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Primary Ideal Theory for Quadratic Jordan Algebras

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The notion of a primary ideal was originally introduced in the study of noetherian rings, i.e., commutative associative rings with unity which have the maximum condition on ideals. In more recent years, this notion has been carried over to noncommutative associative rings and nonassociative rings (see [3]). In this paper, we shall give a definition and basic properties for (inner) primary ideals in a quadratic Jordan algebra based on the concept of prime ideals in Jordan algebras as discussed in [10]. We then give a necessary and sufficient condition for those algebras in which the well-known Lasker–Noether decomposition theorem holds. Finally, the analog of the tertiary ideal is discussed, and we show that in any quadratic Jordan algebra which has the maximum condition on ideals, all ideals can be represented as a finite intersection of tertiary ideals.

1. BASIC DEFINITIONS

A unital quadratic Jordan algebra over a commutative associative ring Φ with unity is a triple $(J, U, 1)$ where J is a unital left Φ-module, 1 is a fixed element in J, and $x \to U_x$ is a quadratic map from J into $\hom_\Phi(J, J)$ such that the identities

(QJ-1) U_1 is the identity of $\hom_\Phi(J, J)$,

(QJ-2) $U_{aU_b} = U_b U_a U_b$ for al $a, b \in J$,

(QJ-3) if, for all $a, b \in J$, $U_{a,b} = U_{a+b} - U_a - U_b$ and $V_{a,b}$ is defined by $xV_{a,b} = aU_{x,b}$ for $x \in J$, then $U_b V_{a,b} = V_{b,a} U_b = U_{aU_b, b}$,

(QJ-4) if P is any commutative associative algebra over Φ and U' is the unique quadratic map from $J_P = J \otimes_\Phi P$ into $\hom_\Phi(J_P, J_P)$ extending U, then U' satisfies (QJ-2) and (QJ-3), are satisfied. When no ambiguity arises, we will let J denote a unital quadratic Jordan algebra over Φ.

A subset K of J is called an inner (outer) ideal of J if K is a Φ-submodule of J and for all $a \in J$ and $k \in K$, $aU_k \in K$ ($kU_a \in K$). K is an ideal of J, written $K \lhd J$, if K is both an inner and outer ideal of J. Moreover, if $K \lhd J$, then the difference algebra $J/K = \bar{J}$ is well-defined and $(\bar{J}, \bar{U}, \bar{1})$ is a quadratic Jordan algebra where, if $\bar{a}_1 = a_1 + K$ and $\bar{a}_2 = a_2 + K$ are elements of $\bar{J}$ then $\bar{a}_1 \bar{U}_{\bar{a}_2} = a_1 U_{a_2}$ (mod K).

For further information concerning quadratic Jordan algebras, the reader is referred to [6 and 8].

PROPOSITION 1.1. *If K is an outer ideal of J, then for all $a, b \in J$ and $k \in K$, $aU_{k,b} \in K$ and $a \cdot k \in K$ where $a \cdot b = aV_b = aV_{b,1}$.*

Proof. If $k \in K$ and a, $b \in J$, then $kU_{a,b} = kU_{a+b} - kU_a - kU_b$, whence $kU_{a,1} = aV_{k,1} = a \cdot k \in K$. Moreover, since $aU_{k,b} + kU_{a,b} = (a \cdot k) \cdot b$ (see [6, p. 1.20]), $aU_{k,b} \in K$.

If A and B are Φ-submodules of J, by AU_B we will mean the Φ-submodule of J generated by the set of all elements of the form aU_b where $a \in A$ and $b \in B$.

PROPOSITION 1.2. *If A and B are ideals of J, then AU_B is an ideal of J.*

Proof. The fact that AU_B is an outer ideal follows from the identity (see [8])

$$aU_bU_x = aU_{x \cdot b} + aU_xU_b + (x^2 \cdot a)\, U_b + aU_{b, bU_x} - (a \cdot x)\, U_{b, b \cdot x},$$

where $x^2 = 1U_x$. Since AU_B is an outer ideal, if d_1, $d_2 \in AU_B$ and $k \in K$, then $kU_{d_1, d_2} \in AU_B$. The fact that AU_B is an inner ideal follows from this and (QJ-2).

Now let $a \in J$. The set $A = \Phi a + JU_a$ is an inner ideal of J containing a, and A is contained in every inner ideal of J that contains a (see [6, p. 1.29]). Let $\langle a \rangle$ be the set of all finite sums of elements of the form $bU_{x_1} \cdots U_{x_k}$, where $b \in A$ and the $x_i \in J$. Then $\langle a \rangle$ is an outer ideal of J containing A and is contained in every outer ideal of J that contains A. We note that $\langle a \rangle$ is also an inner ideal of J. For suppose $w = bU_{x_1} \cdots U_{x_k}$ and $z = cU_{y_1} \cdots U_{y_p}$ are two elements in $\langle a \rangle$ and $d \in J$. By (QJ-2), we have $dU_w \in \langle a \rangle$. Furthermore, since $dU_{w,z} = (d \cdot w) \cdot z - wU_{d,z}$, by Proposition 1.1 we have $dU_{w,z} \in \langle a \rangle$. The fact that $\langle a \rangle$ is an inner ideal follows as in the proof of Proposition 1.2. We will call $\langle a \rangle$ the principal ideal of J generated by the element a.

