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Boolean pairs formed by the Δ_n^0-sets

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Abstract

It will be shown that for all numbers n and m with $n > m \geqslant 1$ the Boolean pairs (Δ_n^0, Δ_m^0) have undecidable elementary theories.

The main subclasses of the arithmetical hierarchy and their mutual positions are well-known. We have the classes Δ_n^0, Σ_n^0 and Π_n^0, $n \geqslant 1$ (Δ_1^0 – the class of all recursive sets) with Σ_n^0, $\Pi_n^0 \subseteq \Sigma_{n+1}^0$, Π_{n+1}^0 and $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$, $n \geqslant 1$.

Let χ be one of the classes: Σ_n^0, Π_n^0 or Δ_n^0, $n \geqslant 1$. Then the following facts are known or easy to see:

(1) $(\chi, \subseteq)$ is a distributive lattice. If $X \in \chi$ and $X =^* Y$ (i.e. $(X - Y) \cup (Y - X)$ is finite) then $Y \in \chi$. (We say that χ is $=^*$-closed). We denote by χ^* the class χ modulo $=^*$ and by X^* the equivalence class of an X resp. to $=^*$. For sets X, Y we write $X \subseteq^* Y$ if $X - Y$ is finite.

(2) If χ is Σ_n^0 or Π_n^0 for any $n \geqslant 1$, then $\mathrm{Th}((\chi, \subseteq))$ (the elementary theory of the structure $(\chi, \subseteq)$) is hereditarily undecidable,[1] see e.g. [4, p. 381]. If χ is Δ_n^0, $n \geqslant 1$, then $(\chi, \subseteq)^*$ is the countable, atomless Boolean algebra. (Hence $\mathrm{Th}((\chi, \subseteq)^*)$ is decidable.)

What can be said about the position of the recursive sets inside the Δ_2^0-sets from the point of view of the lattice theory? How does the sublattice of recursive sets lie inside the lattice of the Δ_2^0-sets? How do the Δ_2^0-sets extend the recursive ones when both are considered as lattices?

We will give an answer, not only for $m = 1$ and $n = 2$, but for the general case $n > m \geqslant 1$. We consider the Boolean pairs $(\Delta_n^0, \Delta_m^0)^*$, i.e. the structures $(\Delta_n^0, \subseteq, R)^*$, where $(\Delta_n^0, \subseteq)^*$ is the Boolean algebra of the Δ_n^0-sets (mod $=^*$) and R a unary relation satisfied exactly for all sets in Δ_m^0 (modulo $=^*$).

[1] Let L be a first-order language (i.e. L is a set of functions $\mathscr{F}$, of relations $\mathscr{R}$ and of constants $\mathscr{C}$). Let T be a theory in L. T is called *hereditarily undecidable* if every subtheory T' of T (in L) is undecidable.

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Now it holds

Theorem. *For all numbers n and m with* $n > m \geqslant 1$, $\mathrm{Th}((\Delta_n^0, \Delta_m^0))$ *is (uniformly) hereditarily undecidable.*

Here uniformly means that for all pairs (Δ_n^0, Δ_m^0) there is an elementary definable (with parameters) structure which can be defined by using the same formulas for each n and m (and this definable structure has a hereditarily undecidable elementary theory). We shall even show that for the same m, but arbitrarily n the same parameters can be used.

Proof. Denote by $\mathscr{E}^n$, $n \geqslant 1$, the lattice of Σ_n^0-sets under inclusion. We shall show that $\mathscr{E}^{n+1}$ is elementary definable (by using three formulas and two parameters) in (Δ_n^0, Δ_m^0) for $n > m \geqslant 1$.[2]

The parameters A and Y: Let A be a hh-simple set with properties described below and Y a small major subset of A (symbolically $Y \subset_{\mathrm{sm}} A$). The set A has two properties:

- $\mathscr{L}^*(A)$ (the r.e. supersets structure of A (modulo $=^*$)) is isomorphic to $\oplus 2$ (i.e. the Boolean algebra formed by all finite and cofinite subsets of $\mathbb{N}$, $\mathbb{N}$ – the set of all numbers). Thus $\oplus 2$ is the completely atomic Boolean algebra.
- A has an r.e. sequence $(A_i)_{i \geqslant 0}$ of r.e. subsets of A with
  - (i) $A_i \cap A_j = \emptyset$ $(i \neq j)$, $\bigcup_{i \geqslant 0} A_i = A$.
  - (ii) ($\forall R$ recursive set) ($(A \cup R)^*$ is an atom in $\mathscr{L}^*(A) \Rightarrow (\exists! k)((A - Y) \cap R =^* (A - Y) \cap A_k)$).
  - (iii) $(\forall i)(\exists R$ recursive set) ($(A \cup R)^*$ – atom in $\mathscr{L}^*(A)$ and $A_i \cup (R - A)$ is recursive).

