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Stability of Difference Schemes in the Maximum-Norm*

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Necessary and sufficient conditions for stability in the maximum-norm of explicity two-level difference schemes with constant coefficients are given. The sufficiency of the conditions has been proved previously by G. Strang.

1. INTRODUCTION AND MAIN RESULT

Consider an explicit difference scheme with constant coefficients for an initial value problem of the form

$$\frac{\partial u}{\partial t} = \rho \frac{\partial^m u}{\partial x^m}, \qquad t > 0, \qquad \rho = \text{constant}, \tag{1.1}$$

$$u(x, 0) = u_0(x).$$

Such a difference scheme can be written

$$v_0(x) = u_0(x), \qquad v_{n+1}(x) = Av_n(x),$$

where A is a linear operator of the form

$$Av(x) = \sum_j a_j v(x + jh), \tag{1.2}$$

where $h > 0$ and only a finite number of the complex numbers a_j are non-zero. Such an operator is evidently bounded in L^p, $1 \leqslant p \leqslant \infty$; we have by Minkowski's inequality

$$\| Av \|_p \leqslant \sum_j | a_j | \, \| v \|_p , \tag{1.3}$$

where $\| \cdot \|_p$ is the ordinary L^p-norm (in case $p = \infty$, the maximum-norm). We say that such a difference scheme, or the corresponding operator A, is stable in L^p ($1 \leqslant p \leqslant \infty$) if there is a constant C such that

$$\| A^n v \|_p \leqslant C \| v \|_p , \qquad n \geqslant 1, \qquad v \in L^p.$$

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I

If we define for a bounded linear operator B in L^p,

$$\| B \|_p = \sup_{v \in L^p} \| Bv \|_p / \| v \|_p ,$$

then stability in L^p can be defined as the uniform boundedness of $\| A^n \|_p$ for all natural numbers n.

We call the trigonometric polynomial

$$a(\theta) = \sum_j a_j e^{ij\theta},$$

the characteristic polynomial of A. It is easy to see that the characteristic polynomial of A^n is $a(\theta)^n$. It is also easy to prove by Parseval's formula that (cf. e.g. [5])

$$\| A \|_2 = \max | a(\theta) | , \tag{1.4}$$

and it follows at once the well-known fact that A is stable in L^2 if and only if $| a(\theta) | \leqslant 1$ for all real θ (von Neumann's condition).

The aim of this paper is to study stability in L^∞, or stability in the maximum-norm. In the sequel we will therefore generally drop the subscript ∞ and write $\| \cdot \|$ instead of $\| \cdot \|_\infty$. We have in this case for the operator defined in (1.2)

$$\| A \| = \| A \|_\infty = \sum_j | a_j | . \tag{1.5}$$

The stability problem is in this case considerably more complicated than in the L^2-case. It was proved by F. John in his paper [3] about difference methods for parabolic equations that the condition

$$| a(\theta) | \leqslant e^{-\gamma\theta^2} \qquad \text{for some} \qquad \gamma > 0 \qquad \text{and all} \qquad | \theta | \leqslant \pi \tag{1.6}$$

is sufficient for stability in L^∞. This condition is evidently not necessary as is seen by the simple example

$$Av(x) = \tfrac{1}{2} (v(x + h) + v(x - h)).$$

In this case we have stability since $\| A \| = 1$. The characteristic polynomial is $a(\theta) = \cos \theta$, and since $a(\pi) = -1$, (1.6) is not satisfied. With methods similar to John's, G. Strang [9] in connection with investigations about hyperbolic equations was able to prove the sufficiency for stability in L^∞ of conditions generalizing John's. It will be shown that Strang's conditions are also necessary conditions. The complete result can be formulated as follows:

THEOREM 1. *The operator A with characteristic polynomial $a(\theta)$ is stable in the maximum-norm if and only if one of the following two conditions is satisfied:*

(a) $a(\theta) = ce^{ij\theta}$, $|c| = 1$,

(b) $|a(\theta)| < 1$ *except for at most a finite number of points* θ_k, $k = 1, \cdots, N$ *in* $|\theta| \leqslant \pi$ *where* $|a(\theta)| = 1$. *For* $k = 1, \cdots, N$ *there are constants* α_k, γ_k, ν_k, *where* α_k *is real,* $\operatorname{Re}\gamma_k > 0$, *and* ν_k *is an even natural number, such that*

$$a(\theta_k + \theta) = a(\theta_k) \exp(i\alpha_k\theta - \gamma_k\theta^{\nu_k}(1 + o(1))) \qquad \text{when} \qquad \theta \to 0.$$

For the sake of completeness we will reproduce Strang's proof of the sufficiency part of Theorem 1 in Section 2. In Section 3, we then prove the necessity of conditions (a) or (b). This will be done by giving estimates to below of the rate of growth of $\| A^n \|$ in the cases when (a) or (b) is not satisfied. In Section 4 we will finally discuss some applications to initial-value problems of the form (1.1). It will, for instance, be proved that the Lax-Wendroff scheme for a hyperbolic equation is not stable in the maximum-norm, and that in this case

$$\| A^n \| \geqslant Cn^{1/12}.$$

This was conjectured by H. Stetter [7] on the basis of numerical data.

The author is indebted to professor L. Hörmander for a conversation on the subject of this paper, and for suggesting application of the results in [2] to this problem. By using this technique it is in fact possible to prove the necessity part of Theorem 1 even in L^p, $1 \leqslant p \leqslant \infty, p \neq 2$. The sufficiency part of Theorem 1 in L^p follows at once from the result in L^∞ and the fact (cf. (1.3) and (1.5)) that

$$\| A^n \|_p \leqslant \| A^n \|_\infty, \qquad 1 \leqslant p \leqslant \infty. \tag{1.7}$$

We hope to return to this point of view in a later paper, but feel that the significance of the particular case $p = \infty$, and the possibility of obtaining the above mentioned estimates, justifies the presentation of the present elementary proof although it does not lend itself to generalization.