2. Prime Ideals

An ideal P of J is prime if it satisfies either of the following equivalent conditions:

(1) if A and B are ideals of J with $AU_B \subseteq P$, then either $A \subseteq P$ or $B \subseteq P$,

(2) if $a, b \in P^c$ (where P^c is the set theoretic complement of P in J), then $\langle a \rangle U_{\langle b \rangle} \cap P^c \neq \emptyset$.

A definition for prime ideals in linear Jordan algebras was given in [10]. We note that the results there can be carried to quadratic Jordan algebras without change.

It is well known that if $\mathfrak{A}$ is an associative algebra with unity over Φ, then $\mathfrak{A}$ can be given the structure of a quadratic Jordan algebra by defining the endomorphism U_x for $x \in \mathfrak{A}$ as the map $y \to xyx$ for all $y \in \mathfrak{A}$. One readily verifies that the map $x \to U_x$ is a quadratic map from $\mathfrak{A}$ into $\hom_\Phi(\mathfrak{A}, \mathfrak{A})$ satisfying (QJ-1)–(QJ-4). We write $\mathfrak{A}^{(q)} = (\mathfrak{A}, U, 1)$ to denote the quadratic structure on $\mathfrak{A}$. Observe that there is not necessarily a one-to-one correspondence between the ideals of $\mathfrak{A}$ and the ideals of $\mathfrak{A}^{(q)}$. However, suppose that P is an ideal of $\mathfrak{A}$ whence P is an ideal of $\mathfrak{A}^{(q)}$. In [5], it is seen that P is prime in $\mathfrak{A}$ (in the usual associative sense) if and only if P is prime in $\mathfrak{A}^{(q)}$.

Remark 1. In the event that our associative algebra is commutative, much more can be said. Specifically, let $\mathfrak{C}$ be a commutative associative Φ-algebra with unity the additional property that if $x \in \mathfrak{C}$, then $\frac{1}{2}x \in \mathfrak{C}$. It is now easy to check that a subset A if $\mathfrak{C}$ is an (associative) ideal of $\mathfrak{C}$ if and only if A is an ideal of $\mathfrak{C}^{(q)}$. Moreover, suppose P is a prime ideal of $\mathfrak{C}$, and A and B are ideals of $\mathfrak{C}^{(q)}$ such that $AU_B \subseteq P$. If $A \not\subseteq P$, then there exists an element $a \in A$, $a \notin P$, but $aU_b = ab^2 \in P$ for all $b \in B$. Since P is prime, $b \in P$ whence $B \subseteq P$. Conversely, suppose P is prime in $\mathfrak{C}^{(q)}$, and A and B are ideals of $\mathfrak{C}$ such that $AB \subseteq P$. Easily $AB \subseteq P$ implies $AU_B \subseteq P$, so either $A \subseteq P$ or $B \subseteq P$. Thus in the commutative case, there is a one-to-one correspondence not only between the ideals of $\mathfrak{C}$ and those of $\mathfrak{C}^{(q)}$, but between the prime ideals too.

A nonempty subset M of J is called a Q-system if, whenever $m_1, m_2 \in M$, $\langle m_1 \rangle U_{\langle m_2 \rangle} \cap M \neq \emptyset$. By definition, an ideal P of J is prime if and only if P^c is a Q-system. Moreover, if M is a Q-system, $A \lhd J$, and $A \cap M = \emptyset$, then there exists a Q-system M' such that $A \cap M' = \emptyset$, $M \subseteq M'$, and $M' = P^c$ for some prime ideal P of J (where, of course, $A \subseteq P$). (See [10, proof of Theorem 1].) If $A \lhd J$, then the Q-radical of A, written $r(A)$, is the set of all $x \in J$ with the property that any Q-system of J that contains x meets

A. The Q-radical of J itself, denoted $r(J)$, is defined to be the Q-radical of the zero ideal. For additional information on Q-systems, the reader is referred to [10].

We now give

PROPOSITION 2.1. *Suppose $A \lhd J$. Then the Q-radical of J is equal to the intersection of those prime ideals of J that contain A.*

PROPOSITION 2.2. *Suppose A and B are ideals of J. Then*

(1) $r(A \cap B) = r(A) \cap r(B)$,

(2) if $A \subseteq B$, $r(A) \subseteq r(B)$,

(3) if $A \subseteq B \subseteq r(A)$, $r(A) = r(r(A)) = r(B)$.

For the proof of Proposition 2.1, we refer the reader to [10]. Proposition 2.2 is verified directly from the definitions and Proposition 2.1.

Now suppose A and B are ideals of J. The subset

$$[B: A] = \{d \in J \mid \langle d \rangle\, U_A \subseteq B\}$$

is called the (inner) quotient of A in B.

PROPOSITION 2.3 *If A and B are ideals of J, then $[B : A]$ is an ideal of J that contains B.*

Proof. Easily $B \subseteq [B : A]$. Now for any two elements $x, y \in J$ we have $\langle x - y \rangle \subseteq \langle x \rangle + \langle y \rangle$, from which it follows that $[B : A]$ is a Φ-submodule of J. That $[B : A]$ is an ideal is now immediate from the fact that $\langle x U_y \rangle \subseteq \langle x \rangle \cap \langle y \rangle$.

PROPOSITION 2.4. *An ideal P of J is prime if and only if for any ideal B of J which properly contains P we have $P = [P : B]$.*

Proof. Suppose there exists an ideal B of J properly containing P and $P \neq [P : B]$. Then $[P : B]\, U_B \subseteq P$ which implies that P is not prime. Conversely, if P is not prime, then there exist ideals A and B of J properly containing P but $A U_B \subseteq P$. Then $P \subseteq A \subseteq [P : B]$, whence $P \neq [P : B]$.