The existence of such a special hh-simple set follows easily from Lachlan's general construction of hh-simple sets. See for this e.g. [3].

Observe that from the properties of the sets A_i and the choice of Y we get that $A_i - Y$ is infinite for every $i \geqslant 0$ and uniquely determined. This means

$$(\forall R \text{ recursive}) \quad (A_i - Y \subseteq^* R \text{ or } (A_i - Y) \cap R =^* \emptyset).$$

Further every set $A_i - Y$ is uniquely connected with exactly one atom $(A \cup R)^*$ of $\mathscr{L}^*(A)$ (by (ii) and (iii)).

Relativization of A and Y: We can relativize these properties to the Σ_m^0-sets. Thus for every $m \geqslant 1$ there are sets $A^{(m)}$ and $Y^{(m)}$ with $Y^{(m)} \subset_{\mathrm{sm}} A^{(m)}$ (defined in the obvious sense using Σ_m^0-sets) and a Σ_m^0-sequence $(A_i^{(m)})_{i \geqslant 0}$ such that $\mathscr{L}^{(m)*}(A^{(m)}) = \{X^*: X \in \Sigma_m^0,\ A^{(m)} \subseteq X\}$ is isomorphic to $\oplus 2$ and for $(A_i^{(m)})_{i \geqslant 0}$ (i)–(iii) hold where in (ii) and (iii) of "recursive" is replaced by "$R \in \Delta_m^0$". (We will need this relativization for the proof that pairs (Δ_n^0, Δ_m^0) are undecidable for $m > 1$).

[2] Indeed one parameter would be sufficient. But by using two the interpretation becomes clearer and easier to understand.

The sets $A^{(m)}$ and $Y^{(m)}$ are Σ^0_m-sets, hence are Δ^0_n-sets for $n > m$. Thus they can be used as parameters in (Δ^0_n, Δ^0_m).

The defining formulas: The elements of $\mathscr{E}^{n+1}$ will be mapped to a definable collection of Δ^0_n-sets defined by the formula φ_0. $\varphi_0(x, Y^{(m)}, A^{(m)})$ is

$$Y^{(m)} \subseteq x \wedge x \subseteq A^{(m)}.$$

We write $\mathscr{L}^{(n)}(Y^{(m)}, A^{(m)})$ for the class

$$\{Z \in \Delta^0_n\colon Y^{(m)} \subseteq Z \subseteq A^{(m)}\}.$$

Let $\varphi_1(x, z, Y^{(m)}, A^{(m)})$ be the (informal) expression:

$$(\forall i)([x \cap (A_i^{(m)} - Y^{(m)}) =^* A_i^{(m)} - Y^{(m)}] \Rightarrow [z \cap (A_i^{(m)} - Y^{(m)}) =^* A_i^{(m)} - Y^{(m)}]). \tag{1}$$

φ_1 will be used for the interpretation of $\subseteq$ from $\mathscr{E}^{n+1}$ into (Δ^0_n, Δ^0_m).

φ_1 is not definable in the language of the Boolean pairs. Later in Lemma 2 we show that φ_1 is equivalent in (Δ^0_n, Δ^0_m) to a definable formula. But for recursion-theoretic estimations (1) will be useful, e.g. in Lemma 1.

The third formula $\varphi_2(x, z, Y^{(m)}, A^{(m)})$ is

$$\varphi_1(x, z, Y^{(m)}, A^{(m)}) \wedge \varphi_1(z, x, Y^{(m)}, A^{(m)}).$$

This defines an equivalence relation on the structure of (Δ^0_n, Δ^0_m). We write $x \subseteq_{\text{cof}} z$ for $\varphi_1(x, z, Y^{(m)}, A^{(m)})$ and $x \approx_{\text{cof}} z$ for $\varphi_2(x, z, Y^{(m)}, A^{(m)})$.

