( Y \) is compact and \( Y \) contains no intervals with length more than \( \varepsilon/8 \), it is routine to show that there are finitely many disjoint closed intervals \( K_1, K_2, \ldots, K_n \) in \( [0, 1] \setminus (\bigcup_{i=1}^{k} J_i) \) with length no more than \( 9\varepsilon/64 \) such that \( Y \subset \bigcup_{j=1}^{n} K_j \). Note that \( \{ J'_1, J'_2, \ldots, J'_k, K_1, K_2, \ldots, K_n \} \) are disjoint closed intervals. Fix a point \( \xi_j \in K_j \), \( j = 1, 2, \ldots, m \). One can define a continuous function \( f : (\bigcup_{i=1}^{k} J'_i) \cup (\bigcup_{s=1}^{n} K_s) \to [0, 1] \) which maps each \( J_i \) onto \( I_i \), \( i = 1, 2, \ldots, k \) and maps \( K_j \) to a single point \( \xi_j \) such that

\[
|f(\xi) - \xi| < \varepsilon/2 \quad \text{for all } \xi \in [0, 1].
\]

Define \( b = f(a) \). We see that \( b \) meets the requirements of the lemma. \( \square \)

2.5. Theorem. Let \( A \) be a unital simple AH-algebra with very slow dimension growth. Then \( A \) is TAI.
Proof. By 1.3.3, 1.3.4 and 4.23 of [23] (and [19]), to show that $A$ is TAI, it suffices to assume that $A = \lim_{n \to \infty} (A_n, \phi_{n,m})$, where $A_n = \bigoplus_{i=1}^{i(n)} M_{n(i)}(C(X_{n,i}))$, $X_{n,i}$ are simplicial complexes and $\phi_{n,m}$ are injective (see also 3.4 below). Moreover, we may also assume that $A$ satisfies the condition of very slow dimension growth.

Let $\varepsilon > 0$, $\mathcal{F} \subset A$ be a finite subset and $e \in A$ be a non-zero projection. To verify (1), (2) and (3') in 2.2, without loss of generality, we may assume that $\mathcal{F} \subset A_1$ and $e \in A_1$. By considering each summand separately, without loss of generality, we may also assume that $A_1 = M_r(C(X))$ for some finite simplicial complex and integer $r \geq 1$. Let $\mathcal{F}_1 \subset C(X)$ be a finite subset such that $\mathcal{F} \subset \{(f_{i,j})_{r \times r}: f_{i,j} \in \mathcal{F}_1\}$.

Let $J > r + 1$ be an integer. Let $\varepsilon/2r^2 > \eta > 0$ such that $|f(x) - f(x')| < \varepsilon/9r^2$ for all $f \in \mathcal{F}_1$ whenever $\text{dist}(x, x') < 2\eta$. Let $\delta > 0$ and $L$ be as in Theorem 4.35 corresponding to $\varepsilon/2r^2$, $\eta$ and $\mathcal{F}_1$ above. Since $A$ is simple, as in 4.36 of [23], each partial map of $\phi_{1,m}$ (for sufficiently large $m$) has the property $\text{sdp}(\eta/32, \varepsilon/2r^2)$. To simplify notation, without loss of generality, we may assume that $A_m = M_k(C(Y))$ and $\text{rank}(\phi_{1,m}(1)) > 2JL^22L^2(\text{dim} \mathcal{X} + \text{dim} \mathcal{Y} + 1)^3$, where $Y$ is a finite simplicial complex. To simplify notation, by considering each summand separately, without loss of generality, we may assume that $Y$ is connected. Since $A$ is simple, by choosing a larger $m$, we further assume that $e \in M_k(C(Y))$ is a non-zero projection which has the rank at least $\text{rank}(\phi_{1,m}(1))/r$.

By applying 4.35 of [23], there are three mutually orthogonal projections $Q_0, Q_1, Q_2 \in A_m$ and homomorphisms $\psi_i: A_1 \to Q_i A_m Q_i$ ($i = 0, 1, 2$) such that:

1. $\phi_{1,m} = Q_0 + Q_1 + Q_3$;
2. $\|\phi_{n,m}(f) - (\phi_0(f) \oplus \phi_1(f) \oplus \phi_2(f))\| < \varepsilon/2$ for all $f \in \mathcal{F}$;
3. $\psi_2$ factors through $M_r(C([0, 1]))$;
4. $\psi_1$ has finite-dimensional range;
5. $J[Q_0] \leq [Q_1]$.

Put $\psi = \psi_1 \oplus \psi_2$. It follows from Lemma 2.4 that there is a unital $C^*$-subalgebra $B_1 \in \mathcal{T}$ of $(Q_1 + Q_2)A_m(Q_1 + Q_2)$ such that

$$\psi(f) \in \varepsilon B_1 \quad \text{for } f \in \mathcal{F}.$$ 

We also have

$$[Q_0] \leq [e].$$

Thus, $A$ is of TAI. \qed

3. Elementary properties of simple TAI $C^*$-algebras

3.1. Lemma. For any $d > 0$ there are $f_1, f_2, \ldots, f_m \in C([0, 1])_+$ with the following properties. For any $n$, and any positive element $x \in B = M_n(C([0, 1]))$ with $\|x\| \leq 1$ if there exist $a_{i,j} \in B$, $i = 1, 2, \ldots, n(j)$, $j = 1, 2, \ldots, m$ with

$$\left\| \sum_{i=1}^{n(j)} a_{i,j} f_j(x) a_{i,j}^* - 1_A \right\| < 1/2, \quad j = 1, 2, \ldots, m,$$
then, for any subinterval $J$ of $[0, 1]$ with $\mu(J) \geq d$ (\(\mu\) is the Lebesgue measure), $\text{sp}(\pi_t(x)) \cap J \neq \emptyset$ for all $t \in [0, 1]$, where $\pi_t : B \to M_n$ is the point evaluation at $t$. Moreover, denote by $N = \max\{n(j): j = 1, 2, \ldots, m\}$,

$$\left|\text{sp}(\pi_t(x)) \cap J\right| \geq 1/N|\text{sp}(\pi_t(x))|,$$

where $|S|$ means the number of elements in the finite set $S$ (counting multiplicities).

**Proof.** Divide $[0, 1]$ into $m$ closed subintervals $\{J_j\}$ each of which has the same length $< d/4$. Let $f_j \in C([0, 1])$ be such that $0 \leq f_j \leq 1$, $f_j(t) = 1$ for $t \in J_j$ and $f_j(t) = 0$ for $\text{dist}(t, J_j) \geq \mu(J_j)$. Note that, for any subinterval $J$ with $\mu(J) \geq d$, there exists $j$ such that $J_j \subset J$. For any $t \in [0, 1]$, set

$$I = \{g \in B: g(t) = 0\}.$$

Then $I$ is a (closed) ideal of $A$. If $\text{sp}(\pi_t(x)) \cap J = \emptyset$, there would be $j$ such that $\pi_t(f_j(x)) = 0$. Therefore $f_j \in I$. But this is impossible, since there is an element $z \in B$ with

$$z\left(\sum_{i=1}^{m} a_{i,j} f_j(x) a_{i,j}^*\right) = 1_B.$$

For the last part of the lemma, fix $t \in [0, 1]$ and an interval $J$ with $t \in J$ and $\mu(J) \geq d$. Let $\pi_t(B) = M_{l(t)}$. Then $|\text{sp}(\pi_t(x))| = l(t)$. Suppose that $J_j \subset J$ so that $f_j(t) = 0$ for all $t \notin J$. Let $q_t$ be the spectral projection of $\pi_t(x)$ in $M_{l(t)}$ corresponding to $J$. Then $q_t \geq f_j(\pi_t(x))$. An elementary linear algebra argument shows that $\text{rank} q_t \geq (1/N) l(i)$. \(\square\)

### 3.2. Theorem

Every unital simple $C^*$-algebra satisfying (1) and (2) in 2.2 has property (SP), i.e., every hereditary $C^*$-subalgebra contains a nonzero projection.

**Proof.** Let $A$ be a unital simple $C^*$-algebra satisfying (1) and (2) and $B \subset A$ be a hereditary $C^*$-subalgebra. We may assume that $A$ is not elementary. Thus $B$ is not elementary. By p. 61 (item 4) in [1], there is $a \in B_+$ such that $\text{sp}(a) = [0, 1]$. It suffices to show that $B_1 = aaB$ has a nonzero projection.

Let $d$ be a positive number with $0 < d < 1/16$. Let $f_1, f_2, \ldots, f_m$ be as in 3.1. Since $A$ is simple, there are $a_{i,j} \in A$ such that

$$\left\|\sum_{i=1}^{m} a_{i,j} f_j(a) a_{i,j}^* - 1_A\right\| < 1/8, \quad j = 1, 2, \ldots, m.$$

Let $g \in C_0((0, 1))$ with $0 \leq g \leq 1$, $g(t) = 1$ if $g \in [1/4, 3/4]$. Denote by $\mathcal{F}$ the subset

$$\{a, g(a), f_j(a), a_{i,j}, a_{i,j}^*: i, j\}.$$

For any $\varepsilon > 0$, there exists a projection $p \in A$ and a $C^*$-subalgebra $C \subset A$ with $1_I = p$ such that

1. $\| [b, p] \| < \varepsilon$ for all $b \in \mathcal{F}$,
2. $pbp \in_c C$ and $\| pap \| \geq 1 - \varepsilon$. 
A standard perturbation argument shows that, for any \( \eta > 0 \), with sufficiently small \( \varepsilon \), there is a homomorphism \( \phi : C^*\langle\langle apa \rangle\rangle \to C \) (where \( C^*\langle\langle apa \rangle\rangle \)) the \( C^* \)-subalgebra generated by \( apa \)) and there are \( b_{ij} \in C \) such that

\[
\| apa - \phi (apa) \| < \eta, \quad \| \phi (f_j (a)) - pf_j (a) \| < \eta, \\
\| \phi (g (a)) - pg (a) \| < \eta, \quad \text{and} \quad \sum_{i=1} \| \phi (f_j (apa)) b_{ij}^* - p \| < 1/4
\]

for \( j = 1, 2, \ldots, m \).

Write \( C = \bigoplus_k M_{m(k)}(C([0, 1])) \) and \( C_k = M_{m(k)}(C([0, 1])) \). By Lemma 3.1, with sufficiently small \( d \), we may assume that

\[
| \text{sp} (\pi_t (\phi_k (apa))) \cap (0, 1) | \geq 4
\]

for each \( t \) and \( k \), where \( \pi_t \) is the point-evaluation at \( t \in [0, 1] \), \( \phi_k \) is the map from \( C([0, 1]) \) to \( M_{m(k)}(C([0, 1])) \) induced by \( \phi \) and where \( | \text{sp} (\pi_t (\phi_k (apa))) | \) is the number of eigenvalues of \( \pi_t (\phi_k (apa)) \) counting multiplicities.

Fix \( k \). For \( t \in [0, 1] \), let \( J_t \subseteq [1/4, 3/4] \) be an open interval with \( \mu (J_t) \geq d \) whose endpoints are not eigenvalues of \( \pi_y (\phi_k (apa)) \). Let \( V_t \) be an open neighborhood of \( t \) such that the end points of \( J_t \) are not eigenvalues of \( \pi_y (\phi_k (apa)) \) for \( y \in V_t \). Let \( \xi_{J_t} \) be the characteristic function on \( J_t \). Then using the continuous functional calculus we can define a continuous projection valued function \( q_t : V_t \to M_{m(k)} \) by \( q_t (y) = \xi_{J_t} (\phi_k (apa)) \). From the previous paragraph, \( \mathrm{rank} (q_t (y)) \geq 4 \). It follows from Proposition 3.2 in [13] that there exists a nonzero projection \( q \in M_{m(k)}(C([0, 1])) \) such that \( q(t) \leq f (\phi (apa))(y) \) for all \( t \in [0, 1] \), where \( f \in C_0((0, 1)) \) with \( 0 \leq f \leq 1 \), \( f (t) = 1 \) if \( t \in [1/4, 3/4] \) (see also the proof of 1.4 of [43]). In particular, \( q f (\phi (apa)) = q \).

Let \( b = f (a) \). We estimate that

\[
\| bq - q \| < 2 \eta.
\]

It is standard that if \( \eta < 1/8 \), there is a projection \( q' \subseteq bAb \) such that

\[
\| q' - q \| < 1/2.
\]

This implies that \( bAb \subseteq aAa \subseteq B \) contains a nonzero projection \( q' \). \( \square \)

3.3. Corollary. Let \( A \) be a unital simple \( C^* \)-algebra satisfying (1) and (2) in 2.2. Then, for any integer \( N \), we may assume that \( I = \bigoplus_{i=1}^k M_{m_i} (C([0, 1])) \oplus \bigoplus_{j=1}^l M_{n_j} \) where \( m_i, n_i \geq N \).

Proof. In the proof of 3.2, we see that if \( 1/2d \geq N \), since \( \text{sp} (\pi_t (\phi (apa)) \cap J \neq \emptyset \) for each \( j \), then \( \pi_t (\phi (apa)) \) has at least \( N \) distinct eigenvalues (see also the proof of 3.1). Therefore, each summand \( C \) in the proof 3.2 has rank at least \( N \). \( \square \)

3.4. Proposition. Let \( A \) be a unital TAI \( C^* \)-algebra and \( e \in A \) be a full projection. Then \( eAe \) satisfies (1) and (2) in 2.2, and for any full positive element \( a \in eAe \), we can have

\[
(3') \quad 1 - p \approx a.
\]
If $A$ is also simple, $eAe$ is TAI.

**Proof.** Fix $\varepsilon > 0$, a finite subset $F \subset eAe$, an integer $n > 0$ and nonzero elements $a, b \in eAe$ with $a \geq 0$ and $b \in F$. Let $\mathcal{F}_1 = \{e\} \cup F$. Since $A$ is TAI, there exists $q \in A$ and a $C^*$-subalgebra $C \in \mathcal{I}$ with $1_C = q$ such that:

(i) $\| [x, q] \| < \varepsilon / 64$ for all $x \in \mathcal{F}$,

(ii) $qxq \in \varepsilon / 64 C$ for all $x \in \mathcal{F}$, and $\| qbq \| \geq \| b \| - \varepsilon / 64$; and,

(iii) $n [1 - q] \leq [q]$ and $1 - q \preceq a$.

Note that, by the second part of (ii), $qeq \neq 0$. We estimate that

$$\| (qe)^2 - e q e \| < \varepsilon / 64 \quad \text{and} \quad \| q e - q e q \| < \varepsilon / 32.$$ 

Therefore there is a projection $p \in eAe$ such that

$$\| p - q e q \| < \varepsilon / 16.$$ 

Consequently, there is a projection $d \in C$ such that

$$\| d - p \| < \varepsilon / 8.$$ 

Note that

$$\| q p - p q \| < \varepsilon / 8 + \| q e q - q e q \| < \varepsilon / 8 + \varepsilon / 32 = 5 \varepsilon / 32,$$

and $B = d C d \in \mathcal{I}$. With $\varepsilon / 2 < 1 / 2$, we obtain a unitary $u \in A$ such that

$$\| u - 1 \| < \varepsilon / 4 \quad \text{and} \quad u^* du = p.$$ 

Set $C_1 = u^* B u$. Then $C_1 \in I$ and $C_1 \subset eAe$. Now $1_{C_1} = p$,

(1) $\| [x, p] \| < \varepsilon / 2$ for all $x \in F$,

(2) $p x p \in \varepsilon / 2 C_1$ for all $x \in F$ and $\| p b p \| \geq \| b \| - \varepsilon / 2$.

We also have

$$\| (e - p) - (1 - q)(e - p)(1 - q) \|$$

$$\leq \| (e - p) - (e - p)(1 - q) + q(e - p)(1 - q) \|$$

$$< \| (e - p)q \| + \varepsilon / 16 < 5 \varepsilon / 32 + \varepsilon / 64 + \| q e q - q p q \| + \varepsilon / 16$$

$$< 5 \varepsilon / 32 + \varepsilon / 64 + \varepsilon / 16 + \| q e q - q e q \| + \varepsilon / 16 < 9 \varepsilon / 32.$$ 

We have (with $\varepsilon < 1$)

(3') $(e - p) \preceq (1 - q) \preceq a$. 

Finally, if we assume that \( A \) is simple, by 3.2 and 3.3 there is a nonzero projection \( p_1 \leq p \) such that \( n[p_1] \leq [p] \). There is a nonzero projection \( p_1 \in aAa \). By applying 3.3, we obtain a nonzero projection \( q_1 \leq p_1 \) such that \( n[q_1] \leq [p_1] \). Applying the first part of the proof to \((e - p)F(e - p)\), we obtain a projection \( p' \leq (e - p) \) and a unital \( C^\ast \)-subalgebra \( C_2 \in \mathcal{I} \) with \( 1_{C_2} = p' \) such that:

\[
(1') \quad \|[e - p]x\(e - p\), p']\| < \varepsilon /2 \text{ for all } x \in \mathcal{F},
\]

\[
(2') \quad p'xp' \in \varepsilon /2 \text{ for all } x \in \mathcal{F}, \quad \text{and}
\]

\[
(3') \quad (e - p - p') \leq p_1.
\]

Now since \( n[p_1] \leq [p] \leq [p + p'] \) and \((e - p - p') \leq (e - p) \leq a\), we obtain

\[
(3) \quad n[(e - p - p')] \leq [p + p'] \text{ and } (e - p - p') \leq a.
\]

We also have \( \|[x, (p + p')]\| < \varepsilon \) and \((p + p')x(p + p') \in \varepsilon C_1 \oplus C_2 \) for all \( x \in \mathcal{F} \). Hence \( eAe \) is TAI. \( \square \)

### 3.5. Corollary
If \( A \) is a unital simple TAI \( C^\ast \)-algebra, then condition (2) can be strengthened to

\[
(2') \quad pxp \in \varepsilon B \text{ and } \|pxp\| \geq \|x\| - \varepsilon \text{ for all } x \in \mathcal{F}.
\]

We omit the proof.

### 3.6. Theorem
Let \( A \) be a unital simple \( C^\ast \)-algebra. Then \( A \) is TAI if and only if \( M_n(A) \) is TAI for all \( n \) (or for some \( n > 0 \)).

**Proof.** If \( M_n(A) \) is TAI, then by identifying \( A \) with a unital hereditary \( C^\ast \)-subalgebra of \( M_n(A) \) and by using 3.4, we know \( A \) is TAI. It remains to prove the “only if” part.

We prove this in two steps. The first step is to prove that \( M_n(A) \) satisfies (1) and (2) in 2.2. To do this, we let \( \varepsilon > 0 \) and \( \mathcal{F} \) be a finite subset of the unit ball of \( M_n(A) \). Set \( \mathcal{G} = \{f_{ij} \in A : (f_{ij})_{n \times n} \in \mathcal{F}\} \). Note that \( \mathcal{G} \subset A \). Since \( A \) is TAI, there exists a projection \( p \in A \) and a unital \( C^\ast \)-subalgebra \( B \in \mathcal{I} \) such that:

\[
(1) \quad \|[x, p]\| < \varepsilon /2n^2,
\]

\[
(2) \quad pxp \in \varepsilon /2n^2 B \text{ for all } x \in \mathcal{G} \text{ and for some } x_1 \in \mathcal{G}, \quad \|px_1p\| \geq \|x_1\| - \varepsilon /2n^2.
\]

Put \( P = \text{diag}(p, p, \ldots, p) \in M_n(A) \) and \( D = M_n(B) \). Then, it is easy to check that

\[
(i) \quad \|[F, P]\| < \varepsilon \quad \text{and}
\]

\[
(ii) \quad PfP \in \varepsilon D \text{ for all } f \in \mathcal{F} \text{ and } \|pf_1p\| \geq \|f_1\| - \varepsilon \text{ (if } f_1 \text{ is prescribed).}
\]

This completes the first step. Now we also know by 3.2 that \( M_n(A) \) has (SP). Let \( a \in M_n(A) \) be given. Choose any nonzero projection \( e \in aM_n(A)a \). Since \( M_n(A) \) is simple and has (SP), by 3.1 in [31], there is a nonzero projection \( q \leq e \) and \( [q] \leq [1_A] \). Applying [31, 3.2], there exists a nonzero projection \( q_1 \leq q \) such that \( (n + 1)[q_1] \leq [q] \). In the first step, we can also require, for any integer \( N > 0 \), that

\[
(3) \quad N[1_A - p] \leq [p] \text{ and } 1_A - p \leq q_1.
\]
This implies that

(iii) \( N[1_{M_n(A)} - P] \subseteq [P] \) and \( (1_{M_n(A)} - P) \preceq q \preceq e. \)

Therefore \( M_n(A) \) is TAI. \( \Box \)

Next we show that every simple TAI \( C^* \)-algebra has the property introduced by Popa [45].

3.7. Proposition. Let \( A \) be a unital simple TAI \( C^* \)-algebra. Then for any finite subset \( \mathcal{F} \subset A \) and \( \varepsilon > 0 \), there exists a projection \( p \in A \) and a finite-dimensional \( C^* \)-algebra \( F \subset A \) with \( 1_F = p \) such that

(P1) \( \|[x, p]\| < \varepsilon \) and

(P2) \( pxp \in \varepsilon F \) for all \( x \in \mathcal{F} \) and \( \|pxp\| \geq \|x\| - \varepsilon \) for all \( x \in \mathcal{F} \).

Proof. By 3.5 it is clear that it suffices to prove the following claim: for any unital \( C^* \)-subalgebra \( B \in I \), the proposition holds for any finite subset \( \mathcal{F} \subset B \subset A \).

This can be further reduced to the case that \( B = C([0, 1]) \otimes M_k \). Moreover, it suffices to prove the claim for the case in which \( B = C([0, 1]) \).

By 3.2, there is a nonzero projection \( e_i \in f_i A f_i \). Note that \( e_i e_j = 0 \) if \( i \neq j \). Set \( p = \sum_{i=1}^n e_i \). We estimate that (see [29, Lemma 2])

\[
\|x - \left( (1 - p)x(1 - p) + \sum_{i=1}^n \xi_i e_i \right)\| < \varepsilon / 2 \quad \text{and} \quad \|[p, x]\| < \varepsilon \text{ by (P1).}
\]

Let \( F_1 \) be the finite-dimensional \( C^* \)-subalgebra generated by \( e_1, e_2, \ldots, e_n \). Then by (P2), \( pxp \in \varepsilon F_1 \) and \( \|pxp\| \geq \|x\| - \varepsilon \). \( \Box \)

4. The structure of simple TAI \( C^* \)-algebras

4.1. Theorem. Every unital separable simple TAI \( C^* \)-algebra is MF [4].

Proof. Let \( A \) be such a \( C^* \)-algebra and let \( \{x_n\} \) be a dense sequence in the unit ball of \( A \). By 3.7, there are projections \( p_n \in A \) and finite-dimensional \( C^* \)-subalgebras \( B_n \) with \( 1_{B_n} = p_n \) such that

(1) \( \|[p_n, x_i]\| < 1/n \), and

(2) \( pxp \in 1/n B_n \) and \( \|p_n x_i p_n\| \geq \|x_i\| - 1/n \) for \( i = 1, 2, \ldots, n \).

Let \( \text{id}_n : B_n \to B_n \) be the identity map and let \( j : B_n \to M_{K(n)} \) be a unital embedding. We note that such \( j \) exists provided that \( K(n) \) is large enough. By [41, 5.2], there exists a completely positive map \( L'_n : p_n A p_n \to M_{K(n)} \) such that \( L'_n |_{B_n} = j \circ \text{id}_n \). Since \( L'_n \) is unital, by [41, 5.9 and
5.10], \( L'_n \) is a contraction. We define \( L_n : A \to M_{K(n)} \) by \( L_n(a) = L'_n(p_n a p_n) \). Let \( y_{i,n} \in B_n \) such that \( \| p_n x_i p_n - y_{i,n} \| < 1/n, n = 1, 2, \ldots \). Then
\[
\| L_n(x_i) - p_n x_i p_n \| \leq \| L_n(x_i - y_{i,n}) - (y_{i,n} - p_n x_i p_n) \| < 2/n \to 0
\]
as \( n \to \infty \). Combining this with (1) above, we see that
\[
\| L_n(ab) - L_n(a)L_n(b) \| \to 0
\]
as \( n \to \infty \).

Define \( \Phi : A \to \prod_{n=1}^{\infty} M_{m(n)} \) by sending \( a \) to \( \{ L_n(a) \} \). Then \( \Phi \) is a completely positive map. Denote by \( \pi : \prod_{n=1}^{\infty} M_{m(n)} \to \prod_{n=1}^{\infty} M_{m(n)} / \bigoplus_{n=1}^{\infty} M_{m(n)} \cdot Then
\[
\pi \circ \Phi : A \to \prod_{n=1}^{\infty} M_{m(n)}/ \bigoplus_{n=1}^{\infty} M_{m(n)}
\]
is a (nonzero) homomorphism. Since \( A \) is simple, \( \pi \circ \Phi \) is injective. It follows from [4, 3.22] that \( A \) is an MF-algebra. □

4.2. Corollary. Every separable unital C*-algebra satisfying (P1) and (P2) is MF.

Proof. We actually proved this above. Note, simplicity is not needed for injectivity since \( \| p_n x_p n \| \to \| x \| \). □

4.3. Proposition. Every nuclear separable simple TAI C*-algebra is quasidiagonal.

Proof. As in [4], a separable nuclear MF C*-algebra is NF, and it is quasidiagonal. In fact it is strong NF (see [5]). □

4.4. Corollary. Every unital separable simple TAI C*-algebra has at least one tracial state.

Proof. It is well known that \( \prod_{n=1}^{\infty} M_{m(n)}/ \bigoplus_{n=1}^{\infty} M_{m(n)} \) has tracial states. Tracial states are defined by weak limits of tracial states on each \( M_{m(n)} \). Let \( \tau \) be such a tracial state. Then, in the proof of 4.2, let \( t(a) = \tau \circ \pi \circ \Phi(a) \). □

4.5. Theorem. A unital simple TAI C*-algebra has stable rank one.

Proof. Let \( A \) be a unital simple C*-algebra. Take a nonzero element \( a \in A \). We will show that \( a \) is a norm limit of invertible elements in \( A \). So we may assume that \( a \) is not invertible and \( \| a \| = 1 \). Since \( A \) is finite, \( a \) is not one-sided invertible. For any \( \varepsilon > 0 \), by [50, 3.2], there is a zero divisor \( b \in A \) such that \( \| a - b \| < \varepsilon / 2 \). We further assume that \( \| b \| \leq 1 \). Therefore, by [50], there is a unitary \( u \in A \) such that \( ub \) is orthogonal to a nonzero positive element \( c \in A \). Set \( d = ub \). Since \( A \) has (SP) (by 3.2), there exists a nonzero projection \( e \in A \) such that \( de = ed = 0 \). Since \( A \) is simple and has (SP) (by 3.2), we may write \( e = e_1 \oplus e_2 \) with \( e_2 \lesssim e_1 \). Note that \( d \lesssim (1 - e) \lesssim (1 - e_1) \). Moreover, \( (1 - e_1) A (1 - e_1) \) is TAI.

Let \( \eta > 0 \) be a positive number. There is a projection \( p \in (1 - e_1) A (1 - e_1) \) and a unital C*-subalgebra \( B \in \mathcal{I} \) with \( 1_B = p \) such that:
(1) \[\| [x, p] \| < \eta, \]
(2) \[pxp \in \eta B \text{ for all } x \in \mathcal{F}, \text{ and} \]
(3) \[[1 - e_1 - p] \leq [e_2], \]

where \(\mathcal{F}\) contains \(d\). Thus, with sufficiently small \(\eta\), we may assume that

\[\| d - (d_1 + d_2) \| < \varepsilon / 16,\]

where \(d_1 \in B\) and \(d_2 \in (1 - e_1 - p)A(1 - e_1 - p)\).

Since \(C^*\)-algebras in \(\mathcal{I}\) have stable rank one and \(B \in \mathcal{I}\), there is an invertible \(d_1' \in B\) such that

\[\| d_1 - d_1' \| < \varepsilon / 8.\]

Let \(v\) be a partial isometry such that \(v^*v = (1 - e_1 - p)\) and \(vv^* \leq e_1\). Set \(e_1' = vv^*\) and \(d_2' = \varepsilon / 8(e_1 - e_1') + (\varepsilon / 8)v + (\varepsilon / 8)v^* + d_2\). Note that \((\varepsilon / 8)v + (\varepsilon / 8)v^* + d_2\) has matrix decomposition

\[
\begin{pmatrix}
0 & \varepsilon / 8 \\
\varepsilon / 8 & d_2
\end{pmatrix}.
\]

Therefore \(d_2'\) is invertible in \((1 - p)A(1 - p)\). This implies that \(d' = d_1' + d_2'\) is invertible in \(A\). We also have

\[\| d_2' - d_2 \| < \varepsilon / 8,\]

whence

\[\| d - d' \| < \| d - (d_1 + d_2) \| + \| (d_1 + d_2) - (d_1' + d_2') \| < \varepsilon / 16 + \varepsilon / 8 + \varepsilon / 8 < 3\varepsilon / 8.\]

We have

\[\| b - u^*d' \| \leq \| u^*u(b - u^*d') \| = \| ub - d' \| < 3\varepsilon / 8.\]

Finally,

\[\| a - u^*d' \| \leq \| a - b \| + \| b - u^*d' \| < \varepsilon / 2 + 3\varepsilon / 8 < \varepsilon.\]

Note that \(u^*d'\) is invertible. □

4.6. Corollary. Every unital simple TAI \(C^*\)-algebra has the cancellation of projections, i.e., if \(p \oplus e \sim q \oplus e\) then \(p \sim q\).

4.7. Theorem. Every unital simple TAI \(C^*\)-algebra has the following Fundamental Comparability [2]: if \(p, q \in A\) are two projections with \(\tau(p) < \tau(q)\) for all tracial states \(\tau\) on \(A\), then \(p \preceq q\).
Proof. Denote by \( T(A) \) the space of all normalized traces. It is compact. There is \( d > 0 \) such that \( \tau(q - p) > d \) for all \( \tau \in T(A) \). It follows from [31, 3.2] that there exists a nonzero projection \( e \leq q \) such that \( \tau(e) < d/2 \) for all \( \tau \in T(A) \). Set \( q' = q - e \). Then \( \tau(q' - p) > d/2 \) for all \( \tau \in T(A) \).

It follows from [8, 6.4] that there exists a nonzero \( a \in A_+ \) such that \( q' - p - (d/4) = a + z \) and there is a sequence \( \{u_n\} \) in \( A \):

\[
z = \sum_n u_n^* u_n - \sum_n u_n u_n^*.
\]

Choose an integer \( N > 0 \) such that

\[
\left\| \sum_n u_n^* u_n - \sum_{n=1}^N u_n^* u_n \right\| < d/128 \quad \text{and} \quad \left\| \sum_n u_n u_n^* - \sum_{n=1}^N u_n u_n^* \right\| < d/128.
\]

Let \( \mathcal{F} = \{p, q, q', e, z, u_n, u_n^*, n = 1, 2, \ldots, N\} \) and let \( 0 < \varepsilon < 1 \). Since \( A \) is TAI, there exists a projection \( P \in A \) and a \( C^* \)-subalgebra \( B \in \mathcal{I} \) with \( 1_B = P \) such that:

1. \( \|[x, P]\| < \varepsilon/2N \),
2. \( PxP \in \varepsilon/2N B \) for all \( x \in \mathcal{F} \),
3. \( (1 - P) \precsim e \).

With sufficiently small \( \varepsilon \), using a standard perturbation argument, we obtain projections \( q'' = q_1 + q_2 \), \( p' = p_1 + p_2 \), where \( q_1, q_2, p_1, p_2 \) are projections, \( p_1, q_1 \in B \) and \( q_2, p_2 \in (1 - P)A(1 - p) \) such that

\[
\|q'' - q'\| < d/32 \quad \text{and} \quad \|p' - p\| < d/32.
\]

Furthermore (with sufficiently small \( \varepsilon \)), we obtain \( v_1, v_2, \ldots, v_N \in B \) such that

\[
\left\| (q_1 - p_1 - (d/4)P) - \left( b + \sum_{n=1}^N v_n^* v_n - \sum_{n=1}^N v_n v_n^* \right) \right\| < d/16,
\]

where \( b \in B_+ \) and \( \|PaP - b\| < \varepsilon/2N \). Denote by \( T(B) \) the space of all normalized traces on \( B \). Then

\[
\tau(q_1 - p_1 - (d/4)P - b) > -d/16
\]

for all \( \tau \in T(B) \). Therefore

\[
\tau(q_1 - p_1) > d/4 - d/16 = 3d/16
\]

for all \( \tau \in T(B) \). This implies that \( p_1 \precsim q_1 \) in \( B \), whence also in \( A \). Since \( p_2 \precsim (1 - P) \precsim e \), we conclude that

\[
[p] = [p_1 + p_2] \precsim [q_1] + [e] \precsim [q]. \quad \Box
\]
4.8. Theorem. Let $A$ be a unital simple TAI $C^*$-algebra. Then $K_0(A)$ is weakly unperforated and satisfies the Riesz interpolation property.

Proof. First we note that, by 3.6, $M_n(A)$ is a unital simple TAI $C^*$-algebra. To show that $K_0(A)$ is weakly unperforated, it suffices to show that if $k[p] > k[q]$ for any projections in $M_n(A)$, then $[p] \not\preceq [q]$, where $k > 0$ is an integer. But $k[p] > k[q]$ implies that $\tau(p) > \tau(q)$ for all traces. This implies that $[p] \not\preceq [q]$ by 4.7. So $K_0(A)$ is weakly unperforated.

Since $A$ has cancellation, to show that $K_0(A)$ has the Riesz interpolation property, it suffices to show the following. If $p \leq q$ are two projections in $A$ and $q = q_1 + q_2$, where $q_1$ and $q_2$ are two mutually orthogonal projections, then $p = p_1 + p_2$ with $p_1 \leq q_1$ and $p_2 \leq q_2$. Without loss of generality, we may assume $q - p \neq 0$. Since $q_1Aq_1$ is a TAI $C^*$-algebra, there is a nonzero projection $q_1' \leq q_1$ such that $[p] \preceq (q_1 - q_1') + q_2$ and $q_1 - q_1' \neq 0$. Let $q' = (q_1 - q_1') + q_2$. Then $p \preceq q'$.

Let $\mathcal{F}$ be a finite subset containing $p, q_1 - q_1', q_2$. For any $\varepsilon > 0$, there exists a unital $C^*$-subalgebra $B \in \mathcal{I}$ and a projection $P \in A$ with $1_B = P$ such that:

1. $\|[, P]\| < \varepsilon$.
2. $PxP \in \varepsilon B$ for all $x \in \mathcal{F}$ and
3. $(1 - P) \preceq q_1'$. \hfill \Box

With sufficiently small $\varepsilon$, without loss of generality, we may assume that $[p, P] = [q_1 - q_1', P] = [q_2, P] = 0$. Write $p'' = PpP, q_1'' = P(q_1 - q_1')P$ and $q_2'' = Pq_2P$. We have $p'' \preceq q_1'' + q_2''$. Note that $M_n$ and $M_n(C([0, 1]))$ have the Riesz interpolation property. So $B$ has the Riesz property. There are $p_1' \preceq q_1''$ and $p_2' \preceq q_2''$ such that $p_1' + p_2' = p''$. Since $p - p'' \preceq (1 - P) \preceq q_1'$, we let $p_1'' = p - p' + p_1'$. Then $p_1'' \preceq q_1' + q_1'' \preceq q_1$. Now $p = p_1' + p_2'$. \hfill \Box

4.9. Let $A$ be a unital separable simple TAI $C^*$-algebra. We summarize some of its properties:
(i) $A$ has stable rank one; (ii) $A$ has at least one tracial state; (iii) $A$ has Fundamental Comparison property; (iv) $A$ has weakly unperforated $K_0(A)$ and satisfies the Riesz interpolation property; (v) $A$ has property (SP); (vi) $A$ is MF; (vii) if $A$ is nuclear, $A$ is also quasidiagonal; (viii) $M_n(A)$ is TAI; (ix) Every quasitrace on $A$ is a trace and $T(A)$ is a (metrizable) Choquet simplex; (x) $A \otimes F$ is TAI for all AF-algebras $F$; (xi) direct limits of TAI $C^*$-algebras are TAI and, in fact, locally TAI $C^*$-algebras are TAI.