3. THE LOWER AND UPPER M COMPONENT OF AN IDEAL

Suppose $A \lhd J$ and M is a Q-system of J not meeting A. Then the (inner) lower M-component of A, denoted A_M, is the set of all $x \in J$ with the property that $\langle x \rangle\, U_{\langle m \rangle} \subseteq A$ for some $m \in M$. It is clear that $A \subseteq A_M$.

LEMMA 3.1. *Suppose $A \lhd J$ and M is a Q-system of J with the property that $M \cap A = \emptyset$. Then $A_M \lhd J$ and $A_M \cap M = \emptyset$.*

Proof. Suppose $x, y \in A_M$ and $c, d \in M$ such that $\langle x\rangle U_{\langle c\rangle} \subseteq A$ and $\langle y\rangle U_{\langle d\rangle} \subseteq A$. Since M is a Q-system, there exists an element $e \in \langle c\rangle U_{\langle d\rangle} \cap M$. As

$$\langle x-y\rangle \subseteq \langle x\rangle + \langle y\rangle, \langle x-y\rangle U_{\langle e\rangle} \subseteq \langle x\rangle U_{\langle c\rangle} + \langle y\rangle U_{\langle d\rangle} \subseteq A.$$

Thus A_M is a Φ-submodule of J. That A_M is an ideal of J follows as in the proof of Proposition 2.3. Finally, suppose $A_M \cap M \neq \emptyset$. Then there exists an element $a \in A_M \cap M$. Now $a \in A_M$ implies that for some $m \in M$, $\langle a\rangle U_{\langle m\rangle} \subseteq A$ whilst $a \in M$ implies that, for this m, $\langle a\rangle U_{\langle m\rangle} \cap M \neq \emptyset$. But this would imply that $M \cap A \neq \emptyset$, a contradiction, whence $A_M \cap M = \emptyset$.

We now extend the concept of the Q-system in the following manner. Suppose M is a Q-system of J. Then a subset N of J is called an (inner) M-n-system if N contains M and if, for every $m \in M$ and $n \in N$, $\langle n\rangle U_{\langle m\rangle} \cap N \neq \emptyset$. Clearly M is an M-n-system associated with itself. Moreover, the set theoretic union of any arbitrary collection of M-n-systems if an M-n-system. Thus it is easy to see that if $A \lhd J$ and M is a Q-system not meeting A, then there exists a maximal M-n-system N which does not meet A, and N is uniquely determined by M and A.

If $A \lhd J$ and M is a Q-system of J which does not meet A, then the (inner) upper M-component of A, denoted A^M, is defined to be the set of all $x \in J$ having the property that every M-n-system which contains x meets A. We note that $A \subseteq A_M \subseteq A^M$. As we have already seen that $A \subseteq A_M$, we now show that $A_M \subseteq A^M$. For if $x \in A_M$, then, for some $m \in M$, $\langle x\rangle U_{\langle m\rangle} \subseteq A$. So if N is an M-n-system that contains x, it follows that $N \cap A \neq \emptyset$, i.e., $x \in A^M$.

In the sequel, we will need the following concepts. Suppose $A \lhd J$. Then an element $a \in J$ is said to be (inner) prime to A if $\langle x\rangle U_{\langle a\rangle} \subseteq A$ implies $x \in A$. An ideal B of J is (inner) prime to A if B contains an element that is prime to A. If M is a Q-system not meeting A, we say that A is related to M is every element of M is prime to A. Easily, if A is related to M, then A^c is an M-n-system, and conversely.

THEOREM 3.2. *Suppose $A \lhd J$ and M is a Q-system not meeting A. Then,*

(1) *$A^M = \bigcap\{I \lhd J \mid A \subseteq I$ and I is related to $M\}$, whence $A^M \lhd J$,*

(2) *$(A^M)^c$ is the uniquely determined maximal M-n-system of J which does not meet A.*

The proof of the theorem is based on the following sequence of lemmas.

LEMMA 3.3. *Suppose M is a Q-system, N is an arbitrary M-n-system, $A \lhd J$, and $A \cap N = \emptyset$. Then A is contained in an ideal A^* where A^* is maximal with respect to $A^* \cap N = \emptyset$.*

Moreover, A^* is related to M.

Proof. Since the union of any linearly ordered (by inclusion) set of ideals which do not meet N is an ideal not meeting N, the existence of A^* is guaranteed by Zorn's lemma.

Now supposee $a \in J$ and $a \in A^*$. Then A^* is properly contained in $A^* + \langle a \rangle$. By the maximality of A^*, $(A^* + \langle a \rangle) \cap N \neq \emptyset$. Let N be an element in this intersection. Thus $n = n_1 + n_2$ where $n_1 \in A^*$ and $n_2 \in \langle a \rangle$. So if $m \in M$ and $\langle a \rangle U_{\langle m \rangle} \subseteq A^*$, it follows that $\langle n \rangle U_{\langle m \rangle} \subseteq A^*$. However, $\langle n \rangle U_{\langle m \rangle} \cap N \neq \emptyset$, which implies that there is an element common both to A^* and N, a contradiction. Consequently, if $\langle x \rangle U_{\langle y \rangle} \subseteq A^*$ for $y \in M$, we must have $x \in A^*$, whence A^* is related to M.

