Electric polarizability of mesons in semirelativistic quark models

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Abstract

The electric polarizability of mesons, in particular, that of the charged pion, is studied in the framework of a semirelativistic description of hadrons as bound states of valence quarks in terms of a Hamiltonian composed of the relativistic kinetic energy as well as a phenomenological potential describing the strong interactions between the quarks. The quark-core contribution to the electric polarizability obtainable in quark models of this kind is in the semirelativistic approaches even smaller than in the nonrelativistic limit.
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1. Introduction

An elementary particle may be considered as fundamental, and thus called "pointlike", if it does not exhibit an experimentally observable internal structure, i.e., if it does not have a discernible spatial extension and is not composed of detectable subcomponents. Nonvanishing electromagnetic polarizabilities of a particle, on the other hand, provide a clear empirical evidence that this particle cannot be fundamental in the above sense. Rather, any particle with this property has to be regarded as a *composite* system. More precisely, it must be built up of subcomponents carrying nonvanishing electric charges.

The electromagnetic polarizabilities of a particle characterize the dipole moments induced by the presence of an external electromagnetic field. They therefore constitute fundamental quantities which represent a measure of the rigidity, stiffness or resistance to deformation of the internal structure of this composite system upon imposition of an external electromagnetic field. Phrased the other way round, they probe the ease with which a composite system can be polarized by the external field. Clearly, more detailed analyses of electromagnetic polarizabilities will then yield still deeper insights into the internal structure of such a composite system. For instance, they will allow to estimate the strength of the basic interactions responsible for the formation of this bound state.

Within the realm of strong interactions, electromagnetic polarizabilities of hadrons play, for the following reasons, an important role for any effective theory proposed as a realization of quantum chromo-

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Table 1
Experimental values of the electric polarizability $\kappa_{\pi\pm}$ of the charged pions $\pi^{\pm}$. These values have been extracted from experiments using two different basic processes: the Serpukhov [1] and the Lebedev [2] experiments used Compton scattering off pions whereas the Mark II [3] result has been deduced from the photon–photon cross-section

Experiment	Reaction	$\kappa_{\pi\pm} [10^{-4} \text{ fm}^3]$
Serpukhov 1983 [1]	$\pi^- + Z \to \pi^- + Z + \gamma$	6.8 ± 1.4 (stat.) ± 1.2 (syst.)
Lebedev 1986 [2]	$\gamma + \text{p} \to \gamma + \text{n} + \pi^+$	20 ± 12 (stat.)
Mark II 1990 [3]	$\text{e}^+ + \text{e}^- \to \text{e}^+ + \text{e}^- + \pi^+ + \pi^-$	2.2 ± 1.6 (stat. + syst.)

dynamics (QCD) at low energies. Chiral symmetry—as implemented, for instance, in chiral perturbation theory (χPT)—allows to formulate a precise, unambiguous prediction for the electric polarizability of the charged pion. This prediction establishes a relation between, on the one hand, the electric polarizability of the charged pion and, on the other hand, the ratio of vector and axial-vector structure constants entering in radiative charged-pion beta decay by the governing weak current. Consequently, the electric polarizability of the charged pion provides a stringent test of the chiral symmetry and hence of (the low-energy limit of) quantum chromodynamics.

1.1. Electric polarizability of pions: theory versus experiment

In spite of the principal importance of the electromagnetic polarizabilities just recalled, experimental data for electric and magnetic polarizabilities of mesons are very rare and the sparse results existing now are still not unambiguous. In particular, concerning the electric polarizability of the charged pion, the experimental situation is far from being consistent; consequently, the present state of the art cannot be regarded as satisfactory. In quantum physics, the appropriate means for the investigation of the electromagnetic polarizabilities of a given composite particle P is the analysis of its Compton scattering $\gamma + P \to \gamma + P$ or of related processes obtained by crossing symmetry, such as the production of a $P\overline{P}$ pair in photon–photon scattering, $\gamma + \gamma \to P + \overline{P}$, or the decay of a $P\overline{P}$ bound state (by the annihilation of P and $\overline{P}$) into two photons, $P + \overline{P} \to \gamma + \gamma$. Our object of desire is the charged pion, i.e., $P = \pi^{\pm}$. The problem, and simultaneously the experimental challenge, is the fact that a pion target is not available. Consequently, the particular process under investigation must be embedded into a more suitable, that is, experimentally accessi-