We have not shown (ix). The only thing that one needs to note is that every quasitrace on $C^*$-algebras in $\mathcal{I}$ is in fact a trace. Then, from condition (3) of Definition 2.2, it is easy to see that every quasitrace is a trace. Note that it was proved in [3] that set of quasitraces on a unital $C^*$-algebra is a Choquet simplex.

We end this section with the following necessary and sufficient condition for a unital simple $C^*$-algebras to be TAI. For the simple case, one could use it as the definition.

4.10. Theorem. Let $A$ be a unital simple $C^*$-algebra. Then $A$ is TAI if and only if the following hold. For any finite subset $\mathcal{F} \subset A$ containing a nonzero element $b$, $\varepsilon > 0$, integers $n > 0$ and $N > 0$, and any nonzero projection $e \in A$, there exist a nonzero projection $p \in A$ and a $C^*$-subalgebra $I = \bigoplus_{i=1}^k M_{n_i}(C([0, 1]))$, with $1_I = p$ and $\min\{n_i: 1 \leq i \leq k\} \geq N$, such that:

1. $\|[x, p]\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) \( pxp \in \varepsilon I \) for all \( x \in \mathcal{F} \) and \( \| pbp \| \geq \| b \| - \varepsilon \), and

(3') \( 1 - p \) is unitarily equivalent to a projection in \( eAe \).

**Proof.** To show that the above is sufficient for \( A \) being TAI we note that \( A \) has property (SP) by 3.2. Then, by [31, 3.2], a result of Cuntz, there exists a projection \( q \in eAe \) such that \( (n + 1)[q] \leq [e] \). Then it is clear that the above (3') implies (3) in 2.2 (if we use the projection \( q \) instead of \( e \)).

To see it is also necessary, we use the fact that simple TAI \( C^* \)-algebras have stable rank one (so they have cancellation). It remains to show that we can make each summand of \( I \) have large rank. But this follows from (the proof of) 3.3. \( \square \)

5. Tracial approximate divisibility and homomorphisms from \( C^* \)-algebras in \( \mathcal{I} \)

The main purpose of this section is to prove 5.8 and 5.9. Theorem 5.8 will be used to prove Theorem 8.6. Theorem 5.9 classifies monomorphisms from a \( C^* \)-algebra in \( \mathcal{I} \) to a unital simple TAI \( C^* \)-algebra.

5.1. Sections 1–6 and 8 and most of 9 were written in a 1998 preprint titled “Classification of simple TAI \( C^* \)-algebras, part I” which was reported at the EU Operator Algebra Conference at Copenhagen in August 1998. The author later introduced the notation of tracial topological rank. When \( A \) is a unital simple \( C^* \)-algebra, \( A \) is a TAI \( C^* \)-algebra if and only if \( A \) has tracial topological rank no more than 1 (see [33, 7.1]).

The following is the definition of tracial topological rank no more than one for simple \( C^* \)-algebras.

5.2. Definition. Let \( A \) be a unital simple \( C^* \)-algebra. Then \( A \) has tracial topological rank no more than one, denote by \( TR(A) \leq 1 \), if the following holds. For any \( \varepsilon > 0 \), and any finite subset \( \mathcal{F} \subset A \) containing a nonzero element \( a \in A_+ \), there is a \( C^* \)-subalgebra \( C \) in \( A \) with \( C = \bigoplus_{i=1}^{k} M_{n_i}(C(X_i)) \), where each \( X_i \) is a finite CW complex with dimension no more than one such that \( 1C = p \) satisfying the following:

(i) \( \| px - xp \| < \varepsilon \) for \( x \in \mathcal{F} \),

(ii) \( pxp \in \varepsilon C \) for \( x \in \mathcal{F} \) and

(iii) \( 1 - p \) is equivalent to a projection in \( eAa \).

In the above definition, if \( C \) can be chosen to be a finite-dimensional \( C^* \)-subalgebra then we write \( TR(A) = 0 \) (see [33]). If \( TR(A) \leq 1 \) but \( TR(A) \neq 0 \) (see [33]) then we will write \( TR(A) = 1 \).

In the light of [33, Theorem 7.1], in what follows, we will replace unital simple TAI \( C^* \)-algebras by unital simple \( C^* \)-algebras with tracial topological rank no more than one and write \( TR(A) \leq 1 \).

5.3. Definition. A unital simple \( C^* \)-algebra \( A \) is said to be **tracially approximately divisible** if for any \( \varepsilon > 0 \), any projection \( e \in A \), any integer \( N > 0 \) and any finite subset \( \mathcal{F} \subset A \), there exists a projection \( q \in A \) and there exists a finite-dimensional \( C^* \)-subalgebra \( B \) with each simple summand having rank at least \( N \) such that:

(1) \( \| qx - xq \| < \varepsilon \) for all \( x \in \mathcal{F} \),
Let $q$ be a unital simple $C^*$-algebra with $\text{TR}(A) \leq 1$. Let $b \in A$ with $\|b\| = 1$ and assume that $b \in \mathcal{F}$. There exist a projection $p \in A$ and a $C^*$-subalgebra $C \subseteq \mathcal{I}$ with $1_C = p$ such that:

1. $\|px - xp\| < \varepsilon/4$ for all $x \in \mathcal{F}$,
2. $pxp \in \varepsilon/4$ and $\|pbp\| \geq \|b\| - \varepsilon/2$, and
3. $\tau(1 - p) < \sigma/2$ for all traces $\tau$ on $A$.

Write $C = \bigoplus C_i$, where $C_i = M_{i(i)}(C[0, 1])$, or $C_i = M_{i(i)}$. It will become clear that, without loss of generality, to simplify notation, we may assume that $C = C_i$ (i.e., there is only one summand). If $C = M_1$, let $\{e_{ij}\}$ be matrix units for $M_1$. Since $A$ is not elementary, there is a positive element $a \in e_{11}A_{e_{11}}$ such that $sp(a) = [0, 1]$ (see [1, p. 6.1]). This implies that $C \subseteq M_1(C([0, 1]))$. So, we may assume that $C = M_1(C([0, 1]))$. Let $G_1 \subseteq C$ be a finite subset such that

$$\text{dist}(pxp, G_1) < \varepsilon/4$$

for all $x \in \mathcal{F}$. Let $G$ be a finite subset of $C$ containing $\{e_{ij}\}$ and $e_{ij}ge_{ij}^*$ for all $g \in G_1$.

Let $\eta > 0$. Denote by $\delta$ the positive number in Theorem 4.3 of [26] corresponding to $\eta$ (instead of $\varepsilon$). Let $\{f_1, f_2, \ldots, f_m\} \subseteq C([0, 1])$ be as in 3.1 with respect to $\delta = \delta$. We identify $C([0, 1])$ with $e_{11}C_{e_{11}}$. Since $e_{11}A_{e_{11}}$ is simple, there are $b_{ij} \in e_{11}A_{e_{11}}$ such that

$$\left\| \sum_{j=1} b_{ij}f_jb_{ij}^* - e_{11} \right\| < 1/16,$$

for $j = 1, 2, \ldots, m$. Let $G_2$ be a finite subset containing $\{f_j, b_{ij}, b_{ij}^*\} \cup \{a_{ij} \in e_{11}A_{e_{11}}: (a_{ij})_{1 \times 1} \in G\}$.

By 3.4, $\text{TR}(e_{11}A_{e_{11}}) \leq 1$. So for any $0 < \sigma < \eta/2$ and any finite subset $G_3 \subseteq G_2$, there exist a projection $q \in e_{11}A_{e_{11}}$ and a $C^*$-subalgebra $C_1 \subseteq e_{11}A_{e_{11}}$ with $1_{C_1} = q$ and $C_1 \subseteq \mathcal{I}$ satisfying the following:

(a) $\|qx - xq\| < \sigma$,
(b) $qxq \in \sigma$ for all $x \in G_3$,
(c) $\tau(e_{11} - q) < \sigma/2l$ for all traces $\tau$.

With sufficiently small $\sigma$ and sufficiently large $G_2$, we may assume that there exists a homomorphism $\phi : C([0, 1]) \to C_1$ such that

(b') $\|\phi(x) - qxq\| < \eta/2$ for all $x \in G_2 \cap C([0, 1])$. 

Of course if $A$ is approximately divisible, then $A$ is tracially approximately divisible (see [7]).
Note that we also have $c_{ij} \subset C_1$ such that
\[
\left\| \sum_{i=1}^j c_{ij} \phi(f_j) c_{ij}^* - q \right\| < 1/8, \quad j = 1, 2, \ldots, m.
\]

We are now applying [26, Theorem 4.3]. It follows from 3.1 that $\text{Sp}(\phi_t)$ is $\delta$-dense in $[0, 1]$. By applying [26, 4.3], there is a homomorphism $\psi : C([0, 1]) \to C_1$ and there is a finite-dimensional $C^*$-subalgebra $F = \bigoplus_i F_i$, where each $F_i$ is simple and $\dim F_i \geq N$, with $1_F = q$ such that
\[
\left\| \psi(f) - \phi(f) \right\| < \eta/2 \quad \text{for all } f \in \mathcal{G}_2 \quad \text{and}
\left\| [\psi(g), b] \right\| = 0
\]
for all $g \in C([0, 1])$ and $b \in F$. Set $F' = \text{diag}(F, F, \ldots, F)$ in $F \otimes M_l$, $\psi' = \psi \otimes \text{id}_{M_l}$, $\phi' = \phi \otimes \text{id}_{M_l}$, and $P = \text{diag}(q, q, \ldots, q) \in M_l(C_1)$. With sufficiently small $\eta$ and large $\mathcal{G}_2$, we have
\[
\left\| \psi'(g) - \phi'(g) \right\| < \varepsilon/2 \quad \text{for } g \in \mathcal{G}.
\]

We also have
\[
\left\| [\psi'(f), c] \right\| = 0 \quad \text{for } f \in C \text{ and } c \in F'.
\]

These imply that
\[
\left\| [PXP, c] \right\| < \varepsilon \quad \text{for all } x \in \mathcal{F} \text{ and } c \in F'.
\]

Note that $1_{F'} = P$. We also have
\[
\tau(1 - P) \leq \sigma/2 + l\sigma/2l = \sigma.
\]

By 4.7, we conclude that $A$ is tracially approximately divisible. When $C = \bigoplus C_i$, it is clear that we can do exactly the same as above for each summand. Let $d_i = 1_{C_i}$. If we find a matrix algebra $F_i \in d_i A d_i$ with rank greater than $N$ which commutes with $C_i$, then $\bigoplus F_i$ commutes with $C$. \hfill $\Box$

5.5. Lemma. Let $A$ be a unital nuclear simple $C^*$-algebra with $\text{TR}(A) \leq 1$. Then for any $\varepsilon > 0$, any $\sigma > 0$, any integer $n > 0$, and any finite subset $\mathcal{F} \subset A$, there exist mutually orthogonal projections $q, p_1, p_2, \ldots, p_n$ with $q \preceq p_1$ and $[p_1] = [p_i]$ ($i = 1, 2, \ldots, n$), a $C^*$-subalgebra $C \in \mathcal{I}$ with $1_C = p_1$ and completely positive linear contractions $L_1 : A \to qAq$ and $L_2 : A \to C$ such that
\[
\left\| x - (L_1(x) \oplus \text{diag}(L_2(x), L_2(x), \ldots, L_2(x))) \right\| < \varepsilon \quad \text{and}
\left\| L_i(xy) - L_i(x)L_i(y) \right\| < \varepsilon,
\]
where $L_2(x)$ is repeated $n$ times, for all $x, y \in \mathcal{F}$ and $\tau(q) < \sigma$ for all $\tau \in T(A)$. 

Proof. From the proof of 5.4, we have the following. For any \( \eta > 0 \), any integer \( K > 0 \), any integer \( N > 4Kn^2 \) and finite subset \( G \subset A \) (containing \( 1_A \)), there exists a projection \( P \in A \) and a finite-dimensional \( C^* \)-subalgebra \( B \) with \( 1_B = P \) such that:

(i) \( \|[P, x]\| < \eta \) for all \( x \in G \);
(ii) every simple summand of \( B \) has rank at least \( N \);
(iii) there is a \( C^* \)-subalgebra \( D \in \mathcal{I} \) with \( 1_D = P \) such that \([d, g] = 0\) for all \( d \in D \), \( g \in B \) and

\[
\text{dist}(x, D) < \eta \quad \text{for} \ x \in G; \quad \text{and}
\]

(iv) \( 5N[(1 - P)] < [P] \).

Let \( \mathcal{F}_1 \subset A \) be a finite subset (containing \( 1_A \)) and \( \sigma > 0 \). Since \( A \) is nuclear, with sufficiently large \( G \) and sufficiently small \( \eta \), by [32, 3.2], there are unital completely positive linear contractions \( L'_1 : A \to (1 - P)A(1 - P) \) and \( L'_2 : A \to D \) such that \( L'_1(a) = (1 - P)a(1 - P) \).

\[
\|x - L'_1(x) \oplus L'_2(x)\| < \sigma \quad \text{and} \quad \|L'_2(x) - PxpP\| < \eta + \sigma
\]

for all \( x \in \mathcal{F}_1 \). It follows that, with sufficiently small \( \sigma \) and \( \eta \),

\[
\|L'_i(xy) - L'_i(x)L'_i(y)\| < \varepsilon
\]

for all \( x, y \in \mathcal{F}_1 \). Write \( B = \bigoplus_{i=1}^k B_i \), where \( B_i \cong M_{l(i)} \) with \( l(i) \geq N \), and denote by \( C \) the \( C^* \)-subalgebra generated by \( D \) and \( B \). Note that \( C \cong \bigoplus_{i=1}^k D_0 \otimes B_i \), where \( D_0 \cong D \). Let \( \pi_i : C \to D_0 \otimes B_i \) be the projection. Denote \( \phi_i = \pi_i \circ L'_2 \). By (iii), we see that we may write \( \phi_i = \text{diag}(\psi_i, \psi_i, \ldots, \psi_i) \), where \( \psi_i : A \to e_i(D_0 \otimes M_{l(i)})e_i \) and \( e_i \) is a minimal rank-one projection of \( M_{l(i)} \). Write \( l(i) = k_in + r_i \), where \( k_i \geq n > 1 \) are integers. We may rewrite

\[
\phi_i = \text{diag}(\Phi'_i, \ldots, \Phi'_i) \oplus \Psi'_i,
\]

where \( \Phi'_i = \text{diag}(\psi_i, \ldots, \psi_i) : A \to D_0 \otimes M_{k_i} \) is repeated \( n \) times and \( \Psi'_i = \text{diag}(\psi_i, \ldots, \psi_i) : A \to D_0 \otimes M_{r_i} \).

Define \( L_2 = \bigoplus_{i=1}^k \Phi'_i \) and \( L_1 = L'_1 \bigoplus_{i=1}^k \Psi'_i \). We estimate that

\[
\tau \left( (1 - P) + \bigoplus_{i=1}^k \Psi'_i(1_A) \right) > (1/5N)\tau(P) + (1/4nK)\tau(P) < (1/2n)\tau(P)
\]

\[
\leq \min(\sigma, \tau([L_2(1_A)])),
\]

provided that \( 1/K < \sigma \). By 4.7, the lemma follows. \( \square \)

The following corollary follows from Lemma 5.5 immediately.

5.6. Corollary. Let \( A \) be a unital separable simple \( C^* \)-algebra \( TR(A) \leq 1 \). Then for any \( \varepsilon > 0 \), any \( \sigma > 0 \), any integer \( n > 0 \), and any finite subset \( \mathcal{F} \subset A \), there exists a \( C^* \)-subalgebra \( C \in \mathcal{I} \) such that

\[
\|x - (1 - p)x(1 - p) \oplus \text{diag}(y, y, \ldots, y)\| < \varepsilon
\]
where \( y \in C \) and \( \text{diag}(y, y, \ldots, y) \in M_n(C) \) and \( p = 1_{M_n(C)} \) for all \( x, \in \mathcal{F} \) and \( \tau((1 - p)) < \sigma \) for all \( \tau \in T(A) \). Moreover, we may require that \( \|(1 - p)x(1 - p)\| \geq (1 - \varepsilon)\|x\| \) for all \( x \in \mathcal{F} \).

**Proof.** Perhaps the last part of the statement needs an explanation. In the proof of 5.5, we know that we may require that \( \|y\| \geq (1 - \varepsilon/2)\|x\| \) for all \( x \in \mathcal{F} \). Thus we may replace \( (1 - p)x(1 - p) \) by \( (1 - p)x(1 - p) \oplus y \) and replace \( (1 - p) \) by \( 1 - p \oplus \text{diag}(1_{C}, 0, \ldots, 0) \). 

5.7. **Lemma.** Let \( B = \bigoplus_{i=1}^{k} B_i \) be a unital \( C^* \)-algebra in \( \mathcal{I} \) (where \( B_i \) is a single summand). For any \( \varepsilon > 0 \), any finite subset \( F \subset B \) and any integer \( L > 0 \), there exist a finite subset \( G \subset B \) depending on \( \varepsilon \) and \( F \) but not on \( L \), and \( \delta = 1/4L \) such that the following holds. If \( A \) is a unital separable nuclear simple \( C^* \)-algebra with \( TR(A) \leq 1 \) and \( \phi_i : B \to A \) are two homomorphisms satisfying the following:

(i) there are \( a_{g,i}, b_{g,j} \in A, i, j \leq L \) with

\[
\left\| \sum_i a_{g,i}^* \phi_i(g)a_{g,i} - 1_A \right\| < 1/16 \quad \text{and} \quad \left\| \sum_j b_{g,j}^* \phi_j(g)b_{g,j} - 1_A \right\| < 1/16
\]

for all \( g \in G \);

(ii) \( \phi_1 = \phi_2 \) on \( K_0(B) \); and,

(iii) if \( \|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)\| < \delta \) for all \( g \in G \), then there exists a unitary \( u \in A \) such that

\[
\|\phi_1(f) - u^* \phi_2(f)u\| < \varepsilon \quad \text{for all} \quad f \in \mathcal{F}.
\]

**Proof.** It is clear that we can reduce the general case to the case in which \( B \) has only one summand. Since the case in which \( B = M_{I(I)} \) is well known to hold, we may assume that \( B = M_{I(I)}([0, 1]) \). Fix any \( d_0 > 0 \). Condition (i), with sufficiently large \( G \), implies that \( \text{Sp}(\phi_1) \) is \( d_0 \)-dense in \([0, 1] \) \( (i = 1, 2) \). By the proof of 3.7, therefore, for any \( d_1 > d_0 \), we may assume that

\[
\phi_i(f) = \phi_i^*(f) \oplus \sum_{j=1}^{N_i} f(t_{(i,j)})q(i, j) \quad i = 1, 2,
\]

where \( \{t_{(i,1)}, t_{(i,2)}, \ldots, t_{(i,N_i)}\} \) \( (i = 1, 2) \) is \( d_1 \)-dense in \([0, 1] \) and \( q(i, 1), \ldots, q(i, N_i) \) are mutually (non-zero) orthogonal projections in \( A \). It is clear that without loss of generality, we may assume that \( t_j = t_{i,j} \) and \( N_1 = N_2 \). Since \( TR(A) \leq 1 \), we can find nonzero projections \( q(i, j) \leq q(i, j) \) such that \( q(1, j)' \) and \( q(2, j) \) are unitarily equivalent. By replacing \( \phi_1 \) by \( \text{ad} z \circ \phi_1 \) for some unitary \( z \), we may assume that \( q(1, j)' = q(2, j)' \). Then, by replacing \( \phi_i^* \) by \( \phi_i^* \oplus \sum_{j=1}^{N_i} f(t_j)(q(i, j) - q(i, j))' \), we may assume that, with \( q_j = q(i, j)' \) and \( N = N_1 \),

\[
\phi_1(f) = \phi_1^*(f) \oplus \sum_{j=1}^{N} f(t_j)q_j \quad (i = 1, 2).
\]

Now let \( Q = 1 - \sum_{j=1}^{N} q_j \). We will apply [26, 5.14]. To do this, we let \( r > 0 \) be as in the statement of [26, 5.14] (but with respect to \( \varepsilon/4 \) and \( L_r \subset B \) (see also [26, 5.2])). Let \( d = 1/r \) and \( \delta = 1/4L \). Set \( G \subset L_r \) such that the functions \( f_1, f_2, \ldots, f_m \) required in 3.1 are all in \( G \).
Fix an integer \( n > 1 \). Let \( e' \leq QAQ \) such that \( n[e'] \leq [q_i], i = 1, 2, \ldots, N \). This is possible since \( A \) is simple and has (SP). Let \( e > 0, e \in QAQ \) be any nonzero projection in \( A \) with \( \tau(e) < 1/2L, [e] \leq [e'] \) and \( K > 0 \) be an integer. Since \( A \) is a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \), there exist a projection \( P \in A \) and a unital \( C^* \)-subalgebra \( C \in \mathcal{I} \) with \( 1_C = P \) such that:

(i) \( \|[\phi'_i(g), P]\| < \eta \),
(ii) \( P\phi_i(g)P \in \eta C \) for all \( g \in G' \) and \( i = 1, 2, \ldots \),
(iii) \( K[Q - P] \leq [P] \) and \( [Q - P] \leq [e] \),

where \( G' \supset G \cup \{a_{g,i}, b_{g,i}, b_{g,j}, a_{g,j}^*, g \in G \text{ and } i, j \leq L \} \). For any \( \sigma > 0 \), with sufficiently small \( \eta \), there exists homomorphism \( \psi_i : B \to C \) such that

\[
\|\psi_i(g) - P\phi'_i(g)P\| < \sigma,
\]

\[
\left\| \sum_{i=1}^{\infty} c_{g,i}^* \psi_1(g)c_{g,i} - P \right\| < 1/8 \text{ and } \left\| \sum_{j=1}^{\infty} d_{g,j}^* \psi_2(g)d_{g,j} - P \right\| < 1/8
\]

for all \( g \in \mathcal{G} \), where \( c_{g,i}, d_{g,j} \in C \). It follows from 3.1 that

\[
|\text{sp}((\psi_i)_t) \cap T| \geq 1/L|\text{Sp}((\psi_i)_t)|
\]

for all \( t \in [0, 1] \) (or, both \( \psi_1 \) and \( \psi_2 \) have the property sdp \((r, 1/L)\) as in [26, 5.13]), where \( T \) has length at least \( 1/r \). We also have, if \( \eta \) and \( \sigma \) are sufficiently small,

\[
\|\tau \circ \psi_1(g) - \tau \circ \psi_2(g)\| < 1/2L
\]

for all \( g \in \mathcal{G} \). It follows from 5.14 in [26] that there exists a unitary \( v \in C \) such that

\[
\|\psi_1(f) - v^* \psi_2(g)v\| < \varepsilon/4 \text{ for all } f \in \mathcal{F}.
\]

We also have

\[
\|\phi'_i(f) - \psi_i(f) - (Q - P)\phi'_i(f)(Q - P)\| < \varepsilon/4
\]

for all \( f \in \mathcal{F} \). Hence,

\[
\left\| \phi_i(f) - \left( \sum_{j=1}^{N} f(t_j)q_j \oplus \psi'_i(f) \oplus (Q - P)\phi''_i(f)(Q - P) \right) \right\| < \varepsilon/2
\]

for all \( f \in \mathcal{F} \). Since

\[
n[Q - P] \leq [q_i],
\]

by, for example, [40, Lemma 8(i)] (this was known earlier), there exists (provided that \( n \) is sufficiently large and \( d_1 \) is sufficiently small, and these two numbers do not depend on \( \phi_i \) or \( A \)) a unitary \( w \in (1 - P)A(1 - P) \) such that
\[
\left\| \sum_{j=1}^{N} f(t_j)q_j \oplus (Q - P)\phi''_1(f)(Q - P) - w^* \left( \sum_{j=1}^{N} f(t_j)q_j \oplus (Q - P)\phi''_2(f) \right)(Q - P)w \right\| < \varepsilon/2
\]

for all \( f \in \mathcal{F} \). Thus, we obtain a unitary \( u \in A \) such that

\[
\left\| \phi_1(f) - u^*\phi_2(f)u \right\| < \varepsilon \quad \text{for all} \quad f \in \mathcal{F}.
\]

5.8. Theorem. Let \( A \) be a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \) and \( C \) be a \( C^* \)-subalgebra of \( A \) in \( \mathcal{I} \). Then for any finite subset \( \mathcal{F} \subset C \) and \( \varepsilon > 0 \), there exist \( \delta > 0 \), \( \sigma > 0 \) and a finite subset \( \mathcal{G} \subset A \) satisfying the following: if \( L_1, L_2 : A \to B \) are two unital \( \mathcal{G} \)-\( \delta \)-multiplicative contractive completely positive linear maps, where \( B \) is a unital simple \( C^* \)-algebra with \( TR(B) \leq 1 \), with \( (L_1|C)_* = (L_2|C)_* \) on \( K_0(C) \) and

\[
|\tau(L_1(g)) - \tau \circ L_2(g)| < \sigma
\]

for all \( g \in \mathcal{G} \) and for all \( \tau \in T(B) \), then there is a unitary \( u \in A \) such that

\[
\left\| L_1(f) - u^*L_2(f)u \right\| < \varepsilon \quad \text{for all} \quad f \in \mathcal{F}.
\]

Proof. Fix \( \varepsilon > 0 \) and a finite subset \( \mathcal{F} \subset A \). Let \( \mathcal{G}_1 \subset C \) be the finite subset required by 5.7 (for a given \( \varepsilon > 0 \) and a given finite subset \( \mathcal{F} \)). Suppose that \( a_{g,i} \in A \) such that

\[
\left\| \sum_{i=1}^{n(g)} a_{g,i}^* g a_{g,i} - 1_A \right\| < 1/64
\]

for all \( g \in \mathcal{G}_1 \). Set \( L = \max\{n(g) : g \in \mathcal{G}\} \). Then, with sufficiently small \( \delta > 0 \) and large \( \mathcal{G} \supset \mathcal{G}_1 \cup \{a_{g,i} : g, i\} \), we have \( b_{g,i,j} \in A \) such that

\[
\left\| \sum_{i=1}^{n(g)} b_{g,i,j}^* L_j(g)b_{g,i,j} - 1_B \right\| < 1/32
\]

for all \( g \in \mathcal{G}_1 \) and \( j = 1, 2 \). Furthermore, for any \( \eta > 0 \), with sufficiently small \( \delta \), there is a homomorphism \( \phi_j : C \to B \) (\( j = 1, 2 \)) such that

\[
\left\| \phi_j(g) - L_j(g) \right\| < \eta \quad \text{and} \quad \left\| \sum_{i=1}^{n(g)} b_{g,i,j}^* \phi_j(g)b_{g,i,j} - 1_B \right\| < 1/16
\]

for \( g \in \mathcal{G}_1 \). We also require that \( \sigma < 1/4L \). Then we see the conclusions of the theorem follow from 5.7 (and its proof) immediately. \( \square \)
5.9. Theorem. Let $A$ be a unital simple $C^*$-algebra with $\text{TR}(A) \leq 1$ and $B \in \mathcal{I}$. Let $\phi_1 : B \to A$ be two monomorphisms such that

$$(\phi_1)_* = (\phi_2)_* : K_0(B) \to K_0(A) \quad \text{and} \quad \tau \circ \phi_1 = \tau \circ \phi_2$$

for all $\tau \in T(A)$. Then there is a sequence of unitaries $u_n \in A$ such that

$$\lim_{n \to \infty} u_n^* \phi_1(x) u_n = \phi_2(x) \quad \text{for all} \quad x \in B.$$

**Proof.** As before, we reduce the general case to the case in which $B = C([0, 1])$. Let $\varepsilon > 0$ and $\mathcal{F} \subset B$ be a finite subset. Let $\mathcal{G} \subset B$ be the finite subset in the statement of 5.7 (it does not depend on $L$). Since $A$ is simple, there exists an integer $L > 0$ and $a_{i,g}, b_{i,g} \in A$, $i = 1, 2, \ldots, L$ (some of them could be zero) such that

$$\left\| \sum a_{i,g}^* \phi_1(g) a_{i,g} - 1 \right\| < 1/16 \quad \text{and} \quad \left\| \sum b_{i,g}^* \phi_2(g) b_{i,g} - 1 \right\| < 1/16$$

for all $g \in \mathcal{G}$. Therefore the theorem follows from 5.7. \qed

6. The unitary group of a simple $C^*$-algebra $A$ with $\text{TR}(A) \leq 1$

We start with the following observations.

6.1. Let $A$ be a unital $C^*$-algebra and $p, a \in A$. Suppose that $p$ is a projection, $\|a\| \leq 1$ and

$$\|a^* a - p\| < 1/16 \quad \text{and} \quad \|a a^* - p\| < 1/16.$$

A standard computation shows that

$$\|pap - ap\| < 3/16 \quad \text{and} \quad \|pa - pap\| < 3/16.$$

Also $\|pa - a\| < 1/2$. Set $b = pap$. Then

$$\|b^* b - p\| \leq \|pa^* a p - pa^* a\| + \|pa^* a - p\| < 1/16 + 1/16 = 1/8.$$

So

$$\|(b^* b)^{-1} - p\| < \frac{1/8}{1 - 1/8} = 1/7 \quad \text{and} \quad \|b^{-1} - p\| < 2/7,$$

where the inverse is taken in $pAp$. Set $v = b |b|^{-1}$. Then $v^* v = p = v v^*$ and

$$\|v - b\| < 2/7.$$

We denote $v$ by $\tilde{a}$. Suppose that $L : A \to B$ is a $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear map, $u$ is a normal partial isometry and a projection $p \in B$ is given so that
\[ \| L(u^*u) - p \| < 1/32. \]

Note if \( v' \) is another unitary in \( pAp \) with \( \| v' - b \| < 1/3, \) then \([v'] = [v] \) in \( U(pAp)/U_0(pAp). \) We define \( L \) as follows. Let \( L(u) = a. \) With small \( \delta \) and large \( G, \) we denote by \( \tilde{L}(u) \) the normal partial isometry (unitary in a corner) \( v \) defined above. This notation will be used later. Note also, if \( u \in U_0(A), \) then, with sufficiently large \( G \) and sufficiently small \( \delta, \) we may assume that \( \tilde{L}(u) \in U_0(B). \)

### 6.2. Definition

Let \( A \) be a unital \( C^* \)-algebra. Let \( CU(A) \) be the closure of the commutator subgroup of \( U(A). \) Clearly that the commutator subgroup forms a normal subgroup of \( U(A). \) It follows that \( CU(A) \) is a normal subgroup of \( A. \) It should be noted that \( U(A)/CU(A) \) is commutative. It is an easy fact that if \( A = M_r(C(X)), \) where \( X \) is a finite CW complex of dimension 1, then \( CU(A) \subset U_0(A). \) If \( K_1(A) = U(A)/U_0(A) \) it is known and easy to verify that every commutator is in \( U_0(A). \) Therefore \( CU(A) \subset U_0(A). \) If \( u \in U(A), \) we will use \( \tilde{u} \) for the image of \( u \) in \( U(A)/CU(A), \) and if \( F \subset U(A) \) is a subgroup of \( U(A), \) then \( \tilde{F} \) is the image of \( F \) in \( U(A)/CU(A). \)

If \( \tilde{u}, \tilde{v} \in U(A)/CU(A) \) define

\[ \text{dist}(\tilde{u}, \tilde{v}) = \inf \{ \| x - y \| : x, y \in U(A) \text{ such that } x = \tilde{u}, \ y = \tilde{v} \}. \]

If \( u, v \in U(A) \) then \( \text{dist}(\tilde{u}, \tilde{v}) = \inf \{ \| u v^* - x \| : x \in CU(A) \}. \) Let \( g = \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1}, \) where \( a_i, b_i \in U(A). \) Let \( G \) be a finite subset of \( A, \) \( \delta > 0 \) and \( L : A \to B \) be a \( G \)-\( \delta \)-multiplicative contractive completely positive linear map, where \( B \) is a unital \( C^* \)-algebra. From 6.1, for \( \varepsilon > 0, \) if \( G \) is sufficiently large and \( \delta \) is sufficiently small,

\[ \left\| L(g) - \prod_{i=1}^{n} a'_i b'_i (a'_i)^{-1} (b'_i)^{-1} \right\| < \varepsilon/2, \]

where \( a'_i, b'_i \in U(B). \) Thus, for any \( g \in CU(A), \) with sufficiently large \( G \) and sufficiently small \( \delta, \)

\[ \left\| L(g) - u \right\| < \varepsilon \]

for some \( u \in CU(B). \) Moreover, for any finite subset \( \mathcal{U} \subset U(B) \) and subgroup \( F \subset U(B) \) generated by \( \mathcal{U}, \) and \( \varepsilon > 0, \) there exists a finite subset \( \mathcal{G} \) and \( \delta > 0 \) such that, for any \( \mathcal{G} \)-\( \delta \)-multiplicative contractive completely positive linear map \( L : A \to B, \) \( L \) induces a homomorphism \( L^\#: F \to U(B)/CU(B) \) such that \( \text{dist}(\overline{L}(u), L^\#(\tilde{u})) < \varepsilon \) for all \( u \in \mathcal{U}. \) Note we may also assume that \( \tilde{F} \cap U_0(A)/CU(A) \subset U_0(B)/CU(B). \)

If \( \phi : A \to B \) is a homomorphism then \( \phi^\#: U(A)/CU(A) \to U(B)/CU(B) \) is the induced homomorphism. It is continuous.

Recall that, for a unitary \( u \in U_0(A) \) in a unital \( C^* \)-algebra \( A, \) we write \( \text{cer}(u) \leq k, \) if \( u = \prod_{j=1}^{k} \exp(ih_j) \) for some self-adjoint elements \( h_j \in A. \) We write \( \text{cer}(u) \leq k + \varepsilon \) if \( u \) is a norm limit of unitaries \( u_n \) with \( \text{cer}(u_n) \leq k. \)

Let \( u \in U_0(A). \) Denote by \( \text{cel}(u) \) the infimum of the length of continuous paths of unitaries in \( A \) from \( u \) to \( 1_A. \)
6.3. Lemma (N.C. Phillips). Let $A$ be a unital $C^*$-algebra and $2 > d > 0$. Let $u_0, u_1, \ldots, u_n$ be $n + 1$ unitaries in $A$ such that

$$u_n = 1_A \quad \text{and} \quad \|u_i - u_{i+1}\| \leq d, \quad i = 0, 1, \ldots, n - 1.$$  

Then there exists a unitary $v \in M_{2n+1}(A)$ with exponential length no more than $2\pi$ such that

$$\| (u_0 \oplus 1_{M_{2n}}(A)) - v \| \leq d.$$  

Moreover, $v$ can be chosen in $CU(M_{2n+1}(A))$.

The following is another version of the above lemma.

6.4. Lemma. Let $A$ be a unital $C^*$-algebra and $u \in U_0(A)$. Then for each $L > 0$, if $u = v \oplus (1 - p)$ and $v \in U_0(pAp)$ with $\text{cel}(v) \leq L$ in $pAp$ and there are $N > 2L$ mutually orthogonal and mutually equivalent projections in $(1 - p)A(1 - p)$ each of which is equivalent to $p$, then $\text{cel}(u) \leq 2\pi + (L/n)\pi$. Furthermore, there is a unitary $w \in CU(A)$ such that $\text{cel}(uw) < (L/n)\pi$.

(See the proof of [44, Theorem 3.8] and also [42, Corollary 5]. It should be noted that a unitary in $M_2(A)$ with the form $\text{diag}(u, u^*)$ is in $CU(M_2(A))$.)