LEMMA 3.4. *Suppose $A \lhd J$ and M is a Q-system not meeting A. A set of element B of J is a minimal ideal containing A and related to M if and only if B^c is a maximal M-n-system which does not meet A.*

Proof. Suppose first that B^c is a maximal M-n-system which does not meet A. Then A is contained in a maximal ideal A^* which does not meet B^c, and moreover A^* is related to M. Hence $(A^*)^c$ is an M-n-system not meeting A, and it follows that $(A^*)^c = B^c$, whence $A^* = B$. Finally, B is minimal in $\Delta = \{C \lhd J \mid A \subseteq C \text{ and } C \text{ is related to } M\}$. If not, then there exists $C \in \Delta$ with $C \subseteq B$, $C \neq B$. Then C^c is an M-n-system not meeting A and properly containing B^c, a contradiction.

Conversely, suppose B is a minimal ideal containing A and related to M. Then B^c is an M-n-system not meeting A, so B^c is contained in a maximal such M-n-system N. From the first part of the proof, N^c is a minimal ideal containing A and N^c is related to M. As $N^c \subseteq B$, $N^c = B$, which completes the proof.

Proof of Theorem 3.2. Let N be the maximal M-n-system of J not meeting A which, by Lemma 3.4, has the property that $B = N^c$ is a minimal ideal of J containing A and B is related to M. As the intersection of any collection of ideals that contain A and are related to M contains A and is related to M, it follows that $B = \bigcap\{I \lhd J \mid A \subseteq I \text{ and } I \text{ is related to } M\}$.

Now $B \subseteq A^M$. For if $x \in B$, $x \in N$, the maximal M-n-system not meeting A. Hence every M-n-system which contains x meets A, whence $x \in A^M$. Conversely, if $x \in A^M$, then x cannot belong to N, so $x \in N^c = B$. Thus $A^M = B$, which proves the theorem.

COROLLARY 3.5. *Suppose $A \lhd J$, M_1 and M_2 are two Q-systems of J not meeting A, and $M_2 \subseteq M_1$. Then $A_{M_2} \subseteq A_{M_1}$ and $A^{M_2} \subseteq A^{M_1}$.*

Proof. Every M_1-n-system is an M_2-n-system. We now observe the following which is a consequence of the corollary and the remarks which follow the definition of a Q-system. Suppose $A \lhd J$ and M' is a Q-system not meeting A. Then there exists a prime ideal P of J such that $A \subseteq P$ and $P \cap M' = \emptyset$. Thus if $M = P^c$, we have $A \subseteq A^{M'} \subseteq A^M$. This fact will be used in the proof of a subsequent theorem.

4. Primary Ideals

Suppose $Q \lhd J$. Then Q is (inner) primary if, whenever A and B are ideals of J with $AU_B \subseteq Q$, then either $A \subseteq Q$ or $B \subseteq r(Q)$. We begin with

PROPOSITION 4.1. *Suppose $Q \lhd J$. Then Q is primary in J if and only if $\langle 0 \rangle$ is primary in J/Q.*

Proof. Let σ be the natural homomorphism from $(J, U, 1)$ onto $(\bar{J}, \bar{U}, \bar{1})$. Then the correspondence $P \to P\sigma$ is a one-to-one map from the set of prime ideals of J containing Q onto the set of prime ideals of $\bar{J}$. Consequently, $r(Q)\sigma = r(\bar{J})$, and the result follows.

THEOREM 4.2. *Suppose $Q \lhd J$. Then the following are equivalent:*

(1) *Q is primary.*

(2) *If M is a Q-system of J with $M \cap Q = \emptyset$, then $Q = Q^M$.*

(3) *If M is a Q-system of J with $M \cap Q = \emptyset$, then $Q = Q_M$.*

(4) *All elements not in $r(Q)$ are prime to Q.*

Proof. By the remark which follows Corollary 3.5, we see that in (1) and (2) we need to consider only those Q-systems M which are the complements of prime ideals containing Q.

(1) $\to$ (2): Suppose Q is primary and P is a prime ideal of J, distinct from J, containing Q. Set $M = P^c$. We claim that Q^c is an M-n-system. For suppose $x \in Q^c$ and $y \in M$. If $\langle x \rangle U_{\langle y \rangle} \cap Q^c = \emptyset$, then $\langle x \rangle U_{\langle y \rangle} \subseteq Q$. As Q is primary, this implies either $x \in Q$ or $y \in P$, either of which is a contradiction, so Q^c is an M-n-system. By Theorem 3.2, $(Q^M)^c$ is the unique maximal M-n-system of J not meeting Q, whence $(Q^M)^c = Q^c$, i.e., $Q = Q^M$.

(2) $\to$ (3): This is immediate from Theorem 3.2.

(3) $\to$ (1): Suppose Q is not primary. Then there exists $b \in Q$ and $c \notin r(Q)$ such that $\langle b \rangle U_{\langle c \rangle} \subseteq Q$. Since $c \notin r(Q)$, there exists a prime ideal P of J such

that $Q \subseteq P$ but $c \in P$ whence $P \neq J$. Now $M = P^c$ is a Q-system not meeting A, and, by definition, $b \in Q_M$. Thus $Q \neq Q_M$.

(1) $\leftrightarrow$ (4): This is clear from the definitions.

Suppose $A \lhd J$. We say that A has an irredundant representation by the ideals $B_1, \dots, B_p$ of J if $A = B_1 \cap \cdots \cap B_p$ and no one of the B_i contains the intersection of the remaining ones.

THEOREM 4.3. *Suppose $K \lhd J$ and K has an irredundant representation by primary ideals $Q_1, \dots, Q_k$ of J. Then K is primary if and only if $r(Q_i) = r(Q_j)$ for $i, j = 1, \dots, k$. Consequently, if K is primary, $r(K) = R(Q_1)$.*

Proof. We may suppose that $k > 1$.