ble, reaction. This circumstance renders all the corresponding measurements very difficult. Compton scattering off charged pions, that is, the process $\gamma + \pi^{\pm} \to \gamma + \pi^{\pm}$, may be experimentally realized either, in form of the Primakoff effect, by *radiative pion scattering* on a nucleus of atomic number Z (involving, of course, a virtual photon), $\pi^- + Z \to \pi^- + Z + \gamma$, as has been done by an experiment using the Sigma spectrometer at Serpukhov [1], or by *radiative pion photoproduction* in photon–nucleon scattering (involving, of course, a virtual pion), $\gamma + \text{p} \to \gamma + \text{n} + \pi^+$, as has been done by an experiment performed at the electron synchrotron PACHRA of the Lebedev Physical Institute [2]. Two-photon production of charged pion pairs, that is, the process $\gamma + \gamma \to \pi^+ + \pi^-$, may be experimentally realized in the *electron–positron collision* (involving two virtual photons) $\text{e}^+ + \text{e}^- \to \text{e}^+ + \text{e}^- + \pi^+ + \pi^-$, as has been done by an experiment with the Mark II detector at the SLAC storage ring PEP [3]. Table 1 summarizes the results for the charged-pion electric polarizability $\kappa_{\pi\pm}$ reported by the above experiments. Our weighted average of these measurements reads

$$\kappa_{\pi\pm}^{\exp} = (4.3 \pm 1.2) \times 10^{-4} \text{ fm}^3. \tag{1}$$

On the theoretical side, electromagnetic polarizabilities of mesons, in particular, those of the charged pion, have been investigated by various approaches, namely, within the framework of a naive static nonrelativistic potential model for the interquark forces [4], of a (generalized) Nambu–Jona-Lasinio model [5], of chiral perturbation theory [6], of a so-called "Dubna quark confinement model" [7], as well as of a relativistic quark model [8] based on the instantaneous but quantum-field-theoretic Bethe–Salpeter formalism. Table 2 attempts to collect the theoretical predictions for the electric polarizability $\kappa_{\pi\pm}$ of the charged pion obtained within the various treatments of the pion just mentioned.

Table 2
Theoretical results for the electric polarizability $\kappa_{\pi^\pm}$ of the charged pions $\pi^\pm$

Quark bound-state approach	Predicted (range of) $\kappa_{\pi^\pm}\left[10^{-4}\ \text{fm}\right]$
Nonrelativistic quark potential model [4]	0.054
(Generalized) Nambu–Jona-Lasinio model [5]	10.5–12.5
Chiral perturbation theory (at two loops) [6]	2.4 ± 0.5
Dubna quark confinement model [7]	3.63
Relativistic Bethe–Salpeter quark model [8]	4.51–6.93

A brief inspection of the theoretical predictions for the electric polarizability $\kappa_{\pi^\pm}$ of the charged pion given in Table 2 allows to draw immediately the following conclusions.

• The most recent experimental determination of $\kappa_{\pi^\pm}$ by the Mark II Collaboration [3] reports a result for $\kappa_{\pi^\pm}$ significantly smaller than the previous measurements. The inclusion of this value results in a considerable reduction of the discrepancy between the experimental average (1) and the predictions of chiral symmetry [6]. When taking into account the next-to-leading order in the chiral expansion, the prediction of chiral perturbation theory for $\kappa_{\pi^\pm}$ is reduced from its one-loop value $\kappa_{\pi^\pm}^{[1]} = (2.7 \pm 0.1) \times 10^{-4}\ \text{fm}^3$ to the two-loop result $\kappa_{\pi^\pm}^{[2]} = (2.4 \pm 0.5) \times 10^{-4}\ \text{fm}^3$. The combined (net) effect of these two shifts is that the value of $\kappa_{\pi^\pm}$ obtained by chiral perturbation theory is still by a factor 1.7 below the experimental average. Considering the quoted errors, at present the predictions of chiral symmetry (and therefore of low-energy QCD) cannot be regarded as consistent with experiment.