2. SUFFICIENCY

We will introduce the class $\mathscr{A}$ of absolutely convergent trigonometric series

$$a(\theta) = \sum_j a_j e^{ij\theta}, \qquad \sum_j | a_j | < \infty.$$

Evidently, if $a \in \mathscr{A}$, then a is a continuous periodic function of the real variable θ with period 2π, and

$$a_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\theta} \, a(\theta) \, d\theta.$$

Further, $\mathscr{A}$ is a normed linear space with the norm

$$\| a \| = \sum_j | a_j | .$$

We have the trivial lemma:

LEMMA 2.1. *If* $a = \sum_j a_j e^{ij\theta} \in \mathscr{A}$ *and* $b = \sum_j b_j e^{ij\theta}$ *is defined by*

$$b(\theta) = \lambda a(\theta_0 + \theta), \qquad | \lambda | = 1,$$

then $| a_j | = | b_j |$. *In particular,* $b \in \mathscr{A}$ *and*

$$\| b \| = \| a \| .$$

The class $\mathscr{A}$ is also closed under multiplication: if $a, b \in \mathscr{A}$, then $ab \in \mathscr{A}$ and

$$\| ab \| \leqslant \| a \| \cdot \| b \| .$$

In particular, if $a \in \mathscr{A}$, and n is any natural number, then $a^n \in \mathscr{A}$. In the sequel, when $a \in \mathscr{A}$ we will denote by $a_{n,j}$ the Fourier-coefficients of a^n, so that

$$a(\theta)^n = \sum_j a_{n,j} e^{ij\theta},$$

$$a_{n,j} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\theta} \, a(\theta)^n \, d\theta. \tag{2.1}$$

Consider an operator A of the form (1.2) with characteristic polynomial a. We then have $a \in \mathscr{A}$ and

$$\| a \| = \| A \| .$$

We say that $a \in \mathscr{B}$ if $a \in \mathscr{A}$ and $\| a^n \|$ is bounded for all natural numbers n. With this notation, A is stable if and only if its characteristic polynomial belongs to $\mathscr{B}$.

The following lemma is due to Beurling:

LEMMA 2.2. *If $a \in \mathscr{A}$, then*

$$\lim_{n\to\infty} \| a^n \|^{1/n} = \sup_\theta | a(\theta) | .$$

Proof. See [6], p. 428.

It follows in particular:

LEMMA 2.3. *If $a \in \mathscr{A}$ and $| a(\theta) | < 1$ for all θ, then there is a $\rho < 1$ such that for sufficiently large n,*

$$\| a^n \| < \rho^n.$$

In particular, $a \in \mathscr{B}$.

We say that $a \in C^p$ if a is p times continuously differentiable.

LEMMA 2.4. *Let $a \in \mathscr{A} \cap C^2$ be such that $| a(\theta) | < 1$ for $0 < | \theta | \leqslant \pi$, $a(0) = 1$. Assume that $a \in C^\nu$ in a neighborhood of $\theta = 0$ and*

$$a(\theta) = \exp(i\alpha\theta - \gamma\theta^\nu(1 + o(1))) \qquad when \qquad \theta \to 0, \tag{2.2}$$

where α is real, $\operatorname{Re}\gamma > 0$, and ν is an even natural number. Then, if $a_{n,j}$ is defined by (2.1), there is a positive constant C independent of n and j such that

$$| a_{n,j} | \leqslant C n^{-1/\nu}, \tag{2.3}$$

$$| a_{n,j} | \leqslant C n^{1/\nu}(j - \alpha n)^{-2}. \tag{2.4}$$

Proof. By (2.2) there is a $\kappa > 0$ such that

$$| a(\theta) | \leqslant \exp(-\kappa\theta^\nu), \qquad | \theta | \leqslant \pi.$$

By (2.1) we get

$$| a_{n,j} | \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} | a(\theta) |^n \, d\theta \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-n\kappa\theta^\nu) \, d\theta$$

$$\leqslant \frac{1}{2\pi n^{1/\nu}} \int_{-\infty}^{\infty} \exp(-\kappa\theta^\nu) \, d\theta,$$

which proves (2.3). To prove (2.4), we define

$$a_\alpha(\theta) = e^{-i\alpha\theta} \, a(\theta),$$

and get

$$a_{n,j} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(j-n\alpha)\theta} \, a_\alpha(\theta)^n \, d\theta.$$

After two partial integrations we get

$$a_{n,j} = -\frac{n}{2\pi(j-\alpha n)^2}\int_{-\pi}^{\pi} e^{-i(j-n\alpha)\theta}\{a_\alpha(\theta)^{n-1}\, a_\alpha''(\theta) + (n-1)\, a_\alpha(\theta)^{n-2}\, a_\alpha'(\theta)^2\}\, d\theta. \tag{2.5}$$

According to (2.2) we have

$$a_\alpha(\theta) = \exp(-\gamma\theta^\nu(1+o(1))) = 1 - \gamma\theta^\nu + o(\theta^\nu) \quad \text{when} \quad \theta \to 0.$$

It follows for $|\theta| \leqslant \pi$,

$$|a_\alpha'(\theta)| \leqslant C|\theta|^{\nu-1},$$
$$|a_\alpha''(\theta)| \leqslant C\theta^{\nu-2}.$$

(Here and in the sequel C denotes a positive constant. When necessary different constants will be distinguished by subscripts.) We thus get

$$|a_{n,j}| \leqslant C\frac{n}{(j-n\alpha)^2}\left\{\int_{-\pi}^{\pi}\theta^{\nu-2}\exp(-n\kappa\theta^\nu)\, d\theta + n\int_{-\pi}^{\pi}\theta^{2\nu-2}\exp(-n\kappa\theta^\nu)\, d\theta\right\},$$

and since

$$\int_{-\pi}^{\pi}\theta^{\nu-2}\exp(-n\kappa\theta^\nu)\, d\theta \leqslant n^{-(1-1/\nu)}\int_{-\infty}^{+\infty}\theta^{\nu-2}\exp(-\kappa\theta^\nu)\, d\theta,$$
$$\int_{-\pi}^{\pi}\theta^{2\nu-2}\exp(-n\kappa\theta^\nu)\, d\theta \leqslant n^{-(2-1/\nu)}\int_{-\infty}^{+\infty}\theta^{2\nu-2}\exp(-\kappa\theta^\nu)\, d\theta,$$

we finally get (2.4).