Suppose first that K is primary and $K = Q_1 \cap \cdots \cap Q_k$ is an irredundant representation of K by primary ideals $Q_1, \dots, Q_k$ of J. Set $R = Q_2 \cap \cdots \cap Q_k$. Since $RU_{Q_1} \subseteq Q_1 \cap R = K$ and $R \nsubseteq K$, we have $Q_1 \subseteq r(K)$. By Proposition 2.2, $r(Q_1) \subseteq r(K) \subseteq \bigcap_{i=1}^{k} r(Q_i)$, so $r(Q_1) \subseteq r(Q_j)$ for $j = 1, \dots, k$. Repeating this argument, we conclude that for all i and j, $i, j = 1, \dots, k$, $r(Q_i) \subseteq r(Q_j)$, and the result follows.

Conversely, suppose $K = Q_1 \cap \cdots \cap Q_k$ is an irredundant representation of K by primary ideals $Q_1, \dots, Q_k$ of J where $r(Q_i) = r(Q_j)$ for $i, j = 1, \dots, k$. Suppose A and B are ideals of J such that $AU_B \subseteq K$. As $r(K) = r(Q_1)$, if $B \nsubseteq r(K)$, then, as $B \nsubseteq r(Q_i)$ for all i, $A \subseteq Q_i$ for all i, i.e., $A \subseteq K$. Thus K is primary with radical $r(Q_1)$.

If $A \lhd J$ and A has an irredundant representation by primary ideals $Q_1, \dots, Q_k$, then this representation is called a normal primary representation if, for all $i \neq j$, $1 \leqslant i, j \leqslant k$, $r(Q_i) \neq r(Q_j)$. As an immediate corollary of Theorem 4.3, we have

COROLLARY 4.4. *If $A \lhd J$ and A can be represented as a finite intersection of primary ideals, then A has a normal primary representation.*

5. QUADRATIC JORDAN ALGEBRAS WITH THE MAXIMUM CONDITION

A quadratic Jordan algebra J is said to have the maximum condition on ideals if every nonempty collection of ideals of J has a maximal element. We will write this "J has max-I".

Suppose $A \lhd J$. Then P is a minimal prime divisor of A if P is a prime ideal of J containing A and if $A \subseteq P' \subseteq P$ where P' is a prime ideal of J, then $P = P'$. We begin with

THEOREM 5.1. *Suppose J has* max-I *and $A \lhd J$. Then A has at most a finite number of minimal prime divisors.*

Proof. Suppose the theorem is false, and A is a maximal counterexample. Clearly, A is not prime, so there exist ideals B and C of J properly containing A and having the property that $BU_C \subseteq A$. By the maximality of A, both B and C have at most a finite number of minimal prime divisors. Now if P is a prime ideal of J containing A, then $BU_C \subseteq P$, so either $B \subseteq P$ or $C \subseteq P$. Hence, if P is a minimal prime divisor of A, P is a minimal prime divisor of either B or C. So if A has an infinite number of minimal prime divisors, this observation leads to a contradiction.

COROLLARY 5.2. *In the setting of the theorem, if A has minimal prime divisors $P_1, \ldots, P_n$ where $P_1 \neq J$, then the following are equivalent:*

(1) *A is primary.*

(2) *$A = A^{(P_i)^c}$ for $i = 1, \ldots, n$.*

(3) *$A = A_{(P_i)^c}$ for $I = 1, \ldots, n$.*

Proof. The proof is immediate from Theorem 4.2 and Corollary 3.5.

Suppose that S is a Φ-submodule of J. We define inductively $D^0(S) = S$ and for integers $k > 0$, $D^k(S) = D^{k-1}(S)\, U_{D^{k-1}(S)}$. If $A \lhd J$, then for all k, $D^k(A) \lhd J$. Moreover, if m and n are non-negative integers, then $D^{m+n}(A) = D^m(D^n(A))$.

THEOREM 5.3. *Suppose J has* max-*I and $A \lhd J$. Then there exists a non-negative integer $k = k(A)$ such that $D^k(r(A)) \subseteq A$.*

Proof. Suppose the theorem is false and A is a maximal counterexample. Easily A is not prime, so there exist ideals B and C of J properly containing A with $BU_C \subseteq A$. By the maximality of A, there exist integers $p = p(B)$ and $q = q(C)$ such that $D^p(r(B)) \subseteq B$ and $D^q(r(C)) \subseteq C$. Set $k = \max\{p, q\}$. As $r(A) \subseteq r(B) \cap r(C)$, this means that $D^k(r(A)) \subseteq B \cap C$. Consequently, $D^{k+1}(r(A)) \subset BU_C \subseteq A$, a contradiction.

COROLLARY 5.4. *If J has* max-*I, then, for some integer k $D^k(r(J)) = 0$.*

We recall that in Section 2, we saw that if $\mathfrak{A}$ is an associative Φ-algebra with unity, then $\mathfrak{A}$ could be given the structure of a unital quadratic Jordan algebra, denoted $\mathfrak{A}^{(q)}$. If $A \lhd \mathfrak{A}$, let $B(A)$ denote the prime radical of A. We recall the definition of primary ideals in $\mathfrak{A}$. Specifically, an ideal I of $\mathfrak{A}$ is right (left) primary if, whenever A and B are ideals of $\mathfrak{A}$ with $AB \subseteq I$, then either $A \subseteq I$ or $B \subseteq B(I)$ (either $B \subseteq I$ or $A \subseteq B(I)$). Then I is primary if it is both left and right primary. Now suppose $A \lhd \mathfrak{A}$. From [5], we have $r(A^{(q)}) \subseteq B(A)$. Hence if $A^{(q)}$ is primary if $\mathfrak{A}^{(q)}$, then A is primary in $\mathfrak{A}$. For suppose B and C are ideals of $\mathfrak{A}$ with $BC \subseteq A$. Then $BU_C \subseteq A$ and $CU_B \subseteq A$, from which it follows that A is both left and right primary. It is not yet known if the converse, i.e., the primary analog of the results in [5], is true.