• The results based on quantum-field-theoretic descriptions of the pion [5–8] lie within a factor 2.3 around the geometrical central value $\kappa_{\pi^\pm} = 5.5 \times 10^{-4}\ \text{fm}^3$:

$$\kappa_{\pi^\pm}^{\text{QFT}} = 5.5 \times 2.3^{\pm 1} \times 10^{-4}\ \text{fm}^3.$$

Thus these approaches yield the correct order of magnitude of the polarizability.

• In contrast to this, a nonrelativistic (potential-model) description [4] of the pion as $q\bar{q}$ bound state yields for the polarizability $\kappa_{\pi^\pm}$ of the charged pion a numerical value which is by a factor 80 smaller than the experimental average (1).

Consequently, we find a rather large discrepancy between the two classes of theoretical approaches to the pion discussed here. In view of this spread of predictions, it does not make much sense to compute the average of all theoretical predictions listed in Table 2.

1.2. Motivation

The brief confrontation of theory and experiment, in Subsection 1.1, demonstrates that the theoretical prediction for the electric polarizability of charged pions computed in a nonrelativistic quark model is smaller than the experimental average (1) by almost two orders of magnitude [4]. In view of this, the authors of Ref. [4] arrive at the conclusion that such potential models can just grasp the effect of the "hard core" of valence quarks but miss completely the by far dominant contribution of the "cloud" of virtual mesons. Here, we study the electric polarizabilities of mesons in *semirelativistic* descriptions of hadrons in order to clarify whether the smallness of the quark core contribution to the pion polarizability is merely an artifact of nonrelativistic kinematics which is removed as soon as the relativistic kinetic energy of the bound-state constituents is taken into account.

2. (Static) electric polarizability of charged pions

Let us start the present analysis from the assumption that the composite system under consideration can be described by a Hamiltonian H_0 and all effects of polarizability can be summarized by some contribution W to the total Hamiltonian H, which thus reads

$$H = H_0 + W. \tag{2}$$

2.1. Electric polarizability of mesons within quantum theory

As mentioned already in the Introduction, an appropriate stage for the investigation of the electromagnetic polarizabilities of some particle P is the Compton scattering off P; the low-energy expansion of the Compton scattering amplitude provides a definition of the electromagnetic polarizabilities of P as the lowest nontrivial expansion coefficients. Here, however, we follow the more intuitive (albeit static) picture drawn in Refs. [4,9], which is based on the displacement of the quarks constituting a meson in electric fields. We use the Heaviside–Lorentz system of units; this means that the electromagnetic fine structure constant α is related to the electric charge unit e by $\alpha \equiv e^2/4\pi \simeq 1/137.036$.

In order to extract the electric polarizability κ of a composite particle, one analyzes the response of this particle to an external electric field. If the electric dipole moment $\boldsymbol{d}$ induced thereby is proportional to the electric field strength $\boldsymbol{E}$ of this external field, the electric polarizability κ is defined as constant of proportionality between $\boldsymbol{E}$ and $\boldsymbol{d}$:

$$\boldsymbol{d} = 4\pi\kappa\boldsymbol{E}.$$

The shift ΔE of some energy level E resulting from this residual interaction is given by

$$\Delta E = -\frac{1}{2}\boldsymbol{d} \cdot \boldsymbol{E} = -2\pi\kappa\boldsymbol{E}^2. \tag{3}$$

Thus the electric polarizability κ may be read off from the contribution of second order in the electric field strength $\boldsymbol{E}$ to the energy E of the composite particle exposed to $\boldsymbol{E}$.