We then have:

COROLLARY. *Under the assumptions of Lemma 2.4, $a \in \mathscr{B}$.*

Proof. With the above notation we have

$$\|a^n\| = \sum_j |a_{n,j}| = \sum_{|j-\alpha n|\leqslant n^{1/\nu}} + \sum_{|j-\alpha n|>n^{1/\nu}}$$
$$\leqslant Cn^{-1/\nu}\sum_{|j-\alpha n|\leqslant n^{1/\nu}} 1 + Cn^{1/\nu}\sum_{|j-\alpha n|>n^{1/\nu}}(j-\alpha n)^{-2}$$
$$\leqslant C_1\left(n^{-1/\nu}n^{1/\nu} + n^{1/\nu}\sum_{|j|>n^{1/\nu}} j^{-2}\right)$$
$$\leqslant C_2(1 + n^{1/\nu}n^{-1/\nu}) = C_3,$$

which proves the Corollary.

We can now prove a theorem which contains the sufficiency of condition (b) in Theorem 1 for stability:

THEOREM 2. *Let $a \in \mathscr{A} \cap C^2$ be such that $| a(\theta) | < 1$ except for at most a finite number of points θ_k, $k = 1, \cdots, N$, in $| \theta | \leqslant \pi$, where $| a(\theta) | = 1$. Assume that for $k = 1, \cdots, N$, there are constants α_k, γ_k, ν_k, where α_k is real,* $\operatorname{Re} \gamma_k > 0$, *and ν_k is an even natural number, such that $a \in C^{\nu_k}$ in a neighborhood of $\theta = \theta_k$ and*

$$a(\theta_k + \theta) = a(\theta_k) \exp (i\alpha_k \theta - \gamma_k \theta^{\nu_k}(1 + o(1))) \quad \text{when} \quad \theta \to 0.$$

Then $a \in \mathscr{B}$.

Proof. Lemma 2.3 proves the theorem if $| a(\theta) | < 1$ for all θ, and the Corollary of Lemma 2.4 proves the theorem for the case $N = 1$, $\theta_1 = 0$, $a(0) = 1$. We now turn to the general case $N \geqslant 1$. Let $\delta > 0$ be less than the distance between any two roots of the equation $| a(\theta) | = 1$, and let $\phi \in \mathscr{A} \cap C^2$ satisfy

$$0 \leqslant \phi(\theta) \leqslant 1, \qquad \text{all} \qquad \theta,$$

$$\phi(\theta) = \begin{cases} 1, & | \theta | \leqslant \delta/4, \\ 0, & \delta/2 \leqslant | \theta | \leqslant \pi. \end{cases} \tag{2.6}$$

The existence of such a periodic function is clear. Let

$$b_k(\theta) = a(\theta) \phi(\theta - \theta_k), \qquad k = 1, \cdots N,$$

and notice that the b_k have disjoint supports. Let further

$$a_0(\theta) = a(\theta) - \sum_{j=1}^{N} b_j(\theta) = a(\theta) \left\{1 - \sum_{j=1}^{N} \phi(\theta - \theta_j)\right\}, \tag{2.7}$$

$$a_k(\theta) = b_k(\theta) + a_0(\theta) = a(\theta) \left\{1 - \sum_{j \neq k} \phi(\theta - \theta_j)\right\}, \qquad k = 1, \cdots, N. \tag{2.8}$$

We obtain

$$a(\theta)^n = \sum_{j=1}^{N} a_j(\theta)^n - (N - 1) a_0(\theta)^n. \tag{2.9}$$

This follows, since if θ belongs to the support of $b_k(\theta)$, then the right side reduces to

$$\sum_{j \neq k} a_0(\theta)^n + a_k(\theta)^n - (N - 1) a_0(\theta)^n = a_k(\theta)^n = a(\theta)^n,$$

and if θ does not belong to the support of any of the b_k, the right side reduces to

$$\sum_{j=1}^{N} a_0(\theta)^n - (N-1)\, a_0(\theta)^n = a_0(\theta)^n = a(\theta)^n.$$

It follows from (2.7) that $| a_0(\theta) | < 1$ for all θ, so that by Lemma 2.3, $a_0 \in \mathscr{B}$. It follows from (2.8) and the assumptions that $a_k(\theta_k + \theta)/a(\theta_k)$, $k \neq 0$, satisfies the conditions in Lemma 2.4. Thus by the Corollary, this function belongs to $\mathscr{B}$. According to Lemma 2.1, this implies that $a_k \in \mathscr{B}$. By (2.9), we can finally conclude that $a \in \mathscr{B}$.

Since evidently the condition (a) is sufficient for stability, this concludes the proof of the sufficiency part of Theorem 1.

3. Necessity

For the proof of the necessity part of Theorem 1, let A be an operator of the form (1.2), and let $a(\theta)$ be its characteristic polynomial. We first notice that Lemma 2.2 implies that in order that $a \in \mathscr{B}$, it is necessary that

$$| a(\theta) | \leqslant 1, \qquad \text{all} \qquad \theta.$$

(This follows also directly from (1.4), and (1.7) with $p = 2$.) Since with $a(\theta)$, $| a(\theta) |^2$ is also a trigonometric polynomial, we can conclude that one of the following two conditions is satisfied for $a \in \mathscr{B}$:

(a′) $| a(\theta) | \equiv 1$,

(b′) $| a(\theta) | < 1$ for all but at most a finite number of points θ_k, $k = 1, \cdots, N$, in $| \theta | \leqslant \pi$.

We first prove that condition (a′) implies condition (a) in Theorem 1:

LEMMA 3.1. *Let $a(\theta)$ be a trigonometric polynomial with $| a(\theta) | \equiv 1$. Then $a(\theta) = ce^{ij\theta}$ where $| c | = 1$ and j is an integer.*

Proof. Let

$$a(\theta) = \sum_{j=P}^{Q} c_j e^{ij\theta}, \qquad c_P, c_Q \neq 0. \tag{3.1}$$

We want to prove that if $| a(\theta) | \equiv 1$, then $P = Q$. Assume that $Q > P$. We would then have

$$| a(\theta) |^2 = a(\theta)\, \overline{a(\theta)} = \sum_{j,k=P}^{Q} c_j \bar{c}_k e^{i(j-k)\theta}.$$

The trigonometric polynomial on the right has the coefficient $c_P\bar{c}_Q$ of $e^{i(P-Q)\theta}$, but since the polynomial is identically 1, this means that $c_P\bar{c}_Q = 0$, which contradicts (3.1), and the lemma follows at once.