Remark 2. Again, in the event that our associative algebra is commutative, much more can be said. Specifically, let $\mathfrak{C}$ be an associative Φ-algebra with unity such that for all $x \in \mathfrak{C}$, $\frac{1}{2}x \in \mathfrak{C}$. In addition, let $\mathfrak{C}$ be noetherian. We recall that an ideal Q of $\mathfrak{C}$ is primary if, whenever $ab \in Q$ for $a, b \in \mathfrak{C}$, then either $a \in Q$ or $b^n \in Q$ for some $n \geqslant 0$. So suppose Q is primary in $\mathfrak{C}$ and $AU_B \subseteq Q$ for ideals A and B of $\mathfrak{C}^{(q)}$. If $A \nsubseteq Q$, then there exists $a \in A$, $a \notin Q$, such that $aU_b = ab^2 \in Q$ for all $b \in B$, whence $b^n \in Q$ for some $n \geqslant 0$. Since $B(Q) = r(Q)$, it follows that $b \in r(Q)$, whence $B \subseteq r(Q)$ and Q is primary in $\mathfrak{C}^{(q)}$. Conversely, if Q is primary in $\mathfrak{C}^{(q)}$ and $ab \in Q$ for $a, b \in \mathfrak{C}$, then, from the definition of $\langle a \rangle$ and $\langle b \rangle$, it follows that for all $a' \in \langle a \rangle$ and $b' \in \langle b \rangle$, $a'U_{b'} \in Q$, whence $\langle a \rangle U_{\langle b \rangle} \subseteq Q$. Then either $\langle a \rangle \subseteq Q$ or $\langle b \rangle \subseteq r(Q)$. Since $\mathfrak{C}$ is noetherian, it follows from Theorem 5.3 that either $a \in Q$ or $b^n \in Q$ for some $n \geqslant 0$. Thus there is a one-to-one correspondence between the primary ideals of $\mathfrak{C}$ and those of $\mathfrak{C}^{(q)}$.

6. The Lasker–Noether Theorem

Suppose $A \lhd J$. Then A is meet irreducible if, whenever A can be represented as the intersection of two ideals B and C of J, then either $A = B$ or $A = C$. Using the same proof as in the associative case, mutatis mutandis, it is easy to see that A is a meet irreducible ideal of J if and only if $\langle 0 \rangle$ is meet irreducible in J/A. The proof of the next lemma is standard.

Lemma 6.1. *If J has* max-I, *then every ideal of J can be represented as a finite intersection of meet irreducible ideals.*

We say that J satisfies the (outer) Artin–Rees property if, whenever A and B are ideals of J, there exists a non-negative integer $h = h(A, B)$ such that $A \cap D^h(B) \subseteq AU_B$. Clearly, to say that J satisfies the Artin–Rees property is tantamount to assuming that an analogue of the well-known Artin–Rees lemma is valid in J. We use the Artin–Rees property to prove

Lemma 6.2. *Suppose J satisfies the Artin–Rees property. Then every meet irreducible ideal of J is primary.*

Proof. Suppose $T \lhd J$ and T is meet irreducible. By the remark preceding Lemma 6.1, it is sufficient to consider the case when $T = 0$. Thus suppose A and B are ideals of J with $AU_B = 0$. Since J satisfies the Artin–Rees property, there exists a non-negative integer h such that $A \cap D^h(B) = 0$. As 0 is meet irreducible, either $A = 0$ or $D^h(B) = 0$. In the case when

$A \neq 0$, then $D^h(B) = 0$, which implies $D^h(B) \subseteq r(J)$. It follows from this fact that $B \subseteq r(J)$. Consequently, 0 is primary, as desired.

THEOREM 6.3. (LASKER–NOETHER). *Suppose J has* max-I. *Then a necessary and sufficient condition that every ideal of J has a normal primary representation is that J satisfies the Artin–Rees property.*

Proof. Suppose first that every ideal of J has a normal primary representation. Thus, if A and B are ideals of J, then AU_B has a normal primary representation by primary ideals $Q_1, \ldots, Q_k$. We shall now demonstrate the existence of an integer h such that $A \cap D^h(B) \subseteq AU_B$. If, for all $i = 1, \ldots, k$, we have $A \subseteq Q_i$, then $D^0(B) \cap A = B \cap A \subseteq A \subseteq Q_1 \cap \cdots \cap Q_k = AU_B$, and we are done. So assume $A \not\subseteq Q_p$ for some p. We rearrange the Q_i so that $A \not\subseteq Q_i$ for $i = 1, \ldots, j$ and $A \subseteq Q_i$ for $i = j + 1, \ldots, k$. Thus

$$A \cap Q_1 \cap \cdots \cap Q_j \subseteq AU_B .$$

Since the Q_i are primary, $AU_B \subseteq Q_i$ for $i = 1, \ldots, j$ implies that $B \subseteq r(Q_i)$. By Theorem 5.3, for each i there exists an integer $n(i)$ such that $D^{n(i)}(B) \subseteq Q_i$. So if $h = \max\{n(i) \mid i = 1, \ldots, j\}$, $D^h(B) \subseteq Q_i$ for $i = 1, \ldots, j$, and we have $A \cap D^h(B) \subseteq AU_B$, as required.