Assume that the meson under consideration is built up of a quark and an antiquark, with masses m_1 and m_2, and electric charges Q_1 and Q_2, located at the coordinates $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. Let the electric field $\boldsymbol{E}$ be generated by a point-like electric charge of magnitude Ze, where e denotes the magnitude of the electron charge. The residual interaction due to the induced electric dipole moment may be found by expanding the potential energy V_e of the two quarks which constitute the meson in this electric field,

$$V_e = Z\alpha\left(\frac{Q_1}{|\boldsymbol{r}_1|} + \frac{Q_2}{|\boldsymbol{r}_2|}\right),$$

according to the relation

$$\frac{1}{|\boldsymbol{R} + \boldsymbol{r}|} \simeq \frac{1}{|\boldsymbol{R}|}\left(1 - \frac{\boldsymbol{R} \cdot \boldsymbol{r}}{|\boldsymbol{R}|^2}\right),$$

over the center-of-momentum coordinate

$$\boldsymbol{R} \equiv \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}$$

of the two-particle system forming the meson. Expressed in terms of the corresponding relative coordinate

$$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2$$

of the two bound-state constituents and the electric field strength $\boldsymbol{E}$ generated by the point-like electric charge Ze,

$$\boldsymbol{E} = \frac{Ze}{4\pi|\boldsymbol{R}|^3}\boldsymbol{R},$$

this potential energy V_e is thus given by

$$V_e = Z(Q_1 + Q_2)\frac{\alpha}{|\boldsymbol{R}|} + \frac{m_1Q_2 - m_2Q_1}{m_1 + m_2}e\boldsymbol{r} \cdot \boldsymbol{E}.$$

It goes without saying that one is only interested in the energy shift ΔE brought about by the possibility to polarize the hadron. Consequently, the perturbation W due to the polarization of this hadron induced by the Coulomb interaction of the quarks with the external point-like charge Ze is given by

$$W = \frac{m_1Q_2 - m_2Q_1}{m_1 + m_2}e\boldsymbol{r} \cdot \boldsymbol{E}.$$

We obtain the energy eigenvalues E of the Hamiltonian H by a standard variational technique [10] around the normalized eigenstates $|\phi_0\rangle$ of the unperturbed Hamiltonian H_0, defined by

$$H_0|\phi_0\rangle = E_0|\phi_0\rangle, \qquad \langle\phi_0|\phi_0\rangle = 1,$$

where E_0 labels the bound-state energies in the absence of the external electric field $\boldsymbol{E}$. Introducing a real variational parameter $\lambda = \lambda^*$, we define a set $\{|\phi_\lambda\rangle\}$ of trial states by

$$|\phi_\lambda\rangle = (1 + \lambda W)|\phi_0\rangle.$$

Just for notational simplicity, in the following all expectation values without explicitly denoted states, that is, all expectation values written in the form $\langle\mathcal{O}\rangle$ for some operator $\mathcal{O}$, have to be understood to be

evaluated with respect to the unperturbed states $|\phi_0\rangle$:

$$\langle \mathcal{O} \rangle \equiv \langle \phi_0 | \mathcal{O} | \phi_0 \rangle .$$

We only investigate the ground state of the composite system under consideration; this state is characterized by vanishing orbital angular momentum and thus spherically symmetric. We note that expectation values $\langle W^{2n+1} \rangle$, $n = 0, 1, \ldots$, of odd powers of W with respect to states corresponding to a vanishing orbital angular momentum vanish:

$$\langle W^{2n+1} \rangle = 0, \quad n = 0, 1, 2, \ldots .$$

Hence, we find, for the square of the norm of the states $|\phi_\lambda\rangle$,

$$\langle \phi_\lambda | \phi_\lambda \rangle = 1 + \lambda^2 \langle W^2 \rangle$$

and, for $\langle \phi_\lambda | H | \phi_\lambda \rangle$,

$$\langle \phi_\lambda | H | \phi_\lambda \rangle = E_0 + 2\lambda \langle W^2 \rangle + \lambda^2 \langle W H_0 W \rangle .$$