We now turn to the case (b′). Consider generally, in the neighborhood of one of the points θ_k, $k = 1, \cdots, N$, an $a \in \mathscr{A}$ satisfying (b′). We say that θ_k is a point of type β for a, or $\theta_k \in \beta$, if there are μ_k, ν_k, α_k, β_k, γ_k, $q_k(\theta)$, where μ_k, ν_k are natural numbers with $1 < \mu_k < \nu_k$, ν_k even, α_k real, $\operatorname{Re} \gamma_k > 0$, $q_k(\theta)$ is a real polynomial with $q_k(0) = \beta_k \neq 0$, such that $a \in C^{\nu_k}$ in a neighborhood of $\theta = \theta_k$ and

$$a(\theta_k + \theta) = a(\theta_k) \exp (i\alpha_k\theta + i\theta^{\mu_k} q_k(\theta) - \gamma_k\theta^{\nu_k}(1 + o(1))) \quad \text{when} \quad \theta \to 0. \tag{3.2}$$

We say that θ_k is a point of type γ for a, or $\theta_k \in \gamma$, if there are ν_k, α_k, γ_k, where ν_k is an even natural number, α_k a real number, and $\operatorname{Re} \gamma_k > 0$, such that $a \in C^{\nu_k}$ in a neighborhood of $\theta = \theta_k$ and

$$a(\theta_k + \theta) = a(\theta_k) \exp (i\alpha_k\theta - \gamma_k\theta^{\nu_k}(1 + o(1))) \qquad \text{when} \qquad \theta \to 0.$$

Since a trigonometric polynomial is an analytic function, then if it satisfies condition (b′), the points θ_k, $k = 1, \cdots, N$, are necessarily points of type β or γ, and $\theta_k \in \beta$ if and only if the first nonvanishing nonlinear term in the MacLaurin expansion of $\log [a(\theta_k + \theta)/a(\theta_k)]$ is purely imaginary.

The fact that for a trigonometric polynomial $a \in \mathscr{B}$, the condition (b′) implies condition (b) in Theorem 1 is contained in the following theorem:

THEOREM 3. *Let $a \in \mathscr{A} \cap C^2$ be such that $|a(\theta)| < 1$ except for a finite number of points θ_k, $k = 1, \cdots, N$, in $|\theta| \leqslant \pi$, which are of type β or γ, and of which at least one is of type β. Let*

$$\mu_0 = \max_{\theta_k \in \beta} \mu_k, \qquad \nu_0 = \max_{\theta_k \in \beta} \nu_k.$$

Then there is a positive constant C such that

$$\| a^n \| \geqslant Cn^{1/\mu_0 - 1/\nu_0},$$

and, in particular, $a \notin \mathscr{B}$.

For the proof we will need a number of lemmas. The first two lemmas are due to van der Corput.

LEMMA 3.2. *Assume that $f(x) \in C^2$ is a real-valued function in $[a, b]$ such that $f'(x)$ is monotone and $|f'(x)| \geqslant \lambda > 0$. Then*

$$\left| \int_a^b \exp (if(x))\, dx \right| \leqslant \frac{4}{\lambda}. \tag{3.3}$$

Proof. Let $F(x) = \exp(if(x))$. We have

$$dF(x) = if'(x) F(x)\, dx,$$

and so

$$iI = i \int_a^b F(x)\, dx = \int_a^b \frac{dF(x)}{f'(x)} = I_1 + iI_2,$$

where I_1 and I_2 are the real and imaginary parts of iI. Let F_1 and F_2 be the real and imaginary parts of F. The second mean-value theorem then gives for certain ξ_j with $a \leqslant \xi_j \leqslant b$,

$$I_j = \frac{1}{f'(a)} \int_a^{\xi_j} dF_j(x) + \frac{1}{f'(b)} \int_{\xi_j}^b dF_j(x)$$

$$= \frac{1}{f'(a)} (F_j(\xi_j) - F_j(a)) + \frac{1}{f'(b)} (F_j(b) - F_j(\xi_j)),$$

and thus

$$iI = \frac{1}{f'(a)} \{[F_1(\xi_1) + iF_2(\xi_2)] - F(a)\} + \frac{1}{f'(b)} \{F(b) - [F_1(\xi_1) + iF_2(\xi_2)]\}$$

$$= \left[\frac{1}{f'(a)} - \frac{1}{f'(b)}\right] [F_1(\xi_1) + iF_2(\xi_2)] + \frac{1}{f'(b)} F(b) - \frac{1}{f'(a)} F(a),$$

which gives (3.3).

LEMMA 3.3. *Assume that $f \in C^2$ is a real-valued function in $[a, b]$ such that $| f''(x) | \geqslant \rho > 0$. Then*

$$\left| \int_a^b \exp(if(x))\, dx \right| \leqslant \frac{8}{\sqrt{\rho}}. \tag{3.4}$$

Proof. Consider the case $f''(x) \geqslant \rho$. The case $f''(x) \leqslant -\rho$ can be treated analogously. The function $f'(x)$ is then increasing. Assume first that $f'(x)$ has a constant sign in (a, b), say $f' \geqslant 0$ (the case $f' \leqslant 0$ can be treated similarly). If $a < c < b$, then $f''(x) \geqslant \rho$ implies

$$f'(x) \geqslant (x - a)\rho \geqslant (c - a)\rho \qquad \text{for} \qquad c \leqslant x \leqslant b,$$

and we get by Lemma 3.2 with the above notation

$$| I | \leqslant \left| \int_a^c F(x)\, dx \right| + \left| \int_c^b F(x)\, dx \right| \leqslant c - a + \frac{4}{(c - a)\rho},$$

or if $c - a = 2/\sqrt{\rho}$,

$$|I| \leqslant \frac{4}{\sqrt{\rho}}. \tag{3.5}$$

(If $b - a < 2/\sqrt{\rho}$, this inequality is trivially valid.) If $f'(x)$ changes sign in (a, b), then (a, b) is the union of two intervals in which $f'(x)$ has constant sign, and (3.5) can be applied on each of these, giving finally (3.4).

LEMMA 3.4. *Let $q(x)$ be a real polynomial with $\beta = q(0) \neq 0$ and μ a natural number > 1. Set*

$$\psi_{n,\alpha}(\theta) = \int_0^\theta \exp(i\alpha x + inx^\mu q(x))\, dx.$$

Then there are positive constants δ and C, independent of n, α, μ, such that for $|\theta| \leqslant \delta$,

$$|\psi_{n,\alpha}(\theta)| \leqslant Cn^{-1/\mu},$$

When $q(x) = $ constant, δ can be chosen arbitrarily.