Now we can show

Lemma 1. *$(\mathscr{L}^{(n)}(Y^{(m)}, A^{(m)}), \subseteq_{\text{cof}})/\approx_{\text{cof}}$ is isomorphic to $\mathscr{E}^{n+1}$.*

Proof. Let X be a Δ^0_n-set such that $Y^{(m)} \subseteq X \subseteq A^{(m)}$. Let S_X be the set

$$\{i \in \mathbb{N}\colon X \cap (A_i^{(m)} - Y^{(m)}) =^* (A_i^{(m)} - Y^{(m)})\}. \tag{2}$$

At first we see that S_X belongs to Σ^0_{n+1}, since (2) has the definition

$$(\exists x)(\forall y > x)(y \in X \cap A_i^{(m)} \Rightarrow y \in Y^{(m)}),$$

this is $\exists\forall((\Delta^0_n \wedge \Sigma^0_m) \Rightarrow \Sigma^0_m)$, hence Σ^0_{n+1}. (Here we use the fact that $(A_i^{(m)})_{i \geqslant 0}$ is a uniformly Σ^0_m-sequence.)

Further, we see that for $X, Z \in \mathscr{L}^{(n)}(Y^{(m)}, A^{(m)})$

$$X \subseteq_{\text{cof}} Z \quad \text{iff } S_X \subseteq S_Z$$

and thus

$$X \approx_{\text{cof}} Z \quad \text{iff } S_X = S_Z.$$

From this we get that the mapping

$$[X]/\approx_{\text{cof}} \to S_X \tag{3}$$

is an embedding of $(\mathscr{L}^{(n)}(Y^{(m)}, A^{(m)}), \subseteq_{\text{cof}})/\approx_{\text{cof}}$ into $\mathscr{E}^{n+1}$.

It remains to show that the mapping (3) is surjective. Let $S \in \Sigma^0_{n+1}$. Then there is a Σ^0_{n-1}-sequence $(U_k)_{k\geqslant 0}$ (of Σ^0_{n-1}-sets) such that

$$k \in S \quad \text{iff } U_k \text{ is cofinite.}$$

This fact we get by an easy relativization of the well-known fact that $\{e\colon W_e =^* \mathbb{N}\}$ is Σ^0_3-complete (see e.g. [4, p. 68]).

Next we have to relativize the following well-known fact:

"If A and B are r.e. sets with $A \subseteq B$ and $B - A$ not co-r.e. then there is an r.e. sequence of finite and disjoint sets $(S_n)_{n\geqslant 0}$ with $B - A \subseteq \bigcup_{n\geqslant 0} S_n \subseteq B$ and $S_n \cap (B-A) \neq \emptyset$ for every n".

(See for this e.g. [4, p. 184] or [3] in the Introduction.)

Since the construction of $(S_n)_{n\geqslant 0}$ is effective, this can be done uniformly if uniformly r.e. sequences $(A_i)_{i\geqslant 0}$ and $(B_i)_{i\geqslant 0}$ are given with $B_i - A_i$ is not co-r.e. for every i.

We relativize this to the sequences $(Y^{(m)})_{i\geqslant 0}$ and $(Y^{(m)} \cup A_i^{(m)})_{i\geqslant 0}$. ($A_i^{(m)} - Y^{(m)}$ are not co-Σ^0_m, since $Y^{(m)} \subset_{\text{sm}} A^{(m)}$ and by the properties (ii) and (iii)). Thus we get a Σ^0_m-sequence $(S_{i,e})_{i,e\geqslant 0}$ such that $Y^{(m)} \cup A_i^{(m)} - Y^{(m)} \subseteq \bigcup_{e\geqslant 0} S_{e,i} \subseteq Y^{(m)} \cup A_i^{(m)}$ and $S_{e,i} \cap (Y^{(m)} \cup A_i^{(m)} - Y^{(m)}) \neq \emptyset$ for all e and every i.

Now let X be the set

$$Y^{(m)} \cup \bigcup \{S_{i,e}\colon e \in U_i,\ i \geqslant 0\}.$$

X is Σ^0_m and so $X \in \mathscr{L}^{(n)}(Y^{(m)}, A^{(m)})$.

Let $i \in S$; then U_i is cofinite, hence $A_i^{(m)} - Y^{(m)} \subseteq^* X$, by definition of X and the finiteness of all sets $S_{e,i}$. Hence by (2) $i \in S_X$.