6.5. Theorem. Let $A$ be a unital simple $C^*$-algebra with $\text{TR}(A) \leq 1$. Let $u \in U_0(A)$. Then, for any $\varepsilon > 0$, there are unitaries $u_1, u_2 \in A$ such that $u_1$ has exponential length no more than $2\pi$, $u_2$ is an exponential and

$$\|u - u_1u_2\| < \varepsilon.$$

Moreover, $\text{cer}(A) \leq 3 + \varepsilon$.

**Proof.** Let $\varepsilon$ be a positive number. Let $v_0, v_1, \ldots, v_n \in U_0(A)$ such that

$$v_0 = u, \quad v_n = 1 \quad \text{and} \quad \|v_i - v_{i+1}\| < \varepsilon /16, \quad i = 0, 1, \ldots, n - 1.$$  

Let $\delta > 0$. Since $\text{TR}(A) \leq 1$, there exists a projection $p \in A$ and a unital $C^*$-subalgebra $B \subset A$ with $B \in \mathcal{I}$ and with $1_B = p$ such that:

1. $\|[v_i, p]\| < \delta, \quad i = 0, 1, \ldots, n$,
2. $pv_i p \in \delta B, \quad 0, 1, \ldots, n$, and
3. $2(n + 1)(1 - p) \leq p$.

There are unitaries $w_i \in (1 - p)A(1 - p)$ with $w_n = (1 - p)$ such that

$$\|w_i - (1 - p)v_i(1 - p)\| < \varepsilon /16, \quad i = 0, 1, \ldots, n$$  

for any given $\varepsilon > 0$, provided $\delta$ is sufficiently small. Furthermore, there is a unitary $z \in B$ such that

$$\|z - v_{i+1}zv_i\| < \varepsilon /16, \quad i = 0, 1, \ldots, n.$$
\[ \| z - \text{pup} \| < \varepsilon /16. \]

Therefore (with \( \delta < \varepsilon /32 \))

\[ \| u - w_1 \oplus z \| < \varepsilon /8. \]

Write \( z_1 = w_1 \oplus p \). Since \( 2(n + 1)[1 - p] \leq p \), by 6.3, there is a unitary \( u_1 \) with exponential length no more than \( 2\pi \) such that

\[ \| z_1 - u_1 \| < \varepsilon /4. \]

Now since \( z \in B \) and it is well known that \( B \) has exponential rank \( 1 + \varepsilon \), there is an exponential \( u_2 \in A \) such that

\[ \| u_2 - (1 - p) - z \| < \varepsilon /3. \]

Therefore

\[ \| u - u_1 u_2 \| < \varepsilon. \]

Since \( \text{cel}(u_1) \leq 2\pi \), it follows from [49] that \( \text{cer}(u_1) \leq 2 + \varepsilon \). Therefore \( \text{cer}(u) \leq 3 + \varepsilon \). So \( \text{cer}(A) \leq 3 + \varepsilon \). \qed

**6.6. Lemma.** Let \( A \) be a unital \( C^* \)-algebra.

1. \( U_0(A)/CU(A) \) is divisible.
2. If \( u \in U(A) \) such that \( u^k \in U_0(A) \). Then there is \( v \in U_0(A) \) such that \( \tilde{v}^k = \tilde{u}^k \) in \( U(A)/CU(A) \).
3. Suppose that \( K_1(A) = U(A)/U_0(A) \) and \( G \subset U(A)/CU(A) \) is finitely generated subgroup. Then one has \( G = G \cap (U_0(A)/CU(A)) \oplus \kappa(G) \), where \( \kappa : U(A)/CU(A) \to U(A)/U_0(A) \) is the quotient map.

**Proof.** Let \( u \in U_0(A) \). Then there are \( a_1, a_2, \ldots, a_n \in A_{sa} \) such that \( u = \prod_{j=1}^n \exp(ia_j) \). For any integer \( k > 0 \), let \( v = \prod_{j=1}^n \exp(ia_j/k) \). Then \( \tilde{v}^k = \tilde{u} \). This proves (1).

To see (2), put \( u^k = \prod_{j=1}^n \exp(ia_j) \), where \( a_j \in A_{sa} \). Let \( v = \prod_{j=1}^n \exp(ia_j/k) \). Thus \( (uv^*)^k = \tilde{1} \). So \( \tilde{v}^k = \tilde{u}^k \). To see (3), we note that (1) implies \( 0 \to U_0(A)/CU(A) \to G + U_0(A)/CU(A) \to \kappa(G) \to 0 \) splits. \qed

**6.7. Theorem.** Let \( A \) be a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \) and \( e \in A \) be a projection. Let \( \gamma : U(eAe)/CU(eAe) \to U(A)/CU(A) \) be defined by \( \gamma(\tilde{u}) = u \oplus (1 - e) \). Then \( \gamma \) is a surjective (contractive) homomorphism.

**Proof.** It is clear that \( \gamma \) is a homomorphism and is contractible. We will show that \( \gamma \) is also surjective. Fix \( u \in U(A) \). Let \( N > 0 \) be an integer such that \( N[\varepsilon] \geq 1 \) in \( A \). Fix \( 1/2 > \varepsilon > 0 \) and \( 0 < \eta < \varepsilon /8(N + 1) \). It follows from 5.6 that there is a unitary \( z_1 = s_0 \oplus s_1 \oplus s_1 \oplus \cdots \oplus s_1 \), where
s_0 \in U((e_0Ae_0)) and s_1 \in U(C) which repeats n + 1 times (n \geq 3), where C \in \mathcal{I} and e_0 \oplus 1_C is equivalent to a subprojection of e such that

\|u - z_1\| < \eta/4.

Note that M_{n+1}(C) is a C*-subalgebra of A. By replacing s_0 by s_0 \oplus s_1 \oplus \cdots \oplus s_1, where s_1 repeats several times, we may assume that 3 \leq n \leq 4N + 1. Without loss of generality, we may also assume that e_0 \oplus 1_C \leq e. Let w = e_0 \oplus s_1^n \oplus s_1^* \oplus s_1^* \oplus \cdots \oplus s_1^*, where s_1^n repeats n times. Then z_1w = s_0 \oplus s_1^{n+1} \oplus 1_C \oplus 1_C \oplus \cdots \oplus 1_C. Put v_1 = s_0 \oplus s_1^{n+1} \oplus [e - (e_0 \oplus 1_C)] and put y = s_1^n \oplus s_1^* \oplus s_1^* \oplus \cdots \oplus s_1^* (s_1^n repeats n times). Then det(y) = 1 (in M_{n+1}(C)). Since U_0(M_{n+1}(C)) = U(M_{n+1}(C)), it follows from [54, 2.4] that y \in CU(M_{n+1}(C)). Hence w \in CU(A). Therefore v_1 \oplus (1 - e) = \bar{z}_1.

Put y_1 = z_1^*u. Then \|y_1 - 1_A\| < \eta/4. We now repeat the same argument. We obtain z_2 = s_0 \oplus s_1^* \oplus \cdots \oplus s_1^* \in U_0(A), where s_0 \in U_0(e_0'\oplus e_0') and where s_1^* repeats n + 1 times, s_1^* \in U_0(C_1), C_1 \in \mathcal{I} and e_0' \oplus 1_{C_1} is equivalent to a subprojection of e such that

\|z_2 - y_1\| < \eta/16.

Without loss of generality, we may further assume that e_0' \oplus 1_{C_1} \leq e. From the fact that \|y_1 - 1_A\| < \eta/4, we may assume that \|s_0 - e_0'\| < \eta/2 and \|s_1^* - 1_{C_1}\| < \eta/2. Put v_2 = s_0 \oplus (s_1^*)^{n+1} \oplus (e - (e_0' \oplus 1_{C_1})). Then (since n < 4N + 1)

\|v_2 - e\| < \eta/2.

As we have shown, we have v_2 \oplus (1 - e) = \bar{z}_2. Note that v_1v_2 \oplus (1 - e) = \bar{z}_1z_2 and

\|z_1z_2 - u\| \leq \|z_1y_1 - u\| + \|z_1y_1 - z_1z_2\| = \|z_1y_1 - z_1z_2\| < \eta/16.

Also

\|v_1v_2 - v_1\| < \eta/2.

Let y_2 = (z_1z_2)^*u. Then \|y_2 - 1_A\| < \eta/16. We can continue the above argument. Consequently, we obtain a sequence of unitaries z_n \in U(A) and a sequence of unitaries v_n \in U(eAe) such that v_1v_2 \cdots v_n \oplus (1 - e) = \bar{z}_1z_2 \cdots z_n,

\|z_1z_2 \cdots z_n - u\| \to 0 \quad \text{and} \quad \|v_1v_2 \cdots v_n - v_1v_2 \cdots v_m\| \to 0

as n, m \to \infty. Therefore we obtain a unitary v \in U(eAe) such that

v \oplus (1 - e) = \bar{u}. \quad \square

6.8. Lemma. Let A be a unital C*-algebra and \mathcal{U} \subset U_0(A) be a finite subset. Then, for any \varepsilon > 0, there is a finite subset \mathcal{G} \subset A and \delta > 0 satisfying the following: for any \mathcal{G}-\delta-multiplicative contractive linear map L : A \to B (for any unital C*-algebra B), there are unitaries v \in B such that

\[ v + (1 - e) = \bar{u} \]
\[ \| L(u) - v \| < \varepsilon/2 \quad \text{and} \quad \text{cel}(v) < \text{cel}(u) + \varepsilon/2 \]

for all \( u \in U \).

**Proof.** Suppose that \( z_0(u) = u, z_j(u) \in U_0(A), \ j = 1, 2, \ldots, n(u) \) such that \( \frac{\text{cel}(u)}{n(u)} \leq 1/4 \) and

\[ \text{cel}(z_j(u)(z_{j-1}(u))^*) < \frac{\text{cel}(u)}{n(u)}, \quad j = 1, 2, \ldots, n(u), \]

for all \( u \in U \). Let \( N = \max\{n(u): u \in U\} \). It follows that (for sufficiently large \( G \) and sufficiently small \( \delta \)) there are unitaries \( w_j(u) \in U(B) \) such that

\[ \| L(z_j(u)) - w_j(u) \| < \varepsilon/8N\pi \]

for all \( j \) and \( u \in U \). Thus for all \( u \in U \),

\[ \| L(u) - w_0(u) \| < \varepsilon/2\pi \quad \text{and} \quad \text{cel}(w_0(u)) < n(u)\left[ \frac{\text{cel}(u)}{n(u)} + (\varepsilon/8N)2\pi \right] < \text{cel}(u) + \varepsilon/2. \]

\[ \square \]

**6.9. Lemma.** Let \( A \) be a unital simple \( C^* \)-algebra with \( \text{TR}(A) \leq 1 \) and let \( u \in CU(A) \). Then \( u \in U_0(A) \) and \( \text{cel}(u) \leq 8\pi \).

**Proof.** We may assume that \( u \) is actually in the commutator group. Write \( u = v_1v_2 \cdots v_k \), where each \( v_i \) is a commutator. We write \( v_i = a_i^*b_i a_i b_i^* \), where \( a_i \) and \( b_i \) are in \( U(A) \). Fix integers \( N > 0 \) and \( K > 0 \). Since \( \text{TR}(A) \leq 1 \), by Corollary 3.3, there is a projection \( p \in A \) and a \( C^* \)-subalgebra \( B \in \mathcal{I} \) with \( 1_p = B \) and \( B = \bigoplus_{i=1}^I M_{m_i}(C([0, 1])) \bigoplus_{j=1}^L M_{n_i} \), where \( m_i, n_i \geq K \) such that

\[ \| a_i - (a_i^* \oplus a_i') \| < \varepsilon/4k, \quad \| b_i - (b_i^* \oplus b_i'') \| < \varepsilon/4k, \quad i = 1, 2, \ldots, k, \]

\[ \| u - \left( \prod_{i=1}^k a_i^*b_i'(a_i')^*(b_i')^* \oplus a_i''b_i''(a_i'')^*(b_i'')^* \right) \| < \varepsilon/8, \]

\( a_i^*, b_i' \in U((1 - p)A(1 - p)), a_i'', b_i'' \in U_0(B) \) and \( N[1 - p] \leq [p] \). Put \( w = \prod_{i=1}^k a_i^*b_i' (a_i')^* (b_i')^* \) and \( z = \prod_{i=1}^k a_i''b_i''(a_i'')^*(b_i'')^* \). Then \( \text{Det}(z) = 1 \). It follows from [43, 3.4] (by choosing \( K \) large) we conclude that \( \text{cel}(z) \leq 6\pi \) in \( pAp \). It is standard to show that \( a_i^*b_i'(a_i')^*(b_i')^* \oplus (1 - p) \oplus (1 - p) \) is in \( U_0(M_4((1 - p)A(1 - p))) \) and it has exponential length no more than \( 4(2\pi) + \varepsilon/8k \).

This implies that (in \( U((1 - p)A(1 - p)) \)) \( \text{cel}(w \oplus (1 - p)) \leq 8k\pi + \varepsilon/2 \). Note the length only depends on \( k \). We can then choose \( N = N(8k\pi + \varepsilon) \) as in 6.4. In this way, \( \text{cel}(w \oplus p) \leq 2\pi + \varepsilon/2 \). It follows that

\[ \text{cel}((w \oplus p)((1 - p) \oplus z)) \leq 8\pi + \varepsilon/2. \]

The fact that \( \| u - (w \oplus p)((1 - p) \oplus z) \| < \varepsilon/8 \) implies that \( \text{cel}(u) \leq 8\pi + \varepsilon. \)  \( \square \)
6.10. Theorem. Let $A$ be a unital simple $C^*$-algebra with $TR(A) \leq 1$. Let $u, v \in U(A)$ such that $[u] = [v]$ in $K_1(A)$ and

$$u^k, v^k \in U_0(A) \quad \text{and} \quad \text{cel}((u^k)^* v^k) < L.$$ 

Then

$$\text{cel}(u^* v) \leq 8\pi + L/k.$$ 

Moreover, there is $y \in U_0(A)$ with $\text{cel}(y) \leq L/k$ such that $u^* v = \bar{y}$ in $U(A)/CU(A)$.

Proof. Suppose that $u^* v = \prod_j \exp(ia_j)$ and $(u^k)^* v^k = \prod_m \exp(ib_m)$,

where $a_j, b_m \in A_{sa}$. Since $\text{cel}(((u^k)^* v^k) < L$, we may assume that $\sum \|b_m\| < L$ (see [49]). Let $M = \sum_j \|a_j\|$. (So $\text{cel}(u^* v) \leq M$.) Since $TR(A) \leq 1$, for any $\delta > 0$ with $\delta/(1 - \delta) < \varepsilon/2(M + L + 1)$ and sufficiently small $\eta > 0$ and with a sufficiently large finite subset $G$ (which contains $a_j, b_m$), there exist a projection $p \in A$ and a unital $C^*$-subalgebra $F \subset A$ with $1_F = p$ and $F \in \mathcal{I}$ such that:

1. $pxp \in \eta F$ for all $x \in G$,
2. $\|u - u_0 \oplus u_1\| < \eta$, $\|v - v_0 \oplus v_1\| < \eta$, and
3. $u_0$ and $v_0$ are unitaries in $(1 - p)A(1 - p)$ and $u_1$ and $v_1$ are unitaries in $pAp$,
4. $\text{cel}((u_0^* v_0) \leq M + 1$ in $(1 - p)A(1 - p)$ and $\text{cel}((u_1^* v_1) < L$ in $F$,
5. $\tau(1 - p) < \delta$ for all $\tau \in T(A)$.

(Note (4) follows from 6.8).

Write $F = \bigoplus_{s=1}^N F_s$, where each $F_s = M_{n(s)}(C([0, 1]))$ or $F_s = M_{n(s)}$. By Corollary 3.3, we may assume that each $n(s) > \max(2\pi^2/\varepsilon, K(1))$, where $K(1)$ is the number described in [43, Lemma 3.4] (with $d = 1$).

First consider the case in which $N = 1$ and $F = M_K(C([0, 1]))$ (so $K > \max(2\pi^2/\varepsilon, K(1))$). Note that $\text{cel}((u_1^* v_1) < L$ in $F$. Therefore, by [43, Lemma 3.3(1)], there exists $a \in F_{sa}$ with $\|a\| < L$ such that

$$\det(\exp(ia)(u_1^* v_1) = 1 \quad \text{(for every } t \in [0, 1]).$$

This implies that

$$\det((\exp(ia/k)u_1^* v_1)^k) = 1 \quad \text{(for every } t \in [0, 1]).$$

Therefore

$$\det(\exp(ia/k)u_1^* v_1) = \exp(i2\pi/k) \quad \text{(for every } t \in [0, 1]).$$
for some \( l = 0, 1, \ldots, k - 1 \). Note since determinant is a continuous function on \([0, 1]\), the above function has to be constant (only one value of \( l \) occurs). Set \( f(t) = -2l\pi/k \). Then \( f \in C([0,1])_{sa} \) and \( \|f\| \leq 2\pi \). Note that \( \exp(if/K) \cdot 1_F \) commutes with \( \exp(ia/k) \) and \( (\exp(if/K) \cdot 1_F) \exp(ia/k) = \exp(i(f/K + a/k)) \). We have
\[
\det((\exp(if/K) \cdot 1_F) \exp(ia/k)u_1^*v_1) = 1.
\]
So, by [43, 3.4 (and 3.1)],
\[
\text{cel}(u_1^*v_1) \leq 2\pi/K + L/k + 6\pi \quad \text{(in } F).\]
Moreover, by [54, 2.4], \( z_1 = (\exp(if/K) \cdot 1_F) \exp(ia/k)u_1^*v_1 \in CU(F) \). Note the above also holds when \( F = M_K \). By considering each summand, the above also holds for the case in which \( N > 1 \).

Moreover, by Lemma 6.4, there is \( y' \in CU(A) \) and \( y'' \in U_0(A) \) such that \( (u_0 \oplus p)^* (v_0 \oplus p) = y'y'' \) and \( \text{cel}(y'') < \varepsilon/2 \). Note that \( ((1 - p) \oplus z_1)y' \in CU(A) \). Therefore
\[
\text{cel}(u^*v) \leq 2\pi/K + L/k + 8\pi + \varepsilon/2 < 8\pi + L/k + \varepsilon. \quad \Box
\]

6.11. Theorem. Let \( A \) be a unital separable simple \( C^* \)-algebra with \( TR(A) \leq 1 \) and \( u \in U_0(A) \). Suppose that \( u^k \in CU(A) \) for some integer \( k > 0 \), then \( u \in CU(A) \). In particular, \( U_0(A)/CU(A) \) is torsion free.

Proof. The proof is essentially the same as that of 6.10. Let \( \varepsilon > 0 \) and let
\[
v = \prod_{j=1}^r a_i b_i (a_i')^{-1} (b_i')^{-1} \quad \text{be such that} \quad \|u^k - v\| < \varepsilon/64.
\]
Put \( l = \text{cel}(u^k) \). Let \( \delta > 0 \) be such that \( 2(l + \varepsilon)\delta < \varepsilon/64\pi \). Fix a finite subset \( \mathcal{G} \subset A \) which contains \( u, u^k, v, a_i, b_i, a_i^{-1}, b_i^{-1} \) among other elements.

Since \( TR(A) \leq 1 \), there is a projection \( p \in A \) and a unital \( C^* \)-subalgebra \( F \in \mathcal{I} \) with \( 1_F = p \) such that:

1. \( p \exp \in \varepsilon/64 \) for all \( x \in \mathcal{G} \),
2. \( \|v - v_0 \oplus 1_F\| < \varepsilon/32 \), \( \|u - u_0 \oplus u_1\| < \varepsilon/32 \), and \( \|u^k - u_0^k \oplus u_1^k\| < \varepsilon/32 \),
3. \( \text{cel}(u_0^k) < l + \varepsilon/32 \) in \( U(((1 - p)A(1 - p)) \) and
4. \( \tau(1 - p) < \delta \) for all \( \tau \in T(A) \).

(Note (3) follows from 6.8 with large \( \mathcal{G} \).)

Here \( u_0, v_0 \in U(((1 - p)A(1 - p)) \) and \( v_1, u_1 \in U(F) \). Moreover, we may assume that there are \( a_i', b_i' \in U(F) \) such that
\[
\left\| u_1^k - \prod_{j=1}^{r} a'_j b'_j (a'_j)^{-1} (b'_j)^{-1} \right\| < \varepsilon / 32.
\]

Put \( w = \prod_{j=1}^{r} a'_j b'_j (a'_j)^{-1} (b'_j)^{-1} \). Since \( U(F) = U_0(F) \), we may write \( w = \prod_{m=1}^{L} \exp(id_m) \) for some \( d_m \in F_{\text{sa}} \). Put \( w_k = \prod_{m=1}^{L} \exp(id_m/k) \). Then \( w_k^k = w \). So
\[
\text{cel}(u_1^k (w_k^k)^k) < \frac{\varepsilon \pi}{32}.
\]

Write \( F = \bigoplus_{s=1}^{N} F_s \), where each \( F_s = M_{r(s)} C([0, 1]) \) or \( F_s = M_{r(s)} \). By 3.3, we may assume that each \( n(s) > \max(16\pi^2 / \varepsilon, K(1)) \), where \( K(1) \) is the number described in [43, Lemma 3.4] (with \( d = 1 \)).

As in the proof of 6.10,
\[
det(\exp(if/K) \exp(ia/k)u_1 w_k^*) = 1
\]
for some \( a, f \in F_{\text{sa}} \) with \( \|f\| \leq 2\pi \) and \( \|a\| < \varepsilon \pi / 32 \) (with \( K > \max(16\pi^2 / \varepsilon, K(1)) \)). By [54, 2.4], \( \exp(if/K) \exp(ia/k)u_1 w_k^* \in CU(F) \). We also have
\[
\| \exp(if/K) \exp(ia/k) - 1_F \| < \varepsilon / 8 + \varepsilon / 32.
\]
Thus
\[
\text{dist}(u_1 w_k^*, \bar{1}) < \varepsilon / 8 + \varepsilon / 32.
\]
Since \( \det(w) = 1 \), as in the proof of 6.10, we also have
\[
det(\exp(ig/K)w_k^*) = 1
\]
for some \( g \in F_{\text{sa}} \) with \( \|g\| \leq 2\pi \). Again, \( \exp(ig/K)w_k^* \in CU(F) \). But
\[
\| (\exp(ig/K)w_k) w_k^* - 1 \| \leq \| \exp(ig/K) - 1 \| < \varepsilon / 4.
\]
So
\[
\text{dist}(\bar{w_k}, \bar{1}) < \varepsilon / 4.
\]
Therefore
\[
\text{dist}(\bar{u_1}, \bar{1}) \leq \text{dist}(\bar{u}, \bar{w_k}) + \text{dist}(\bar{w_k}, \bar{1}) < \varepsilon / 8 + \varepsilon / 32 + \varepsilon / 4 < \varepsilon / 2
\]
in \( U(F)/CU(F) \). On the other hand, by 6.4 and the choice of \( \delta \),
\[
\text{cel}(u_0 \oplus p)z < \varepsilon / (8\pi)
\]
for some \( z \in CU(A) \). Thus
We have that \( G \) is identified with \( S \) (which is bounded by 4) (see [25]). Suppose that \( B_n \) is divisible follows from the fact that each \( B_n \) can be represented by unitaries in \( M_K(z)(B_n) \) for some integer \( K(z) > 0 \). Then the kernel of the map

\[
K_1\left( \prod_n B_n \right) \to \prod_n K_1(B_n) \to 0
\]

is a divisible and torsion free subgroup of \( K_1(\prod_n B_n) \).

**Proof.** By 6.5, the exponential rank of each \( B_n \) is bounded by 4. Therefore the kernel is divisible follows from the fact that each \( B_n \) has stable rank one (and has exponential rank bounded by 4) (see [25]). Suppose that \( \{u_n\} \in U(M_K(\prod_n B_n)) \) such that \( \{[u_n]\} \) is in the kernel and \( k([u_n]) = 0 \). By changing notation (with different \( \{u_n\} \) and larger \( K \)), we may assume that \( \{u_n^k\} \in U_0(M_K(\prod_n B_n)) \). Also each \( u_n \in U_0(B_n) \). This implies that there is \( L > 0 \) such that \( \text{cel}(u_n^k) \leq L \) for all \( n \). It follows from 6.10 that

\[
\text{cel}(u_n) \leq 8\pi + L + L/k + \pi/4 \quad \text{for all } n.
\]

This implies (see for example [25]) \( \{u_n\} \in U_0(M_L(\prod_n B_n)) \). Therefore \( \{[u_n]\} = 0 \) in \( K_1(\prod_n B_n) \). So the kernel is torsion free. \( \square \)

**6.12. Corollary.** Let \( B_n \) be a sequence of unital simple \( C^* \)-algebra with \( TR(B_n) \leq 1 \). Let \( \prod_n K_1(B_n) \) be the set of sequences \( z = \{z_n\} \), where \( z_n \in K_1(B_n) \) and \( z_n \) can be represented by unitaries in \( M_K(z)(B_n) \) for some integer \( K(z) > 0 \). Then the kernel of the map

\[
K_1\left( \prod_n B_n \right) \to \prod_n K_1(B_n) \to 0
\]

is a divisible and torsion free subgroup of \( K_1(\prod_n B_n) \).

**Remark.** When \( \dim X \leq 1 \), we believe that the conclusion of [43, 3.4] can be improved and \( \text{cel}(u) \) should be no more than \( D(u) + 2\pi \). This could be achieved by a modification of Phillips’ argument as we were informed by N.C. Phillips. Consequently, in 6.10, \( 8\pi \) could be replaced by \( 4\pi \) and in 6.9, \( 8\pi \) could be replaced by \( 4\pi \). However, we will not use these better estimates.

**7. Homomorphisms from \( U(C)/CU(C) \) to \( U(B)/CU(B) \)**

**7.1. Definition.** Let \( Y \) be a connected finite CW complex with dimension no more than three with torsion \( K_1(C(Y)) \) and set \( C' = PM_r(C(X)) P \), where \( X = S^1 \lor S^1 \lor \cdots \lor S^1 \lor Y \) and \( P \in M_r(C(Y)) \) is a projection and \( P \) has rank \( R \geq 6 \). We assume that \( S^1 \) is repeated \( s \geq 0 \) times. Note that the above includes the case that \( X = Y = [0, 1] \). Then \( K_1(C') = tor(K_1(C')) \oplus G_1 \), where \( G_1 \) is \( s \) copies of \( \mathbb{Z} \). Denote by \( \xi \) the point in \( X \) where each \( S^1 \) and \( Y \) meet. Rename each \( S^1 \) by \( \Omega_i, i = 1, 2, \ldots, s \). Denote by \( z_\xi^i \) the identity map on \( i \)th \( S^1 \) (which is identified with \( S^1 \)) and \( z_\xi^i(\xi) = \xi \) for all \( \xi \notin \Omega_i \). There is an obvious homomorphism \( \Pi : PM_r(C(X)) P \to D' = \bigoplus_{i=1}^k E_i \), where \( E_i \cong M_R(C(S^1)) \). Note that if \( s \geq 2 \), then \( \Pi \) is not surjective. We have that \( G_1 = \Pi_1(D') \).

\[
7.13. \text{Remark.} \quad \text{When } \dim X \leq 1, \text{ we believe that the conclusion of [43, 3.4] can be improved and } \text{cel}(u) \text{ should be no more than } D(u) + 2\pi. \text{ This could be achieved by a modification of Phillips’ argument as we were informed by N.C. Phillips. Consequently, in 6.10, } 8\pi \text{ could be replaced by } 4\pi \text{ and in 6.9, } 8\pi \text{ could be replaced by } 4\pi. \text{ However, we will not use these better estimates.}
\]

\[
7. \text{Homomorphisms from } U(C)/CU(C) \text{ to } U(B)/CU(B)
\]

\[
7.1. \text{Definition.} \quad \text{Let } Y \text{ be a connected finite CW complex with dimension no more than three}
\]

\[
\text{with torsion } K_1(C(Y)) \text{ and set } C' = PM_r(C(X)) P, \text{ where } X = S^1 \lor S^1 \lor \cdots \lor S^1 \lor Y \text{ and } P \in M_r(C(Y)) \text{ is a projection and } P \text{ has rank } R \geq 6. \text{ We assume that } S^1 \text{ is repeated } s \geq 0 \text{ times. Note that the above includes the case that } X = Y = [0, 1]. \text{ Then } K_1(C') = tor(K_1(C')) \oplus G_1, \text{ where } G_1 \text{ is } s \text{ copies of } \mathbb{Z}. \text{ Denote by } \xi \text{ the point in } X \text{ where each } S^1 \text{ and } Y.\text{ Meet. Rename each } S^1 \text{ by } \Omega_i, i = 1, 2, \ldots, s. \text{ Denote by } z_\xi^i \text{ the identity map on } \text{i}th \ S^1 \text{ (which is identified with } S^1 \text{)) and } z_\xi^i(\xi) = \xi \text{ for all } \xi \notin \Omega_i. \text{ There is an obvious homomorphism } \Pi : PM_r(C(X)) P \to D' = \bigoplus_{i=1}^k E_i, \text{ where } E_i \cong M_R(C(S^1)). \text{ Note that if } s \geq 2, \text{ then } \Pi \text{ is not surjective. We have that } G_1 = \Pi_1(D'). \text{ We also use } \Pi_i : PM_r(C(X)) P \to E_i \text{ which is the composition of } \Pi \text{ with the projection from } D' \text{ to } E_i. \text{ Let } z \text{ be the identity map on } S^1.
\]
We may write

\[ P(t) = \begin{pmatrix} P_1 & 0 \\ 0 & I \end{pmatrix} \]

where \( P_1 \) is a projection with rank 3 and \( I = \text{diag}(1, 1, \ldots, 1) \) with 1 repeating rank(\( P \)) = 3 times.

Note that, since rank(\( P \)) \( \geq 6 \), \( \text{tsr}(C(X)) = 2 \) and \( \text{csr}(C(X)) \leq 2 + 1 \) (by 4.10 of [47]). It follows that \( \text{csr}M_2(C') \leq 2 \) (by 4.7 in [48]). It then follows from 5.3 of [48] that \( U(C')/U_0(C') = K_1(C') \). In particular, \( CU(C') \subset U_0(C') \).

Denote by \( u_i = \text{diag}(z_i^r, 1, \ldots, 1) \), where 1 is repeated \( r - 4 \) times. If we write \( z_i \in U(C) \), we mean the unitary

\[ z_i(t) = \begin{pmatrix} P_1 & 0 \\ 0 & u_i \end{pmatrix}. \]

If we write \( z_i \in E_i \), we mean \( \Pi_i(z_i^r) \). Note that in this case, \( z_i \) has the form \( \text{diag}(1, \ldots, 1, z, 1, \ldots, 1) \), where \( z \) is in the 4th position and there are \( R - 1 \) many 1’s.

Now let \( C = \bigoplus_{j=1}^{l+1} C(j) \), where \( C(j) \) is either of the form \( P_jM_{r(j)}(C(X_j))P_j \) for \( j \leq l \), where \( X_j \) is of the form \( X \) described above, \( C^{(j)} = M_{r(j)} \), or \( C^{(j)} = P_jM_{r(j)}(C(Y_j))P_j \), where \( Y_j \) is a finite CW complex with dimension no more than 3, rank of \( P_j \) is \( R(j) \geq 6 \) and \( K_1(Y_j) \) is finite for \( l + 1 \leq j \leq l + 1 \). Let \( D^{(j)} \) be as \( D' \) above for each \( j \leq l \). Let \( \Pi^{(j)} \) be as \( \Pi \) above for \( C^{(j)} = P_jM_{r(j)}(C(X_j))P_j \). Put \( D = \bigoplus_{j=1}^{l+1} D^{(j)} \) and \( \Pi = \bigoplus_{j=1}^{l+1} \Pi^{(j)} \). Since \( K_1(C) \) is finitely generated and \( U_0(C)/CU(C) \) is divisible (see 6.6), we may write

\[ U(C)/CU(C) = U_0(C)/CU(C) \oplus K_1(D) \oplus \text{tor}(K_1(C)). \]

Let \( \pi_1 : U(C)/CU(C) \to K_1(D) \), \( \pi_0 : U(C)/CU(C) \to U_0(C)/CU(C) \), \( \pi_2 : U(C)/CU(C) \to \text{tor}(K_1(C)) \) be fixed projection maps associated with the above decomposition. To avoid possible confusion, by \( \pi_1(U(C)/CU(C)) \), we mean a subgroup of \( U(C)/CU(C) \), \( i = 0, 1, 2 \). We also assume that \( \pi_1(\bar{z}_i) = \bar{z}_i \) in \( U(C)/CU(C) \).

It is worth pointing out that one could have \( X = Y = [0, 1] \).

The notation established above will be used in the rest of this section.

**7.2. Lemma.** Let \( C = \bigoplus_{i=1}^{l+1} C_i \) be as above and \( \mathcal{U} \subset U(C) \) be a finite subset and \( \mathcal{F} \) be the group generated by \( \mathcal{U} \). Suppose that \( G \) is a subgroup of \( U(C)/CU(C) \) which contains \( \mathcal{F} \), \( \pi_2(U(C)/CU(C)) \) and \( \pi_1(U(C)/CU(C)) \). Suppose that the composition map \( \gamma : \bar{F} \to U(D)/CU(D) \to U(D)/U_0(D) \) is injective and \( \gamma(\bar{F}) \text{ is free.} \) Let \( B \) be a unital C*-algebra and \( \Lambda : G \to U(B)/CU(B) \) be a homomorphism such that \( \Lambda(G \cap U_0(C)/CU(C)) \subset U_0(B)/CU(B) \). Then there are homomorphisms \( \beta : U(D)/CU(D) \to U(B)/CU(B) \) and \( \theta : \pi_2(U(C)/CU(C)) \to U(B)/CU(B) \) such that

\[ \beta \circ \Pi^B \circ \pi_1(\bar{w}) = \Lambda(\bar{w})(\theta \circ \pi_2(\bar{w})) \]

for all \( \bar{w} \in \mathcal{F} \) and such that \( \theta(g) = \Lambda|_{\pi_2(U(C)/CU(C))}(g^{-1}) \) for \( g \in \pi_2(U(C)/CU(C)) \). Moreover, \( \beta(U_0(D)/CU(D)) \subset U_0(B)/CU(B) \).
If furthermore $A$ is a simple $C^*$-algebra with $TR(B) \leq 1$ and $\Lambda(U(C)/CU(C)) \subset U_0(B)/CU(B)$, then $\beta \circ \Pi^\sharp \circ (\pi_1)|_F = \Lambda|_F$.