Proof. Consider the polynomial $q_1(x) = x^\mu q(x)$. We have

$$q_1''(x) = x^{\mu-2} q_2(x), \qquad q_2(0) = \mu(\mu - 1)\beta = 2\rho \neq 0.$$

Let δ be so small that $|q_2(x)| \geqslant \rho$ for $|x| \leqslant \delta$. This is true for any $\delta > 0$ if $q(x) = $ constant. If $0 < |\theta_0| \leqslant |\theta| \leqslant \delta$, we have for $|\theta_0| \leqslant |x| \leqslant |\theta|$,

$$\left| \frac{d^2}{dx^2} (\alpha x + nx^\mu q(x)) \right| \geqslant n\rho\, |\theta_0|^{\mu-2},$$

and so by Lemma 3.3,

$$|\psi_{n,\alpha}(\theta)| \leqslant \left| \int_0^{\theta_0} \right| + \left| \int_{\theta_0}^{\theta} \right| \leqslant |\theta_0| + \frac{8}{\sqrt{n\rho\, |\theta_0|^{\mu-2}}}.$$

This inequality is evidently valid also if $|\theta| \leqslant |\theta_0|$. With $|\theta_0| = n^{-1/\mu}$, we get

$$|\psi_{n,\alpha}(\theta)| \leqslant \left(1 + \frac{8}{\sqrt{\rho}}\right) n^{-1/\mu},$$

which proves the lemma.

LEMMA 3.5. *Assume that $a \in \mathscr{A} \cap C^1$ satisfies condition* (b') *with $N = 1$, $\theta_1 = 0$, $a(0) = 1$, and that $\theta_1 = 0$ is a point of type β for a. Then, if $a_{n,j}$ are defined by (2.1), there is a constant C independent of n and j such that*

$$|a_{n,j}| \leqslant Cn^{-1/\mu_1}. \tag{3.6}$$

Proof. We drop all subscripts 1 and define

$$\tilde{a}(\theta) = \exp\left[-i\alpha\theta - i\theta^{\mu}q(\theta)\right] a(\theta),$$

and get

$$\tilde{a}(\theta) = \exp\{-\gamma\theta^{\nu}[1 + o(1)]\} \qquad \text{when} \qquad \theta \to 0.$$

It follows that there are positive constants κ and C_1 such that

$$| \tilde{a}(\theta) | \leqslant \exp(-\kappa\theta^{\nu}), \qquad | \theta | \leqslant \pi,$$
$$| \tilde{a}'(\theta) | \leqslant C_1 | \theta |^{\nu-1}, \qquad | \theta | \leqslant \pi.$$

We get with the notation of Lemma 3.4,

$$a_{n,j} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ij\theta} a(\theta)^n \, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi'_{n,n\alpha-j}(\theta)\, \tilde{a}(\theta)^n \, d\theta$$
$$= \int_{|\theta|\leqslant\delta} + \int_{\delta\leqslant|\theta|\leqslant\pi} = I_1 + I_2 . \tag{3.7}$$

For the first term we obtain by partial integration

$$I_1 = [\psi_{n,n\alpha-j}\tilde{a}(\theta)^n]_{-\delta}^{\delta} - n\int_{-\delta}^{\delta} \psi_{n,n\alpha-j}(\theta)\, \tilde{a}'(\theta)\, \tilde{a}(\theta)^{n-1} \, d\theta,$$

and thus by Lemma 3.4,

$$| I_1 | \leqslant C_2 \left(n^{-1/\mu} + n^{1-1/\mu}\int_{-\delta}^{\delta} | \theta |^{\nu-1} \exp(-n\kappa\theta^{\nu})\, d\theta\right)$$
$$\leqslant C_3 n^{-1/\mu}\left(1 + \int_{-\infty}^{+\infty} | \theta |^{\nu-1} \exp(-\kappa\theta^{\nu})\, d\theta\right) \leqslant C_4 n^{-1/\mu}. \tag{3.8}$$

For the second term in (3.7) we get

$$| I_2 | \leqslant \exp(-n\kappa\delta^{\nu}) \leqslant C_5 n^{-1/\mu}. \tag{3.9}$$

Together the estimates (3.8) and (3.9) imply (3.6).

LEMMA 3.6. *Assume that* $a \in \mathscr{A} \cap C^1$ *satisfies condition* (b′), *and that the points* θ_k, $k = 1, \cdots, N$, $N \geqslant 1$, *are points of type* β *for* a. *Then, if* $a_{n,j}$ *is defined by (2.1) and* μ_k *by (3.2), there is a constant* C *independent of* n *and* j *such that*

$$| a_{n,j} | \leqslant C n^{-1/\mu_0}, \qquad \mu_0 = \max_k \mu_k . \tag{3.10}$$

Proof. Exactly as in the proof of Theorem 2, we can write

$$a(\theta)^n = \sum_{k=1}^{N} a_k(\theta)^n - (N-1)\, a_0(\theta)^n,$$

where for $k = 1, \cdots, N$, $| a_k(\theta) | < 1$ for $0 < | \theta - \theta_k | \leqslant \pi$, $a_k(\theta) \equiv a(\theta)$ in a neighborhood of θ_k, and $| a_0(\theta) | < 1$ for all θ. It is then easy to conclude (3.10) by applying Lemma 3.5 to each of the functions $a_k(\theta_k + \theta)/a(\theta_k)$, $k = 1, \cdots, N$, and using Lemma 2.1 and Lemma 2.3.

LEMMA 3.7. *Let $a \in \mathscr{A}$ be such that $| a(\theta) | < 1$ for $0 < | \theta | \leqslant \delta$, and*

$$| a(\theta) |^2 = \exp(-\kappa\theta^\nu(1 + o(1))) \qquad when \qquad \theta \to 0, \tag{3.11}$$

where $\kappa > 0$. Then there is a positive constant C such that

$$\int_{-\delta}^{\delta} | a(\theta) |^{2n}\, d\theta = n^{-1/\nu}(C + o(1)) \qquad when \qquad n \to \infty.$$

Proof. We have

$$\int_{-\delta}^{\delta} | a(\theta) |^{2n}\, d\theta = n^{-1/\nu} \int_{-\delta n^{1/\nu}}^{\delta n^{1/\nu}} | a(\theta n^{-1/\nu}) |^{2n}\, d\theta.$$