Let $i \in S_X$. Then $A_i^{(m)} - Y^{(m)} \subseteq^* X$. Thus $U_i =^* \mathbb{N}$, but this means $i \in S$.

It remains to show that φ_1 can be defined elementary with parameters.

Lemma 2. *The expression $\varphi_1(x, z, Y^{(m)}, A^{(m)})$ is equivalent (in (Δ^0_n, Δ^0_m)) to the elementary formula*

$$(\forall R \in \Delta^0_m) \quad [(A^{(m)} \cup R)^* \text{ is an atom in } \mathscr{L}^{(m)*}(A^{(m)})$$
$$\Rightarrow^3 \{(A^{(m)} - Y^{(m)}) \cap R \subseteq^* x \Rightarrow (A^{(m)} - Y^{(m)}) \cap R \subseteq^* z\}]. \tag{4}$$

Proof. (1) $\Rightarrow$ (4): Suppose (1) is satisfied for sets X and Z (in place of x and z, respectively). Let R be such that $(A^{(m)} \cup R)^*$ is an atom in $\mathscr{L}^{(m)*}(A^{(m)})$. Then

[3] "$(A^{(m)} \cup R)^*$ is an atom in $\mathscr{L}^{(m)*}(A^{(m)})$" means

$$(\forall T \in \Delta^0_m) \quad (T \cap (A^{(m)} \cup R) =^* \emptyset \vee A^{(m)} \cup R \subseteq^* A^{(m)} \cup T).$$

$(A^{(m)} - Y^{(m)}) \cap R =^* A_i^{(m)} - Y^{(m)}$ for some unique i, by (ii). Thus the second line in (1) implies also the second one in (4).

(4) $\Rightarrow$ (1): Suppose (4) is satisfied for sets X and Z from Δ_n^0. If for some i $X \cap (A_i^{(m)} - Y^{(m)}) =^* A_i^{(m)} - Y^{(m)}$ then by (iii) there is a Δ_m^0-set R with $R \cap (A^{(m)} - Y^{(m)}) =^* A_i^{(m)} - Y^{(m)}$ (see the sentence before: Relativization of A and Y, after (iii)). Thus by (4) $Z \cap (A_i^{(m)} - Y^{(m)}) =^* -Y^{(m)}$.

It is easy to see that "finiteness" is elementary definable in (Δ_n^0, Δ_m^0), hence also the relation $=^*$. For $X \in \Delta_n^0$ $X =^* \emptyset$ means

$$(\forall Y \in \Delta_n^0) \quad (Y \subseteq X \Rightarrow Y \in \Delta_m^0)). \tag{5}$$

If X is finite then obviously every subset of X belongs to Δ_m^0.

If X is infinite and not from Δ_m^0 then $Y = X$ negates (5). If X is from Δ_m^0 then X has a Σ_m^0-subset (hence is from Δ_n^0), not from Δ_m^0.

Thus we have

Corollary. *For all numbers n and m with $n > m \geqslant 1$ the theory* $\mathrm{Th}((\Delta_n^0, \Delta_m^0))$ *is undecidable.*

Proof. It was shown that $\mathscr{E}^{n+1}$ is elementary definable with parameters in (Δ_n^0, Δ_m^0) and $\mathrm{Th}(\mathscr{E}^{n+1})$ is hereditarily undecidable. Thus this holds also for $\mathrm{Th}((\Delta_n^0, \Delta_m^0))$.

Final remarks. The following problems remain still open:

(1) Are $(\Delta_{n_1}^0, \Delta_{m_1}^0)$ and $(\Delta_{n_2}^0, \Delta_{m_2}^0)$ elementary equivalent? We should note that Harrington and Nies have shown that $\mathscr{E}^{n_1}$ and $\mathscr{E}^{n_2}$ are not elementary equivalent for $n_1 \neq n_2$. We believe this supports our conjecture that $(\Delta_{n_1}^0, \Delta_{m_1}^0)$ and $(\Delta_{n_2}^0, \Delta_{m_2}^0)$ are not elementary equivalent for $n_1 \neq n_2$ or $m_1 \neq m_2$.

(2) Let Δ_0^0 be the class of primitive recursive sets. Do the Boolean pairs (Δ_n^0, Δ_0^0), $n \geqslant 1$, have undecidable theories?

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