**Proof.** Let $\kappa_1 : U(D)/CU(D) \to K_1(D)$ be the quotient map. Let $\eta : \pi_1(U(C)/CU(C)) \to \eta \circ K_1(U(C)/CU(C))$. Note that $\eta$ is an isomorphism. Since $\gamma$ is injective and $\gamma(F)$ is free, we conclude that $\kappa_1 \circ \Pi^\sharp \circ \pi_1$ is also injective on $F$. From this fact and the fact that $U_0(C)/CU(C)$ is divisible (6.6, we obtain a homomorphism $\lambda : K_1(D) \to U_0(C)/CU(C)$ such that

$$\lambda|_{\kappa_1 \circ \Pi^\sharp \circ \pi_1(F)} = \pi_0 \circ ((\kappa_1 \circ \Pi^\sharp \circ \pi_1)|_F)^{-1}.$$  

Now define $\beta = \Lambda((\eta^{-1} \circ \kappa_1) \oplus (\lambda \circ \kappa_1))$. Then for any $\tilde{w} \in F$,

$$\beta(\Pi^\sharp \circ \pi_1(\tilde{w})) = \Lambda(\eta^{-1}(\kappa_1 \circ \Pi^\sharp(\pi_1(\tilde{w}))) \oplus \lambda \circ \kappa_1(\Pi^\sharp(\pi_1(\tilde{w})))) = \Lambda(\pi_1(\tilde{w}) \oplus \pi_0(\tilde{w})).$$

Now define $\theta : \pi_2(U(C)/CU(C)) \to U(B)/CU(B)$ by $\theta(x) = \Lambda(x^{-1})$ for $x \in \pi_2(U(C)/CU(C))$. Then

$$\beta(\Pi^\sharp(\pi_1(\tilde{w}))) = \Lambda(\tilde{w}) \theta(\tilde{w}) \text{ for } w \in F.$$  

To see the last statement, we assume $\Lambda(U(C)/CU(C)) \subset U_0(B)/CU(B)$. Then $\Lambda(\pi_2(U(C)/CU(C)))$ is a torsion subgroup of $U_0(B)/CU(B)$. By 6.11, $U_0(B)/CU(B)$ is torsion free. Therefore $\theta = 0$. □

**7.3. Lemma.** Let $A$ be a unital separable simple $C^*$-algebra with $TR(A) \leq 1$ and $C$ be as described in 7.1. Let $\mathcal{U} \subset U(A)$ be a finite subset and $F$ be the subgroup generated by $\mathcal{U}$ such that $(\kappa_1)|_F$ is injective and $\kappa_1(F)$ is free, where $\kappa_1 : U(A)/CU(A) \to K_1(A)$ is the quotient map. Suppose that $\alpha : K_1(C) \to K_1(A)$ is an injective homomorphism and $L : \tilde{F} \to U(C)/CU(C)$ is an injective homomorphism with $L(\tilde{F} \cap U_0(A)/CU(A)) \subset U_0(C)/CU(C)$ such that $\pi_1 \circ L$ is injective (see 7.1 for $\pi_1$) and

$$\alpha \circ \kappa'_1 \circ L(g) = \kappa_1(g) \text{ for all } g \in F,$$

where $\kappa'_1 : U(C)/CU(C) \to K_1(C)$ is the quotient map. Then there exists a homomorphism $\beta : U(C)/CU(C) \to U(A)/CU(A)$ with $\beta(U_0(C)/CU(C)) \subset U_0(A)/CU(A)$ such that

$$\beta \circ L(\tilde{w}) = \tilde{w} \text{ for } w \in F.$$  

**Proof.** Let $G$ be the preimage of $\alpha \circ \kappa'_1(U(C)/CU(C))$ under $\kappa_1$. So we have the following short exact sequence:

$$0 \to U_0(A)/CU(A) \to G \to \alpha \circ \kappa'_1(U(C)/CU(C)) \to 0.$$  

Since $U_0(A)/CU(A)$ is divisible, there exists an injective homomorphism

$$\gamma : \alpha \circ \kappa'_1(U(C)/CU(C)) \to G$$
such that $\kappa_1 \circ \gamma(g) = g$ for $g \in \alpha \circ \kappa'_1(U(C)/CU(C))$. Since $\alpha \circ \kappa'_1 \circ L(f) = \kappa_1(f)$ for all $f \in \tilde{F}$, we have $\tilde{F} \subset G$. Moreover, $(\gamma \circ \alpha \circ \kappa'_1 \circ L(f))^{-1} f \in U_0(A)/CU(A)$ for all $f \in \tilde{F}$. Define $\psi : L(\tilde{F}) \to U_0(A)/CU(A)$ by

$$
\psi(x) = \gamma \circ \alpha \circ \kappa'_1 \circ L(\overline{[\gamma^{-1}(x)]^{-1}}L^{-1}(x))
$$

for $x \in L(\tilde{F})$. Since $U_0(A)/CU(A)$ is divisible, there is homomorphism $\tilde{\psi} : U(C)/CU(C) \to U_0(A)/CU(A)$ such that $\tilde{\psi}|_{L(\tilde{F})} = \psi$. Now define

$$
\beta(x) = \gamma \circ \alpha \circ \kappa'_1(x)\tilde{\psi}(x).
$$

Hence $\beta(L(f)) = f$ for $f \in \tilde{F}$. □

7.4. Lemma. Let $B$ be a unital separable simple $C^*$-algebra with $TR(B) \leq 1$ and $C$ be as in 7.1. Let $F$ be a group generated by a finite subset $\mathcal{U} \subset U(C)$ such that $(\pi_1)|_{\tilde{F}}$ is injective. Let $G$ be a subgroup containing $\tilde{F}$, $\pi_0(\tilde{F})$, $\pi_1(U(C)/CU(C))$ and $\pi_2(U(C)/CU(C))$. Suppose that $\alpha : U(C)/CU(C) \to U(B)/CU(B)$ is a homomorphism with $\alpha(U(C)/CU(C)) \subset U_0(B)/CU(B)$. Then for any $\varepsilon > 0$ there is $\delta > 0$ satisfying the following: if $\phi = \phi_0 \oplus \phi_1 : C \to B$ is a $G$-$\eta$-multiplicative contractive completely positive linear map such that:

1. both $\phi_0$ and $\phi_1$ are $G$-$\eta$-multiplicative and $\phi_0$ maps the identity of each summand of $C$ into a projection,
2. $G$ is sufficiently large and $\eta$ is sufficiently small which depend only on $C$ and $F$ (so that $\phi^\delta$ is well defined on $G$),
3. $\phi_0$ is homotopically trivial (see (vi) in Section 1), $\phi_0|_{\bar{\pi}_0(\tilde{F})}$ is a well-defined homomorphism and $[\phi]|_{\bar{\pi}_1(\tilde{F})} = \alpha|_{\bar{\pi}_1(\tilde{F})}$, where $\alpha_* : K_1(C) \to K_1(B)$ induced by $\alpha$ and $\kappa_1 : U(C)/CU(C) \to K_1(C)$ is the quotient map,
4. $\tau(\phi_0(1_C)) < \delta$ for all $\tau \in T(B)$,

then there is a homomorphism $\Phi : C \to e_0 Be_0$ ($e_0 = \phi_0(1_C)$) such that:

1. $\Phi$ is homotopically trivial and $\Phi|_{\mathcal{U}} = (\phi_0)|_{\mathcal{U}}$ and
2. $\alpha(\overline{w})^{-1}(\Phi \oplus \phi_1)^\delta(\overline{w}) = \overline{g_w}$, where $g_w \in U_0(B)$ and $\text{cel}(g_w) < \varepsilon$ for all $w \in \mathcal{U}$.

Proof. By Lemma 7.2, there are homomorphisms $\beta_1, \beta_2 : U(D)/CU(D) \to U(B)/CU(B)$ with $\beta_i(U_0(D)/CU(D)) \subset U_0(B)/CU(B)$ ($i = 1, 2$) and homomorphisms

$$
\theta_1, \theta_2 : \pi_2(U(C)/CU(C)) \to U(B)/CU(B)
$$

such that

$$
\beta_1 \circ \Pi^\delta(\pi_1(\overline{w})) = \alpha(\overline{w})\theta_1(\overline{\pi_2(\overline{w})}) \quad \text{and} \quad \beta_2 \circ \Pi^\delta(\pi_1(\overline{w})) = \phi_1^\delta(\overline{\phi^*}(\overline{w}))\theta_2(\overline{\pi_2(\overline{w})})
$$

for all $\overline{w} \in \tilde{F}$. Moreover $\theta_1(g) = \alpha(g^{-1})$ and $\theta_2(g) = \phi_1^\delta(g)$ if $g \in \pi_2(\tilde{F})$. Since $\phi_0$ is homotopically trivial,

$$
\theta_1(g)\theta_2(g) \in U_0(B)/CU(B) \quad \text{for all } g \in \pi_2(\tilde{F}).
$$
Since $\pi_2(U(C)/CU(C)$ is torsion and $U_0(B)/CU(B)$ is torsion free, we conclude that

$$\theta_1(g)\theta_2(g) = \bar{1} \quad \text{for all } g \in \pi_2(\bar{F}).$$

To simplify notation, without loss of generality, we may assume that $C = \bigoplus_{j=1}^{1+l_1} C^{(j)}$ (with $l = 1$) such that $C^{(1)} = P M_r(C(X))P$ as described in 7.1 and $C^{(j)}$ is also as described in 7.1 for $2 = l + 1 \leq j \leq l_1 + 1$. Let $D$ be as described in 7.1.

For each $w \in U(C)$, we may write $w = (w_1, w_2, \ldots, w_{l_1+1})$ according to the direct sum $C = \bigoplus_{j=1}^{1+l_1} C^{(j)}$. Note that $\pi_1(w) = \pi_1(w_1)$. Let $\pi_1(\bar{w}) = (\bar{z}_1^{k(1,w)}, \bar{z}_2^{k(2,w)}, \ldots, \bar{z}_5^{k(s,w)})$, where $k(i, w)$ is an integer (here $z_i$ is described in 7.1). Then $\Pi_1^b(\pi_1(\bar{w})) = z_i^{k(i,w)}$. On the other hand, we may also write $\Pi_1^b(\bar{w}) = z_i^{k(i,w)} g_{i,w}$ for some $g_{i,w} \in U_0(C(S^1, M_R))$.

Let $l = \max\{\text{cel}(g_{i,w}): w \in U, 1 \leq i \leq s\}$. Choose $\delta$ so that $(2+l)\delta < \varepsilon/4$. Let $e_0 = \phi_0(1_C)$ and $e_1 = \phi_1(1_C)$. Write $e_0 = E_1 \oplus E_2 \oplus \cdots \oplus E_{l+1}$, where $E_j = \phi_0(1_{C^{(j)}})$, $j = 1, 2, \ldots, l + 1$. Recall that $P$ has rank $R$. Since $\phi_0$ is homotopically trivial (see (vi) in Section 1), we may also write $E_1 = e_01 \oplus \cdots \oplus e_0 R$, where $\{e_0 i: 1 \leq i \leq R\}$ is a set of mutually orthogonal and mutually equivalent projections. Since $e_0 B e_0$ is simple and has the property (SP), $e_0$ can be written as a sum of $s$ mutually orthogonal projections. Thus $E_1 = p_1 \oplus p_2 \oplus \cdots \oplus p_s$, where each $p_i$ can be written as a direct sum of $R$ mutually orthogonal and mutually equivalent projections $\{q_1, \ldots, q_i, R\}$. For each $q_i$, we write $q_i = q_{i1} \oplus q_{i2}$, where both $q_{i1}$ and $q_{i2}$ are zero. Let $q = \sum_{i=1}^{l} q_i$. We may view $E_1 B E_1 = \bigoplus_{i=1}^{l} M_R(q_{i1} B q_{i2}) = M_R(q B q)$.

Let $z_i$ be as in 7.1. Put $x_i' \in U(q_{i1} B q_{i2})$ such that $x_i' = \beta_1(z_i)$. And $y_i' \in U(q_{i1} B q_{i2})$ such that $y_i' = \beta_2(z_i)$, $i = 1, 2, \ldots, s$. This is possible because of 6.7. Put $x_i = x_i' \oplus q_{i2}$, $y_i = y_i' \oplus q_{i1}$, $i = 1, 2, \ldots, s$. Note that $x_i y_i = y_i' x_i$. Define $\Phi_1: D \rightarrow M_R(q B q) = \bigoplus_{i=1}^{l} M_R(q_{i1} B q_{i2})$ by $\Phi_1(f) = \sum_{i=1}^{l} f_i(x_i y_i)$, where $f = (f_1, f_2, \ldots, f_s)$, $f_i \in C(S^1, M_R)$. Define $h(g) = \Phi_1(\Pi(g)) \oplus \phi_1(g)$ for $g \in C$. We compute that

$$h(w) = \prod_{i=1}^{l} x_i^{k(i,w)} g_{i,w}(x_i y_i) y_i^{k(i,w)} \phi_1^b(\bar{w})$$

$$= \beta_1(\Pi^b(\pi_1(\bar{w}))) \Phi_1^b \left( \bigoplus_{i=1}^{l} g_{i,w} \right) \beta_2(\Pi^b(\pi_1(\bar{w}))) \phi_1^b(\bar{w})$$

$$= \alpha(\bar{w}) \theta_1(\pi_2(\bar{w}))) \theta_2(\pi_2(\bar{w})) \phi_1^b \left( \bigoplus_{i=1}^{l} g_{i,w} \right)$$

$$= \alpha(\bar{w}) \phi_1^b \left( \bigoplus_{i=1}^{l} g_{i,w} \right)$$

for all $w \in U$. Put $g'_{w'} = \Phi_1(\bigoplus_{i=1}^{l} g_{i,w}) \oplus (1 - \phi_0(1_C))$. Since $\tau(\phi_0(1_C)) < \delta$, by the choice of $\delta$, we conclude from Lemma 6.4 that there exists $w' \in CU(B)$ such that

$$\text{cel}(w' g'_{w'}) < \varepsilon/2 \quad \text{for all } w \in U.$$
\[ g_w'' = \Phi_2(w) \oplus (1 - (e_0 - E_1)). \] Since \( \Phi_2(\sum_{j=2}^{1+L_2} C(j)) \) is finite-dimensional, \( \text{cel}(\Phi_2(w)) \leq 2\pi \) (in \( U_0((e_0 - E_1)B(e_0 - E_1)) \) for all \( w \in \mathcal{U} \)). By the choice of \( \delta \), we conclude that there is \( w'' \in CU(B) \) such that \( \text{cel}(w''g_w'') < \varepsilon /2 \) (see 6.4). Put \( g_w = w'g_w'w''g_w'' \). We have, for all \( w \in \mathcal{U} \),

\[ \alpha(\tilde{w})^{-1}(\Phi \oplus \phi_1)(\tilde{w}) = \tilde{g}_w \quad \text{with} \quad g_w \in U_0(B) \quad \text{and} \quad \text{cel}(g_w) < \varepsilon. \]

**7.5. Lemma.** Let \( B \) be a unital separable simple \( C^* \)-algebra with \( TR(B) \leq 1 \) and \( C \) be as described in 7.1. Let \( \mathcal{U} \subset U(B) \) be a finite subset and \( F \) be the subgroup generated by \( \mathcal{U} \) such that \( \kappa_1(\tilde{F}) \) is free, where \( \kappa_1 : U(B)/CU(B) \to K_1(B) \) is the quotient map. Let \( \phi : C \to B \) be a homomorphism such that \( \kappa_{*1} \) is injective. Suppose that \( j, L : \tilde{F} \to U(C)/CU(C) \) are two injective homomorphisms with \( j(\tilde{F} \cap U_0(B)/CU(B)), L(\tilde{F} \cap U_0(B)/CU(B)) \subset U_0(C)/CU(C) \) such that \( \kappa_1 \circ \phi \circ L = \kappa_1 \circ \phi \circ j = \kappa_1|_F \) and all three are injective.

Then, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \phi = \phi_0 \oplus \phi_1 : C \to B \), where \( \phi_0 \) and \( \phi_1 \) are homomorphisms satisfying the following:

1. \( \tau(\phi_0(1_C)) < \delta \) for all \( \tau \in T(B) \) and
2. \( \phi_0 \) is homotopically trivial,

then there is a homomorphism \( \psi : C \to e_0Be_0 \) (\( e_0 = \phi_0(1_C) \)) such that:

1. \( [\psi] = [\phi_0] \) in \( KL(C, B) \) and
2. \( (\phi \circ j(\tilde{w}))^{-1}(\psi \oplus \phi_1)(L(\tilde{w})) = g_w \), where \( g_w \in U_0(B) \) and \( \text{cel}(g_w) < \varepsilon \) for all \( w \in \mathcal{U} \).

**Proof.** The first part of the proof is essentially the same as that of 7.3. Let \( \kappa_1' : U(C)/CU(C) \to K_1(C) \) be the quotient map and let \( G \) be the preimage of \( \phi_{*1} \circ \kappa_1'(U(C)/CU(C)) \) under \( \kappa_1 \). Since \( U_0(B)/CU(B) \) is divisible, there exists an injective homomorphism \( \gamma : \phi_{*1} \circ \kappa_1'(U(C)/CU(C)) \to G \) such that \( \kappa_1 \circ \gamma(g) = g \) for \( g \in \phi_{*1} \circ \kappa_1'(U(C)/CU(C)) \). Since \( \phi_{*1} \circ \kappa_1'(U(f) = \kappa_1(f) = \kappa_1(\phi \circ j(f)) \) for all \( f \in \tilde{F} \), we have \( \tilde{F} \subset G \). Moreover,

\[ \left[ \gamma \circ \phi_{*1} \circ \kappa_1'(U(f) \right]^{-1} \phi \circ j(f) \in U_0(B)/CU(B) \]

for all \( f \in F \). Define \( \psi : L(\tilde{F}) \to U_0(B)/CU(B) \) by

\[ \psi(x) = \left[ \gamma \circ \phi_{*1} \circ \kappa_1'(x) \right]^{-1} \left[ \phi \circ j \circ L^{-1}(x) \right] \]

for all \( x \in L(\tilde{F}) \). Since \( U_0(B)/CU(B) \) is divisible, there is a homomorphism

\[ \tilde{\psi} : U(C)/CU(C) \to U_0(B)/CU(B) \]

such that \( \tilde{\psi}|_{L(\tilde{F})} = \psi \).

Define \( \alpha : U(C)/CU(C) \to U(B)/CU(B) \) by \( \alpha(x) = \gamma \circ \phi_{*1} \circ \kappa_1'(x) \tilde{\psi}(x) \) for all \( x \in U(C)/CU(B) \). Note that

\[ \alpha(L(f)) = \phi \circ j(f) \quad \text{for all} \quad f \in \tilde{F}. \]

Now the lemma follows from 7.4. \( \Box \)
8. A uniqueness theorem and automorphisms on simple $C^*$-algebras with $\text{TR}(A) \leq 1$

8.1. Definition. Let $A$ and $B$ be $C^*$-algebras. Two homomorphisms $\phi, \psi : A \to B$ are said to be stably unitarily equivalent if for any monomorphism $h : A \to B$, $\varepsilon > 0$ and finite subset $F \subset A$, there exists an integer $n > 0$ and a unitary $U \in M_{n+1}(B)$ (or in $M_{n+1}(B)$, if $B$ is unital) such that

$$\|U^* \text{diag}(\phi(a), h(a), h(a), \ldots, h(a)) - \text{diag}(\psi(a), h(a), h(a), \ldots, h(a))\| < \varepsilon$$

for all $a \in F$, where $h(a)$ is repeated $n$ times on both diagonals.

Let $A$ and $B$ be $C^*$-algebras and $\phi, \psi : A \to B$ be (linear) maps. Let $F \subset A$ and $\varepsilon > 0$. We write

$$\phi \sim \varepsilon \psi \text{ on } F,$$

if there exists a unitary $u \in B$ such that

$$\|\text{ad}(u) \circ \phi(a) - \psi(a)\| < \varepsilon \text{ for all } a \in F.$$

We write

$$\phi \approx \varepsilon \psi \text{ on } F, \text{ if } \|\phi(a) - \psi(a)\| < \varepsilon \text{ for all } a \in F.$$

8.2. Definition. Let $A$ be a $C^*$-algebra.

(i) Denote by $\mathbf{P}(A)$ the set of all projections and unitaries in $M_{\infty}(\tilde{A} \otimes C_n)$, $n = 1, 2, \ldots$, where $C_n$ is an abelian $C^*$-algebra so that

$$K_i(A \otimes C_n) = K_i^*(A; \mathbb{Z}/n\mathbb{Z}).$$

One also has the following exact sequence:

$$
\begin{array}{ccc}
K_0(A) & \longrightarrow & K_0(A, \mathbb{Z}/k\mathbb{Z}) \\
\downarrow k & & \downarrow k \\
K_1(A) & \leftarrow & K_1(A, \mathbb{Z}/k\mathbb{Z})
\end{array}
$$

(see [53]). As in [12], we use the notation

$$K(A) = \bigoplus_{i=0,1,n \in \mathbb{Z}^+} K_i(A; \mathbb{Z}/n\mathbb{Z}).$$

By $\text{Hom}_A(K(A), K(B))$ we mean all homomorphisms from $K(A)$ to $K(B)$ which respect to the direct sum decomposition and the so-called Bockstein operations (see [12]). Denote by $\text{Hom}_A(K(A), K(B))^{++}$ those $\alpha \in \text{Hom}_A(K(A), K(B))$ with the property that $\alpha(K_0(A) \setminus \{0\}) \subset K_0(B) \setminus \{0\}$. It follows from [12] that if $A$ satisfies the Universal Coefficient Theorem, then $\text{Hom}_A(K(A), K(B)) \cong KL(A, B)$. Moreover, one has the following short exact sequence,
0 \to \text{Pext}(K_*(A), K_*(B)) \to KK(A, B) \to KL(A, B) \to 0.

A separable C*-algebra $A$ is said to satisfy Approximate Universal Coefficient Theorem (AUCT) if

$$KL(A, B) \cong \text{Hom}_A(K(A), K(B))$$

for any $\sigma$-unital C*-algebra $B$ (see [39]). A separable C*-algebra $A$ which satisfies the UCT must satisfy the AUCT. If $A$ satisfies the AUCT, for convenience, we will use $KL(A, B)^{++}$ for $\text{Hom}_A(K(A), K(B))^{++}$.

(ii) Let $L : A \to B$, be a contractive completely positive linear map. We also use $L$ for the extension from $A \otimes K \to B \otimes K$ as well as maps from $A \otimes C_n \to B \otimes C_n$ for all $n$. Given a projection $p \in \mathcal{P}(A)$, if $L : A \to B$ is an $\mathcal{F}$-$\delta$-multiplicative contractive completely positive linear map with sufficiently large $\mathcal{F}$ and sufficiently small $\delta$, $\|L(p) - q\| < 1/4$ for some projection $q'$. Define $[L](p) = [q']$ in $\tilde{K}(B)$. It is easy to see this is well defined (see [30]). Suppose that $q$ is also in $\mathcal{P}(A)$ with $[q] = k[p]$ for some integer $k$. By adding sufficiently many elements (partial isometries) in $\mathcal{F}$, we can assume that $[L](q) = k[L](p)$. Similarly as in 6.1, one can do the same for unitaries. Let $\mathcal{P} \subset \mathcal{P}(A)$ be a finite subset. We say $[L]|_{\mathcal{P}}$ is well defined if $[L](p)$ is well defined for every $p \in \mathcal{P}$ and if $[p'] = [p]$ and $p' \in \mathcal{P}$, then $[L](p') = [L](p)$. This always occurs if $\mathcal{F}$ is sufficiently large and $\delta$ is sufficiently small. In what follows we write $[L]|_{\mathcal{P}}$ when $[L]$ is well defined on $\mathcal{P}$.

(iii) Let $A = \bigoplus_{i=1}^n A_i$, where each $A_i$ is a unital C*-algebra. Suppose that $L : A \to B$ is a $\mathcal{G}$-$\varepsilon$-multiplicative contractive completely positive linear map. For any $\eta > 0$, if $\mathcal{G}$ is large enough and $\varepsilon$ is small enough, we may assume that

$$\|L(1_{A_i}) - p_i\| < \eta, \quad \|p_jL(1_{A_i})\| < \eta \quad \text{and} \quad \|L(1_{A_i})p_j\| < \eta$$

for some projection $p_i \in B$ and $i \neq j$. Let $b = p_1L(1_{A_1})p_1$. Then, with sufficiently small $\eta$, we may assume that $b$ is invertible in $p_1BP_1$. Define $L_1(a) = b^{-1/2}p_1L(a)p_1b^{-1/2}$. Then $L_1(1_{A_1}) = p_1$. Consider $(1 - p_1)L(1 - p_1)$. It is clear that, for any $\delta > 0$, by induction and choosing a sufficiently large $\mathcal{G}$ and sufficiently small $\eta$ and $\varepsilon$,

$$\|L - \Psi\| < \delta,$$

where $\Psi(a) = \bigoplus_{i=1}^n L_i(1_{A_i}a1_{A_i})$ for $a \in A$. So, to save notation in what follows, we may assume that $L = \bigoplus_{i=1}^n L_i$, where each $L_i : A_i \to B$ is a completely positive contraction which maps $1_{A_i}$ to a projection in $B$ and $(L_1(1_{A_1}), L_2(1_{A_2}), \ldots, L_n(1_{A_n}))$ are mutually orthogonal.

Throughout the rest of this section, $\mathcal{A}$ denotes the class of separable nuclear C*-algebras satisfying the Approximate Universal Coefficient Theorem.

8.3. Lemma. (See [35, Theorem 4.4].) Let $B$ be a unital C*-algebra and let $A$ be a unital C*-algebra in $\mathcal{A}$ which is a unital C*-subalgebra of $B$. Let $\alpha : A \to B$ and $\beta : A \to B$ be two homomorphisms. Then $\alpha$ and $\beta$ are stably approximately unitarily equivalent if $[\alpha] = [\beta]$ in $KK(A, B)$ and if $A$ is simple or $B$ is simple.
The following is a modification of [35, Theorem 4.8]. A proof was given in the earlier version of this paper. Since then a more general version of the following appeared in [39, Theorem 3.9]. We will omit the original proof and view the following as a special case.

### 8.4. Theorem
(Cf. [35, Theorem 4.8].) Let $B$ be a $C^*$-algebra with stable rank one and $\text{cel}(M_m(B)) \leq k$ for some $k \geq \pi$ and for all $m$, and let $A$ be a unital simple $C^*$-algebra in $A$ which is a $C^*$-subalgebra of $B$. Let $\alpha : A \to B$ and $\beta : A \to B$ be two homomorphisms. Then $\alpha$ and $\beta$ are stably approximately unitarily equivalent if $[\alpha] = [\beta]$ in $KL(A, B)$.

The following uniqueness theorem is a modification of [35, Theorem 5.3].

### 8.5. Theorem
(See [35, 5.3].) Let $A$ be a unital simple $C^*$-algebra in $A$ and $L : U(M_\infty(A)) \to R_+$ be a map. For any $\varepsilon > 0$ and any finite subset $F \subset A$ there exist a positive number $\delta > 0$, a finite subset $G \subset A$, a finite subset $P \subset P(A)$ and an integer $n > 0$ satisfying the following: for any unital simple $C^*$-algebra $B$ with $\text{Tr}(B) \leq 1$, if $\phi, \psi, \sigma : A \to C$ are three $G$-$\delta$-multiplicative contractive completely positive linear maps with

$$[\phi]|_P = [\psi]|_P,$$

$$\text{cel}(\tilde{\phi}(u)^* \tilde{\psi}(u)) \leq L(u)$$

for all $u \in U(A) \cap P$ and $\sigma$ is unital, then there is a unitary $u \in M_{n+1}(B)$ such that

$$\|u^* \text{diag}((\phi(a), \sigma(a), \ldots, \sigma(a))) - \text{diag}((\psi(a), \sigma(a), \ldots, \sigma(a)))\| < \varepsilon$$

for all $a \in F$, where $\sigma(a)$ is repeated $n$ times.

**Proof.** Suppose that the theorem is false. Then there are $\varepsilon_0 > 0$ and a finite subset $F \subset A$ such that there are a sequence of positive numbers $\{\delta_n\}$ with $\delta_n \downarrow 0$, an increasing sequence of finite subsets $\{G_n\}$ whose union is dense in the unit ball of $A$, a sequence of finite subsets $\{P_n\}$ of $P(A)$ with $\bigcup_{n=1}^\infty P_n = P(A)$ and with $U_n = U(A) \cap P_n$, a sequence $\{k(n)\}$ of integers $k(n) \not\to \infty$ and sequences $\{\phi_n\}$, $\{\psi_n\}$ and $\{\sigma_n\}$ of $G_n$-$\delta_n$-multiplicative positive linear maps from $A$ to $B_n$ with $[\phi_n]|_{P_n} = [\psi_n]|_{P_n}$ and

$$\text{cel}(\tilde{\phi}_n(u)^* \tilde{\psi}_n(u)) \leq L(u)$$

for all $u \in U_n$ satisfying the following:

$$\inf\{\sup\{\|u^* \text{diag}((\phi_n(a), \sigma_n(a), \ldots, \sigma_n(a))) - \text{diag}((\psi_n(a), \sigma_n(a), \ldots, \sigma_n(a)))\| : a \in F\}\} \geq \varepsilon_0$$

where $\sigma_n(a)$ is repeated $k(n)$ times and the infimum is taken over all unitaries in $M_{k(n)+1}(B_n)$.

Set $D_0 = \bigoplus_{n=1}^\infty B_n$ and $D = \prod_{n=1}^\infty B_n$. Define $\Phi, \Psi, \Sigma : A \to D$ by $\Phi(a) = \{\phi_n(a)\}$, $\Psi(a) = \{\psi_n(a)\}$ and $\Sigma(a) = \{\sigma_n(a)\}$ for $a \in A$. Let $\pi : D \to D/D_0$ be the quotient map and set $\Phi = \pi \circ \Phi$, $\Psi = \pi \circ \Psi$ and $\Sigma = \pi \circ \Sigma$. Note that $\Phi$, $\Psi$ and $\Sigma$ are monomorphisms. For any $u \in U_k$,

$$\text{cel}(\tilde{\phi}_n(u)^* \tilde{\psi}_n(u)) \leq L(u)$$
for all sufficiently large \( n > k \). This implies that there is an equi-continuous path \( \{v_n(t)\} \) \( (t \in [0, 1]) \) such that

\[
v_n(0) = \tilde{\phi}_n(u) \quad \text{and} \quad v_n(1) = \tilde{\psi}_n(u)
\]

(see, for example, [25, Theorem 1.1]). Therefore, we conclude that

\[
[\tilde{\Phi}]|_{K_1(A)} = [\tilde{\Psi}]|_{K_1(A)}.
\]

Given an element \( p \in P_k \setminus U_k \) \((\text{for some } k)\), we claim that

\[
[\bar{\Phi}(p)] = [\bar{\Psi}(p)].
\]

We have (see [25, Proposition 2.1])

\[
K_0 \left( \prod B_n \right) = \prod K_0(B_n) \quad \text{and} \quad K_0(D/D_0) = \prod K_0(B_n)/\bigoplus K_0(B_n),
\]

where \( \prod K_0(B_n) \) is the sequences of elements \( \{[p_n] - [q_n]\} \). where \( p_n \) and \( q_n \) can be represented by projections in \( M_L(B_n) \) for some integer \( L \). Since each \( TR(B_n) \leq 1 \), \( B_n \) has stable rank one and \( K_0(B_n) \) is weakly unperforated. By [25, Proposition 2.2], each \( B_n \) has \( K_i \)-divisible rank \( T \) with \( T(n, k) = 1 \). By 6.5, \( cer(M_k(B_n)) \leq 4 \) for all \( k \) and \( n \), and the kernel of the map from \( K_1(\prod B_n) \) to \( \prod K_1(B_n) \) is divisible and torsion free (see 6.12). By the proof of [25, Theorem 2.1, part (2)], we also have

\[
K_i \left( \prod B_n, \mathbb{Z}/m\mathbb{Z} \right) \subset \prod K_i(B_n, \mathbb{Z}/m\mathbb{Z}), \quad m = 2, 3, \ldots
\]

(In fact, by 6.10, each \( B_n \) has exponential length divisible rank \( E \) with \( E(L, k) = 8/\pi + L/k + 1 \) so that [25, Theorem 2.1(2)] can be applied directly. See also [25, Corollary 2.1, part (2)].)

Since \( [\phi_n(p)] = [\psi_n(p)] \) in \( K_0(B_n) \) or in \( K_i(B_n, \mathbb{Z}/m\mathbb{Z}) \) \((i = 0, 1, m = 2, 3, \ldots)\) for large \( n \),

\[
[\bar{\Phi}(p)] = [\bar{\Psi}(p)].
\]

Then \( \bar{\Phi}_* = \bar{\Psi}_* \). Therefore \([\bar{\Phi}] = [\bar{\Psi}] \) in \( KL(A, \prod_b B_n/\bigoplus B_n) \).

By applying 8.4, we obtain an integer \( N \) and a unitary \( u \in M_{N+1}(D/D_0) \) such that

\[
\|u^* \text{diag}(\Phi(a), \tilde{\Sigma}(a), \ldots, \tilde{\Sigma}(a))u - \text{diag}(\Psi(a), \tilde{\Sigma}(a), \ldots, \tilde{\Sigma}(a))\| < \varepsilon_0/3
\]

for all \( a \in \mathcal{F} \), where \( \tilde{\Sigma}(a) \) is repeated \( N \) times. It is easy to see (see [30, 1.3] for example) there is a unitary \( U \in M_{N+1}(D) \) such that \( \pi(U) = u \) and for each \( a \in \mathcal{F} \) there exists \( c_a \in M_{N+1}(D_0) \) such that

\[
\|U^* \text{diag}(\Phi(a), \Sigma(a), \ldots, \Sigma(a))U - \text{diag}(\Psi(a), \Sigma(a), \ldots, \Sigma(a)) + c_a\| < \varepsilon_0/3
\]

where \( \Sigma(a) \) is repeated \( N \) times. Write \( U = \{u_n\} \), where \( u_n \in M_{N+1}(B_n) \) are unitaries. Since \( c_a \in M_{N+1}(D_0) \) and \( \mathcal{F} \) is finite, there is \( N_0 > 0 \) such that for \( n \geq N_0 \)

\[
\|u_n^* \text{diag}(\Phi(a), \Sigma(a), \ldots, \Sigma(a))u_n - \text{diag}(\Psi(a), \Sigma(a), \ldots, \Sigma(a))\| < \varepsilon_0/3
\]
\[ \left\| u_n^* \text{diag}(\phi_n(a), \sigma_n(a), \ldots, \sigma_n(a))u_n - \text{diag}(\psi_n(a), \sigma_n(a), \ldots, \sigma_n(a)) \right\| < \varepsilon_0/2 \]

for all \( a \in \mathcal{F} \), where \( \sigma_n \) is repeated \( N \) times. This contradicts the assumption that the theorem is false. \( \square \)

### 8.6. Theorem

Let \( A \) be a separable unital nuclear simple C*-algebra with \( TR(A) \leq 1 \) satisfying the AUCT and let \( L : U(A) \to \mathbb{R}_+ \). Then for any \( \varepsilon > 0 \) and any finite subset \( \mathcal{F} \subset A \), there exist \( \delta_1 > 0 \), an integer \( n > 0 \), a finite subset \( \mathcal{P} \subset \mathcal{P}(A) \), a finite subset \( \mathcal{S} \subset A \) satisfying the following:

(i) there exist mutually orthogonal projections \( q, p_1, \ldots, p_n \) with \( q \leq p_1 \) and \( p_1, \ldots, p_n \) mutually unitarily equivalent, and there exists a C*-subalgebra \( C \subset \mathcal{I} \) with \( 1_C = p_1 \) and unital \( S_\delta_1/2 \)-multiplicative contractive completely positive linear maps \( \phi_0 : A \to qAq \) and \( \phi_1 : A \to C \) such that

\[ \left\| x - \left( \phi_0(x) \oplus \left( \phi_1(x), \phi_1(x), \ldots, \phi_1(x) \right) \right) \right\| < \delta_1/2 \]

for all \( x \in S \), where \( \phi_1(x) \) is repeated \( n \) times; moreover, there exist a finite subset \( \mathcal{G}_0 \subset A \), a finite subset \( \mathcal{P}_0 \) of projections in \( M_\infty(C) \), a finite subset \( \mathcal{H} \subset A_{sa} \), \( \delta_0 > 0 \) and \( \sigma > 0 \) (which depend on the choices of \( C \));

for any unital simple C*-algebra \( B \) with \( TR(B) \leq 1 \) and any two \( S \cup \mathcal{G}_0 \)-\( \delta \)-multiplicative completely positive linear contractions \( L_1, L_2 : A \to B \) for which the following hold (with \( \delta = \min\{\delta_1, \delta_0\} \)):

(ii) \( [L_1]|_{\mathcal{P} \cup \mathcal{P}_0} = [L_2]|_{\mathcal{P} \cup \mathcal{P}_0} \);

(iii) \( |\tau \circ L_1(g) - \tau \circ L_2(g)| < \sigma \) for all \( g \in \mathcal{H} \) and \( \tau \in T(A) \);

(iv) \( e = L_1 \circ \phi_0(1_A) = L_2 \circ \phi_0(1_A) \) is a projection;

(v) \( \text{cel}(L_1(\phi_0(u))) \leq L(u) (\text{in } U(eBe)) \) for all \( u \in U(A) \cap \mathcal{P} \),

there exists a unitary \( U \in B \) such that

\[ \text{ad}(U) \circ L_1 \approx_\varepsilon L_2 \quad \text{on } \mathcal{F}. \]

Note that (i) holds as long as \( TR(A) \leq 1 \) and does not depend on \( L, \varepsilon \) and \( \mathcal{F} \).