The expectation values of the Hamiltonian H with respect to the above trial states $|\phi_\lambda\rangle$ yields a set of variational approximations $E(\lambda)$ to the exact energy eigenvalue E of H:

$$\begin{aligned}
E(\lambda) &\equiv \frac{\langle \phi_\lambda | H | \phi_\lambda \rangle}{\langle \phi_\lambda | \phi_\lambda \rangle} \\
&= E_0 + 2\lambda \langle W^2 \rangle + \lambda^2 \left(\langle W H_0 W \rangle - E_0 \langle W^2 \rangle \right) \\
&\quad + \mathcal{O}(W^4) \\
&= E_0 + 2\lambda \langle W^2 \rangle + \lambda^2 \langle [W, H_0] W \rangle + \mathcal{O}(W^4) .
\end{aligned}$$

The minimum energy eigenvalue $\overline{E}$ is determined by the requirement of stationarity of the latter expression with respect to the variational parameter λ:

$$\overline{E} = E_0 - \frac{\langle W^2 \rangle^2}{\langle [W, H_0] W \rangle} .$$

The second term on the right-hand side of this result represents the energy shift ΔE.

Moreover, for spherically symmetric states, the expectation value in the numerator of the above energy shift may be simplified with the help of

$$\langle (\boldsymbol{r} \cdot \boldsymbol{E})^2 \rangle = \frac{1}{3} \boldsymbol{E}^2 \langle r^2 \rangle, \qquad r \equiv |\boldsymbol{r}| .$$

A final simplification of this energy shift is achieved by choosing our spatial coordinate frame such that the (external) electric field strength $\boldsymbol{E}$ is aligned along the

z-axis of our coordinate frame; this choice entails

$$\boldsymbol{r} \cdot \boldsymbol{E} = z |\boldsymbol{E}| .$$

When comparing the energy shift thus obtained within the above variational technique with the ansatz (3), we find that the electric polarizability κ of mesons is given by the general expression

$$\kappa = \frac{2}{9} \left(\frac{m_1 Q_2 - m_2 Q_1}{m_1 + m_2} \right)^2 \alpha \frac{\langle r^2 \rangle^2}{\langle [z, H_0] z \rangle} . \tag{4}$$

2.2. Relativistic kinematics: the "spinless Salpeter equation"

Let us perform for the unperturbed Hamiltonian H_0 introduced in Eq. (2) the first step in an attempt to reconcile the potential-model approach to bound states of quarks with relativity. The semirelativistic Hamiltonian H_0 governing the dynamics of two particles with masses m_1 and m_2 and relative momentum $\boldsymbol{p}$ in their center-of-momentum frame involves the relativistic kinetic energies $T_i(p)$, $i = 1, 2$, of the bound-state constituents,

$$T_i(p) \equiv \sqrt{\boldsymbol{p}^2 + m_i^2}, \quad p \equiv |\boldsymbol{p}|, \quad i = 1, 2,$$

and the relevant interaction potential $V(\boldsymbol{r})$, which depends on the relative coordinate $\boldsymbol{r}$ of the bound-state constituents and is responsible for the formation of the bound state:

$$H_0 = T_1(p) + T_2(p) + V(\boldsymbol{r}) . \tag{5}$$

In the case of mesons, this potential $V(\boldsymbol{r})$ arises from the strong interactions described, according to our common understanding, by quantum chromodynamics: $V(\boldsymbol{r}) = V_s(r)$.

The eigenvalue equation for a Hamiltonian H which is the sum of the "square-root" operator of the relativistic expression for the free (i.e., kinetic) energies of the particles constituting the physical system under consideration and a static interaction potential is called the "spinless Salpeter equation". It may be obtained from the Bethe–Salpeter equation [11]—which describes bound states in (relativistic) quantum field theory—by assuming the interaction to be instantaneous (which yields the Salpeter equation [12]) and by neglecting all spin degrees of freedom of the involved bound-state constituents. On the other hand, it

can be regarded as the simplest generalization of the Schrödinger equation of standard nonrelativistic quantum theory towards relativistic kinematics.

