Some extensions of a property of linear representation functions

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ABSTRACT

Let \( A = \{a_1, a_2, \ldots\}(a_1 < a_2 < \cdots) \) be an infinite sequence of nonnegative integers. Let \( k \geq 2 \) be a fixed integer and for \( n \in \mathbb{N} \), let \( R_k(A, n) \) be the number of solutions of \( a_1 + a_2 + \cdots + a_k = n \) with \( a_1, a_2, \ldots, a_k \in A \). The other one predicted that \( R_2(A, n) \) cannot be asymptotically well approximated by its average value; more precisely, \( \sum_{n=0}^{\infty} R_2(A, n) = cN + O(1) \) cannot hold for any constant \( c > 0 \). In this paper, we obtain the analogous results for \( R_k(A, n) \), \( R_k^{(1)}(A, n) \) and \( R_k^{(2)}(A, n) \).

1. Introduction

In 1941, Erdős and Turán [2] made two conjectures in additive number theory, which have had an important impact on the field. One of them conjectured that if \( A \) is a basis of \( \mathbb{N} \), then \( R_2(A, n) \) is unbounded, which is called the Erdős–Turán conjecture. The other one predicted that \( R_2(A, n) \) cannot be asymptotically well approximated by its average value; more precisely, \( \sum_{n=0}^{\infty} R_2(A, n) = cN + O(1) \) cannot hold for any constant \( c > 0 \). Fifteen years later, Erdős and Fuchs [1] proved that \( \sum_{n=0}^{\infty} R_2(A, n) = cN + o(N^{1/4}(\log N)^{-1/2}) \) cannot hold for any constant \( c > 0 \). The importance of the Erdős–Fuchs theorem is based on the fact that the special case \( A = \{1^2, 2^2, \ldots\} \) of it corresponds to the circle problem. Up until now, this original result had been extended in various directions [3,5,6]. Recently, Horváth [4] proved that if \( d > 0 \) is an integer, then there does not exist \( n_0 \) such that \( d \leq R_2^{(2)}(A, n) \leq d + \left[ \sqrt{2d} + \frac{1}{2} \right] \) for \( n > n_0 \).

In this paper, we obtain the analogous results for \( R_k(A, n) \), \( R_k^{(1)}(A, n) \), and \( R_k^{(2)}(A, n) \).

Theorem 1. Let \( d > 0 \) be an integer. If \( R_k(A, n) \geq d \) for all sufficiently large integers \( n \), then \( R_k(A, n) \geq d + 2\sqrt{d} + 1 \) for infinitely many integers \( n \).

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Theorem 2. If there exist two positive integers $d$ and $n_0$ such that
\[ d \leq R_k^{(1)}(A, n) \leq d + \left[ \frac{2\sqrt{k}d + 1}{k!} \right] \text{ for all } n > n_0, \tag{1}\]
then $R_k(A, n)$ is unbounded.

Theorem 3. Let $d > 0$ be an integer. If $R_k^{(2)}(A, n) \geq d$ for all sufficiently large integers $n$, then $R_k^{(2)}(A, n) \geq d + \frac{2\sqrt{k}d + 1}{k!}$ for infinitely many integers $n$.

Remark. On the representation function $R_k(A, n)$, Horváth even [3] proved that
\[ \sum_{n=0}^{N} R_k(A, n) = cN + o(N^{1/4}(\log N)^{1-\frac{3}{4}}) \tag{2}\]
cannot hold for any constant $c > 0$.

Motivated by the Erdős–Turán conjecture, we pose the following conjecture.

Conjecture. (i) If $R_k(A, n) \geq 1$ for all sufficiently large $n$, then $R_k(A, n)$ is unbounded.

(ii) If $R_k^{(1)}(A, n) \geq 1$ for all sufficiently large $n$, then $R_k^{(1)}(A, n)$ is unbounded.

For $k = 2$, Conjecture (i) is the Erdős–Turán conjecture. Since $R_k^{(2)}(A, n) \leq R_k(A, n) \leq k!R_k^{(2)}(A, n)$, Conjecture (i) is equivalent to

Conjecture. (iii) If $R_k^{(2)}(A, n) \geq 1$ for all sufficiently large $n$, then $R_k^{(2)}(A, n)$ is unbounded.

2. Proofs

Lemma ([4]). For large $N$, we have
\[ \int_{0}^{1} \frac{1}{|1 - z|} d\alpha \ll \log N, \tag{3}\]
where $z = re^{2\pi i\alpha}$, $r = 1 - \frac{1}{N}$ and $\alpha$ is a real variable.