**Proof.** Since \( TR(A) \leq 1 \), 8.5 applies. Fix a finite subset \( \mathcal{F} \subset A, \varepsilon > 0 \) and \( L \). Let \( \delta_1 > 0, \mathcal{G}_1 \subset A, \mathcal{P} \subset \mathcal{P}(A) \) and \( n \) be as required by 8.5 corresponding to \( L, \varepsilon/4 \) and \( \mathcal{F} \).

Let \( S = \mathcal{G}_1, \eta = \min(\delta_1/2, \varepsilon/4) \). Let \( q, p_1, \ldots, p_n \), \( \phi_0 \) and \( \phi_1 \) satisfy (i) given by 5.5. Let \( \delta_0 > 0, \sigma_1 > 0 \), and a finite subset \( \mathcal{G}_0 \subset A (\mathcal{G}_2 \supset \phi_0(\mathcal{G}_1)) \) be as required by 5.8 corresponding to \( C \), the finite subset \( \phi_1(S) \) and \( \eta \). Let \( \mathcal{P}_0 \) contain a finite set of projections in \( C \) which generates \( K_0(C) \).

Now choose a finite subset \( \mathcal{G}_0'' \subset C \) and \( \delta_2 > 0 \) so that any \( C''_\delta \)-multiplicative contractive completely positive linear map \( L \) from \( C \) to any C*-algebra gives a well defined map from \( K_0(C) \).

Let \( \mathcal{G}_0 = \mathcal{G}_0' \cup \mathcal{G}_0'' \), \( \delta_0 = \min(\delta_2, \delta_0) \) and \( \sigma = \sigma_1/2(n+1) \), and let \( \mathcal{H} = \{ a_{+}^\sigma : a_{+}^\sigma : a \in \mathcal{G}_0'' \} \).

Let \( L_i : A \to B \) be two \( S \cup \mathcal{G}_0 \)-\( \delta \)-multiplicative contractive completely positive linear maps \( (i = 1, 2) \) which satisfy (ii)–(v). Note that \( [L_1]|_{K_0(C)} = [L_2]|_{K_0(C)} \). Let \( e_1 = \phi_1(1_A) \). By assumption,
$e_1$ is a projection in $B$. By the choice of $\delta_2$, $\sigma_1$ and $\mathcal{G}_2$, applying 5.8, we obtain a unitary $v \in e_1Be_1$ such that

$$\|L_1(x) - v^*L_2(x)v\| < \eta$$

for all $x \in \phi_1(S)$. Therefore,

$$\|L_1\phi_1(a) - \text{ad}(v) \circ L_2 \circ \phi_1(a)\| < \eta$$

for all $a \in \mathcal{G}_1$. To simplify notation, without loss of generality, we may assume that $L_2 \circ \phi_1 = \text{ad}(v) \circ L_2 \circ \phi_1$.

Now, by applying 8.5, we have

$$L_1 \circ \phi_0 \oplus (L_1 \circ \phi_1, L_1 \circ \phi_1, \ldots, L_1 \circ \phi_1) \sim_{\varepsilon/4} L_2 \circ \phi_0 \oplus (L_1 \circ \phi_1, L_1 \circ \phi_1, \ldots, L_1 \circ \phi_1)$$

on $\mathcal{F}$. From the above ($\eta < \varepsilon/4$), we obtain

$$L_2 \circ \phi_0 \oplus (L_1 \circ \phi_1, L_1 \circ \phi_1, \ldots, L_1 \circ \phi_1) \sim_{\varepsilon/2} L_2 \circ \phi_0 \oplus (L_2 \circ \phi_1, \ldots, L_2 \circ \phi_1)$$

on $\mathcal{F}$. Therefore

$$L_1 \sim_{\varepsilon} L_2 \text{ on } \mathcal{F}.$$

It turns out that when $K_1(A)$ is torsion the “uniqueness theorem” can be stated in a much more easy way.

**8.7. Theorem.** Let $A$ be a unital separable simple $C^*$-algebra with $\text{TR}(A) \leq 1$ and with torsion $K_1(A)$. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there exist $\delta > 0$, $\sigma > 0$, a finite subset $\mathcal{P} \subset \mathcal{P}(A)$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any unital simple $C^*$-algebra $B$ with $\text{TR}(B) \leq 1$, any two $\mathcal{G}$-$\delta$-multiplicative completely positive linear contractions $L_1, L_2 : A \to B$ with

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}$$

and

$$\sup_{\tau \in T(B)} \{ |\tau \circ L_1(g) - \tau \circ L_2(g)| \} < \sigma$$

for all $g \in \mathcal{G}$, there exists a unitary $U \in B$ such that

$$\text{ad}(U) \circ L_1 \approx_{\varepsilon} L_2 \text{ on } \mathcal{F}.$$ 

**Proof.** Define $L : U(A) \to \mathbb{R}_+$ as follows. If $u \in U_0(A)$, we let $L(u) = 2(\text{cel}(u) + \pi/16)$; if $u \in U(A) \setminus U_0(A)$, but $[u]$ has order $k > 1$ in $K_1(A)$ (therefore $u^k \in U_0(A)$ since $A$ has stable rank one), let $L(u) = 8\pi + \frac{2\text{cel}(u^k)}{k} + \pi/16$. 

Let $G_1 = G(F, \varepsilon/2)$, $\delta_1 = \delta(F, \varepsilon/2)$, $n = n(F, \varepsilon/2)$, $S_1 = S(F, \varepsilon/2)$ and $\mathcal{P}_1 = \mathcal{P}(F, \varepsilon/2)$ be as required in Theorem 8.6 (corresponding to $F, \varepsilon/2$ and $L$). Choose a finite subset $G_2$ and a positive number $\eta < \min(\varepsilon/2, \delta_1/2)$ satisfying the following: if $H_i : A \to B$ are both $G_2$-$\eta$-multiplicative with

$$H_1 \approx_\eta H_2 \quad \text{on } G_2$$

then $[H_1]|_{\mathcal{P}_1} = [H_2]|_{\mathcal{P}_1}$.

If $u \in U(A)$, denote by $k(u)$ the smallest integer for which $[u^{k(u)}] \in U_0(A)$. Let $K = \max[k(u): u \in \mathcal{P}_1 \cap U(A) \setminus U_0(A)]$. Let $\mathcal{P}_2 = \mathcal{P}_1 \cup \{u^{k(u)}: u \in U(A) \cap \mathcal{P}_1\}$. Set $V = \{v^1, v^2, \ldots, v^K, v \in \mathcal{P}_1 \cap (U(A) \setminus U_0(A))\}$. We may assume that $G_2 \supset V \cup G_1 \cup S_1$. Set $0 < \delta_2 < \min(\delta_1/2, \eta/4, \varepsilon/4, 1/K^2512)$.

Since $A$ is a nuclear simple $C^*$-algebra with $TR(A) \leq 1$, from Lemma 5.5, there exist a $C^*$-subalgebra $F \in \mathcal{T}$ of $A$ with $p_1 = 1$ and $G_2$-$\delta_2$-multiplicative completely positive linear contraction $\phi_1 : A \to F$ such that

$$\text{id}_A(x) \approx_{\delta_2} qxq \oplus \text{diag}(\phi_1(x), \phi_1(x), \ldots, \phi_1(x)) \quad \text{for all } x \in G_2,$$

where $\phi_1$ is repeated $n$ times and $(1 - q)A(1 - q) = M_n(p_1 A p_1)$. Set $\phi_0(x) = qxq$ for $x \in A$. Now let $\mathcal{P}_0, G_0, \delta_0 > 0, \sigma > 0$ and finite subset $H \subset A_{sa}$ be as required in of 8.6(i).

By 6.8 (with perhaps larger $G_2$), we also have (in $U(q A q)$)

$$\text{cel}(\tilde{\phi}_0(u)) \leq \text{cel}(u) + \pi/128$$

for $u \in U_0(A) \cap \mathcal{P}_1$; and

$$\text{cel}(\tilde{\phi}_0(u^k)) \leq \text{cel}(u^k) + \pi/128$$

and

$$\|\tilde{\phi}_0(u^k) - \tilde{\phi}_0(u^k)\| < 1/128$$

for $u \in (U(A) \setminus U_0(A)) \cap \mathcal{P}_1$ and $[u]$ has order $k$ in $K_1(A)$.

Let $G \supset G_2 \cup \phi_0(G_2) \cup G_0$ be a finite subset and $0 < \delta < \delta_2$. Let $\mathcal{P} = \mathcal{P}_1 \cup \{q, p_1\}$. Suppose that $L_i$ are two unital $G$-$\delta$-multiplicative contractive completely positive linear maps that satisfy

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}} \quad \text{and} \quad \sup_{\tau \in T(B)} \{\|\tau \circ L_1(g) - \tau \circ L_2(g)\|\} < \sigma/2$$

for all $g \in H$.

For any (finite) $G_3 \subset G_2 \cup \phi_0(G_2)$ and $0 < \delta_3 < \delta_2/2$, by considering $(L_i)|_{q B q \oplus (1-q) B(1-q)}$ from 8.2(iii), with possibly larger $G$ and smaller $\delta$, there are $G_3$-$\delta_3$-multiplicative contractive completely positive linear maps $L_i' : A \to B$ ($i = 1, 2$) such that

$$\|L_i(g) - L_i'(g)\| < \delta_3 \quad \text{for all } g \in G_3$$

and $L_i'(q)$ is a projection ($i = 1, 2$). Furthermore, we may assume that

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}} \quad \text{and} \quad |\tau \circ L_1'(g) - \tau \circ L_2'(g)| < \sigma$$
for all \( g \in \mathcal{H} \) and \( \tau \in T(B) \). Replacing \( L_2' \) by \( \text{ad}(W) \circ L_2' \) for a suitable unitary \( W \in B \), we may assume that \( e = L_1'(q) = L_1'(\phi_0(1_A)) = L_2'(q) \).

Let \( \Lambda_i = L_i' \circ \phi_0 \). With sufficiently large \( G_3 \) and sufficiently small \( \delta_3 \) (applying 6.8), we may assume that

\[
\text{cel}(\tilde{\Lambda}_i(u)) \leq \text{cel}(u) + \pi/64,
\]

\( i = 1, 2, u \in U_0(A) \cap P_1; \)

\[
\|\tilde{\Lambda}_i(u^k) - \tilde{\Lambda}_i(u^k)\| < 1/64 \quad \text{and} \quad \text{cel}(\tilde{\Lambda}_i(u^k)) \leq \text{cel}(u^k) + \pi/64
\]

for \( u \in (U(A) \setminus U_0(A)) \cap P_1 \) but \( [u] \) has order \( k \) in \( K_1(A) \). Then, for \( u \in U_0(A) \cap P_1 \)

\[
\text{cel}(\tilde{\Lambda}_1(\phi_0(u)) \ast \tilde{\Lambda}_2(\phi_0(u))) \leq 2(\text{cel}(u) + \pi/64) \leq L(u);
\]

and, for \( u \in (U(A) \setminus U_0(A)) \cap P_1 \) with order \( k \), since

\[
\text{cel}(\tilde{\Lambda}_1(\phi_0(u^k)) \ast \tilde{\Lambda}_2(\phi_0(u^k))) < 2[\text{cel}(u^k) + \pi/64].
\]

by 6.10,

\[
\text{cel}(\tilde{\Lambda}_1(\phi_0(u)) \ast \tilde{\Lambda}_2(\phi_0(u))) \leq 8\pi + \frac{2[\text{cel}(u^k) + \pi/64]}{k} + \pi/64 < L(u).
\]

Now Theorem 8.6 provides a unitary \( U \in B \) such that

\[
\text{ad}(U) \circ L_2' \approx_{\varepsilon/2} L_1' \quad \text{on } \mathcal{F}.
\]

This implies that

\[
L_2 \sim_{\varepsilon} L_1 \quad \text{on } \mathcal{F}. \quad \square
\]

The following is a characterization of approximately inner automorphisms (for the case in which \( K_1(A) \) is torsion).

**8.8. Theorem.** Let \( A \) be a unital nuclear simple \( C^* \)-algebra with \( \text{TR}(A) \leq 1 \) and with torsion \( K_1(A) \) which satisfies the AUCT. Then an automorphism \( \alpha : A \to A \) is approximately inner if and only if \([\alpha] = [\text{id}_A]\) in \( KL(A, A) \) and \( \tau \circ \alpha(x) = \tau(x) \) for all \( x \in A \) and \( \tau \in T(A) \).

**Proof.** If \( \alpha \) is approximately inner, then it is clear that

\[
\tau \circ \alpha(x) = \tau(x)
\]

for all \( x \in A \) and \( \tau \in T(A) \). The “only if” part follows from [35, 4.5]. It is also clear that the “if part” follows from 8.7. \( \square \)
9. The existence theorems

9.1. Definition. Let $A$ and $B$ be two unital stably finite $C^*$-algebras and let $\alpha : K_0(A) \to K_0(B)$ be a positive homomorphism and $\Lambda : T(B) \to T(A)$ be a continuous affine map. We say $\Lambda$ is compatible with $\alpha$ if $\Lambda(\tau)(x) = \tau(\alpha(x))$ for all $x \in K_0(A)$, where we view $\tau$ as a state on $K_0(A)$. Let $S$ be a compact convex set. Denote by $\text{Aff}(S)$ the set of all (real) continuous affine functions on $S$. Let $\Lambda : S \to T$ be a continuous affine map from $S$ to another compact convex set $T$. We denote by $\Lambda_\circ : \text{Aff}(T) \to \text{Aff}(S)$ the unital positive linear continuous map defined by $\Lambda_\circ(f)(s) = f(\Lambda(s))$ for $f \in \text{Aff}(T)$.

A positive linear map $\xi : \text{Aff} T(A) \to \text{Aff} T(B)$ is said to be compatible with $\alpha$ if $\xi(\hat{p})(\tau) = \tau(\alpha(p))$ for all $\tau \in T(B)$ and any projection $p \in M_\infty(A)$. Let $A$ be a unital $C^*$-algebra (with at least one normalized trace). Define $Q : A_{\text{sa}} \to \text{Aff} T(A)$ by $Q(a)(\tau) = \tau(\alpha(a))$ for all $a \in A$. Then $Q$ is a unital positive linear map.

A $C^*$-algebra $A$ is said to be $KK$-attainable for a class of stably finite $C^*$-algebras $\mathcal{C}$, if for any $C^*$-algebra $B \in \mathcal{C}$, any $\alpha \in \text{Hom}_A(K_0(A), K_0(B))_{++}$ (see 8.2) and any finite subset $\mathcal{P} \subset \mathcal{P}(A)$ with $[1_A] \subset \mathcal{P}$, there exists a sequence of completely positive linear contractions $L_n : A \to B \otimes \mathcal{K}$ such that

$$\|L_n(a)L_n(b) - L_n(ab)\| \to 0 \quad \text{and} \quad \|L_n\|_\mathcal{P} = \alpha|\mathcal{P} \quad \text{for all} \ a, b \in A.$$ 

For the rest of the paper, when we say a $C^*$-algebra $A$ is $KK$-attainable, we mean that $A$ is $KK$-attainable for unital separable simple $C^*$-algebras with tracial rank no more than 1.

As in [32], if for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a $C^*$-subalgebra $A_1$ of $A$ which is $KK$-attainable such that $\mathcal{F} \subset_{\varepsilon} A_1$, then $A$ is $KK$-attainable.

A unital nuclear separable simple $C^*$-algebra $A$ with $TR(A) \leq 1$ is said to be pre-classifiable if it satisfies the Universal Coefficient Theorem and is $KK$-attainable, and, in addition to the above, for any unital separable nuclear simple $C^*$-algebra with $TR(A) \leq 1$ and any continuous affine map $A : T(B) \to T(A)$ compatible with $\alpha$,

$$\sup_{\tau \in T(B)} \left\{\|\Lambda(\tau)(a) - \tau \circ L_n(a)\|\right\} \to 0 \quad \text{for all} \ a \in A.$$ 

Or, equivalently, for any contractive positive linear map $\xi : \text{Aff} T(A) \to \text{Aff} T(B)$ compatible with $\alpha$,

$$\sup_{\tau \in T(B)} \left\{\|\xi(Q(a))(\tau) - \tau \circ L_n(a)\|\right\} \to 0 \quad \text{for all} \ a \in A_{\text{sa}}.$$ 

9.2. If $h : A \to B$ is a unital homomorphism, then $h$ induces a unital positive affine map $h_\circ : \text{Aff} T(A) \to \text{Aff} T(B)$. The map $h_\circ$ is contractive. Suppose that $Y$ is a compact metric space and $P \in M_1(C(Y))$ is a non-zero projection with constant rank. It is known and easy to see that

$$\text{Aff} T(P M_1(C(Y)) P) = \text{Aff} T(M_1(C(Y)) = C_\mathcal{R}(Y).$$ 

9.3. Theorem. Let $A$ be a simple unital $C^*$-algebra with at least one tracial state. Then for any affine function $f \in \text{Aff}(T(A))$ with $\|f\| \leq 1$ and any $\varepsilon > 0$, there exists an element $a \in A_{\text{sa}}$ with $\|a\| < \|f\| + \varepsilon$ such that $\tau(a) = f(\tau)$ for all $\tau \in T(A)$. Furthermore, if $f > 0$, we can choose $a > 0$. 
Proof. We prove this using the results in [8]. By [8, 2.7], we may identify $T(A)$ with the real part of the unit sphere of $(A^q)^*$ (see [8] for the notation). By [8, 2.8], it suffices to consider those $f \in \text{Aff}(T(A))$ with $f(\tau) > 0$ for all $\tau \in T(A)$. There is an element $b \in (A^q)^{**}$ such that $b(\tau) = f(\tau)$ for all $\tau \in T(A)$. Since $f$ is (weak-*) continuous, $b \in A^q$. Since $b(\tau) > 0$ for all $\tau \in T(A)$, by [8, 6.4], there is $c \in A_+$ and $z \in A_{sa}$ with $\tau(z) = 0$ for all $\tau \in T(A)$ (i.e., $z \in A_0$ using the notation in [8]) such that $b = c + z$. Now the theorem follows from [8, 2.9].  

9.4. Lemma. Let $A$ be a separable unital $C^*$-algebra. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. Then there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any unital separable $C^*$-algebra $C$ with at least one tracial state and any unital contractive positive linear maps $L : A \to C$ which is $\mathcal{G}$-$\delta$-multiplicative, then, for any $t \in T(C)$ there is a trace $\tau \in T(A)$ such that

$$|\tau(a) - t(L(a))| < \varepsilon \quad \text{for all } a \in \mathcal{F}. $$

Proof. Otherwise, there would be an $\varepsilon_0 > 0$ and a finite subset $\mathcal{F} \subset A$, a sequence of unital separable $C^*$-algebra $C_n$, a sequence of unital contractive positive linear map $L_n : A \to C_n$ such that

$$\lim_{n \to \infty} \left\|L_n(a)L_n(b) - L_n(ab)\right\| = 0 \quad \text{for all } a, b \in A,$$

and a sequence $t_n \in T(C_n)$ such that

$$\inf\left\{\max\left\{|t(a) - t_n(L_n(a))| : a \in \mathcal{F}\right\} : t \in T(A)\right\} \geq \varepsilon_0$$

for all $n$. Let $s_n$ be a state of $A$ which extends $t_n \circ L_n$. Let $\tau$ be a weak limit of $\{s_n\}$. So there is a subsequence $\{n_k\}$ such that $\tau(a) = \lim_{n \to \infty} s_{n_k}(a)$ for all $a \in A$. It is a routine to check that $\tau \in T(A)$. Therefore, there exists $K > 0$, such that

$$|\tau(a) - t_{n_k}(L_{n_k}(a))| < \varepsilon_0/2$$

for all $k \geq K$. We obtain a contradiction.  

9.5. Lemma. Let $A = C(X)$, where $X$ is a path connected finite CW-complex. Let $B$ be a unital separable nuclear non-elementary simple $C^*$-algebra with $\text{TR}(B) \leq 1$ and $\Lambda : T(B) \to T(A)$ be a continuous affine map. Then, for any $\sigma > 0$ and any finite subset $\mathcal{H} \subset \text{Aff} T(A)$, there exists a unital monomorphism $h : A \to B$ such that the image of $h$ is in a $C^*$-subalgebra $B_0 \in \mathcal{I}$ and

$$\left\|h_\sharp(f) - \Lambda_\sharp(f)\right\| < \varepsilon$$

for all $f \in \mathcal{H}$, where $h_\sharp, \Lambda_\sharp : \text{Aff} T(A) \to \text{Aff} T(B)$ are the maps induced by $h$ and $\Lambda$, respectively.

Moreover, if there is positive homomorphism $\alpha : K_0(A) \to K_0(B)$ with $\alpha([1_A]) = [1_B]$ and $\Lambda_\sharp$ is compatible with $\alpha$, then the above is also true for $A = \mathcal{P}\mathcal{M}_f(C(X))P$, where $P \in \mathcal{M}_f(C(X))$ is a projection in $\mathcal{M}_f(C(X))$. Furthermore, if $X$ is contractible, we can also require that $h_{|\mathcal{A}_0} = \alpha$.  

Proof. We will apply [27, 2.5]. We may assume that the identity of \( A \) is contained in \( H \). Fix \( \varepsilon > 0 \). By 9.3, for each \( f \in A_{\neq}(H) \), there is \( a_f \in B_{sa} \) with \( \|a_f\| \leq \|f\| \) such that

\[
\sup_{\tau \in T(B)} \{ |\tau(a_f) - \tau(f)| \} < \varepsilon / 32.
\]

Let \( G = \{ a_f : f \in A_{\neq}(H) \} \). Let \( N(2) \) be the number described in [27, 2.5] corresponding to \( \varepsilon / 4 \) and \( H \).

Since \( TR(B) \leq 1 \), by 4.10, for any \( \delta > 0 \) and any finite subset \( G_1 \subset A \), (we assume that \( G \subset G_1 \)), there exist a nonzero projection \( p \in B \) and a unital \( C^* \)-subalgebra \( C \in I \) with \( 1_c = p \) such that:

1. \( \|[x, p]\| < \delta \) for all \( x \in G_1 \),
2. \( pxp \in \delta C \) for all \( x \in G_1 \),
3. \( \tau(1 - p) < \varepsilon / 64 \), and
4. \( C = \bigoplus_{i=1}^{n} C_i \) with \( C_i = M_{m(i)}(C([0, 1])) \) and \( m(i) \geq N(2) \).

We have

\[
\left| \tau(pxp) - \tau(x) \right| < \varepsilon / 32 \quad \text{for all } \tau \in T(B) \text{ and for all } x \in G.
\]

Since \( B \) is nuclear, by [32, 3.2], there exists a unital completely positive contraction \( L' : pBp \to C \) such that

\[
\left\| L'(pxp) - pxp \right\| < \varepsilon / 32 \quad \text{for all } x \in G.
\]

Define \( L : B \to C \) by \( L(b) = L'(pbp) \) for \( b \in B \). With sufficiently small \( \delta \), we have

\[
\sup_{\tau \in T(B)} \left\{ \left| t \circ L(x) - \tau(x) \right| : \tau \in T(B) \text{ and } t = \tau / \tau(p) \right\} < \varepsilon / 16
\]

for all \( x \in G \). Choose an integer \( N > 0 \) so that

\[
\left| g(t') - g(t) \right| < \varepsilon / 32 \quad \text{if } |t' - t''| \leq 1/N \text{ for all } g \in A_{\neq}(H).
\]

Here we view \( \text{Aff}(C) = \bigoplus_{i=1}^{n} C_R([0, 1]) \) and \( t' \) and \( t' \) are points in the same \( [0, 1] \). Let \( 0 = t_{0,0} < t_{1,1} < \cdots < t_{i,N} = 1 \) such that \( t_{i,k} - t_{i,k+1} = 1/N \) and \( i \) indicates the \( i \)th interval. We also view them as tracial states of \( C \), i.e., \( t_{i,k}(f) = \text{tr}(f (t_{i,k})) \) for \( f \in \bigoplus_{i=1}^{n} C([0, 1], M_{m(i)}) \), where \( \text{tr} \) is the normalized trace on \( M_{m(i)} \).

By applying Lemma 9.4, with sufficiently large \( G_1 \) and small \( \delta \), we may assume that there are \( \tau_{i,k} \in T(B) \) such that

\[
\left| \tau_{i,k}(a) - t_{i,k}(L(a)) \right| < \varepsilon / 32
\]

for all \( a \in G \). Let \( \Delta \) be the convex hull of \( t_{i,k} \) in \( T(C) \). Define \( \gamma_1 : \Delta \to T(B) \) by \( \gamma_1(\sum_{i,k} \alpha_{i,k} t_{i,k}) = \sum_{i,k} \alpha_{i,k} \tau_{i,k} \) for \( \alpha_{i,k} \geq 0 \) and \( \sum_{i,k} \alpha_{i,k} = 1 \). Therefore

\[
\left| \gamma_1(t)(a) - t(L(a)) \right| < \varepsilon / 32 \quad \text{for all } a \in G.
\]
and \( t \in \Delta \). For each \( \tau \in T(C) \), one has, for \( f \in \bigoplus_{i=1}^n C([0, 1], M_{m(i)}) \), that

\[
\tau(f) = \sum_{i=1}^n \int t \cdot \text{tr}(f(t)) d\mu_i,
\]

where \( \mu_i \) is a Borel measure on \([0, 1]\). Define \( \gamma_2 : T(C) \to \Delta \) by

\[
\gamma_2(\tau)(f) = \sum_{i=1}^n \sum_{k=0}^N \alpha_{i,k} \text{tr}(f(t_{i,k}))
\]

for \( f \in C \), where \( \alpha_{i,k} = \mu_i(A_k) \) and \( A_k = [t_{i,k}, t_{i,k+1}] \), \( k = 0, 1, 2, \ldots, N \). Put \( \gamma = \gamma_1 \circ \gamma_2 \). It is clear that \( \gamma \) is an affine continuous map from \( T(C) \) into \( T(B) \). Moreover,

\[
\left| \gamma(t)(a) - t(L(a)) \right| < \varepsilon / 16
\]

for all \( f \in \mathcal{G} \) and \( t \in T(C) \).

By [27, 2.5], there is a unital homomorphism \( h_1 : A \to C \) such that

\[
\left\| (h_1)_* (g) - \gamma_2 \circ \Lambda_z(g) \right\| < \varepsilon / 16
\]

for all \( g \in \mathcal{H} \). In the following estimation, if \( \tau \in T(B) \), we denote \( \tilde{\tau} = (1/\tau(p))\tau|_C \), where \( C \) is regarded as a subalgebra of \( A \). We also note that \( \Lambda_z(g)(\gamma(\tilde{\tau})) = \gamma(\tilde{\tau})(a_f) \) for \( f = \Lambda_z(g) \) and \( g \in \text{Aff}(T(A)) \). We estimate, that, for \( g \in \mathcal{H} \), \( f = \Lambda_z(g) \), and for any \( \tau \in T(B) \),

\[
\left| (h_1)_* (g)(\tilde{\tau}) - \Lambda_z(g)(\tau) \right| \leq \left| (h_1)_* (g)(\tilde{\tau}) - \gamma_2 \circ \Lambda_z(g)(\tilde{\tau}) \right| + \left| \gamma(\tilde{\tau})(a_f) - \tilde{\tau}(L(a_f)) \right| + \left| \tilde{\tau}(L(a_f) - \Lambda_z(g)(\tau) \right|
\]

for \( g \in \mathcal{H} \). Each of the first two terms on the right-hand side of the inequality is no more than \( \varepsilon / 16 \). For the last term, we note that

\[
\Lambda_z(g)(\tau) = \tau(a_f) \quad \text{and} \quad \left| \tilde{\tau}(L(a_f)) - \tau(a_f) \right| < \varepsilon / 16
\]

for all \( g \in \mathcal{H} \). Thus we have

\[
\left| (h_1)_* (g)(\tilde{\tau}) - \Lambda_z(g)(\tau) \right| < 3\varepsilon / 16 \quad \text{for all } g \in \mathcal{H}.
\]

Since \((1 - p)B(1 - p)\) is non-elementary and simple, by [1, p. 61], there exists a positive element \( b \in (1 - p)B(1 - p) \) with \( \text{sp}(b) = [0, 1] \). From this we know that there is a unital \( C^* \)-subalgebra \( C_0 \) of \((1 - p)B(1 - p)\) such that \( C_0 \cong C([0, 1]) \). It is well known that there is a unital monomorphism \( h_2 : A \to C_0 \). Finally, we let \( h = h_1 + h_2 \). It is clear that \( h \) meets the requirements of the conclusion of the (first part of) lemma.

For the second part, we note there is an integer \( N > 0 \) and a projection \( e \in M_N(A) \) such that \( eM_N(A)e \cong C(X) \). Let \( d \in B \) be a projection such that \( \alpha([e]) = [d] \). Since \( \Lambda_z \) is compatible with \( \alpha \), \( \Lambda_z \) induces a unital positive map \( \xi : \text{Aff}_T(eAe) \to \text{Aff}_T(dBd) \). From what we have proved, we obtain a homomorphism \( h_1 : eAe \to dBd \) as required (with \( \delta \cdot \varepsilon \), where \( \delta = \inf \{ \tau(d) \) :
exists a unital monomorphism $\phi$ from $eAe \otimes M_l$ to $dAd \otimes M_1$. Since $eAe \cong C(X)$, we may assume that $P \in eAe \otimes M_l$. Let $p \in dAd \otimes M_1$ such that $p = h_2(P)$. Since $\Lambda_2$ is compatible with $\alpha$, $[p] = [1_B]$. Since $B$ has stable rank one, $p$ is unitarily equivalent to $1_B$. Therefore $p(dAd \otimes M_1) p \cong A$. The second part of the lemma follows.

For the last part of the lemma, we note that $PM_l(C(X))p = M_s(C(X))$ for some integer $0 < s < l$, since $X$ is contractible. Let $e_{11}$ be a minimal projection of $A$. Choose a projection $d \in B$.

For any finite subset $\mathcal{H}_l \subset (e_{11}Ae_{11})_{sa}$, from what we have shown, we obtain a homomorphism $h' : e_{11}Ae_{11} \to dBd$ such that

$$\|h'_{e_{11}}(f) - \Lambda_2(f)\| < \epsilon/s.$$ 

View $M_r(dBd)$ as a unital hereditary $C^*$-subalgebra of $B$. Put $h = h' \otimes \text{id}_{M_r}$. It is clear that $h$ meets the requirements of the lemma. $\square$

9.6. Corollary. Let $A \in \mathcal{I}$, $B$ be a unital separable nuclear simple $C^*$-algebra with $TR(B) \leq 1$, $\gamma : K_0(A) \to K_0(B)$ be a positive homomorphism and $\Lambda : T(B) \to T(A)$ be an affine continuous map which is compatible with $\gamma$. Then, for any $\sigma > 0$ and any finite subset $\mathcal{G} \subset A$, there exists a unital monomorphism $\phi : A \to B$ such that

$$\sup_{\tau \in T(B)} \left\{ |\tau \circ \phi(g) - \Lambda(\tau)(g)| \right\} < \sigma$$

for all $g \in \mathcal{G}$ and $\phi_\sigma = \gamma$.

Proof. Note that 9.5 holds for $A = M_n$. It is then clear that, by considering each summand of $A$, the corollary follows from 9.5. $\square$

9.7. Proposition. Every $KK$-attainable, unital separable nuclear simple $C^*$-algebra $A$ with $TR(A) \leq 1$ which satisfies the AUCT is pre-classifiable.

Proof. Let $A$ be a $KK$-attainable separable nuclear simple $C^*$-algebra with $TR(A) \leq 1$ satisfying the AUCT and $B$ be a unital nuclear separable simple $C^*$-algebra with $TR(B) \leq 1$.

Let $\alpha \in \text{Hom}_A(K(A), K(B))^{++}$, $P \subset P(A)$ be a finite subset containing $[1_A]$, and $\Lambda : T(B) \to T(A)$ be a continuous map which is comparable to $\alpha|_{K_0(A)}$. Suppose that $e \in B$ is a projection such that $\alpha(1_A) = e$. To save notation, without loss of generality, we may assume that $B = e(B \otimes K)e$.

Let $\{\delta_n\}$ be a decreasing sequence of positive numbers with $\lim_{n \to \infty} \delta_n = 0$. For each $n$, since $A$ is a unital simple $C^*$-algebra with $TR(A) \leq 1$, there are nonzero projections $p_n \in A$ and a $C^*$-subalgebra $C_n \in \mathcal{I}$ with $1_{C_n} = p_n$, and a sequence of unital completely positive linear contractions $\Phi_n : A \to C_n$ such that:

1. $\|x, p_n\| < \delta_n$,
2. $\|p_n x p_n - \Phi_n(x)\| < \delta_n$,
3. $\|x - (p_n x p_n \oplus \Phi_n(x))\| < \delta_n$ for all $x \in A$ with $\|x\| \leq 1$ and
4. $\tau(1 - p_n) < 1/2n$ for all $\tau \in T(A)$.

Denote by $\Psi_n(x) = (1 - p_n)x(1 - p_n) + \Phi_n(x)$ (for $x \in A$). Note that
\[ \| \Psi_n(ab) - \Psi_n(a)\Psi_n(b) \| \to 0 \quad \text{and} \quad \| \Phi_n(ab) - \Phi_n(a)\Phi_n(b) \| \to 0 \]

for all \( a, b \in A \) as \( n \to \infty \).