The function f_n defined by

$$f_n(\theta) = \begin{cases} | a(\theta n^{-1/\nu}) |^{2n}, & | \theta | \leqslant \delta n^{1/\nu}, \\ 0 \quad , & | \theta | > \delta n^{1/\nu}, \end{cases}$$

then by (3.11) has the property that for all θ,

$$\lim_{n\to\infty} f_n(\theta) = \exp(-\kappa\theta^\nu).$$

Since by the assumptions, for some $\lambda > 0$, and for $| \theta | \leqslant \delta$, $| a(\theta) |^2 \leqslant \exp(-\lambda\theta^\nu)$, it follows with the same λ and for all θ that $f_n(\theta) \leqslant \exp(-\lambda\theta^\nu)$. Thus by Lebesgue's theorem on dominated convergence,

$$\lim_{n\to\infty} \int_{-\delta n^{1/\nu}}^{\delta n^{1/\nu}} | a(\theta n^{-1/\nu}) |^{2n}\, d\theta = \lim_{n\to\infty} \int_{-\infty}^{+\infty} f_n(\theta)\, d\theta = \int_{-\infty}^{+\infty} \exp(-\kappa\theta^\nu)\, d\theta = C,$$

which proves the lemma.

LEMMA 3.8. *Let $a \in \mathscr{A}$ be such that $| a(\theta) | < 1$ except for at most a finite number of points θ_k, $k = 1, \cdots, N, N \geqslant 1$, in $| \theta | \leqslant \pi$ where $| a(\theta) | = 1$. Assume that for $k = 1, \cdots, N$, there are constants κ_k, ν_k, where $\kappa_k > 0$ and ν_k is an even natural number such that*

$$| a(\theta_k + \theta) |^2 = \exp(-\kappa_k\theta^{\nu_k}(1 + o(1))) \qquad when \qquad \theta \to 0.$$

Then if $a_{n,j}$ *is defined by (2.1) and* $\nu_0 = \max_k \nu_k$, *there is a positive constant* C *such that*

$$\sum_j |a_{n,j}|^2 = n^{-1/\nu_0}(C + o(1)) \qquad \textit{when} \qquad n \to \infty.$$

Proof. Let $\delta > 0$ be smaller than half the distance between any two roots of $| a(\theta) | = 1$. We have by Parseval's formula

$$2\pi \sum_j | a_{n,j} |^2 = \int_{-\pi}^{\pi} | a(\theta) |^{2n} \, d\theta$$

$$= \sum_{k=1}^{N} \int_{-\delta}^{\delta} | a(\theta_k + \theta) |^{2n} \, d\theta + \int_S | a(\theta) |^{2n} \, d\theta,$$

where S is a subset of $[-\pi, \pi]$ such that

$$q = \sup_S | a(\theta) |^2 < 1.$$

We therefore have by Lemma 3.7 that

$$\sum_j | a_{n,j} |^2 = \sum_{k=1}^{N} n^{-1/\nu_k}(C_k + o(1)) + O(q^n) \qquad \text{when} \qquad n \to \infty,$$

which proves the lemma.

Proof of Theorem 3. We first prove the theorem under the assumption that there are no points of type γ. We can apply Lemma 3.8 with $\kappa_k = 2 \operatorname{Re} \gamma_k$, and get with the notation (2.1) for sufficiently large n,

$$\sum_j | a_{n,j} |^2 \geqslant C_1 n^{-1/\nu_0}.$$

On the other hand, by Lemma 3.6, we obtain

$$| a_{n,j} | \leqslant C_2 n^{1/\mu_0},$$

and so since

$$\sum_j | a_{n,j} |^2 \leqslant \max_j | a_{n,j} | \cdot \sum_j | a_{n,j} | = \max_j | a_{n,j} | \cdot \| a^n \| ,$$

we get

$$\| a^n \| \geqslant C_1 n^{-1/\nu_0}(C_2 n^{-1/\mu_0})^{-1} = C_3 n^{1/\mu_0 - 1/\nu_0},$$

which proves to theorem in this case.

We now turn to the general case. As in the proof of Theorem 2, let $\phi \in \mathscr{A} \cap C^2$ be a function satisfying (2.6). Also, let

$$\phi_\beta(\theta) = \sum_{\theta_k \in \beta} \phi(\theta - \theta_k),$$

$$\phi_\gamma(\theta) = \sum_{\theta_k \in \gamma} \phi(\theta - \theta_k),$$

$$\phi_0(\theta) = 1 - \phi_\beta(\theta) - \phi_\gamma(\theta).$$

We set

$$a_0(\theta) = a(\theta)\,\phi_0(\theta),$$
$$a_\beta(\theta) = a(\theta)\,(1 - \phi_\gamma(\theta)),$$
$$a_\gamma(\theta) = a(\theta)\,(1 - \phi_\beta(\theta)),$$

and obtain in the same manner as before

$$a(\theta)^n = a_\beta(\theta)^n + a_\gamma(\theta)^n - a_0(\theta)^n. \tag{2.12}$$

Since $| a_0(\theta) | < 1$ for all θ, $a_0 \in \mathscr{B}$. Further $a_\gamma(\theta) \equiv a(\theta)$ in the neighborhood of all points of a of type γ and $| a_\gamma(\theta) | < 1$ for all other points. Hence by Theorem 2, $a_\gamma \in \mathscr{B}$. But $a_\beta(\theta) \equiv a(\theta)$ in the neighborhood of all points of a of type β, and $| a_\beta(\theta) | < 1$ for all other points, and so by the above,

$$\| a_\beta^n \| \geqslant C_3 n^{1/\mu_0 - 1/\nu_0},$$

and so the desired result follows at once from (3.12). This concludes the proof of Theorem 3.

4. Applications

We return to the initial value problem

$$\frac{\partial u}{\partial t} = \rho \frac{\partial^m u}{\partial x^m}, \qquad t > 0, \qquad \rho = \text{constant}, \tag{4.1}$$

$$u(x, 0) = u_0(x), \tag{4.1}$$

and define recursively approximations $v_n(x)$ to $u(x, nk)$ by means of

$$v_0(x) = u_0(x),$$

$$v_{n+1}(x) = \sum_j a_j v_n(x + jh) = A v_n(x). \tag{4.2}$$

Here h and k are considered variable but the ratio $\lambda = k/h^m$ is kept constant.