The converse is clear by virtue of Lemmas 6.1, 6.2, and Corollary 4.4.

The following example shows that the Artin–Rees property is not valid in all unital quadratic Jordan algebras with max-I. Specifically, there exist J with max-I having ideals that do not have a normal primary representation.

EXAMPLE 6.4. Suppose R is the field of real numbers and $\mathfrak{A}$ is the associative algebra of 2×2 upper triangular matrices, i.e., if $a \in \mathfrak{A}$, then $a = \begin{pmatrix} \alpha & \beta \\ 0 & \sigma \end{pmatrix}$ where $\alpha, \beta, \sigma \in R$. Let $\mathfrak{A}^{(q)}$ denote the unital quadratic Jordan algebra over R generated by $\mathfrak{A}$. One readily computes that the nontrivial ideals of $\mathfrak{A}^{(q)}$ are

$$A_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \,\middle|\, \alpha, \beta \in R \right\},$$

$$A_2 = \left\{ \begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix} \,\middle|\, \alpha, \beta \in R \right\},$$

$$A_3 = \left\{ \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \,\middle|\, \alpha \in R \right\}.$$

For all nonnegative integers n, $D^n(A_1) = A_1$ and $D^n(A_2) = A_2$, while $A_3 U_{A_3} = 0$ and $A_i U_{A_j} = 0$ $i, j = 1, 2, 3$, $i \neq j$. It is easy to see that A_1 and A_2 are prime ideals of $\mathfrak{A}^{(q)}$. Moreover, $A_1 \cap A_2 = A_3$. Consequently, for all integers $n \geqslant 0$, $A_1 \cap D^n(A_2) = A_3$ but $A_1 U_{A_2} = 0$. So $\mathfrak{A}^{(q)}$ does not satisfy the Artin–Rees property. Easily the zero ideal provides an example of an ideal that does not have a normal primary representation.

7. Tertiary Ideals

We have seen that if J has max-I and satisfies the Artin–Rees property, then ideals in J have normal primary representations. The assumption of the Artin–Rees property is a disagreeable feature of this theory. We will now show that it is possible to obtain an analogue for quadratic Jordan algebras of the tertiary ideal as given in [7], and we give a decomposition theorem for all quadratic Jordan algebras with max-I.

Suppose $A \lhd J$. The set $t(A) = \{a \in J \mid$ for any $b \notin A$ there exists $c \in \langle b \rangle$, $c \notin A$, such that $\langle c \rangle U_{\langle a \rangle} \subseteq A\}$ is called the tertiary radical of A. Clearly, $A \subseteq t(A)$. An ideal T of J is a tertiary ideal if, whenever A and B are ideals of J with $AU_B \subseteq T$, then either $A \subseteq T$ or $B \subseteq t(T)$.

Lemma 7.1. *Suppose $A \lhd J$ has an irredundant representation by tertiary ideals $T_1,..., T_k$. Then $t(A) = t(T_1) \cap \cdots \cap t(T_k)$.*

Proof. We may assume that $k > 1$.

Suppose $a \in t(T_1) \cap \cdots \cap t(T_k)$ and $b \notin A$. We may also assume that $b \notin T_1$. Then there exists $b_1 \in \langle b \rangle$, $b_1 \notin T_1$ such that $\langle b_1 \rangle U_{\langle a \rangle} \subseteq T_1$. If $b_1 \in T_2$, then $b_1 \notin T_1 \cap T_2$ but $\langle b_1 \rangle U_{\langle a \rangle} \subseteq T_2$. On the other hand, if $b_1 \notin T_2$, we find $b_1' \in \langle b_1 \rangle$ with $b_1' \notin T_2$ and $\langle b_1' \rangle U_{\langle a \rangle} \subseteq T_2$. In either case, there is an element $b_2 \in \langle b \rangle$, $b_2 \notin T_1 \cap T_2$ with $\langle b_2 \rangle U_{\langle a \rangle} \subseteq T_1 \cap T_2$, i.e., $a \in t(T_1 \cap T_2)$. Since k is finite, by repeating this argument, we conclude that $a \in t(A)$.

Conversely, suppose $a \in t(A)$. By irreducibility, there exists an element b such that $b \in T_2 \cap \cdots \cap T_k$, $b \notin A$. Then we find an element $c \in \langle b \rangle$, $c \notin A$, with $\langle c \rangle U_{\langle a \rangle} \subseteq A$. As $c \notin T_1$, $a \in t(T_1)$. Similarly $a \in t(T_j)$ for $j = 2,..., k$, and we have $t(A) \subseteq t(T_1) \cap \cdots \cap t(T_k)$.

Lemma 7.2. *If $T_1,..., T_k$ are tertiary ideals of J such that $t(T_1) = \cdots = t(T_k)$, then $T = T_1 \cap \cdots \cap T_k$ is tertiary and $t(T) = t(T_1)$.*

Proof. Suppose $a, b \in J$ and $\langle a \rangle U_{\langle b \rangle} \subseteq T$. If

$$\langle b \rangle \not\subseteq t(T_1), \langle a \rangle \subseteq T_1 \cap \cdots \cap T_k = T.$$

That $t(T) = t(T_1)$ follows from Lemma 7.1.