By substituting the unperturbed Hamiltonian H_0 of Eq. (5) into our general result (4) (and by taking into account that $[z, V(\boldsymbol{r})] = 0$), we find that here the magnitude of the electric polarizability κ is determined by the following ratio of expectation values:

$$R \equiv \frac{\langle r^2 \rangle^2}{\langle [z, H_0] z \rangle} = \frac{\langle r^2 \rangle^2}{\langle [z, T_1(p) + T_2(p)] z \rangle}.$$

The question is whether this ratio R is greater or less than its nonrelativistic limit R_{NR}.

Let the ground state of the meson under consideration be described by a real wave function $\psi(p)$. The evaluation of the expectation value $\langle [z, T_i(p)] z \rangle$ is straightforward:

$$\langle [z, T_i(p)] z \rangle$$
$$= \frac{2\pi}{3} \int_0^\infty \mathrm{d}p \, p \left(2 \frac{\partial T_i(p)}{\partial p} + p \frac{\partial^2 T_i(p)}{\partial p^2} \right) \psi^2(p)$$
$$= \frac{1}{6} \int \mathrm{d}^3 p \, \frac{2p^2 + 3m_i^2}{(p^2 + m_i^2)^{3/2}} \psi^2(p).$$

It is rather easy to convince oneself that the expectation value $\langle [z, T_i(p)] z \rangle$ is bounded from above by its nonrelativistic limit $\langle [z, T_i(p)] z \rangle_{\mathrm{NR}}$:

$$\langle [z, T_i(p)] z \rangle \leqslant \langle [z, T_i(p)] z \rangle_{\mathrm{NR}} = \frac{1}{2m_i}.$$

This expectation value enters into the denominator of κ. Hence, this inequality tends to increase the ratio R and the value of κ compared with the nonrelativistic situation. Note that—in contrast to the general semirelativistic case—in the nonrelativistic limit the expectation value $\langle [z, T_i(p)] z \rangle$ reduces to a constant (namely, the inverse of $2m_i$), that is, it depends no longer on the wave function ψ which describes the bound state.

Within the present approach it suffices to model the hadrons by a static, spherically symmetric interaction potential $V_{\mathrm{s}}(r)$. For recent reviews of the description of hadrons as bound states of quarks in terms of both nonrelativistic and semirelativistic potential models see, for instance, Refs. [9,13]. The prototype of a large class of quark–antiquark interaction potentials is the

"Coulomb-plus-linear" (or, in view of its shape, "funnel") potential. This static potential is a linear combination of a Coulomb term (arising from one-gluon exchange between the strongly interacting quarks) and a linearly rising term (which is assumed to summarize all nonperturbative effects of the strong interactions):

$$V_{\mathrm{s}}(r) = -\frac{4}{3} \frac{\alpha_{\mathrm{s}}}{r} + ar. \tag{6}$$

For the actual application of the semirelativistic quark model introduced above, we have to specify the numerical values of the parameters entering in the Hamiltonian H_0. In the case of the pion, we obviously deal with bound-state constituents of equal masses $m = m_1 = m_2$. For this common mass, we adopt the canonical value for the constituent mass of light nonstrange quarks: $m = m_{\mathrm{u}} = m_{\mathrm{d}} = 0.336$ GeV [9]. The two parameters of the interaction potential $V_{\mathrm{s}}(r)$ are taken from a (nonrelativistic) fit [14] of the meson spectrum. We use, for the strong fine structure constant α_{s} characterizing the coupling strength of the Coulomb-like term in the funnel potential, the value $\alpha_{\mathrm{s}} = 0.31$ and, for the slope a of the linear contribution to the funnel potential, the value $a = 0.15$ GeV^2. (One might be tempted to question the choice of a rather large constituent quark mass m for the description of the comparatively light pion. However, one has to bear in mind that the numerical values of the mass and coupling parameters used here have emerged from a satisfactory simultaneous fit [14] of both the meson and the baryon mass spectra within a specific *nonrelativistic* quark-potential model. It goes without saying that any semirelativistic description of hadrons will require slightly different numerical values of the parameters involved. In principle, all parameters, that is, the quark mass m and the two couplings α_{s} and a, should be readjusted for the present *semirelativistic* treatments of bound states of quarks. Moreover, the interquark potential employed in Ref. [14] is of a more complicated parametric shape than the simple funnel potential $V_{\mathrm{s}}(r)$ in Eq. (6). It appears, however, very unlikely that such—comparatively minor—modifications are able to induce drastic changes in the resulting predictions for the electric polarizability. At least, they will hardly alter the order of magnitude of the computed polarizability.)