Since \( A \) is KK-attainable, for each \( n \), there exists a sequence of completely positive linear contractions \( L_n : A \to B \otimes K \) such that

\[
[\Psi_n]_P = [\text{id}]_P, \quad [L_n]_P = \alpha|_P, \quad [L_n \circ \Psi_n]_P = \alpha|_P, \quad [L_n \circ \Phi_n]_P = \alpha|_P,
\]

\[ \| L_n \circ \Psi_n(ab) - L_n \circ \Psi_n(a)L_n \circ \Psi_n(b) \| \to 0 \quad \text{and} \quad \| L_n \circ \Phi_n(ab) - L_n \circ \Phi_n(a)L_n \circ \Phi_n(b) \| \to 0 \]

as \( n \to \infty \) for all \( a, b \in A \). Suppose that \( C_n = \bigoplus_{i=1}^{t(n)} D_{n,i} \), where \( D_{n,i} \cong M_{n,i} \) or \( D_{n,i} \cong M_{(n,i)}(C([0, 1])) \). Let \( d_{n,i} = \text{id}_{D_{n,i}} \). We may also assume that, for each \( n \), \( L_n(d_{n,i}) \) is a projection (see 8.2(iii)) and

\[ [L_n](\alpha(d_{n,i})) = \alpha(\alpha(d_{n,i})) \quad \text{for all} \quad i, \]

Let \( \gamma_n : T(A) \to T(C_n) \) be defined by \( \gamma_n(\tau) = (1/\tau(p_n)) \tau|_{C_n} \). Let \( \mathcal{G}_n \) be a finite subset (containing generators) of \( C_n \) and let \( \{d_n\} \) be a decreasing sequence of positive numbers with \( \lim_{n \to \infty} d_n = 0 \). For large \( n \), by applying 9.6, we obtain a homomorphism \( h_n : C_n \to q_n B q_n \), where \( [d_{n,i}] = \alpha([d_{n,i}] \), such that

\[ |\tau \circ h_n(g) - \gamma_n \circ \Lambda(\tau)(g)| < d_n \quad \text{for all} \quad \tau \in T(B) \]

for all \( g \in \mathcal{G}_n \). Put \( \phi_n(x) = L_n \circ ((1 - p_n)x(1 - p_n)) + h_n \circ \Phi_n(x) \) for \( x \in A \). It is easy to see, by choosing a large \( n \), \( \phi_n : A \to B \) meets the requirements of Definition 9.1. \( \square \)

9.8. Lemma. Let \( A \) be a unital \( C^\ast \)-algebra, \( B \) be a unital separable simple \( C^\ast \)-algebra with \( TR(B) \leq 1 \) and \( F \in T \) be a \( C^\ast \)-subalgebra of \( B \). Let \( G \) be a subgroup generated by a finite subset of \( \text{P}(A) \). Suppose that there is an \( \mathcal{F} - \mathcal{D} \)-multiplicative contractive completely positive linear map \( \psi : A \to F \subset B \) such that \( [\psi]|_G \) is well defined. Then for any \( \varepsilon > 0 \), there exists a finite-dimensional \( C^\ast \)-subalgebra \( C \subset B \) and an \( \mathcal{F} - \mathcal{D} \)-multiplicative contractive completely positive linear map \( L : A \to C \subset B \) such that

\[ [L]|_{C \cap K_0(A, Z/kZ)} = [\psi]|_{C \cap K_0(A, Z/kZ)}, \quad \text{and} \quad \tau(1_C) < \varepsilon \]

for all tracial states \( \tau \) in \( T(B) \) and for all \( k \geq 1 \) so that \( G \cap K_0(A, Z/kZ) \neq \{0\} \), where \( L \) and \( \psi \) are viewed as maps to \( B \). Furthermore, if \( [\psi]|_{C \cap K_0(A)} \) is positive, so is \( [L]|_{C \cap K_0(A)} \).

Proof. This is a minor modification of the proof of [32, 4.2]. Let \( 0 < \varepsilon < 1 \). Without loss of generality, we may assume that \( F = C([0, 1]) \otimes M_n \). Let \( q_1 \in F \) be a minimal projection. Suppose that

\[ G \cap K_0(A, Z/kZ) = \{0\} \quad \text{for} \quad k > K. \]

By 5.5, with \( m = 2|K| ! + 1 \) and \( 1/l < \varepsilon/n \), we may write \( q_1 = q + \sum_{i=1}^{m} p_i \), where \( [q] \leq [p_1] \), \( q, p_1, \ldots, p_m \) are mutually orthogonal projections, \( [p_1] = [p_i] \), \( i = 1, 2, \ldots, m \) and \( \tau(p_1) < \varepsilon/n \).
1/2l < ε/2n. Set \( e_1 = q + p_1 \) and \( q_0 = \sum_{j=2}^{2l+1} p_j \). Then \([e_1] + K!\{q_0\} = [q_1]\) in \( K_0(B)\) and 
\( \tau(e_1) < \epsilon/n \) for all tracial states \( \tau \) on \( B \). From this we obtain a \( C^*\)-subalgebra \( C \) of \( B \) such that 
\( C \cong M_n \) and its minimal projection is equivalent to \( e_1 \). In particular, \( \tau(1_C) < \epsilon \). Let \( \phi : F \to M_n \to C \) be a unital homomorphism, where the map \( F \to M_n \) is a point-evaluation. Let 
\[ L = \phi \circ \psi, \quad j : F \to B \text{ and } j_2 : C \to B \text{ be embeddings. By the choice of } q_1, \quad [e_1] \text{ and } [q_1] \]
have the same image in \( K_0(B)/kK_0(B) \) for \( k = 1, 2, \ldots, \). Therefore \((j_1)_* = (j_2 \circ \phi)_*\) on 
\( K_0(F, Z/kZ) \) for all \( k \leq K \). Since \( K_1(F) = K_1(C) = 0 \), by the six-term exact sequence in 
8.2 (see [32, 1.6]), both \([L]\) and \([\psi]\) map \( K_0(A, Z/kZ) \to K_0(B)/kK_0(B) \) and factor through 
\( K_0(F, Z/kZ) \). Therefore 
\[ [L]_{G \cap K_0(A, Z/kZ)} = [\psi]_{G \cap K_0(A, Z/kZ)}, \quad k = 1, 2, \ldots, \.) \]
The general case in which \( F \) is a direct sum of \( M_1(C([0, 1])) \) follows immediately. \( \square \)

9.9. Lemma. Let \( C = \bigoplus_{j=1}^n C_j \), where each \( C_j = P_j M_{s(j)}(C(X_j)) P_j \). \( P_j \) is a projection in 
\( M_{s(j)}(C(X_j)) \) and \( X_j \) is a path connected compact metric space with finitely generated \( K_i(C_j) \), 
\( K_0(C(X_j)) = Z \oplus \text{tor}(K_0(C_j)) \), \( K_1(C(X_j)) \) and \( K_0(C_j) \) \( \subset \{ (z, x) : z \in N, \text{ or } (z, x) = (0, 0) \} \). Then \( C \) is \( KK\)-attainable.

Proof. Clearly, by considering each summand separately, we may assume that \( C \) has only one summand. It is also clear that one can reduce the general case to the case in which \( C = M_k(C(X)) \) and 
\( K_i(C) \) satisfies the condition described in the lemma.

Let \( A \) be a unital simple \( C^*\)-algebra with \( TR(A) \leq 1 \) and \( \alpha \in KL(C, A)^{++} \). Let \( \mathcal{P} \subset \mathcal{P}(A) \) be a finite subset and \( G \) be the subgroup generated by \( \mathcal{P} \). By [11], for any finite subset \( G \subset C \) and 
\( \eta > 0 \), there exists a \( G\)-\( \eta/2\)-multiplicative contractive completely positive linear map \( \psi : C \to M_N(B) \) for some large integer \( N \) such that 
\[ [\psi]_G = \alpha|_G + h|_G \]
for some one-point evaluation (at \( \zeta \)) \( h : C \to M_N(B) \). Since \( TR(M_N(B)) \leq 1 \), for any \( \sigma_1 > 0 \) and \( \epsilon/N > \sigma_2 > 0 \), there exist a projection \( p \in M_N(B) \) and a unital \( C^*\)-subalgebra \( F \subset M_N(B) \) with 
\( F \in \mathcal{I} \) and with \( 1_F = p \) such that:

1. there are \( G\)-\( \eta\)-multiplicative contractive completely positive linear maps \( L_1 : C \to F \) and 
\( L_2 : C \to (1 - p)M_N(B)(1 - p) \) such that
\[ \| \psi(x) - L_1(x) \oplus L_2(x) \| < \sigma_1 \]
for all \( x \in G \), and
2. \( \tau(1 - p) < \sigma_2 \) for all \( \tau \in T(M_N(B)) \).

With sufficiently small \( \sigma_1 \), we may assume that 
\[ [\psi]_G = [L_1]_G + [L_2]_G. \]
Suppose that
\[ G \cap K_0(C, \mathbb{Z}/k\mathbb{Z}) = \{0\} \text{ for } k > K. \]

By Lemma 9.8, there exist a projection \( e \leq p \) with \( \tau(e) < \sigma_2 \) for all \( \tau \in T(M_N(B)) \) and a unital \( G \)-\( \eta \)-multiplicative contractive completely positive linear map \( L'_1 : C \to F_1 \), where \( F_1 \) is a \( C^* \)-subalgebra of \( pM_N(B)p \) with \( 1_{F_1} = e \) such that \( \dim F_1 < \infty \) and

\[
\left[ L'_1 \right]|_{G \cap K_0(C, \mathbb{Z}/k\mathbb{Z})} = [L_1]|_{G \cap K_0(C, \mathbb{Z}/k\mathbb{Z})}
\]

for all \( k \) so that \( G \cap K_0(C, \mathbb{Z}/k\mathbb{Z}) \neq \{0\} \). So, in particular,

\[
\left( \left[ L'_1 \right] + [L_2] \right)|_{G \cap K_1(C)} = 0, \quad \left( \left[ L'_1 \right] + [h_0] \right)|_{G \cap K_1(C, \mathbb{Z}/k\mathbb{Z})} = 0 \quad \text{and} \quad \left( \left[ L'_1 \right] + [h_0] \right)|_{G \cap \text{tor}(K_0(C))} = 0.
\]

Hence we compute that

\[
[\Psi]|_{G \cap K_i(C)} = \alpha|_{G \cap K_i(C)}, \quad i = 0, 1, \quad \text{and} \quad [\Psi]|_{G \cap K_1(C, \mathbb{Z}/k\mathbb{Z})} = \alpha|_{G \cap K_0(C, \mathbb{Z}/k\mathbb{Z})}.
\]

The proof is complete if we can show, in addition, that \([\Psi]|_{G \cap K_0(C, \mathbb{Z}/k\mathbb{Z})} = \alpha|_{G \cap K_0(C, \mathbb{Z}/k\mathbb{Z})}\). We note that there is an (unnatural) splitting short exact sequence

\[
0 \to K_0(C)/kK_0(C) \to K_0(C, \mathbb{Z}/k\mathbb{Z}) \to \text{Tor}(K_1(C, \mathbb{Z}/k\mathbb{Z})) \to 0.
\]

From \([\Psi]|_{G \cap K_0(C)} = \alpha|_{G \cap K_0(C)}\), we conclude that

\[
[\Psi]|_{G \cap K_0(C)/kK_0(C)} = \alpha|_{G \cap K_0(C)/kK_0(C)}.
\]

On the other hand, it is easy to compute that

\[
K_0(C, \mathbb{Z}/k\mathbb{Z}) = \mathbb{Z}/k\mathbb{Z} \oplus K_0(C_0(Y), \mathbb{Z}/k\mathbb{Z}),
\]

where \( Y \) is the locally compact space formed by taking the point \( \xi \) away from \( X \) and the summand \( \mathbb{Z}/k\mathbb{Z} \subset K_0(C)/kK_0(C) \). Since both \( h \) and \( h_0 \) are point-evaluation at \( \xi \), we have

\[
[h]|_{K_0(C_0(Y), \mathbb{Z}/k\mathbb{Z})} = [h_0]|_{K_0(C_0(Y), \mathbb{Z}/k\mathbb{Z})} = 0.
\]

It follows that
\[[\Psi]G \cap K_0(\mathbb{C}\mathbb{Z}/k\mathbb{Z}) = \alpha|G \cap K_0(\mathbb{C}\mathbb{Z}/k\mathbb{Z})].\]

This proves the lemma. □

Let \( X \) be a connected compact metric space and let \( A = C(X) \). Denote by \( \rho_A : K_0(A) \to \mathbb{Z} \) the rank map. In 9.9, we assume that \( \ker \rho_{C(X_j)} \) is a finite group. One should note that the proof of 9.9 does not work when \( \ker \rho_{C(X_j)} \) contains an infinite cyclic subgroup. This happens because \( L_1' \) would not kill \( \rho_{C(X_j)} \). In 9.10 below, we apply a result of L. Li and a result of Villadsen to avoid this problem.

Recall a \( C^* \)-algebra \( A \) is said to be locally AH if for any \( \epsilon > 0 \) and any finite subset \( \mathcal{F} \subset A \), there exists a \( C^* \)-subalgebra \( B \subset A \) with \( B = PM_l(C(X))P \) for some compact metric space \( X \) and where \( P \) is a projection in \( M_l(C(X)) \), such that

\[ \text{dist}(x, B) < \epsilon \quad \text{for all} \ x \in \mathcal{F}. \]

9.10. Proposition. Let \( A \) be a separable unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \). If \( A \) is locally AH, then \( A \) is pre-classifiable.

Proof. It follows from [39] that \( A \) satisfies the AUCT. We may assume that \( A = \bigcup_{n=1}^{\infty} A_n \), where each \( A_n \) is a finite direct sums of \( P_{n,i}M_{r(n,i)}C(X_{n,i})P_{n,i} \) and \( X_{n,i} \) is a path connected finite CW complex. One may assume that \( \lambda_{A_n} = 1_A \). Put \( j_n : A_n \to A \) the embedding. Consider \( j_n \times \alpha \). If \( A_n \) has only one summand, then \( K_0(A_n) = \mathbb{Z} \oplus \ker \rho_{A_n} \). Since \( \alpha \in KL(A, B)^{++} \), \( j_n \times \alpha \in KL(A_n, B)^{++} \). By considering each summand separately, we may assume \( A_n \) has only one summand. Since \( A \) is simple, by 9.7 and 9.1, it suffices to show that, \( A = C(X) \) is \( KK \)-attainable for every path connected finite CW complex \( X \).

Let \( \alpha \in KK(A, B)^{++} \). Suppose that \( \alpha(1_A) = [p] \neq 0 \), where \( p \in M_l(B) \) is a projection. Fix a unital nuclear simple \( C^* \)-algebra \( B \) with \( TR(B) \leq 1 \). By [57], there is a unital simple \( C^* \)-algebra \( C \) which is direct limit of \( C^* \)-algebras in 9.9 such that

\[ (K_0(C), K_0(C)_+, [1_C], K_1(C)) = (K_0(B), K_0(B)_+, [1_A], K_1(B)). \]

By [52], there exists \( \beta \in KK(C, B) \) which gives the above isomorphism.

Let \( \alpha \in KL(A, B)^{++} \) and \( \gamma = \alpha \times \beta^{-1} \in KL(A, C)^+ \). Since \( K_l(C(X)) \) is finitely generated, \( KL(A, C) = KK(A, C) \). In particular, \( \gamma(K_0(A)_+ \setminus \{0\}) \subset K_0(C)_+ \setminus \{0\} \). By [28], there is a homomorphism \( h : A \to PM_l(C)p \) such that \( [h] = \gamma \). But by 9.9, since each \( C^* \)-algebra described in 9.9 is \( KK \)-attainable, \( C \) is \( KK \)-attainable (see 9.1). Let \( \epsilon > 0 \) and fix finite subsets \( \mathcal{F} \subset A \) and \( \mathcal{P} \subset \mathcal{P}(A) \). Let \( \mathcal{G} = h(\mathcal{F}) \subset \mathcal{C} \) and \( \mathcal{Q} = [h](\mathcal{P}) \subset \mathcal{P}(C) \). Let \( A : C \to B \) be a \( G-\epsilon \)-multiplicative contractive completely positive linear map such that

\[ [A]|_{\mathcal{Q}} = \beta|_{\mathcal{Q}}. \]

Define \( L = A \circ h \). Then \( L : A \to B \) is a \( \mathcal{F}-\epsilon \)-multiplicative contractive completely positive linear map such that

\[ [L]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}. \]

So \( A \) is \( KK \)-attainable. □
9.11. Lemma. Let $A$ be a unital separable C*-algebra, $\{F_k\}$ be an increasing sequence of finite subsets of the unit ball of $A$ such that $\bigcup_k F_k$ is dense in the unit ball of $A$, and let $\Phi_n : A \to A$ be a sequence of unital contractive completely positive linear maps such that $\lim_{n \to \infty} \|\Phi_n(a)\| = \|a\|$ for all $a \in A$ and

$$\sum_{k=n}^\infty \|\Phi_k(ab) - \Phi_k(a)\Phi_k(b)\| < \sum_{k=n}^\infty \delta_n,$$

for all $a, b \in G_n$, and for any finite subset $\mathcal{P} \subset \mathcal{P}(A)$, $[\Phi_n]\|\mathcal{P} = [\id]\|\mathcal{P}$ for all sufficiently large $n$, where $G_1 = \mathcal{F}_1$, $G_{n+1} \supset \bigcup_{k=1}^n \Phi_k(F_n) \cup F_n \cup \Phi_n(G_n)$, $n = 1, 2, \ldots$, and where $\sum_{n=1}^\infty \delta_n < \infty$. Let $B = \lim_{n \to \infty}(A, \Phi_n)$ be the generalized inductive limit in the sense of [4]). Then $\{\Phi_n\}$ induces an isomorphism

$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Proof. The proof is standard. We sketch here. Write $K_1(A) = \bigcup_{n=1}^\infty G_n^{(i)}$, where each $G_n^{(i)}$ is a finitely generated subgroup of $K_0(A)$. Let $\Phi_{n,m} = \Phi_{n+m} \circ \Phi_{n+m-1} \circ \cdots \circ \Phi_n$ and $\Psi_n : A \to B$ be the map induced by the inductive system which maps the $n$th $A$ to $B$. For each $G_n^{(i)}$, we may assume that $[\Psi_m]|_{G_n^{(i)}}$ is well defined for all $m \geq n$. The assumption that $[\Phi_n]|_{\mathcal{P}} = [\id]|_{\mathcal{P}}$ for all sufficiently large $n$ implies that $[\Psi_m]|_{G_n^{(i)}} = [\Psi_m']|_{G_n^{(i)}}$ for all $m, m' \geq n$. This gives a homomorphism $\beta_i : K_1(A) \to K_1(B)$ ($i = 0, 1$).

Suppose that $p_1, p_2, v \in M_l(B)$ such that $v^* v = p_1$ and $v v^* = p_2$. There is a sequence $\{v_{n,k}(a_k)\}$, where $a_k \in M_l(A)$, such that it converges to $v$. Since $v^* v = p_1$, we have $\Psi_{n,k}(a_k^* a_k) \to p_1$ and $\Psi_{n,k}(a_k a_k^*) \to p_2$. Therefore we may assume that

$$\|\Psi_{n,k}(a_k^* a_k - (a_k^* a_k)^2)\| < 1/2^{k+1} \quad \text{and} \quad \|\Psi_{n,k}(a_k^* a_k) - p_1\| < 1/2^{k+1}.$$

Since $\|\Psi_m(x)\| = \lim sup\|\Phi_{m,n}(x)\|$ for all $x \in A$ and $m \geq 1$, by passing to a subsequence and possibly replacing $a_k$ by $\Phi_{n,k,m,k}(a_k)$, $\Psi_{n,k}$ by $\Psi_{m,k}$, if necessary, we may assume that

$$\|a_k^* a_k - (a_k^* a_k)^2\| < 1/2^k, \quad k = 1, 2, \ldots.$$

It is standard that there is a partial isometry $v_k$ and a projection $q_k \in A$ such that

$$v_k^* v_k = q_k \quad \text{and} \quad \|v_k - a_k\| < 1/2^{k-1}$$

for all large $k$. Let $q_k' = v_k v_k^*$. Note also, for any $\epsilon > 0$, we have

$$\|\Psi_{n,k}(q_k) - p_1\| < \epsilon \quad \text{and} \quad \|\Psi_{n,k}(q_k') - p_2\| < \epsilon$$

for all large $k$. Hence $[\Psi_{n,k}](q_k) = [p_1]$ and $[\Psi_{n,k}](q_k') = [p_2]$. This, in particular, implies that $[p_1]$ is in the image of $\beta_0$. It follows that $\beta_0$ is surjective. Note also that $[q_k] = [q_k']$ in $K_0(A)$. It follows that $\beta_0$ is also injective. It is also easy to check from the definition that $\beta_0$ preserves the order.
In the above, if we let \( w^*w = p_1 \) and \( ww^* \leq p_2 \), then exactly the same argument shows that there are partial isometries \( v_k \in A \) such that \( v_k v_k^* = q_k \), \( v_k v_k^* \leq q_k' \) and \( \Psi_{n_k}(v_k) \to v \), \( \Psi_{n_k}(q_k) \to p_1 \) and \( \Psi_{n_k}(q_k') \to vv^* \leq p_2 \). These imply that \( \beta_0 \) is an order isomorphism.

A similar argument shows that \( \beta_1 \) is an isomorphism and \( K_1(A) = K_1(B) \).  

9.12. Theorem. Let \( A \) be a unital separable nuclear simple \( C^* \)-algebra with \( TR(A) \leq 1 \) satisfying the AUCT. Then there exists a unital separable nuclear simple \( C^* \)-algebra \( B \) with \( TR(B) = 0 \) satisfying AUCT and the following:

(1) \( (K_0(A), K_0(A)_+, [1_A], K_1(A)) = (K_0(B), K_0(B)_+, [1_B], K_1(B)) \),

(2) there exists a sequence of contractive completely positive linear maps \( \Phi_n : A \to B \) such that:

(i) \( \lim_{n \to \infty} \| \Phi_n(ab) - \Phi_n(a)\Phi_n(b) \| = 0 \) for \( a, b \in A \),

(ii) For each finite subset \( P \subset \mathcal{P}(A) \) there exists an integer \( N > 0 \) such that

\[ [\Phi_n]_P = [\alpha]_P \]

for all \( n \geq N \), where \( \alpha \in KL(A, B) \) which gives an identification in (1) above.

Proof. Let \( \{ \mathcal{P}_n \} \) be an increasing sequence of finite subsets of \( \mathcal{P}(A) \) such that the union is dense in \( \mathcal{P}(A) \). In particular the union of the subgroups generated by the images of \( \mathcal{P}_n \) in \( K(A) \) is \( K(A) \). Let \( \{ \mathcal{F}_n \} \) be an increasing sequence of finite subsets of the unit ball of \( A \) whose union is dense in the unit ball of \( A \). Let \( \{ \delta_n \} \) be a decreasing sequence of positive numbers such that \( \sum_{n=1}^{\infty} \delta_n < \infty \).

Without loss of generality, we may assume that any \( \mathcal{F}_n^{-\delta_n} \)-multiplicative contractive completely positive linear map \( L_n \) defined on \( A \) well defines \( [L_n]_{\mathcal{P}_n} \). Furthermore we may assume that \( (\mathcal{F}_n, \mathcal{F}_n, \mathcal{P}_n, \delta_n) \) (for some finite subsets \( \mathcal{G}_n \) and \( \mathcal{P}_n \)) forms a 5-tuple as defined in 6.8 of [39] for \( l = 1, b \geq \pi \) and \( M = 1 \). We may also assume that \( \mathcal{F}_n \subset \mathcal{G}_n \).

Let \( G'_1 \) be the union of \( G_1 \) and \( \{ (a + a^*)_+, (a + a^*)_-, (a - a^*)_+, (a - a^*)_- : a \in \mathcal{F}_1 \} \). Since \( A \) is simple, if \( b \in (G'_1)_+ \) is a nonzero element, there are \( x_i(b) \in A \) such that

\[ \sum_{i=1}^{n(b)} x_i(b)^* b x_i(b) = 1_A. \]

Let \( G''_1 \) be the union of \( G'_1 \) and \( \{ x_i(b), x_i(b)^*: b \in (G'_1)_+ \} \). Since \( TR(A) \leq 1 \), there exists a \( C^* \)-subalgebra \( C_1 \in \mathcal{I} \) with \( 1_{C_1} = p_1 \) such that:

(i) \( \| p_1 a - a p_1 \| < \delta_1/4 \) for all \( a \in G''_1 \),

(ii) \( p_1 a p_1 \in \delta_1/4 C_1 \) and \( \| (1 - p_1)a(1 - p_1) \| \geq (1 - \delta_1/4)\|a\| \) for all \( a \in G''_1 \) (see 5.6) and

(iii) \( \tau(1 - p_1) < \delta_1/4 \) for all \( \tau \in T(A) \).

We may also assume that:

(iv) \( \| \sum_{i=1}^{n(b)} (1 - p_1) x_i(b)^* (1 - p_1) b (1 - p_1) x_i(b)(1 - p_1) - (1 - p_1) \| < 1/8 \) and there are \( z_i(b), b' \in C_1 \) such that \( \| b - b' \| < \delta_1/2 \) and \( \| \sum_{i=1}^{n(b)} z_i(b)^* b' z_i(b)^* - p_1 \| < 1/8 \) for all \( b \in (G'_1)_+ \).
Write $C_1 = M_{d(1,1)}(C([0,1])) \oplus \cdots \oplus M_{d(s_1,1)}(C([0,1])) \oplus M_{d(s_1 + 1,1)} \oplus \cdots \oplus M_{d(t_1,1)}$. Put $F_1 = M_{d(1,1)} \oplus \cdots \oplus M_{d(t_1,1)}$. Note that dim $F_1 < \infty$. Define $\pi_1 : C_1 \to F_1$ to be a surjective point-evaluation map. We identify $F_1$ with the unital $C^*$-subalgebra of $C_1$ (scalar matrices). So $F_1 \subset C_1 \subset p_1 Ap_1$. Note $K(C_1) = K(F_1)$ and $[\pi_1]$ gives the identification. Since $A$ is nuclear, there is a contractive completely positive linear map $L_1' : p_1 Ap_1 \to C_1$ such that

$$\|L_1'(p_1 ap_1) - p_1 ap_1\| < \delta_1/4 \quad \text{for all } a \in G_1''.$$

Define $L_1(a) = \pi_1 \circ L_1'(p_1 ap_1)$ for $a \in A$. Then $L_1$ is $G_1''$-$\delta_1/2$-multiplicative. Define $\phi_1 : A \to A$ by $\phi_1(a) = (1 - p_1)a(1 - p_1) \oplus L_1(a)$ for $a \in A$. It is clear that $\phi_1$ is $F_1$-$\delta_1/2$-multiplicative. Furthermore,

$$[\phi_1]|_{P_1} = [id_A]|_{P_1}, \quad \|\phi_1(a)\| \geq (1 - \delta_1/2)\|a\| \quad \text{for } a \in F_1 \quad \text{and} \quad \left\| \sum_{i=1}^{n(b)} \phi_1(x_i(b))^* \phi_1(b) \phi_1(x_i(b)) - 1_A \right\| < 1/4.$$

Let $G_2'$ be the union of $G_2$, $\phi_1(G_2'')$ and $\{a + a^*\_+, (a + a^*)_\_-, (a - a^*)_+, (a - a^*)_\_-, \in F_2 \cup \phi_1(G_2'')\}$. Since $A$ is simple, for each nonzero positive element $b \in G_2'$, there are $x_1(b), \ldots, x_n(b) \in A$ such that

$$\sum_{i=1}^{n(b)} x_i(b)^* bx_i(b) = 1_A.$$

Now let $G_2''$ be a finite subset of $A$ containing $G_2'$, $\{1 - p_1, p_1\}$, a generating set of $F_1$ and $\{x_i(b), x_i(b)^* : b \in (G_2')_+\}. Let Q_2 be the finite subset which is the union of $P_2$, $1 - p_1, p_1$ and contains at least one minimal projection of each summand of $F_1$. By choosing a possibly larger $G_2''$ and smaller $\delta_2$ we may assume that any $G_2''$-$\delta_2$-multiplicative contractive completely positive linear map $L$ well defines $[L]|_{Q_2}$. Since $TR(A) \leq 1$, there is a $C^*$-subalgebra $C_2 \in \mathcal{I}$ with $1_{C_2} = p_2$ such that:

(i) $\|p_2 a - p_2\| < \delta_2/4$ for all $a \in G_2''$,

(ii) $p_2 a p_2 \in \beta_{\delta_2/4} C_2$ and $\|(1 - p_2)a(1 - p_2)\| \geq (1 - \delta_2/4)\|a\|$ for all $a \in G_2''$ (see 5.6),

(iii) $\tau(1 - p_2) < \delta_2/4$ for all $\tau \in T(A)$, and

(iv) $\|\sum_{i=1}^{n(b)} (1 - p_2)x_i(b)^*(1 - p_2)bx_i(b)(1 - p_2) - (1 - p_2)\| < 1/32$ and there are $z_i(b), b' \in C_2$ such that $\|b - b'\| < \delta_1/2$ and $\|\sum_{i=1}^{n(b)} z_i(b)^* b' z_i(b) - p_2\| < 1/32$ for all $b \in (G_2')_+$.

Write $C_2 = M_{d(1,2)}(C([0,1])) \oplus \cdots \oplus M_{d(s_2,2)}(C([0,1])) \oplus M_{d(s_2 + 1,2)} \oplus \cdots \oplus M_{d(t_2,2)}$. Put $F_2 = M_{d(1,2)} \oplus \cdots \oplus M_{d(t_2,2)}$. Note that dim $F_2 < \infty$. Define $\pi_2 : C_2 \to F_2$ to be a surjective point-evaluation map. We identify $F_2$ with the unital $C^*$-subalgebra of $C_2$ (scalar matrices). So $F_2 \subset C_2 \subset p_2 Ap_2$. Note that $K(C_2) = K(F_2)$ and $[\pi_2]$ gives the identification. Since $A$ is nuclear, there is a contractive completely positive linear map $L_2' : p_2 Ap_2 \to C_2$ such that

$$\|L_2'(p_2 ap_2) - p_2 ap_2\| < \delta_1/4 \quad \text{for all } a \in G_2''.$$
Define \( L_2(a) = \pi_2 \circ L'_2(p_2 ap_2) \) for \( a \in A \). Then \( L_2 \) is \( \mathcal{F}_2 - \delta_2 / 2 \)-multiplicative. Define \( \phi_2 : A \to A \) by \( \phi_2(a) = (1 - p_2)a(1 - p_2) \oplus L_2(a) \) for \( a \in A \). It is clear that \( \phi_2 \) is \( \mathcal{F}_2 - \delta_2 / 2 \)-multiplicative. Furthermore,

\[
[\phi_2]|_{\mathcal{Q}_2} = [\text{id}_A]|_{\mathcal{Q}_2}, \quad \|\phi_2(a)\| \geq (1 - \delta_2 / 4)\|a\| \quad (a \in \mathcal{G}'_2) \quad \text{and}
\]

\[
\left\| \sum_{j=1}^{n} (b)\phi_2(x_j(b))\phi_2(b)\phi_2(x_j(b)) - 1_{A} \right\| < 1 / 16.
\]

Since \( \dim F_1 < \infty \), we may also assume that there exists an injective homomorphism \( h_2 : F_1 \to A \) such that

\[
\| h_2 - (L_2)|_{F_1} \| < 1 / 4
\]

(see for example [37, 2.3]). We continue the construction of \( L_n, \pi_n \) and \( \phi_n \) in this fashion.

Let \( \mathcal{G}'\n+1 \) be the union of \( \mathcal{G}_{n+1}, \phi_1(\mathcal{G}'_n), \ldots, \phi_n(\mathcal{G}'_n), \) and \( \{(a + a^\ast)_+, (a - a^\ast)_- : a \in \mathcal{F}_n \cup \bigcup_{k=1}^{n} \phi_k(\mathcal{G}'_n)\} \). Since \( A \) is simple, for each nonzero positive element \( a \in \mathcal{G}'\n+1 \), there are \( x_1(a), \ldots, x_{n}(a) \in \mathcal{A} \) such that \( \sum_{i=1}^{n(a)} x_i(a)^{\ast} x_i(a) = 1_{A} \).

Now let \( \mathcal{G}''_n \) be a finite subset of \( A \) containing \( \mathcal{G}'\n+1, \{1 - p_n, p_n\} \), a generating set of \( F_n \) and \( \{x_i(b), x_i(b)^{\ast} : b \in (\mathcal{G}'\n+1)^{\ast}\} \). Let \( \mathcal{Q}_{n+1} \) be the finite subset which is the union of \( \mathcal{P}_n, 1 - p_n, p_n, 1 \) and at least one minimal projection of each summand of \( F_n \). By choosing a possibly larger \( \mathcal{G}''_n \) and smaller \( \delta_n \), we may assume that any \( \mathcal{G}''_n - \delta_n \)-multiplicative contractive completely positive linear map \( \mathcal{L} \) well defines \( [\mathcal{L}]|\mathcal{Q}_n \).

Since \( TR(A) \leq 1 \), there is a \( \mathcal{C}^{\ast} \)-subalgebra \( C_{n+1} \in \mathcal{I} \) with \( 1_{C_{n+1}} = p_{n+1} \) such that:

\[
\begin{align*}
(i_{n+1}) & \quad \| p_{n+1}a - ap_{n+1} \| < \delta_{n+1} / 4 \quad \text{for all } a \in \mathcal{G}'_n, \\
(ii_{n+1}) & \quad p_{n+1}a p_{n+1} \in \mathcal{E}_{n+1} \quad \text{and } \|(1 - p_{n+1})a(1 - p_{n+1})\| \geq (1 - \delta_{n+1} / 4)\|a\| \quad \text{for all } a \in \mathcal{G}'_n, \\
(iii_{n+1}) & \quad \tau (1 - p_{n+1}) < \delta_n / 4 \quad \text{for all } \tau \in (T(A) \text{ and } \\
(iv_{n+1}) & \quad \| \sum_{i=1}^{n(b)} (1 - p_{n+1})x_i(b)^{\ast} (1 - p_{n+1})b(1 - p_{n+1})x_i(b)(1 - p_{n+1}) - (1 - p_{n+1}) \| < 1 / 2^{n+3} \quad \text{and there are } z_i(b), b' \in \mathcal{C}_2 \text{ such that }
\]

\[
\| b - b' \| < \delta_{n+1} / 2 \quad \text{and } \left\| \sum_{i=1}^{n(b)} z_i(b)^{\ast} b' z_i(b)^{\ast} - p_{n+1} \right\| < 1 / 2^{n+3} \quad \text{for all } b \in (\mathcal{G}'_{n+1})^{\ast}.
\]

Write \( C_{n+1} = M_{d(1, n+1)}(C([0, 1])) \oplus \cdots \oplus M_{d(n_{1}, n+1)}(C([0, 1])) \oplus M_{d(n_{1} + 1, n+1)}(C([0, 1])) \oplus \cdots \oplus M_{d(n_{2}, n+1)} \). Put \( F_{n+1} = M_{d(1, n+1)}(1) \oplus \cdots \oplus M_{d(n_{1}, n+1)}(1) \). Note that \( \dim F_{n+1} < \infty \). Define \( \pi_{n+1} : C_{n+1} \to F_{n+1} \) to be a surjective point-evaluation map. We identify \( F_{n+1} \) with the unital \( \mathcal{C}^{\ast} \)-subalgebra of \( C_{n+1} \) (scalar matrices). So \( F_{n+1} \subset C_{n+1} \subset p_{n+1} A p_{n+1} \). Note that \( K(C_{n+1}) = K(F_{n+1}) \) and \( [\pi_{n+1}] \) gives the identification. Since \( A \) is nuclear, there is a contractive completely positive linear map \( L'_{n+1} : p_{n+1} A p_{n+1} \to C_{n+1} \) such that

\[
\| L'_{n+1}(p_{n+1} a p_{n+1}) - p_{n+1} a p_{n+1} \| < \delta_n / 4 \quad \text{for all } a \in F_{n+1}.
\]
Define $L_{n+1}(a) = \pi_{n+1} \circ L'_{n+1}(p_{n+1} a p_{n+1}), a \in A$. Then $L_{n+1}$ is $\mathcal{F}_{n+1} - \mathcal{F}_{n+1}/2$-multiplicative. Define $\phi_{n+1} : A \to A$ by $\phi_{n+1}(a) = (1 - p_{n+1}) a (1 - p_{n+1}) \oplus L_{n+1}(a)$. Then $\phi_{n+1}$ is $\mathcal{G}''_{n+1} - \mathcal{G}''_{n+1}$-multiplicative. Furthermore,

$$\phi_{n+1}|_{Q_{n+1}} = [\text{id}_A]|_{Q_{n+1}}, \quad \left\| \phi_{n+1}(a) \right\| \geq (1 - \delta_{n+1}/2) \left\| a \right\| \quad \text{for all } a \in \mathcal{G}''_{n+1}$$

and

$$\left\| \sum_{j=1}^{n(b)} \phi_{n+1}(x_j(b)) \phi_{n+1}(b) \phi_{n+1}(x_j(b)) - 1_A \right\| < 1/2^{n+1}.$$  

Again, we may also assume that there exists an injective homomorphism $h_{n+1} : F_n \to A$ such that

$$\left\| h_{n+1} - (L_{n+1})|_{F_n} \right\| < 1/2^{n+1}.$$  

We then define $B = \lim_n (A, \phi_n)$. This is a generalized inductive limit in the sense of [4] (but $\{\phi_n\}$ is in fact asymptotically multiplicative). Note that $B$ is a unital separable $C^*$-algebra. By [4, 5.13], $B$ is nuclear. From $(i, v_{n+1})$ and the construction above, it is easy to check that $B$ is simple. Since for each $n$

$$\phi_{n+1}|_{Q_{n+1}} = [\text{id}_A]|_{Q_{n+1}},$$

by 9.11, we have that

$$(K_0(B), K_0(B)_+ + [1_B], K_1(B)) = (K_1(A), K_1(A)_+ + [1_A], K_1(A)).$$

From the construction it is also standard to show that $TR(B) = 0$ (see for example the proof of [31, 4.3] and also [37]). Let $\phi_{n,\infty} : A \to B$ be the map from the $n$th $A$ to $B$ induced by the inductive limit system. Set also $\Phi_n = \phi_{n,\infty}$. It is clear $\{\Phi_n\}$ satisfies (i) and (ii). To see that $B$ satisfies AUCT, we note that $B$ satisfies the property (P) described in [39, 6.8]. It follows from [39, 6.13 and 6.16] that $B$ also satisfies the AUCT. □

10. The classification theorem

To establish the classification of separable nuclear simple $C^*$-algebras with tracial rank no more than one, we first present, for each Elliott invariant satisfying the abstract conditions described in 10.1 below, a concrete AH-algebra (with special form) with no dimension growth realizing the given Elliott invariant (see Theorem 10.1). We then show that a separable simple nuclear $C^*$-algebra $A$ with $TR(A) \leq 1$ has an Elliott invariant which coincides with the Elliott invariant of one of the aforementioned concrete AH-algebras, and prove that if $A$ satisfies (AUCT), then it is in fact isomorphic to the said concrete AH-algebra.