Suppose $A \lhd J$ has an irredundant representation by tertiary ideals $T_1,..., T_k$. Then this representation is called a normal tertiary representation if, for all $i \neq j$, $t(T_i) \neq t(T_j)$.

COROLLARY 7.3. *If $A \lhd J$ and A has a finite representation by tertiary ideals, then A has a normal tertiary representation.*

8. TERTIARY REPRESENTATIONS

We begin with

LEMMA 8.1. *Every meet irreducible ideal of J is tertiary.*

Proof. Suppose A is an ideal of J and A is not tertiary. Then there exist elements $b, c \in J$, $b \notin A$, $c \notin t(A)$, with $\langle b \rangle U_{\langle c \rangle} \subseteq A$. Now $c \notin t(A)$ implies the existence of an element $d \in J$, $d \notin A$, such that whenever $d' \in \langle d \rangle$ and $\langle d' \rangle U_{\langle c \rangle} \subseteq A$, then $d' \in A$. Clearly, the ideals $D_1 = A + \langle d \rangle$ and $D_2 = A + \langle b \rangle$ properly contain A. Hence let $x = a_1 + d' = a_2 + b' \in D_1 \cap D_2$, where $a_1, a_2 \in A$, $d' \in \langle d \rangle$, and $b' \in \langle b \rangle$. Then $d' = b' + a'$ where $a' = a_2 - a_1$. Then $\langle d' \rangle U_{\langle c \rangle} = \langle b' + a' \rangle U_{\langle c \rangle} \subseteq \langle b' \rangle U_{\langle c \rangle} + \langle a' \rangle U_{\langle c \rangle} \subseteq A$. Hence $d' \in A$, which implies $x \in A$. Thus $A = D_1 \cap D_2$, i.e., A is not meet irreducible.

THEOREM 8.2. *If J has* max-*I*, *then every ideal A of J has a normal tertiary representation. If $A = T_1 \cap \cdots \cap T_m$ and $A = S_1 \cap \cdots \cap S_n$ are two normal tertiary representations of A, then $m = n$ and it is possible to arrange the components in such a way that $t(T_i) = t(S_i)$ for $i = 1,..., m$.*

Proof. By Lemmas 6.2, 8.1, and Corollary 7.3, it follows that every ideal of J has a normal tertiary representation.

To prove the rest of the theorem, we first suppose that $I \lhd J$ and $I = T_1 \cap B_1 = T_2 \cap B_2$ where T_1 and T_2 are tertiary ideals of J and B_1 and B_2 are ideals of J. Suppose $t(T_1) \neq t(T_2)$, and $b \in t(T_1)$, $b \notin t(T_2)$. If $a \in B_1 \cap B_2$ but $a \notin A$, then $a \notin T_1$, so there exists $c \in \langle a \rangle$ such that $c \notin T_1$ and $\langle c \rangle U_{\langle b \rangle} \subseteq T_1$. Hence $\langle c \rangle U_{\langle b \rangle} \subseteq T_1 \cap B_1 \subseteq T_2$. Therefore, $c \in T_2$ implying $c \in B_2 \cap T_2 \subseteq T_1$, a contradiction. This fact, together with the inclusion $I \subseteq B_1 \cap B_2$ implies $I = B_1 \cap B_2$.

Now suppose $A \lhd J$ and $A = T_1 \cap \cdots \cap T_k = S_1 \cap \cdots \cap S_p$ where these representations are normal tertiary representations. To prove the theorem, it is sufficient to show that for some j, $1 \leqslant j \leqslant p$, $t(T_1) = t(S_j)$. So suppose $t(T_1) \neq t(S_j)$ for $j = 1,..., p$. By what we have shown above, $A = T_1 \cap \cdots \cap T_k = T_2 \cap \cdots \cap T_k \cap S_2 \cap \cdots \cap S_p$. By repeating this process, we obtain $T_1 \cap \cdots \cap T_k = T_2 \cap \cdots \cap T_k$, a contradiction. Therefore, for some j, $t(T_1) = t(S_j)$, and the result follows.

Finally, the question arises as to the relationship between tertiary ideals and primary ideals in J. We have

PROPOSITION 8.3. *Suppose J satisfies the Artin–Rees property. Then every tertiary ideal of J is primary. Conversely, if J has* max-I *and if every tertiary ideal of J is primary, then J satisfies the Artin–Rees property.*

Proof. Suppose T is a tertiary ideal of J, $a, b \in J$, $a \notin T$, and $\langle a \rangle U_{\langle b \rangle} \subseteq T$. Since J has the Artin–Rees property, there exists a nonnegative integer h such that $[T : \langle b \rangle] \cap D^h(\langle b \rangle) \subseteq [T : B] U_{\langle b \rangle} \subseteq T$. We claim that $D^h(\langle b \rangle) \subseteq T$. If $b \in T$, we are done. So suppose $b \notin T$. If $D^h(\langle b \rangle) \nsubseteq T$, we choose $c \in D^h(\langle b \rangle)$, $c \notin T$. Since $b \in t(T)$, there is a $d \in \langle c \rangle$, $d \notin T$, such that $\langle d \rangle U_{\langle b \rangle} \subseteq T$. Thus $d \in [T : \langle b \rangle]$ which implies $d \in T$, a contradiction. Hence $D^h(\langle b \rangle) \subseteq T$, from which it follows that $\langle b \rangle \subseteq r(T)$, i.e., T is primary.

Conversely, suppose J has max-I and that tertiary ideals of J are primary. The result follows from Theorems 8.2 and 6.3.

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