The discrete eigenvalues and corresponding eigenstates of the operator represented by the semirelativis-

tic Hamiltonian H_0 of Eq. (5) are (approximately) determined with the help of the well-known minimum–maximum principle [15]. The trial space required by this technique is spanned by the "Laguerre" basis states defined in Refs. [16–18].

The relativistic virial theorem derived in Ref. [19] (for a very comprehensive review, see Ref. [20]) allows to define a precise quantitative measure [21, 22] for the quality of the results obtained within the framework of variational techniques. According to the analysis presented in Refs. [21,22], for a generic Hamiltonian operator H consisting of a (momentum-dependent) kinetic term $T(\boldsymbol{p})$ and a (coordinate-dependent) interaction potential $V(\boldsymbol{x})$, that is,

$$H = T(\boldsymbol{p}) + V(\boldsymbol{x}),$$

the accuracy of a trial state $|\varphi\rangle$ which approximates the exact bound state under study may be quantitatively estimated by the deviation from zero of the quantity

$$\nu \equiv \frac{\langle\varphi|\boldsymbol{p}\cdot\frac{\partial}{\partial\boldsymbol{p}}T(\boldsymbol{p})|\varphi\rangle}{\langle\varphi|\boldsymbol{x}\cdot\frac{\partial}{\partial\boldsymbol{x}}V(\boldsymbol{x})|\varphi\rangle} - 1.$$

The main advantage of this measure for the accuracy of approximate eigenstates $|\varphi\rangle$ is that it does not require any information on the solutions of the investigated eigenvalue problem other than the one provided by the variational approximation technique itself.

Clearly, any enlargement of the dimension d of the minimum–maximum trial space will, in general, improve the obtained approximation. For our choice of $d = 25$ (that is, when truncating the expansion of the approximate eigenstates $|\varphi\rangle$ after 25 basis states) we find for the above quality measure ν the numerical value $\nu = 2 \times 10^{-4}$. Beyond doubt, the precision achieved in this way may be regarded as sufficient for the present purpose.

In order to estimate the effects of relativistic kinematics, we compare the outcome of the above semirelativistic treatment with the corresponding nonrelativistic solution. The nonrelativistic counterparts of the semirelativistic eigenvalues and eigenstates are found with rather high accuracy by solving the Schrödinger equation with a (standard) numerical integration procedure developed in Ref. [23] precisely for this purpose.

The resulting expectation values $\langle r^2\rangle$ and $\langle[z, H_0]z\rangle$, obtained for both relativistic and nonrelativistic kine-

Table 3
Expectation values $\langle r^2\rangle$ and $\langle[z, H_0]z\rangle$ determining the electric polarizabilities of mesons, obtained within semirelativistic and nonrelativistic descriptions of the pion, for a funnel-shaped quark interaction potential $V_S(r) = -\frac{4}{3}\alpha_s/r + ar$ with a light-quark constituent mass $m = 0.336$ GeV and coupling constants $\alpha_s = 0.31$ and $a = 0.15$ GeV^2

Kinematics	$\langle r^2\rangle$ [GeV^{-2}]	$\langle[z, H_0]z\rangle$ [GeV^{-1}]
Relativistic	11.7	1.66
Nonrelativistic	17.8	2.98

matics along the lines sketched above, are presented in Table 3. These two expectation values determine the ratio R and, in consequence of the general result (4), the theoretical prediction for the (static) electric polarizability κ of mesons. Insertion of the numerical values of Table 3, however, leads to the (disappointing) ratio

$$\frac{\kappa}{\kappa_{\mathrm{NR}}} = \frac{R}{R_{\mathrm{NR}}} = \frac{\langle r^2\rangle^2}{\langle r^2\rangle_{\mathrm{NR}}^2}\frac{\langle[z, H_0]z\rangle_{\mathrm{NR}}}{\langle[z, H_0]z\rangle} = 0.77 < 1.$$

This tells us that merely replacing, in the Hamiltonian H, nonrelativistic by relativistic kinematics reduces the valence-quark contribution to the pion's electric polarizability.