The following theorem was proved by J. Villadsen. The extra conditions (1) and (2) are not new either. It has appeared implicitly in several places including Villadsen’s proof.

If $X$ is a convex set, the extremal points of $X$ are denoted by $\partial_e(X)$.

10.1. Theorem. (Cf. [57].) Suppose that $G$ is a countable, partially ordered abelian group which is simple, weakly unperforated with the Riesz interpolation property, that $G/tor(G)$ is non-cyclic,
\(u \in G_+,\ H\) is a countable abelian group, \(\Delta\) is a metrizable Choquet simplex and \(\lambda : \Delta \to S(G, u)\) is a continuous affine map with \(\lambda(\partial_{\Delta}) = \partial(S(G, u))\). Then there is a simple AH-algebra 
\(A = \lim_{n \to \infty}(A_n, h_n)\) with \(TR(A) \leq 1\) and with \(A_n = C_1 \oplus C_2 \oplus \cdots \oplus C_{m(n)}\), where \(C_1\) is of the form as described in 7.1 (a single summand) and \(C_j\) is of the form \(C([0, 1]) \otimes M_{m(j)}\) (for \(j > 1\)).

\[
(1) \quad h_n = h_n^{(0)} \oplus h_n^{(1)} \oplus h_n^{(2)}, \text{ where } h_n^{(0)}, h_n^{(1)} \text{ factor through a } C^*\text{-algebra in } I, \text{ and } h_n \text{ is injective, in particular, } h_n^{(0)} \text{ is homotopically trivial,} \\
(2) \quad \tau \circ h_{n+1} \circ h_n^{(0)}(1_{A_n}) \to 0 \text{ uniformly on } T(A), \\
(3) \quad \tau \circ h_{n+1} \circ h_n^{(2)}(1_{A_n}) \to 0 \text{ uniformly on } T(A), \\
(4) \quad (h_n)_{n \geq 1} \text{ is injective and} \\
(5) \quad (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A) = (G, G_+, u, H, \Delta, \lambda).
\]

**Proof.** The proof of this is a combination of Villadsen’s proof of the main theorem in [57] and the proof of [36, 1.5]. Let \(A = \lim_{n \to \infty}(A_n, h_n)\) be as in [36, 1.5]. This algebra \(A\) satisfies (3), (4) and \(K_0(A), K_0(A)_+, [1_A], K_1(A) = (G, G_+, u, H)\). Moreover each \(A_n\) can be chosen so that it has the form as required. Here one needs one modification and one explanation. In the proof of [36, 1.5] we use \(C_j = M_{m(j)}\) for \(j > 1\). But we can map \(C([0, 1], M_{m(j)})\) into \(M_{m(j)}\) by a one point-evaluation and then map \(M_{m(j)}\) into \(C([0, 1], M_{m(j)})\) (as constant functions). So we can assume that \(A_n\) has the required form. Note also that the new \(A_n\) has the same \(K\)-theory as the old one. If \(K_1(A) = F = \lim_{n \to \infty}(F_n, \gamma_n)\), in the proof of [36, 1.5], \(K_1(A_n) = F_n\) and the map \(\Phi_{n,n+1}\) in the proof of [36, 1.5] has the property \((\Phi_{n,n+1})_{n \geq 1} = \gamma_n\). However, since \(F\) is a countable abelian group, one can always assume that \(F_n\) is finitely generated and \(\gamma_n\) is injective (by choosing \(F_n\) as subgroups and \(\gamma_n\) as embeddings) so that (4) holds.

We will revise the map \(h_n\) to meet the other requirements. Villadsen’s proof in [57] is to replace \(h_n\) by \(\phi_n\) without changing its \(K\)-theory in such a way that one gets \(\Delta\) as tracial space and \(\lambda\) as pairing. We will follow his proof with a minor modification. Each \(h_n\) may be written as \(h_n' \oplus h_n''\), where \(h_n''\) is a point-evaluation, as in [36, 1.5]. Note that [57, Lemma] holds when \(X_q^1\) is a compact connected CW complex with dimension at least one but no more than three. Following Villadsen’s proof, by applying [57, Lemma] and its proof, one can replace \(h_n''\) to achieve exactly what [57, Lemma] achieved. It should be noted that Villadsen’s proof of the main theorem in [57] works when \(X_q^1\) has lower dimension (but at least one), since the required maps \(i_q^1 : [0, 1] \to X_q^1\) and \(k_q^1 : X_q^1 \to [0, 1]\) still exist. The new map obtained from Villadsen’s proof has the form \(\tilde{\psi}_n = h_n' \oplus h_n''\), where \(h_n''\) is homotopically trivial. Furthermore, it can be chosen so that it factors through a \(C^*\)-algebra in \(I\). The construction of Villadsen then gives a simple AH-algebra \(B\) with \(TR(B) \leq 1\) and satisfies (5). Moreover, one has \(\tau \circ \tilde{\psi}_{n+1} \circ h_n(1_{A_n}) \to 0\) uniformly on \(T(B)\). The construction does not change (3) and (4). It is also easy to get (1) and (2). For example, consider \(h_n' \circ h_n' \oplus h_n'' \circ h_n'' \circ \tilde{\psi}_n\). Note that \(h_n' \circ h_n''\) and \(h_n'' \circ \tilde{\psi}_n\) are homotopically trivial and \(\tau \circ \tilde{\psi}_{n, \infty} \circ h_n' \circ h_n''(1_{A_n}) \to 0\) uniformly on \(T(B)\).

**10.2. Definition.** Let \(C\) be a unital \(C^*\)-algebra. We denote by \(S_u(K_0(C))\) the set of states on \(K_0(C)\), i.e., the set of order and unit preserving homomorphisms from \(K_0(C)\) to (the additive group) \(\mathbb{R}\). There is an affine map \(\lambda : T(C) \to S_u(K_0(C))\) such that \(\lambda(t)([p]) = t(p)\) for all projections \(p \in M_{\infty}(C)\) and \(t \in T(C)\). Suppose that \(C\) is stably finite. It was proved in [51, Theorem 6.1] (for the simple case) and [6, Theorem 3.5] that each state in \(S_u(K_0(C))\) is induced
by a quasitrace \( t \in QT(C) \). If \( C \) is exact, or if it is both simple and of tracial rank at most one, then all quasitraces on \( C \) are traces (see (ix) in 4.9).

Let \( A \) and \( B \) be two unital \( C^* \)-algebras. We say
\[
\gamma : (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \to (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B))
\]
is an order isomorphism if there is an order isomorphism
\[
\gamma_0 : (K_0(A), K_0(A)_+) \to (K_0(B), K_0(B)_+)
\]
which maps \([1_A]\) to \([1_B]\), there is an isomorphism \( \gamma_1 : K_1(A) \to K_1(B) \) and an affine homeomorphism \( \gamma_2 : T(A) \to T(B) \) such that \( \gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x)) \) for all \( \tau \in T(B) \) and \( x \in K_0(A) \), where we view \( \tau \) as a state on \( K_0(A) \).

10.3. Theorem. Let \( A \) and \( B \) be two unital separable nuclear simple \( C^* \)-algebras with \( TR(A) \leq 1 \) and \( TR(B) \leq 1 \) satisfying AUCT such that
\[
(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)) = (K_0(A), K_0(A), [1_A], K_1(A), T(A))
\]
in the sense of 10.2. Then there is a sequence of contractive completely positive linear maps \( \{\Psi_n\} \) from \( A \) to \( B \) such that:

(i) \( \lim_{n \to \infty} \|\Psi_n(ab) - \Psi_n(a)\Psi_n(b)\| = 0 \) for all \( a, b \in A \),
(ii) for any finite subset set \( \mathcal{P} \subset \mathcal{P}(A) \),
\[
[\Psi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}},
\]
for all sufficiently large \( n \), where \( \alpha \in KL(A, B)^{++} \) (see 8.2) gives the above identification on \( K \)-theory and
(iii) \( \lim_{n \to \infty} \sup_{\tau \in T(B)} \{||\tau \circ \Psi_n(a) - \xi(Q(a))(\tau)||\} = 0 \)
for all \( a \in A_{sa} \), where \( \xi : \text{Aff} T(A) \to \text{Aff} T(B) \) is the affine isometry given above.

Proof. It follows from Theorem 9.12 that there is a unital separable simple nuclear \( C^* \)-algebra \( C \) with \( TR(C) = 0 \) satisfying the AUCT such that
\[
(K_0(A), K_0(A), [1_A], K_1(A)) = (K_0(C), K_0(C)_+, [1_C], K_1(C))
\]
and a sequence of contractive completely positive linear maps \( L_n : A \to C \) satisfying condition (2) in 9.12. In particular, \( [L_n]|_{\mathcal{P}} = \beta|_{\mathcal{P}} \), for any finite subset \( \mathcal{P} \) and all sufficiently large \( n \), where \( \beta \in KL(A, C)^{++} \) gives the above identification on \( K \)-theory. It follows from [38] that there is a unital separable simple AH-algebra \( C_1 \) such that \( C_1 \cong C \). To simplify notation, we may assume that \( C_1 = C \).

It follows from 9.10 that there exists a sequence of contractive completely positive linear maps \( \Phi'_n : C \to B \) such that:

(i') \( \lim_{n \to \infty} \|\Phi'_n(ab) - \Phi'_n(a)\Phi'_n(b)\| = 0 \) for all \( a, b \in C \),
(ii') for any finite subset $Q \subset P(C)$,

$$[\Phi'_n]|_Q = (\beta^{-1} \times \alpha)|_Q, \text{ for all sufficiently large } n.$$  

Thus by choosing a subsequence $\{k(n)\}$ and defining $\Psi_n = \Phi'_{k(n)} \circ L_n : A \to B$ we see that $\Psi_n$ satisfies (i) and (ii). (In fact one can show that $A$ is KK-attainable.) We then apply the proof of 9.7, to obtain a (new) sequence $\{\Phi_n\}$ which also satisfies (iii). \hfill \Box

Using the argument of [38], Zhuang Niu gives a different proof of the above theorem.

**10.4. Theorem.** Let $A$ and $B$ be two unital separable nuclear simple $C^*$-algebras with $\text{Tr}(A) \leq 1$ and $\text{Tr}(B) \leq 1$ satisfying the AUCT. Suppose that $\lambda(\partial_e(T(A))) = \partial_e(S_u(K_0(A)))$ and $\lambda(\partial_e(T(B))) = \partial_e(S_u(K_0(B)))$. Then $A$ is isomorphic to $B$ if and only if there exists an order isomorphism

$$\gamma = (\gamma_0, \gamma_1, \gamma_2) : (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \to (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)),$$

where $\gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x))$ for all $\tau \in T(B)$ and $x \in K_0(A)$ (see 10.2).

**Proof.** Let $A$ be as in the theorem. Note that $T(A)$ is a Choquet simplex (see 4.9(ix) for example). By 10.1 and 10.3 (as well as 4.8), there is a unital simple AH-algebra $B = \lim_{n \to \infty}(B_n, \phi_{n,n+1})$ with

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)),$$

where $B_n$ is as described in 10.1. Here $\phi_{n,n+1}$ is a homomorphism from $B_n$ to $B_{n+1}$. Put $\phi_{n,m} = \phi_{m-1,m} \circ \cdots \circ \phi_{n,n+1}$. Denote by $\psi_n : B_n \to B$ the homomorphism induced by the inductive system. As in 10.1, we will assume that $\phi_{n,m}$ and $\psi_n$ are injective and $(\phi_{n,m})_+$ is injective for all $m > n$. In what follows, when it is convenient, we may identify $B_n$ with $\psi_n(B_n)$ without further warning. We also assume that the inductive limit satisfies the conditions (1)–(5) in 10.1. To prove the theorem it suffices to prove that $A$ is isomorphic to this specially constructed $B$. In what follows, $\kappa$ is the quotient map from $U(D)/CU(D)$ to $K_1(D)$ for a $C^*$-algebra $D$.

Let $\xi : \text{Aff } T(A) \to \text{Aff } T(B)$ be the affine isometry induced by the above identification. Since both $A$ and $B$ satisfy the AUCT, there is $\alpha \in KL(A,B)^{++}$ which gives the isomorphism $(K_0(A), K_0(A)_+, [1_A], K_1(A)) \to (K_0(B), K_0(B)_+, [1_B], K_1(B))$. Let $L : U(B) \to \mathbb{R}_+$ be defined as follows: if $u \in U_0(B)$, $L(u) = 2\text{cel}(u) + 8\pi + \pi/16$; if $u \in U(B) \setminus U_0(B)$ and $u^k \in U_0(B)$ (where $k$ is smallest such positive integer) $L(u) = 16\pi + 2\text{cel}(u^k)/k + \pi/16$, and if $u \in U(B) \setminus U_0(B)$ and $|u|$ is not of finite order in $K_1(B)$, $L(u) = 16\pi + \pi/16$. Fix any $\varepsilon > 0$ ($\varepsilon < \pi/128$) and finite subset $\mathcal{F} \subset B$. Let $\delta' > 0$, integer $n > 0$, finite subsets $\mathcal{P} \subset P(B)$, $\mathcal{S} \subset B$ be as required in Theorem 8.6 (corresponding to $B$, $\mathcal{L}$, $\varepsilon/2$ and $\mathcal{F}$). There are mutually orthogonal projections $q, p_1, p_2, \ldots, p_n$ with $q \preceq p_i$, and with $[p_i] = [p_1]$, $i = 1, 2, \ldots, n$ and there is a $C^*$-subalgebra $C'_1 \in \mathcal{I}$ with $1_{C'_1} = q$ and there are unital $\mathcal{S}$-$\delta'/4$-multiplicative contractive completely positive linear maps $h_0 : B \to \mathcal{Q}Bq$ and $h_1 : B \to C'_1$ such that $h_0(x) = qxq$ and

$$\|x - (h_0(x) \oplus h_1(x) \oplus \cdots \oplus h_1(x))\| < \delta'/16.$$
for all $x \in S$, where $h_1$ is repeated $n$ times. Put $C = M_n(C'_1) \subset (1 - q)B(1 - q)$. Let $\mathcal{P}_0$, $\mathcal{G}_0$, $\mathcal{H}$, $\delta_0 > 0$ and $\sigma_1 > 0$ also be as required by 8.6 and let $\delta = \min\{\delta_0, \delta'\}$.

We may assume that $\mathcal{P}_0$ contains at least one minimal projection of each summand of $C$. Note that, without loss of generality, we may also assume that $q$ commutes with each element in $S$ and $\mathcal{H}$.

Without loss of generality (by omitting a possible error of $\delta/16$), we may also assume that each $u \in U(B) \cap \mathcal{P}$ has the form $uq \oplus (1 - q)u(1 - q)$, where $uq \oplus qU(Bq)$ and $(1 - q)u(1 - q) \in U(C)$. We may further assume that $q \in B_1$ and $uq \in U(B_1)$. Let $U' = \{uq: u \in (U(B) \cap \mathcal{P})\}$ and let $F$ be the subgroup of $U(Bq)$ generated by $U'$. Let $\tilde{F}$ be the image of $F$ in $U(qBq)/CU(qBq)$. By 6.6(3), we write $\tilde{F} = \tilde{F} \cap U_0(qBq)/CU(qBq) \oplus \tilde{F}_0 \oplus \tilde{F}_1$, where $\tilde{F}_0$ is torsion and $\tilde{F}_1$ is free. Note $\kappa(\tilde{F}_1) \cong \tilde{F}_1$. We may assume that $U' = U_0 \cup U_1$ such that $\tilde{U}_0$ generates $\tilde{F} \cap U_0(qBq)/CU(qBq) \oplus \tilde{F}_0$ and $\tilde{U}_1$ generates $\tilde{F}_1$. Note that modulo unitaries in $CU(qBq)$, we can always make this assumption and it will cost us no more than $8\pi$ in the estimation of the exponential length (see 6.9). Without loss of generality, we may also assume that $q \in B_1$ and $U_0, U_1 \subset qB_1$. Note we also assume that $K_1(B_m) = K_1(B_{m+1}) \to K_1(B)$ is injective.

Let $\mathcal{G}_1'$ be a finite subset which contains $S$, $\mathcal{G}_0$ and $\mathcal{H}$ as well as $q, p_1, \ldots, p_n$ and a finite generating set of $C'_1$. It also contains $U'$. Without loss of generality, we may further assume that $\mathcal{G}_1' \subset B_1$. Note we still have that $q$ commutes with all elements in $\mathcal{G}_1'$. It follows from 10.3 (or 9.10) that there is a $\mathcal{G}_1'-\delta/4$-multiplicative contractive completely positive linear map $L_1 : B \rightarrow A$ such that

$$\sup_{\tau \in T(B)} \|\tau \circ L_1(a) - \xi^{-1}(Q(a)) (\tau)\| \leq \frac{\sigma}{2}$$

for all $a \in \mathcal{H}$,

where $Q : \mathcal{A}_{sa} \rightarrow Aff(T(A))$ is the evaluation map.

We assume that $L_1^\dagger$ is well defined on $\tilde{F} (\subset U(qBq)/CU(qBq))$ (see 6.2). Define $L_1(A) \rightarrow \mathbb{R}_+$ exactly in the same way as $L$ above. Let $\mathcal{F}_1$ be a finite subset of $A$. Let $\delta'_1 > 0$, integer $n_1 > 0$, finite subsets $\mathcal{P}_1 \subset \mathcal{P}(A)$, $S_1 \subset A$ for $A$, $L_1, \varepsilon/4$ and $\mathcal{F}_1$ be as required in Theorem 8.6. There are mutually orthogonal projections $q', p'_1, p'_2, \ldots, p'_{n_1}$ with $q' \lesssim p'_i$, and with $[p'_i] = [p'_1]$ ($i = 1, 2, \ldots, n_1$) and there is a $C^*$-subalgebra $C'_2 \subset \mathcal{I}$ with $1_{C'_2} = q'$ and there are unital $S_1$-$\delta'_1/4$-multiplicative contractive completely positive linear maps $h'_0 : A \rightarrow q'Aq'$ and $h_1 : A \rightarrow C'_2$ such that $h_0(x) = q'xq'$ and

$$\left\|x - (h'_0(x) \oplus h'_1(x) \oplus \cdots \oplus h'_1(x))\right\| < \delta'_1/16$$

for all $x \in S_1$,

where $h'_1$ is repeated $n_1$ times. We may assume that $L_1(S) \subset S_1$. Put $C_1 = M_{n_1}(C'_1) \subset (1 - q)A(1 - q)$. Let $\mathcal{P}_0, \mathcal{G}_0, \mathcal{H}_1, \delta_0$, and $\sigma_0$ also be as required by 8.6. Let $\delta_1 = \min\{\delta'_1, \delta_0\}$. We may assume that $\delta_1 < \delta/2$, $\sigma_1 < \sigma/4$ and $\mathcal{P}_0$ contains at least one minimal projection in each summand of $C_1$. Furthermore, without loss of generality, we may also assume that $q'$ commutes with each element in $\mathcal{H}_1$ and $S_1$, and that $[\mathcal{P}_1] \supset [L_1][\mathcal{P}] \supset \mathcal{P}_0$.

Without loss of generality, we may assume that each $u \in U(A) \cap \mathcal{P}_1$ has the form $q'uq' + (1 - q')u(1 - q')$, where $q'uq' \in U(q'Aq')$ and $(1 - q')u(1 - q') \in U((1 - q')C_1(1 - q'))$. Put $\mathcal{V}' = \{q'uq': u \in U(A) \cap \mathcal{P}_1\}$. Let $\mathcal{F}'$ be the subgroup of $U(q'Aq')$ generated by $\mathcal{V}'$. By 6.6(3), we write $\mathcal{F}' = \mathcal{F}' \cap U_0(q'Aq')/CU(q'Aq') \oplus \mathcal{F}_0' \oplus \mathcal{F}_1'$, where $\mathcal{F}_0'$ is torsion and $\mathcal{F}_1'$ is
free. Note \( \kappa(F'_1) \cong F'_1 \). We may also assume that \( F' \supset L^\perp(\bar{F}) \). Since \( A \) is simple and separable, \( K_1(q'Aq') = K_1(A) \). Without loss of generality, we may also assume that \( \mathcal{V}' = \mathcal{V}_0 \cup \mathcal{V}_1 \), where \( \mathcal{V}_0 \) generates \( \bar{F}' \cap U_0(q'Aq')/CU(q'Aq') \oplus \bar{F}'_0 \) and \( \mathcal{V}_1 \) generates \( \bar{F}'_1 \). Note again this assumption will cost us no more than \( 8\pi \) when we estimate the exponential length later (see 6.9).

Let \( G'_2 \) be a finite subset which contains \( S_1, G_{01}, H_1 \) and \( L_1(G'_1) \) as well as \( q', \rho'_1, \ldots, \rho'_{n_1} \) and a finite generating set of \( C_2 \). It also contains \( \mathcal{V}' \). Note \( q' \) commutes with all elements in \( G'_2 \). It follows from 10.3 that there is a \( G'_2-\delta_1/2 \)-multiplicative contractive completely positive linear map \( \Phi'_1 : A \to B \) such that

\[
\Psin \circ L_1(\tau) - q < \delta/4.
\]

Define \( \Lambda(b) = aq[\Phi_1 \circ L_1(qbq)]qa \) (for \( a = (\Phi_1 \circ L_1(qq)^{-1/2} \)) and \( b \in qBq \). Note that

\[
\| \Phi'_1 \circ L_1(q) - q \| < \delta/2.
\]

Write \( B_n = \bigoplus_{j=1}^m B_n(j) \), where each \( B_n(j) \) has the form \( C^{(j)} \) as described in 7.1. According to this direct sum decomposition, we may write \( q = q_1 \oplus q_2 \oplus \cdots \oplus q_l \) with \( 0 \leq l \leq m \) and \( q_j \neq 0 \), \( 1 \leq j \leq l \). Choose an integer \( N_1 > 0 \) such that \( N_1[q_j] \geq \{1, 2, \ldots, p_n(n)\} \) for \( j \leq l \). Note that we may assume that \( q_j \) has rank at least 6. By applying an inner automorphism, we may assume that \( \bigoplus_{j=1}^m B_n(j) \) is a hereditary \( C^* \)-subalgebra of \( M_{N_1}(B'_n) \). Since \( F_1 \) is finitely generated, with sufficiently large \( n \), we obtain (see 6.2) a homomorphism \( j : \bar{F}_1 \to U(qB'_nq)/CU(qB'_nq) \) such that \( \psi_n^\perp \circ j = \text{id} \bar{F}_1 \). Then (since \( K_1(B_1) \to K_1(B_n) \to K_1(B) \) is injective),

\[
\kappa_1 \circ \psi_n^\perp \circ (\Phi_1 \circ L_1)^{\perp} \big|_{\bar{F}_1} = \kappa_1 \circ (\psi_n^\perp) \circ j = (\kappa_1)\big|_{\bar{F}_1},
\]

where \( \kappa_1 : U(qBq)/CU(qBq) \to K_1(qBq) \) is the quotient map. Note that \( K_1(qBq) = K_1(B) \). Let \( \Delta_1 \) be \( \delta(\varepsilon/4) \) as described in 7.5. We may assume that \( \Delta_1 < \sigma_1/4 \). To simplify notation, without loss of generality, we may assume that \( \psi_n(q) = q \). By the assumption on \( B \), we may write that \( \psi_n|_{B'_n} = (\psi_n)_{0} \oplus (\psi_n)_{1} \), where

(1) \( (\psi_n)_{0}(1_{B_n}) < \Delta_1/2(N_1 + 1)^2 \) for all \( \tau \in T(B) \) and

(2) \( (\psi_n)_{0} \) is homotopically trivial (but nonzero)

(see 10.1).
It follows from 7.5 that there is a homomorphism $h : B'_n \to e_0 B e_0$ such that:

(i) $[h] = [(\psi_n)]_0$ in $KL(B'_n, B)$ and

(ii) $(\psi_n \circ j(\bar{w}))^{-1} (h \oplus (\psi_n)_1 \circ (\Lambda(\bar{w}) - 1)) = \overline{g_w}$, where $g_w \in U_0(q B q)$ and $\text{cel}(g_w) < \varepsilon/4$ (in $U(q B q)$) for all $w \in \mathcal{U}_1$.

Define (we have assumed that $B_n \subset MN_1(B'_n)$)

$$h' = (\left( (h \oplus (\psi_n)_1) \otimes \text{id}_{MN_1} \right) \mid_{\bigoplus_{j=1}^{\infty} B_n(j)}$$

and define $\Psi' = h' \oplus (\psi_n) \mid_{\bigoplus_{j=1}^{\infty} B_n(j)}$. Let $\Phi_1 = \Psi' \circ \Phi_1'$. It is clear that (since $\Delta_1 < \sigma_1/4$)

$$[\Phi_1]_{\mathcal{P}_1 \cup \mathcal{P}_{01}} = [\Phi_1']_{\mathcal{P}_1 \cup \mathcal{P}_{01}} \quad \text{and} \quad |\tau \circ \Phi_1(a) - \tau \circ \Phi_1'(a)| < \sigma_1/2$$

for all $a \in A_{sa}$ and $\tau \in T(B)$. For all $w \in \mathcal{U}_1$, we have, by (ii) above,

$$\text{cel}(w^*(\Phi_1 \circ L_1(w))) < 8\pi + \varepsilon/4 \quad \text{in } U(q B q).$$

For $w \in \mathcal{U}_0$, by 6.8, 6.10 and 6.9, we also have

$$\text{cel}(w^*(\Phi_1 \circ L_1(w))) < 2\text{cel}(w) + \pi/64 \quad \text{(or } < 8\pi + 2\text{cel}(w)/k + \pi/16) \quad \text{in } U(q B q)$$

(depending if $[w] = 0$ or $[w]$ has order $k$ in $K_1(B)$). (Recall the definition of $h_0$ earlier in this proof in the next estimate.) Therefore (even after we add $8\pi$ for the decomposition assumption of $\mathcal{F}$)

$$\text{cel}(\text{id}_B(h_0(u))^{-1}(\Phi_1 \circ L_1(h_0(u))) < L(u) \quad \text{in } U(q B q)$$

for all $u \in U(B) \cap \mathcal{P}_1$. Since we also have

$$[\text{id}_B]_{\mathcal{P}_1 \cup \mathcal{P}_0} = [\Phi_1 \circ L_1]_{\mathcal{P}_1 \cup \mathcal{P}_0} \quad \text{and} \quad \sup_{\tau \in T(B)} \{ ||\tau(a) - \tau(\Phi_1 \circ L_1(a))|| \} < \sigma$$

for all $a \in \mathcal{H}$, by 8.6, we obtain a unitary $W \in U(B)$ such that

$$\text{ad } W \circ \Phi_1 \approx \sigma/2 \text{id}_B \quad \text{on } \mathcal{F}.$$

Replacing $\Phi_1$ by $\text{ad } W \circ \Phi_1$, we may assume that

$$\Phi_1 \circ L_1 \approx \sigma/2 \text{id}_B \quad \text{on } \mathcal{F}.$$

Now let $\mathcal{F}_2 \subset B$. We may assume that $\mathcal{F}_2 \subset B_{m_1'}$ ($m_1' > n$). Let $\delta_2 > 0$, integer $n_2 > 0$, finite subsets $\mathcal{P}_2 \subset \mathcal{P}(B)$, $S_2 \subset B$, be as required by Theorem 8.6 (for $B$, $L$, $\varepsilon/16$ and $\mathcal{F}_2$). There are mutually orthogonal projections $q''$, $p_1''$, $p_2''$, ..., $p_n''$ with $q'' \lesssim p_i''$, and with $[p_i''] = [p_i']$ (i =
1, 2, . . . , n) and there is a $C^*$-subalgebra $C'_3 \in \mathcal{I}$ with $1_{C'_3} = q''$, and there are unital $S_2-\delta'_2/4$-multiplicative contractive completely positive linear maps $h''_0 : B \to q''Bq''$ and $h''_1 : B \to C'_3$ such that $h''_0(x) = q''xq''$ and

$$
\|x - (h''_0(x) \oplus h''_1(x) \oplus \cdots \oplus h''_1(x))\| < \delta'_2/4
$$

for all $x \in S_2$, where $h''_1$ is repeated $n$ times. We assume that $S_2 \supset \Phi_1(S_1)$. Put $C_2 = M_{n2}(C'_3) \subset (1 - q''B)(1 - q'').$ Let $\mathcal{P}_{02}, \mathcal{G}_{02}, \mathcal{H}_{2}, \delta_{02} > 0$ and $\sigma_2 > 0$ also be as required by 8.6. Let $\delta_2 = \min(\delta'_2, \delta_{02})$. We may assume that $\sigma_2 < \sigma_1/4$ and $\delta_2 < \delta_1/4$. We may also assume that $\mathcal{P}_{02}$ contains at least one minimal projection of each summand of $C_2$ and $[\mathcal{P}_2] \supset [\Phi_1(\mathcal{P}_1 \cup \mathcal{P}_{01})]$. Furthermore, we may assume that each $u \in U(B) \cap \mathcal{P}_2$ has the form $q''uq'' \oplus (1 - q'')u(1 - q'')$, where $q''uq'' \in U(q''Bq'')$ and $(1 - q'')u(1 - q'') \in U(C_2)$. Put $\mathcal{W} = \{q''uq'' : u \in U(B) \cap \mathcal{P}_2\}$. Let $F''$ be the subgroup generated by $\mathcal{W}$. Write $F'' = F''_0 \oplus F''_1$, where $F''_0$ is torsion and $F''_1$ is free. We may also assume that $F'' \supset \Phi_1^2(F')$. We may further assume, without loss of generality, that $F'' = \mathcal{W}_0 \cup \mathcal{W}_1$, where $\mathcal{W}_0$ generates $F''_0 \oplus \mathcal{U}_0(q''Bq'') \oplus F''_1$ and $\mathcal{W}_1$ generates $F''_1$.

Let $\mathcal{G}_3'$ be a finite subset which contains $\mathcal{S}_2, \mathcal{G}_{02}, q'', p_1, . . . , p_{n1}, \mathcal{H}_2, \Phi_1(\mathcal{G}_3')$, a generating set of $C_2$ and $\mathcal{W}$. Without loss of generality, we may assume (see [31, 6.2]) that $\Phi_1(A) \subset B_m$ ($m > m_1 > n$). By choosing a larger $m$, we may assume that there is a contractible completely positive linear map $J : B \to B_m$ such that

$$
\|J(a) - \text{id}(a)\| < \delta_2/8 \quad \text{for all } a \in \mathcal{G}_3'.
$$

There is a projection $\tilde{q}' \in B_m$ such that

$$
\|\Phi_1(q') - \tilde{q}'\| < \delta_2/2.
$$

We may write $B_m = \bigoplus_{j=1}^s B_m(j)$. As above, by choosing a possibly larger $m$, we may also assume that $\tilde{q}'$ has rank at least 6 at each point. We write that $\tilde{q}' = q'_1 \oplus q'_2 \oplus \cdots \oplus q'_s$ according to the direct sum decomposition ($q'_j \neq 0$ for $1 \leq j \leq l$ and $l \leq s$). Suppose that $N_2 > 0$ is an integer such that $N_2[q'_j] > 3[1_{B_m(j)}]$ for $1 \leq j \leq l$. Set $B'_m = \tilde{q}'B_m\tilde{q}'$. Note $\Phi_1^{\frac{1}{2}}$ is injective on $\tilde{F}'_1$. We may further assume that $\mathcal{G}_3'$ contains $q'_1, q'_2, . . . , q'_s$ and a generating set of $B'_m$ and $B_m$.

Now let $L'_2 : B \to A$ be a $\mathcal{G}_3'-\delta_2/16(N_2 + 1)^2$-multiplicative contractive completely positive linear map (10.3) such that

$$
[L'_2]_{\mathcal{P}_{22}} = \alpha^{-1}1_{\mathcal{P}_{22} \cup \mathcal{P}_{02}} \quad \text{and} \quad \sup_{\tau \in T(A)} \{||\tau \circ L'_2(a) - \xi^{-1}(Q(a))(\tau)||\} < \sigma_2/4 \quad \text{for all } a \in \mathcal{H}_2 \cup \Phi_1(\mathcal{H}_1).
$$

We may assume that $(L'_2)^{\frac{1}{2}}$ is well defined on $F''$. Suppose that $e \in A$ is a projection such that

$$
\|L'_2 \circ \Phi_1(q') - e\| < \delta_2/4.
$$

Since $q' \in \mathcal{P}_1$, $[e] = [q']$ in $K_0(A)$. Thus, by replacing $L'_2$ by $adu' \circ L'_2$ for some unitary $u' \in A$, we may assume that $e = q'$. Without loss of generality, to simplify notation, we may assume $L'_2 \circ \Phi_1(q') = q'$. Note $\tilde{F}'_1$ is free and $(\phi_{m,M})_h^1$ ($M > m$) is injective. We compute that
that there is \( e \overline{\epsilon} \) for all \( g \). Homomorphism \( \beta \) homotopically trivial and \( \tau(\psi_M) = (\psi_M) \) for all \( w \) is homotopically trivial. \( \Phi \) playing the role of \( \alpha \) in 7.3, by applying 7.3, we obtain a homomorphism \( \beta : U(B_m')/CU(B_m') \to U(q'Aq') \) with \( \beta(U_0(B_m')/CU(B_m')) \subset U_0(q'Aq')/CU(q'Aq') \) such that

\[
\beta \circ (\Phi_1^\dagger)(\tilde{w}) = \tilde{w}
\]

for all \( \tilde{w} \in \tilde{F}_1 \). Let \( \Delta_2 = \delta(\epsilon/16) \) be as described in 7.4. It follows from the assumption on \( B \) that there is \( M \gg m \) such that \( \phi_{m,M} = \phi_{m,M}^{(0)} \oplus \phi_{m,M}^{(1)} : B_m \to B_M \) such that \( \phi_{m,M}^{(0)} \) is (nonzero) homotopically trivial and \( \tau(\psi_M \circ \phi_{m,M}^{(0)}(B_m')) < \Delta_2^2/4(N_2 + 1)^2 \) for all \( \tau \in T(B) \). To simplify notation, without loss of generality, we may also assume that \( e_0' = L_2' \circ \psi_M \circ \phi_{m,M}^{(0)}(B_m') \) and \( e_1' = L_2' \circ \psi_M \circ \phi_{m,M}^{(1)}(B_m') \) are mutually orthogonal projections (see 8.2(ii)). Note that \( \psi_M \circ \phi_{m,M} = \psi_M \cdot \psi_M \circ \phi_{m,M} \). It follows from 7.4 (by the choice of \( \Delta_2 \) and with \( \beta \) playing the role of \( \alpha \)), \( L_2' \circ \psi_M \circ \phi_{m,M} \) playing the role of \( \phi_0 \) and \( L_2' \circ \psi_M \circ \phi_{m,M} \) playing the role of \( \phi_1 \) in 7.4, we obtain a homomorphism \( \Phi' : B_m' \to \psi_M \circ \phi_{m,M}^{(1)}(B_m') \). Define \( L_2' = (\Phi' \oplus L_2' \circ \psi_M \circ \phi_{m,M}^{(1)}) \circ J \). It is clear that

\[
[L_2']_{|\mathcal{P}_2 \cup \mathcal{P}_0} = [L_2']_{|\mathcal{P}_2 \cup \mathcal{P}_0} = \alpha^{-1}|_{\mathcal{P}_2 \cup \mathcal{P}_0}.
\]

Given the choice of \( \Delta_2 \), we also have

\[
|\tau \circ L_2(a) - \tau(L_2'(a))| < \sigma_2/4
\]

for all \( a \in A_{sa} \) and \( \tau \in T(A) \). In particular,

\[
\sup_{\tau \in T(A)} \left| \tau \circ L_2 \circ \phi_1(a) - \tau(a) \right| < \sigma_1/2
\]

for all \( a \in H_1 \). Since \( \beta \circ \Phi_1^\dagger(\tilde{w}) = \tilde{w} \) for all \( w \in \mathcal{V}_1 \), by (i'') and 6.9, we have

\[
\text{cel}(id_A(h_0^t(w^*)))L_2(\Phi_1(h_0^t(w))) < 8\pi + \text{cel}(g_w) + \epsilon/4 < 8\pi + \epsilon/2 \quad \text{in} \ U(q'Aq')
\]

for all \( w \in \mathcal{V}_1 \). We also have, by 6.8–6.10,
\[ \text{cel}(\text{id}_A(h^*_0(w^*)))L_2(\Phi_1(h^*_0(w))) < 2 \text{cel}(w) + \pi/16 \]

(or \( < 8\pi + 2 \text{cel}(w^k)/k + \pi/16 \)) in \( U(q^{'A}q') \)

for all \( w \in V_0 \) (depends on if \([w] = 0 \) in \( K_1(A) \) or \([w] \) has order \( k \)). It follows that (by adding \( 8\pi \))

\[ \text{cel}(\text{id}(h^*_0(u^*)))L_2(\Phi_1(h^*_0(u))) < L(u) \]

for all \( u \in U(A) \cap \mathcal{P}_2 \) (in \( q^{'A}q' \)).