We say that the explicit difference scheme (4.2) (with mesh-ratio λ), or the corresponding operator A, is consistent with the differential equation (4.1) if for solutions u of (4.1) in C^∞,

$$u(x, t + k) = Au(x, t) + o(k) \qquad \text{when} \qquad k \to 0, \tag{4.3}$$

and that the order of accuracy is p if p is the largest integer such that for all such u,

$$u(x, t + k) = Au(x, t) + O(h^{m+p}) \qquad \text{when} \qquad h \to 0. \tag{4.4}$$

It follows that A is consistent with (4.1) if and only if the order of accuracy is at least 1.

As is well known, these definitions can also be expressed in terms of the characteristic polynomial $a(\theta)$ of A:

LEMMA 4.1. *The difference scheme (4.2) is consistent with the equation (4.1) if and only if*

$$a(\theta) = \exp\,(\rho\lambda(i\theta)^m + o(\theta^m)) \qquad \textit{when} \qquad \theta \to 0, \tag{4.5}$$

and its order of accuracy is p if and only if there is a $\gamma \neq 0$ such that

$$a(\theta) = \exp\,(\rho\lambda(i\theta)^m - \gamma\theta^{m+p} + o(\theta^{m+p})) \qquad \textit{when} \qquad \theta \to 0. \tag{4.6}$$

Proof. We have for any solution of (4.1) in C^∞,

$$u(x, t + k) - Au(x, t) = u(x, t) + k\,\frac{\partial u}{\partial t}(x, t) + o(k) - \sum_j a_j u(x + jh, t)$$

$$= u(x, t) + \rho\lambda h^m \frac{\partial^m u}{\partial x^m}(x, t) - \sum_{s=0}^{m} \frac{h^s}{s!}\left(\sum_j j^s a_j\right)\frac{\partial^s u}{\partial x^s}(x, t) + o(h^m) \text{ when } h \to 0,$$

and thus (4.3) is satisfied if and only if

$$\frac{1}{s!}\sum_j j^s a_j = \begin{cases} 1, & s = 0, \\ 0, & s = 1, \quad \cdots, \quad m - 1, \quad (\text{if } m > 1), \\ \rho\lambda, & s = m \end{cases} \tag{4.7}$$

On the other hand,

$$\exp\,(\rho\lambda(i\theta)^m) - a(\theta) = 1 + \rho\lambda(i\theta)^m - \sum_{s=0}^{m} \frac{(i\theta)^s}{s!}\left(\sum_j j^s a_j\right) + o(\theta^m) \text{ when } \theta \to 0,$$

and so

$$a(\theta) = \exp\,(\rho\lambda(i\theta)^m) + o(\theta^m) \qquad \text{when} \qquad \theta \to 0, \tag{4.8}$$

if and only if the conditions (4.7) are satisfied. But (4.8) and (4.5) are equivalent. This proves the statement about the consistency. The statement about the order of accuracy is proved similarly.

We can now express a number of consequences of Theorem 1 in terms of the concepts introduced in this section:

THEOREM 4. *In order that (4.1) admits a stable, consistent, explicit difference scheme, it is necessary that* (i) *m even*, Re $(-1)^{m/2}\rho < 0$ *(parabolic case) or* (ii) $m = 1$, ρ *real (hyperbolic case). In case* (ii) *a stable explicit scheme has necessarily an odd order of accuracy. On the other hand, in the cases* (i) *and* (ii) *there exist stable, consistent, explicit difference schemes. If the operator A in the scheme (4.2) has the characteristic polynomial* $a(\theta)$ *and if* $|a(\theta)| < 1$ *for* $0 < |\theta| \leqslant \pi$, *then the scheme is stable in case* (i) *if it is consistent, and in case* (ii) *if* $a(\theta)$ *satisfies (4.6) with* $m = 1$, Re $\gamma > 0$, *and p odd.*

Proof. The necessity and sufficiency of the conditions follow at once from Theorem 1 and Lemma 4.1. It remains only to prove the existence of stable consistent operators in the two cases. It is then sufficient to consider the schemes with the characteristic polynomials.

(i) $a(\theta) = 1 + (-1)^{m/2}\rho\lambda\, 2^{m/2}(1 - \cos\theta)^{m/2}$ with λ such that $|1 + (-1)^{m/2}\rho\lambda\, 2^m| < 1$.

(ii) $a(\theta) = \cos\theta + i\rho\lambda\sin\theta$ with λ such that $|\rho|\lambda < 1$.

Further examples in case (ii) are the operators of odd order $2p - 1$ of accuracy based on $2p$ points considered by Strang [*8*, *9*] (of which the above scheme of Friedrichs [*1*] is a special case). On the other hand, the explicit schemes of even order $2p$ of accuracy using $2p + 1$ points, also investigated by strang in [*8*], and shown there to be stable in L^2, cannot according to our results be stable in the maximum-norm. A special example of these operators is ($m = 1$) the Lax-Wendroff scheme (cf. [*4*]), defined by

$$a(\theta) = (\rho\lambda)^2\cos\theta - i\rho\lambda\sin\theta + 1 - (\rho\lambda)^2.$$

The instability of schemes of even order of accuracy in the hyperbolic case does not necessarily imply that such schemes are useless for numerical purposes. For any explicit scheme with characteristic polynomial

$$a(\theta) = \sum_{|j|\leqslant P} a_j e^{ij\theta},$$

where $|a(\theta)| \leqslant 1$ for all θ, so that the scheme is stable in L^2, we have by the Cauchy inequality and Parseval's formula,

$$\|a^n\|^2 = \Big(\sum_{|j|\leqslant nP} |a_{n,j}|\Big)^2 \leqslant (2nP + 1)\sum_{|j|\leqslant nP} |a_{n,j}|^2$$
$$= (2nP + 1)\frac{1}{2\pi}\int_{-\pi}^{\pi} |a(\theta)|^{2n}\, d\theta \leqslant 2nP + 1.$$

Thus

$$\| A^n \| \leqslant \sqrt{2nP + 1},$$

and so does not grow too fast with n.