2.3. Relativistic corrections to the static interaction potential

The static potential may be regarded as the lowest term in a (relativistic) expansion of the interaction of the bound-state constituents over the nonrelativistic limit. Of course, one might ask whether the inclusion of (the totality of all) relativistic corrections to the interaction potential will save the day. The derivation of the relativistic corrections to the static interaction potential from the corresponding two-particle scattering problem has been thoroughly analyzed in Refs. [24–26] and extensively reviewed in Ref. [13].

Since the pion is a quark–antiquark bound state corresponding to vanishing relative orbital angular momentum of its constituents, only the spin–spin term will contribute. Moreover, in the ultrarelativistic case of *massless* bound-state constituents, that is, for $m_1 = m_2 = 0$, we encounter a comparatively simple situation. In this case, at least, it is possible to make analytic statements about the influence of semirelativistic treatments of both kinetic terms and interaction-potential

terms on the pion electric polarizability. To this end, we take advantage of explicit results presented in Section VII of Ref. [24]. Within the variational evaluation employed for the example studied there, it is easy to convince oneself that, for the expectation value of r^2 entering into the numerator of the ratio R, the value $\langle r^2 \rangle_{\mathrm{RC}}$ obtained from semirelativistic Hamiltonians that include the relativistic corrections to the interaction potential is always smaller than the result $\langle r^2 \rangle$ derived from the corresponding operators with just the static potential: $\langle r^2 \rangle_{\mathrm{RC}} \leqslant \langle r^2 \rangle$. A closer inspection reveals that, expressed in terms of the coupling parameters α_{s} and a of the funnel potential (6), these quantities differ, for hydrogen-like trial functions, by

$$\langle r^2 \rangle - \langle r^2 \rangle_{\mathrm{RC}} = \frac{64\alpha_s}{3\pi^2 a} > 0;$$

this may be translated into the inequality

$$\langle r^2 \rangle_{\mathrm{RC}} \leqslant \left(1 - \frac{2\alpha_{\mathrm{s}}}{\pi}\right)\langle r^2 \rangle.$$

Concerning the expectation value $\langle [z, H_0]z \rangle$ entering in the denominator of the ratio R as well as the more general case of nonvanishing values of the bound-state constituents' masses, we did not succeed to derive rigorous analytical statements on the influence of relativistic corrections to the interaction potential on the pion's electric polarizability. A straightforward, entirely numerical, phenomenological analysis shows that inclusion of these relativistic corrections entails a significant further reduction of the value of the pion electric polarizability compared with the spinless-Salpeter value of Subsection 2.2.

3. Conclusions

This analysis has been devoted to the determination of the valence-quark contribution to the electric polarizability of mesons (for the experimentally most relevant particular example of the charged pion $\pi^{\pm}$) within the framework of semirelativistic quark models based on a conventional Hamiltonian description of hadrons as bound states of quarks. Recently, one observes a revival of interest in the experimental study of the electric and magnetic polarizabilities of mesons via processes such as, e.g., the Primakoff effect, cf. the proposal for the NA58 (COMPASS) experiment at the

CERN SPS collider [27,28]. This motivates the continued search for a satisfactory theoretical understanding of this rather fundamental quantity and the present attempt to reconcile different approaches.

From this investigation we are forced to conclude that even within a semirelativistic description of hadrons the valence-quark contribution constitutes only a minor fraction of the total electric polarizability of mesons. This point of view is, in fact, supported by a study [29] of the *nucleon* electromagnetic polarizabilities within a chiral quark model. This model considers any hadron explicitly as a core formed by the appropriate valence quarks which is surrounded by clouds of virtual hadrons, in particular, of virtual pions.

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