By applying 8.6, we obtain a unitary \( Z \in U(A) \) such that

\[ \text{ad} Z \circ L_2 \circ \Phi_1(a) \approx \varepsilon/16 \text{id}_A \text{ on } \mathcal{F}_1. \]

Therefore, by replacing \( L_2 \) by \( \text{ad} Z \circ L_2 \), we obtain the following “approximate intertwining” diagram:

\[
\begin{array}{c}
B \\
| id_B \\
\downarrow L_1 \\
A \\
\downarrow id_A \\
\end{array}
\begin{array}{c}
\uparrow \Phi_1 \\
B \\
\uparrow L_2 \\
A \\
\end{array}
\]

Since this process continues, we see that \( L_1 \) is recursively \( \mathcal{F} \)-invertible (and \( \Phi_1 \) is recursively \( \mathcal{F}_1 \)-invertible—see [32, 3.6]). It follows from an argument of Elliott (see [32, Theorem 3.6], for example) that \( A \) is isomorphic to \( B \). \( \square \)

10.5. Remark. If \( K_1(A) \) and \( K_1(B) \) are torsion groups, then one can use the “uniqueness theorem” 8.7. Since we do not need to control exponential length in this case, Section 7 is not needed. Furthermore, we do not need to assume \( B \) is AH. Consequently, we do not need to assume the condition on the \( \partial_e(S_0(K_0(A))) \) either. The whole proof is much shorter.

Since simple AH-algebras with very slow dimension growth have tracial rank one or zero, we have the following.

10.6. Theorem. (See [20] and [23].) Let \( A \) and \( B \) be two unital simple AH-algebras with very slow dimension growth and with torsion \( K_1(A) \). Then \( A \) is isomorphic to \( B \) if and only if

\[
(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B))
\]

in the sense of 10.2.

Note that 2.5 only uses [23, Section 4]. Thus 10.6 does not use [23, Sections 5 and 6] nor it uses [20].

10.7. Let \( C \) be a stably finite non-unital \( C^* \)-algebra with an approximate identity consisting of projections \( \{e_n\} \). Let \( T(C) \) denotes the set of traces \( \tau \) on \( C \) such that \( \sup_n \tau(e_n) = 1 \). We refer
to these traces as tracial states on \( C \), and to \( T(\mathcal{C}) \) the tracial state space of \( C \). Note that each tracial state extends to a tracial state on \( \tilde{\mathcal{C}} \). Therefore \( T(\tilde{\mathcal{C}}) \) is the set of convex combinations of \( \tau \in T(\mathcal{C}) \) and the tracial state which vanishes on \( C \). We also denote by \( S_w'(K_0(C)) \) the set of those order preserving homomorphisms from \( K_0(C) \) to \( \mathbb{R} \) such that \( \sup_n s([e_n]) = 1 \). Then each element in \( S_w'(K_0(\tilde{\mathcal{C}})) \) is the convex combination of \( s \in S_w'(K_0(C)) \) and the state which vanishes on \( j_s(K_0(C)) \), where \( j : C \to \tilde{\mathcal{C}} \) is the embedding.

10.8. Lemma. Let \( A \) be a unital separable simple \( C^* \)-algebra with \( TR(A) \leq 1 \). Then there is a \( C^* \)-algebra \( C = \lim_{n \to \infty} (C_n, \phi_n) \), where \( C_n \in \mathcal{I} \), satisfying the following:

(i) each \( C_n \) is a \( C^* \)-subalgebra of \( A \) and \( \{\phi_n, (1_{C_n})\} \) forms an approximate identity for \( C \);
(ii) there is a sequence of contractive completely positive linear maps \( L_n : A \to C \) such that
\[
\lim_{n \to \infty} \| L_n(ab) - L_n(a)L_n(b) \| = 0, \quad a, b \in A;
\]
(iii) there is an affine continuous (face-preserving) isomorphism \( \tau^* : T(\mathcal{A}) \to T(\mathcal{C}) \) such that
\[
\tau^*(\phi_n, (b)) = \lim_{k \to \infty} \tau(\phi_n, (k)(b)) \quad \text{for all } b \in C_n \text{ and } \tau \in T(\mathcal{A});
\]
(iv) there is an affine continuous (face-preserving) isomorphism \( r_x : S_w'(K_0(C)) \to S_w'(K_0(A)) \) such that
\[
r_x(s)([p]) = \lim_{n \to \infty} \tau_s(L_n(p)) \quad \text{for all } s \in S_w'(K_0(C)) \text{ and projection } p \in A,
\]
where \( \tau_s \) is the trace which induces \( s \).

Proof. Let \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots \) be a sequence of finite subsets of \( A \) such that \( \bigcup_n \mathcal{F}_n \) is dense in \( A \). Since \( TR(A) \leq 1 \), there is a \( C^* \)-subalgebra \( C_1 \subset A \) with \( C_1 \in \mathcal{I} \) and \( 1_{C_1} = p_1 \) such that:

(1') \( \| ap_1 - p_1a \| < 1/2 \) for all \( a \in \mathcal{F}_1 \),
(2') \( \text{dist}(p_1, p_1) < 1/2 \) for all \( a \in \mathcal{F}_1 \),
(3') \( \tau_1(1 - p_1) < 1/4 \) for all \( \tau \in T(\mathcal{A}) \).

Let \( 1 > \eta_1 > 0 \). By (3') and 4.7, there is a projection \( e_{(1,1)} \leq p_1 \) such that \( e_{(1,1)} \) is equivalent to \( 1 - p_1 \). Since \( \tau_1(p_1 - e_{(1,1)}) > 1/2 > \tau_1(1 - p_1) \) for all \( \tau \in T(\mathcal{A}) \), by 4.7 again, we obtain mutually orthogonal projections \( e_{(1,1)} , e_{(1,2)} \) such that \( e_{(1,i)} \leq p_1 \), \( [e_{(1,1)}] = [e_{(1,2)}] \geq [1 - p_1] \). There are \( x_{(1,1)} , x_{(1,2)} \in A \) such that \( x_{(1,i)} x_{(1,i)} \geq 1 - p_1 \) and \( x_{(1,i)} x_{(1,i)}^{*} = e_{(1,i)} \). Let \( G' \) be a finite set of generators of \( C_1 \) and \( G_2 = \mathcal{F}_2 \cup G' \cup \{ x_{(1,i)}, x_{(1,i)}^{*} , e_{(1,i)} : 1 \leq i \leq 2 \} \). There is a \( C^* \)-subalgebra \( C_2 \subset A \) with \( C_2 \in \mathcal{I} \) and \( 1_{C_2} = p_2 \) such that:

(1'') \( \| ap_2 - p_2a \| < \eta_1/4 \) for all \( a \in G_2 \),
(2'') \( \text{dist}(p_2, p_2) < \eta_1/4 \) for all \( a \in G_2 \),
(3'') \( \tau_1(1 - p_2) < 1/8 \) for all \( \tau \in T(\mathcal{A}) \).
By 2.1(iii), with sufficiently small $\eta_1$, there is a homomorphism $\phi_1 : C_1 \to C_2$ such that

$$\|\phi_1(b) - p_2bp_2\| < 1/4 \quad \text{for all } b \in F_1 \cup G'_1.$$ 

Put $q_2 = \phi_1(1_{C_1})$. With sufficiently small $\eta_1$, since $x_{(1,i)} \in G_2$, we may also assume that $2[p_2 - q_2] \leq [q_2]$ in $K_0(C_2)$. Note that $q_2 \leq p_2$.

We continue in this fashion. Suppose that $C_n \subset A$ is a unital $C^*$-subalgebra which is in $\mathcal{I}$ has been constructed. If $\tau(1 - p_n) < 1/2^{n+1}$ for all $\tau \in T(A)$, there are partial isometries $x_{(n,i)} \in A$ such that $p_n = 1_{C_n}$, $x_{(n,i)}^*x_{(n,i)} = 1 - p_n$, $x_{(n,i)}x_{(n,i)}^* = e_{(n,i)} \leq p_n$, $e_{(n,i)}e_{(n,i)} = 0$ if $i \neq j$ and $[e_{(n,i)}] = \{[e_{(n,i)}]\} \geq [1 - p_n]$, $1 \leq i \leq 2^n$. Let $G'_n$ be a finite set which contains a set of generators of $C_n$, $\phi_{i,n}(G_i)$ and $\phi_{i,n}(p_i)$, $i = 1, 2, \ldots, n - 1$, where $\phi_{i,n} = \phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_i$. (Note that $C_i \subset A$.)

Let $G_{n+1} = F_{n+1} \cup G'_n \cup \{e_{(n,i)}, x_{(n,i)}, x_{(n,i)}^*; 1 \leq i \leq 2^n\}$. Let $1 > \eta_{n+1} > 0$ be a positive number to be determined (but it depends only on $C_n$ and $F_n \cup G'_n$). Since $A$ has tracial topological rank one, there exist a $C^*$-subalgebra $C_{n+1} \subset A$ with $C_{n+1} \subset \mathcal{I}$ and a projection $p_{n+1}$ with $1_{C_{n+1}} = p_{n+1}$ such that:

1. $\|ap_{n+1} - p_{n+1}a\| < \eta_{n+1}/2^{n+1}$ for all $a \in G_{n+1}$,
2. dist$(p_{n+1}a p_{n+1}, C_{n+1}) < \eta_{n+1}/2^{n+1}$ for all $a \in G_{n+1}$ and
3. $\tau(1 - p_{n+1}) < 1/2^{n+2}$ for all $\tau \in T(A)$.

We choose $\eta_{n+1}$ so small that there exist a homomorphisms $\phi_n : C_n \to C_{n+1}$ (by 2.1(iii)) such that

$$\|\phi_n(b) - p_{n+1}bp_{n+1}\| < 1/2^{n+1} \quad \text{for all } b \in F_n \cup G'_n.$$ (e1)

Put $q_{n+1} = \phi_n(p_n)$. It is useful to note that $q_{n+1} \leq p_{n+1}$. Since $x_{(n,i)} \in G_{n+1}$, we may further assume that

4. $2^n[p_{n+1} - q_{n+1}] \leq [q_{n+1}]$ in $K_0(C_{n+1})$.

Set $C = \lim_{n \to \infty}(C_n, \phi_n)$. Since each $C_n$ is nuclear $C^*$-subalgebra of $A$, there is a contractive completely positive linear map $L'_n : A \to C_n$ (see, for example, [34, 2.3.13]) such that

$$\lim_{n \to \infty}\|L'_n(a) - p_nap_n\| = 0$$

for all $a \in A$. Note that, by (1),

$$\lim_{n \to \infty}\|L'_n(ab) - L'_n(a)L'_n(b)\| = 0 \quad \text{for all } a, b \in A.$$

Define $L_n = \phi_{n,\infty} \circ L'_n$. It is clear that $L_n$ satisfies (ii). Put $\phi_{n,n+1} = \phi_n$ and for $k > n + 1$, $\phi_{n,k} = \phi_{k-1} \circ \cdots \circ \phi_n$. Define $r^\sharp : T(A) \to T(C)$ as follows. For each $b \in C_n$, define

$$r^\sharp(\tau)(\phi_{n,\infty}(b)) = \lim_{k \to \infty} \tau(\phi_{n,k}(b)) \quad \text{for } \tau \in T(A).$$
Note that $\phi_{n,k}(b) \in C_k \subset A$. We will show that the right-hand side above converges. Since we may replace $b$ by $\phi_{n,k}(b)$ (replacing $n$ by a larger integer if necessary), without loss of generality, we may also assume that $b \in F_n$. From (e1), one obtains that

\[
\|\phi_{n,k+j+1}(b) - p_{k+j+2}\phi_{n,k+j}(b) p_{k+j+2}\| < 1/2^{k+j+2}. \tag{e2}
\]

On the other hand, for any integer $k \geq 0$,

\[
\begin{align*}
\left| \tau(p_{k+j+2}\phi_{n,k+j}(b) p_{k+j+2}) - \tau(\phi_{n,k+j}(b)) \right| \\
\leq \left| \tau((1 - p_{k+j+2})\phi_{n,k+j}(b)(1 - p_{k+j+2})) \right| + \left| \tau(p_{k+j+2}\phi_{n,k+j}(b)(1 - p_{k+j+2})) \right| \\
+ \left| \tau((1 - p_{k+j+2})\phi_{n,k+j}(b) p_{k+j+2}) \right| \\
< 3\|b\| \tau(1 - p_{k+j+2}) \leq 3\|b\|/2^{k+j+2}. \tag{e3}
\end{align*}
\]

It follows from (e2) and (e3) that

\[
\left| \tau(\phi_{n,k+j+1}(b)) - \tau(\phi_{n,k+j}(b)) \right| < 1/2^{k+j+2} + 3\|b\|/2^{k+j+2}.
\]

Therefore, for any $m \geq 1$,

\[
\left| \tau(\phi_{n,k}(b)) - \tau(\phi_{n,k+m}(b)) \right| < \sum_{j=0}^{m} 1/2^{k+j+2} + 3\|b\| \sum_{j=0}^{m} 1/2^{k+j+2} \to 0,
\]

as $k \to \infty$. We conclude that $\lim_{k \to \infty} \tau(\phi_{n,k}(b))$ converges. To see $r^\tau$ is well defined, we let $c \in C_m$ so that $\phi_{m,\infty}(c) = \phi_{n,\infty}(b)$. Then, for any $\varepsilon > 0$, there exists $N > \max\{n, m\}$ such that

\[
\|\phi_{n,k}(b) - \phi_{m,k}(c)\| < \varepsilon \quad \text{(in } C_k)\]

for all $k \geq N$. It follows that $(C_k \subset A)$

\[
\left| \tau(\phi_{n,k}(b)) - \tau(\phi_{m,k}(c)) \right| < \varepsilon
\]

for all $\tau \in T(A)$ and $k \geq N$. It follows that $r^\tau$ is well defined on $\bigcup_{n=1}^{\infty} \phi_{n,\infty}(C_n)$. Since $|\tau(\phi_{n,k}(b))| \leq \|\phi_{n,k}(b)\|$, $r^\tau(\tau)$ is bounded linear functional on $\bigcup_{n=1}^{\infty} \phi_{n,\infty}(C_n)$. It defines (uniquely) a bounded linear functional on $C$. One then easily sees that $r^\tau(\tau)$ is a state. Moreover, one checks that it is a tracial state. Thus $r^\tau$ is well defined. It is then easy to see that $r^\tau$ is an affine continuous map. Define $r^{\tau^{-1}}: T(C) \to T(A)$ by

\[
r^{\tau^{-1}}(t)(a) = \lim_{n \to \infty} t(L_n(a)) = \lim_{n \to \infty} t(\phi_{n,\infty}(L'_n(a))) \quad \text{for all } t \in T(C) \text{ and } a \in A.
\]

To justify the definition, we first need to show that $\lim_{n \to \infty} t(\phi_{n,\infty}(L'_n(a)))$ exists. Let $a \in F_n$ and $k > 0$. Define
\[ b_{n,k,1}(a) = (p_{n+k+2} - q_{n+k+2})L'_{n+k+2}(a)(p_{n+k+2} - q_{n+k+2}), \]
\[ b_{n,k,2}(a) = (p_{n+k+2} - q_{n+k+2})L'_{n+k+2}(a)q_{n+k+2}, \]
and
\[ b_{n,k,3}(a) = q_{n+k+2}L'_{n+k+2}(a)(p_{n+k+2} - q_{n+k+2}). \]

and define
\[ b_{n,k,1}(a)' = p_{n+k+2}(1 - p_{n+k+1})L'_{n+k+2}(a)(1 - p_{n+k+1})p_{n+k+2}, \]
\[ b_{n,k,2}(a)' = p_{n+k+2}(1 - p_{n+k+1})L'_{n+k+2}(a)p_{n+k+1}p_{n+k+2} \]
and
\[ b_{n,k,3}(a)' = p_{n+k+2}p_{n+k+1}L'_{n+k+2}(a)(1 - p_{n+k+1})p_{n+k+2}. \]

Note that \( b_{n,k,i}(a) \in C_{n+k+2}, \ i = 1, 2, 3. \) By (e1) and (1) above, in \( A, \)
\[
\left\| \left[ \phi_{n+1} \circ L'_{n+k+1}(a) - L'_{n+k+2}(a) \right] - \left[ b_{n,k,1}(a)' + b_{n,k,2}(a)' + b_{n,k,3}(a)' \right] \right\| < 5/2^{n+k+2} + 5\eta_{n+k+2}/2^{n+k+2}. \tag{e4} \]

We also estimate:
\[
\left\| (b_{n,k,1}(a)' + b_{n,k,2}(a)' + b_{n,k,3}(a)') - (b_{n,k,1}(a) + b_{n,k,2}(a) + b_{n,k,3}(a)) \right\| < 3/2^{n+k+2}.
\]

It follows that (in \( C_{n+k+2} \))
\[
\left\| (\phi_{n+1} \circ L'_{n+k+1}(a) - L'_{n+k+2}(a)) - (b_{n,k,1}(a) + b_{n,k,2}(a) + b_{n,k,3}(a)) \right\| < 8/2^{n+k+2} + 5\eta_{n+k+2}/2^{n+k+2}. \tag{e5} \]

By (e5),
\[
\left\| (L_{n+k+1}(a) - L_{n+k+2}(a)) - \phi_{n+k+2,\infty}(b_{n,k,1}(a) + b_{n,k,2}(a) + b_{n,k,3}(a)) \right\| < 1/2^{n+k-1} + 5\eta_{n+k+2}/2^{n+k+2}. \tag{e6} \]

By (4), in \( K_0(C), \)
\[
2^{n+k+1}[\phi_{n+k+2,\infty}(p_{n+k+2} - q_{n+k+2})] \leqslant [\phi_{n+k+2,\infty}(q_{n+k+2})].
\]

It follows that, for any \( t \in T(C), \)
\[
t(\phi_{n+k+2,\infty}(p_{n+k+2} - q_{n+k+2})) < 1/2^{n+k+1}.
\]

From this, we estimate that
\[
t(\phi_{n+k+2,\infty}(b_{n,k,i}(a))) \leqslant \|a\|/2^{n+k+1}, \quad i = 1, 2, 3
\]
for all \( t \in T(C). \) Combing this with (e6), we have
\[ |t(L_{n+k+1}(a)) - t(L_{n+k+2}(a))| < 1/2^{n+k-1} + 5\eta_{n+k+2}/2^{n+k+2} + 3\|a\|/2^{n+k+1}. \]

Hence
\[ |t(L_{n+1}(a)) - t(L_{n+m}(a))| < \sum_{k=0}^{m} (1/2^{n+k-1} + 5\eta_{n+k+2}/2^{n+k+2} + 3\|a\|/2^{n+k+1}) \to 0 \]
as \( n \to \infty \). This proves that \( \lim_{n \to \infty} t(\phi_{n,\infty}(L'_n(a))) \) exists. Then one shows that \( r^{\sharp -1}(t) \) is well defined. By (ii), which we have shown, \( r^{\sharp -1}(t) \) is a trace on \( A \). It is then clear that \( r^{\sharp -1} \) is an affine continuous map. It should be noted that even if \( a \in C_m \) (for \( m < n \)), \( L'_n(a) \in C_n \).

Now let \( \tau \in T(A) \) and \( a \in A \). To show that \( (r^{\sharp -1} \circ r^{\sharp})(\tau)(a) = \tau(a) \), we note that
\[ (r^{\sharp -1} \circ r^{\sharp})(\tau)(a) = \lim_{n \to \infty} r^{\sharp}(\tau)(\phi_{n,\infty}(L'_n(a))) = \lim_{k \to \infty} \tau(\phi_{n,k}(L'_n(a))) \]
for all \( a \in A \) and \( \tau \in T(A) \). Let \( \varepsilon > 0 \). Without loss of generality, we may assume that \( a \in F_n \) for some integer \( n > 0 \). Moreover, with sufficiently large \( n \), we may assume that \( 1/2^n < \varepsilon /8 \) and
\[ \|L'_n(a) - p_n a p_n\| < \varepsilon /4. \]

One estimates, by (e1) (with \( k > n \)),
\[ \|\phi_{n,k}(L'_n(a)) - p_k p_{k-1} \cdots p_{n+1} p_n a p_n p_{n+1} \cdots p_{k-1} p_k\| < \sum_{j=1}^{k-n} 1/2^{n+j} + \varepsilon /4 < \varepsilon /2. \]

By (3) and as in (e3), one has
\[ |\tau(p_k p_{k-1} \cdots p_{n+1} p_n a p_n p_{n+1} \cdots p_{k-1} p_k) - \tau(a)| < 3\|a\| \sum_{j=n}^{k-n} 1/2^{n+j} < 3\|a\|\varepsilon /8 \quad \text{for all } \tau \in T(A). \]

It follows that
\[ |\tau(\phi_{n,k}(L'_n(a))) - \tau(a)| < 3\|a\|\varepsilon /8 + \varepsilon /2 \quad \text{for all } \tau \in T(A) \]
if \( k > n \). Therefore
\[ (5) \quad \tau(a) = \lim_{n \to \infty} (\lim_{k \to \infty} \tau(\phi_{n,k}(L'_n(a)))) \quad \text{for all } a \in A \text{ and } \tau \in T(A). \]

This also proves \( r^{\sharp -1} \circ r^{\sharp}(\tau)(a) = \tau(a) \) for all \( a \in A \) and \( \tau \in T(A) \). Therefore \( r^{\sharp -1} \circ r^{\sharp} = \text{id}_{T(A)} \).

Suppose that \( t \in T(C) \) and \( b \in C_n \). Then
\[ r^{\sharp} \circ r^{\sharp -1}(t)(\phi_{n,\infty}(b)) = \lim_{k \to \infty} r^{\sharp -1}(t)(\phi_{n,k}(b)) = \lim_{k \to \infty} \lim_{m \to \infty} t(\phi_{m,\infty}(L'_m(\phi_{n,k}(b)))) \]

Fix $\varepsilon > 0$. Choose $k > n$ such that $1/2^k < \varepsilon/32$. We may assume that $\|b\| \leq 1$. For any $m > k$, put $r_j = \phi_{j,m}(p_j)$, $j = k, \ldots, m - 1$. Since $\phi_j(p_j) \leq p_{j+1}$, $r_j \leq r_{j+1}$. By choosing a larger $k$, applying (1) and (2) above, we may assume that there is $c_1 \in A$ such that (we view $\phi_{n,k}(b) \in C_k \subset A$)

$$r_j c_1 = c_1 r_j, \quad k + 1 \leq j \leq m - 1, \quad \text{and} \quad \|c_1 - \phi_{n,k}(b)\| < \varepsilon/8.$$ 

We also have

$$\|\phi_{n,m}(b) - \sum_{j=1}^{m-k} 2^k j < \varepsilon/8.$$ 

Put $c_2 = \sum_{j=1}^{m-k} 2^k j$. It then follows that

$$c_3 = L'_m(c_1) - c_2 \leq 2(p_m - r_k).$$ 

Since each $C_j$ has stable rank one, by (4), there are $y_i \in C_m$ such that $y_i^* s_i = p_m - r_k$ and $y_i s_i^* (1 \leq i \leq 2^m)$ are mutually orthogonal. Let $z_i = \phi_{m,\infty}(y_i)$, $i = 1, 2, \ldots, 2^m$. Then $z_i^* z_i = \phi_{m,\infty}(p_m - r_k)$ and $z_i z_i^* (1 \leq i \leq 2^m)$ are mutually orthogonal. It follows that

$$t(\phi_{m,\infty}(c_3)) \leq 2(1/2^m) < \varepsilon/8$$ 

for all $t \in T(C)$. On the other hand, from the above estimates,

$$\left\|L_m'\left(\phi_{n,k}(b)\right) - \phi_{n,\infty}(b)\right\| - L_m'\left(\phi_{n,k}(b)\right) - \phi_{n,\infty}(b)\right\| - \phi_{n,\infty}(c_3)\right\|$$

$$\leq \left\|L_m'(\phi_{n,k}(b)) - L_m'(c_1)\right\| + \|\phi_{n,m}(b) - c_2\| + \left\|L_m'(c_1) - c_2\right\|$$

$$\leq \varepsilon/8 + (\varepsilon/8 + \varepsilon/8) + 0 = 3\varepsilon/8.$$ 

It follows that

$$\left|t(\phi_{m,\infty}(L_m'(\phi_{n,k}(b)))) - t(\phi_{n,\infty}(b))\right| < 3\varepsilon/8 + t(\phi_{m,\infty}(c_3)) < 3\varepsilon/8 + \varepsilon/8 < \varepsilon$$ 

for all $t \in T(C)$ if $m > k$. Thus

$$t(\phi_{n,\infty}(b)) = \lim_{k \to \infty}(\lim_{m \to \infty} t(\phi_{m,\infty}(L_m'(\phi_{n,k}(b))))$$ 

for all $b \in C_n$ and $t \in T(C)$. Thus

$$r^z \circ r^{z-1} = \text{id}_{T(C)}.$$ 

It follows that $r^z$ is an affine continuous surjective map with an affine continuous inverse $r^{-1}$. To see $r^z$ is face-preserving, let $\tau \in T(A)$, $t_1, t_2 \in T(C)$ and $0 \leq a \leq 1$ for which

$$r^z(\tau) = at_1 + (1-a)t_2.$$ 

Let $\tau_1, \tau_2 \in T(A)$ such that $r^z(\tau_i) = t_i$, $i = 1, 2$. Then, since $r^{z-1}$ is the inverse of $r^z$, we see that
\[ \tau = a\tau_1 + (1-a)\tau_2. \]

Fix a projection \( p \in A \) and \( s \in S_u(K_0(C)) \). Here we will use the notation from 10.7 and 10.2. One obtains a sequence of projections \( e_n \in C_n \) such that

\[
\lim_{n \to \infty} \| p_n p - e_n \| = 0
\]

or equivalently

\[
\lim_{n \to \infty} \| L_n'(p) - e_n \| = 0.
\]

We have shown that, for each \( t \in T(C) \), \( \lim_{n \to \infty} t(\phi_n,\infty(L_n'(p))) \) exists. So \( \lim_{n \to \infty} t(\phi_n,\infty(e_n)) \) exists. If \( p \in MK(A) \) for some integer \( K > 0 \), by replacing \( C \) by \( MK(C) \) and \( p_n \) by \( \text{diag}(p_n,\ldots,p_n) \), we also obtain a projection \( e_n \in C_n \) such that

\[
\lim_{n \to \infty} t(\phi_n,\infty(L_n'(p))) = \lim_{n \to \infty} t(\phi_n,\infty(e_n)). \tag{e7}
\]

Since \( C \) is an inductive limit of \( C^* \)-algebras in \( I \), there exists \( \sigma_s \in T(C) \) such that \( s([e]) = \sigma_s(e) \) for any projection \( e \in MK(C) \) and for any integer \( K \geq 1 \) (recall that we use \( \sigma_s \otimes Tr \)). Suppose that \( \sigma_s, \tau_t \in T(C) \) such that \( \sigma_s(e) = \tau_t(e) \) for all projections \( e \in MK(C) \) (for all integer \( K \geq 1 \)). For any projection \( p \in MK(C) \), let \( e_n \) be a projection in \( C_n \) for which (e7) holds. Then

\[
\lim_{n \to \infty} \sigma_s(\phi_n,\infty \circ L_n'(p)) = \lim_{n \to \infty} \sigma_s(\phi_n,\infty(e_n)) = \lim_{n \to \infty} \tau_t(\phi_n,\infty(e_n)) = \lim_{n \to \infty} \tau_t(\phi_n,\infty \circ L_n'(p)).
\]

It follows that the map

\[
\tau_t(s)[p] = r^{\sigma_s-1}(\sigma_s)(p) = \lim_{n \to \infty} \sigma_s(\phi_n,\infty(L_n'(p))) = \lim_{n \to \infty} s(\phi_n,\infty(e_n)) \tag{e8}
\]

is independent of the choices of \( \sigma_s \) and is well defined from \( S_u(K_0(C)) \) to \( S_u(K_0(A)) \). (Here we extend \( L_n' \) and \( \phi_n,\infty \) to \( MK(A) \) and \( MK(C) \) in the obvious way.) It is clear that \( \tau_t \) is affine.

Let \( t \in S_u(K_0(A)) \). Since \( A \) is a simple \( C^* \)-algebra with \( TR(A) \leq 1 \) (by 10.2), there exists \( \tau_t \in T(A) \) such that \( \tau_t \) induces \( t \). Suppose that \( \sigma_t \in T(A) \) such that \( \tau_t(p) = \sigma_t(p) \) for all projections \( p \in MK(A) \) (for all integer \( K \geq 1 \)). Let \( e \in MK(C_n) \) be a projection. Then (note that \( C_k \subset A \))

\[
\lim_{k \to \infty} \tau_t(\phi_n,k(e)) = \lim_{k \to \infty} \sigma_t(\phi_n,k(e)).
\]

It follows that the map

\[
r_t(s)([e]) = r^{\tau_t}(\phi_n,\infty)(e) = \lim_{k \to \infty} \tau_t(\phi_n,k(e)) = \lim_{k \to \infty} t([\phi_n,k(e)])
\]

is independent of the choice of \( \tau_t \) and it is well-defined affine map (where we view \( C_n \) as a \( C^* \)-subalgebra of \( A \)).

Now let \( p \in A \) be a projection and \( t \in S_u(K_0(A)) \). By 10.2, \( t \) is induced by a trace \( \tau_t \in T(A) \). One has, by (5) and (e8),
Let
\[ r_\tau^\prime(r^\prime_\tau)([p]) = \lim_{n \to \infty} r_\tau^\prime(r^\prime_\tau([\phi_n,\infty(L_n^\prime(p))])) = \lim_{n \to \infty} \left( \lim_{k \to \infty} \tau_t(\phi_n,k(L_n^\prime(p))) \right) = \tau_t(p) = \tau([p]). \]
It follows that \( r_\tau \circ r_\tau^\prime = \text{id}_{S_\tau(K_0(A))} \). On the other hand, let \( e \in M_K(C_n) \) be a projection and \( s \in S_\tau(K_0(C)) \). Let \( \sigma_s \in T(C) \) which induces \( K_0(C) \) and \( \text{or}_e \). By \([56, 1.11]\), the map from \( T(C) \) onto \( S_\tau(K_0(C)) \) maps extremal points onto extremal points. Let \( t_0 \in T(C) \) be the trace such that \( t_0(c) = 0 \) for all \( c \in C \) and let \( s_0 \in S_\tau(K_0(C)) \) such that \( s_0(x) = 0 \) for all \( x \in j_s(K_0(C)) \), where \( j : C \to \tilde{C} \) is the embedding. Note that \( T(C) \) is the set of convex combinations of \( \tau \in T(A) \) and \( t_0 \) and \( S_\tau(K_0(C)) \) is the set of convex combinations of \( s \in S_\tau(K_0(C)) \) and \( s_0 \). Suppose that \( \tau \in \partial_e(T(A)) \). Then, by 10.8, \( r_\tau^\prime(\tau) \in \partial_e(T(C)) \subseteq \partial_e(T(\tilde{C})) \). It follows that \( r_\tau^\prime(\tau) \) gives an extremal state \( s_\tau \) in \( S_\tau(K_0(\tilde{C})) \). It follows that \( s_\tau \in \partial_e(S_\tau(K_0(C))) \). Note that \( \lambda(\tau) = r_\tau^\prime(s_\tau) \). By 10.8, this shows that \( \lambda(\partial_e(T(A))) \subseteq \partial_e(S_\tau(K_0(\tilde{C}))) \). To see that \( \lambda(\partial_e(T(A))) = \partial_e(S_\tau(K_0(C))) \), let \( s \in \partial_e(S_\tau(K_0(\tilde{C}))) \). Set
\[ F = \{ \tau \in T(A) : \lambda(\tau) = s \}. \]
It is clear that \( F \) is a closed and convex subset of \( T(A) \). Furthermore it is a face. By the Krein–Milman theorem, it contains an extremal point \( t \). Since \( F \) is a face, \( t \in \partial_e(T(A)) \). Thus \( \lambda(\partial_e(T(A))) = \partial_e(S_\tau(K_0(\tilde{C}))) \).

To see \( K_0(A)/\text{tor}(K_0(A)) \not\cong \mathbb{Z} \) when \( A \) is infinite-dimensional, we note that \( A \) has (SP) by 3.2. Since \( A \) is simple, we obtain, for any integer \( n > 0 \), \( n + 1 \) mutually orthogonal nonzero projections (see for example 5.5) \( p_1, p_2, \ldots, p_n \) and \( q \) in \( A \) for which \( 1 = q + \sum_{i=1}^n p_i \), \( [p_i] = [p_i] \) \( (i = 1, 2, \ldots, n) \) and \( [q] \leq [p_1] \). This implies that \( K_0(A)/\text{tor}(K_0(A)) \not\cong \mathbb{Z} \).

The last statement follows from 10.1 and the above.

Now by 10.4 and 10.9 we have the following.

10.10. Theorem. Let \( A \) and \( B \) be two unital separable nuclear simple \( C^* \)-algebras with \( TR(A) \leq 1 \) and \( TR(B) \leq 1 \) which satisfy the AUCT. Then \( A \cong B \) if and only if
\[
(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B))
\]
in the sense of 10.2.
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