As an illustration, we consider in some detail the Lax-Wendroff scheme for the equation (4.1) with $m = 1$ and ρ real. We assume that $0 < | \rho | < 1$ and $\lambda = 1$ so that $k = h$. We obtain

$$\begin{aligned} a(\theta) &= \rho^2 \cos \theta - i\rho \sin \theta + 1 - \rho^2 \\ &= \exp\left(-\rho i\theta + \tfrac{1}{6}\rho(1-\rho^2)\, i\theta^3 - \tfrac{1}{8}\rho^2(1-\rho^2)\,\theta^4 + o(\theta^4)\right) \\ &= \exp\left(i\alpha\theta + i\beta\theta^3 - \gamma\theta^4 + o(\theta^4)\right) \qquad \text{when} \qquad \theta \to 0, \end{aligned} \tag{4.9}$$

$$| a(\theta) |^2 = 1 - 4\rho^2(1-\rho^2)\sin^4(\theta/2),$$

and so $| a(\theta) | < 1$ for $0 < | \theta | \leqslant \pi$. The point $\theta = 0$ is a point of type β with $\mu = 3$, $\nu = 4$, and so by Theorem 3,

$$\| A^n \| \geqslant Cn^{1/3-1/4} = Cn^{1/12}.$$

This estimate was conjectured by H. Stetter [7] on the basis of numerical evidence.

It follows from Lemma 3.5 that

$$| a_{n,j} | \leqslant Cn^{-1/3}. \tag{4.10}$$

We can also prove:

LEMMA 4.2. *If $a(\theta)$ is defined by (4.9) where ρ is real and $0 < | \rho | < 1$, then*

$$| a_{n,j} | \leqslant Cn^{2/3}(j + \rho n)^{-2}. \tag{4.11}$$

Proof. We introduce in the same manner as previously

$$a_\alpha(\theta) = e^{-i\alpha\theta}\, a(\theta) = \exp\left(i\beta\theta^3 - \gamma\theta^4 + o(\theta^4)\right) \qquad \text{when} \qquad \theta \to 0,$$

and obtain by (2.5)

$$a_{n,j} = -\frac{n}{2\pi(j-\alpha n)^2}\int_{-\pi}^{\pi} e^{-i(j-n\alpha)\theta}\{a_\alpha(\theta)^{n-1}\, a_\alpha''(\theta) + (n-1)\, a_\alpha(\theta)^{n-2}\, a_\alpha'(\theta)^2\}\, d\theta.$$

With

$$\begin{aligned} \tilde{a}(\theta) = \exp(-i\beta\theta^3)\, a_\alpha(\theta) &= \exp(-i\alpha\theta - i\beta\theta^3)\, a(\theta) \\ &= \exp(-\gamma\theta^4 + o(\theta^4)) \qquad \text{when} \qquad \theta \to 0, \end{aligned}$$

$$\psi_{n,\alpha}(\theta) = \int_0^\theta \exp(i\alpha x + in\beta x^3)\, dx,$$

this can be written

$$a_{n,j} = -\frac{n}{2\pi(j-\alpha n)^2}\int_{-\pi}^{\pi} \psi'_{n-1,n\alpha-j}(\theta)\,\tilde{a}(\theta)^{n-1}\,a_\alpha''(\theta) + (n-1)\,\psi'_{n-2,n\alpha-j}(\theta)\,\tilde{a}(\theta)^{n-2}\,a_\alpha'(\theta)^2\}\,d\theta.$$

Integrating by parts, and using the facts that for $|\theta| \leqslant \pi$,

$$| a_\alpha^{(k)}(\theta) | \leqslant C_1 | \theta |^{3-k}, \qquad k = 1, 2, 3,$$
$$| \tilde{a}(\theta) | \leqslant \exp(-\kappa\theta^4), \qquad \kappa > 0,$$
$$| \tilde{a}'(\theta) | \leqslant C_2 | \theta |^3,$$

and that by Lemma 3.4,

$$| \psi_{n,\alpha}(\theta) | \leqslant C_3 n^{-1/3},$$

we obtain

$$| a_{n,j}| \leqslant C_4 \frac{n^{2/3}}{(j-\alpha n)^2}\Big\{ n\exp(-n\kappa\pi^4) + \int_{-\pi}^{\pi}[1 + n|\theta|^3 + n\theta^4 + n^2|\theta|^7]\exp(-\kappa n\theta^4)\,d\theta\Big\}$$
$$\leqslant Cn^{2/3}(j-\alpha n)^{-2},$$

which proves the lemma since $\alpha = -\rho$.

Together, (4.10) and (4.11) imply:

COROLLARY. *For the Lax-Wendroff scheme defined by (4.9) with ρ real and $0 < |\rho| < 1$, we have*

$$\| A^n \| \leqslant Cn^{1/6}.$$

Proof. We have by (4.10) and (4.11),

$$\| A^n \| = \sum_j | a_{n,j} | = \sum_{|j+\rho n| \leqslant \sqrt{n}} + \sum_{|j+\rho n| > \sqrt{n}}$$
$$\leqslant C_1 \Big(n^{1/2} n^{-1/3} + n^{2/3} \sum_{|j+\rho n| > \sqrt{n}} (j+\rho n)^{-2} \Big)$$
$$\leqslant C_2(n^{1/6} + n^{2/3}n^{-1/2}) = Cn^{1/6}.$$

This improves a result by H. Stetter [7] who has proved that in this case

$$\| A^n \| \leqslant Cn^{1/4}.$$

Thus the rate of growth of $\| A^n \|$ is in this case very small from a numerical point of view, and it is still true that the solution of the discrete problem converges rapidly to the solution of the continuous counterpart when $h \to 0$. More precisely, let $u \in C^3$ be a solution of (4.1) and assume that u has bounded third derivatives (or equivalently, assume that $u_0(x)$ has a bounded third derivative), Let v_n be the solution of the corresponding discrete problem and let $T > 0$ be fixed. Then

$$\sup_{x, nh \leqslant T} | u(x, nh) - v_n(x) | \leqslant C h^{11/6}. \tag{4.12}$$

This follows since

$$u(x, (n+1)h) = Au(x, nh) + \epsilon_n(x), \qquad \sup_{x, nh \leqslant T} | \epsilon_n(x) | \leqslant C_1 h^3,$$

and so with $w_n(x) = u(x, nh) - v_n(x)$,

$$w_0(x) = 0,$$
$$w_{n+1}(x) = Aw_n(x) + \epsilon_n(x).$$

That is

$$w_n(x) = \sum_{j=0}^{n-1} A^j \epsilon_{n-j-1}(x),$$

and so for $nh \leqslant T$,

$$\sup_{x, nh \leqslant T} | w_n(x) | \leqslant C_2 h^3 \sum_{j=0}^{n-1} j^{1/6} \leqslant C_3 n^{7/6} h^3 \leqslant C_3 T^{7/6} h^{11/6},$$

which proves (4.12).

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