Proof of Theorem 1. Suppose that there is a number $n_0$ such that $d \leq R_k(A, n) < d + 2\sqrt{d} + 1$ for $n > n_0$. Let $l$ be the largest integer that is less than $2\sqrt{d} + 1$. For $n > n_0$ we have
\[ R_k(A, n) = d + v(n), \quad \text{where } 0 \leq v(n) \leq l. \tag{4}\]
Let $c$ be a real number such that $d + l - \sqrt{d} + 1 < c < d + \sqrt{d}$. It is easy to obtain that
\[ (d + j - c)^2 < d + j \quad \text{for } j = 0, 1, \ldots, l. \tag{5}\]
Thus we can choose some $\delta > 0$ such that
\[ \frac{(d + j - c)^2}{d + c} \leq 1 - \delta \quad \text{for } j = 0, 1, \ldots, l. \tag{6}\]
So by (4) and (6) we have
\[ (R_k(A, n) - c)^2 \leq (1 - \delta)R_k(A, n) \quad \text{for } n > n_0. \tag{7}\]
Let $z = re^{\alpha}$, where $r = 1 - \frac{1}{N}(N \in \mathbb{N}$ is “large”) and $e^{\alpha} = e^{2\pi i\alpha}(\alpha \text{ real})$, and write
\[ F(z) = \sum_{a \in A} z^a. \]
Then
\[ F^k(z) = \sum_{n=0}^{\infty} R_k(A, n)z^n \quad \text{for } |z| < 1. \tag{8}\]
Let
\[ J = \int_0^1 \left| F^k(z) - c \frac{1}{1 - z} \right| \, d\alpha. \] (9)

By the triangle inequality we have
\[ J \geq \int_0^1 |F^k(z)| \, d\alpha - |c| \int_0^1 \frac{1}{|1 - z|} \, d\alpha. \] (10)

Let
\[ J_1 = \int_0^1 |F^k(z)| \, d\alpha. \]

By Hölder’s inequality and Parseval’s formula, we have
\[ J_{2/k} = (\int_0^1 |F(z)|^k \, d\alpha)^{2/k} \cdot (\int_0^1 1 \, d\alpha)^{1-2/k} \]
\[ \geq \int_0^1 |F(z)|^2 \, d\alpha = \sum_{a \in A} r^{2n} = F(r^2). \]

Thus
\[ J_1 \geq F^{k/2}(r^2). \] (11)

By (4) and (8) we have
\[ F^k(r^2) = \sum_{n=0}^{\infty} R_k(A, n) r^{2n} \geq d \sum_{n>n_0} r^{2n} > (r^2)^{n_0+1} \frac{1}{1 - r^2} \gg N. \] (12)

Thus by (12) and the lemma we have
\[ \int_0^1 \frac{1}{|1 - z|} \, d\alpha = o\left( F^{k/2}(r^2) \right) \quad \text{(as } N \to +\infty). \] (13)

Therefore by (10), (11) and (13) we have
\[ J \geq (1 - o(1)) F^{k/2}(r^2). \] (14)

By (8) and (9), Cauchy’s inequality and Parseval’s formula
\[ J \leq \left( \int_0^1 \left| F^k(z) - c \frac{1}{1 - z} \right|^2 \, d\alpha \right)^{1/2} \]
\[ = \left( \int_0^1 \left| \sum_{n=0}^{\infty} (R_k(A, n) - c) z^n \right|^2 \, d\alpha \right)^{1/2} \]
\[ \leq \left( \sum_{n=0}^{\infty} (R_k(A, n) - c)^2 r^{2n} \right)^{1/2}. \] (15)

Thus, by (7) and (8) and the assumption, we have
\[ J \leq (1 - \delta) R_k(A, n) r^{2n} + \sum_{n \leq n_0} ((R_k(A, n) - c)^2 - (1 - \delta) R_k(A, n)) r^{2n} \]
\[ = \left( (1 - \delta) F^k(r^2) + O(1) \right)^{1/2}. \] (16)

So
\[ J \leq (1 - \delta + o(1))^{1/2} F^{k/2}(r^2). \] (17)

From (14) and (17), we have \( (1 - o(1)) F^{k/2}(r^2) \leq (1 - \delta + o(1))^{1/2} F^{k/2}(r^2) \), i.e., \( 1 - o(1) \leq 1 - \delta \), this cannot hold for sufficiently large \( N \).

This completes the proof of Theorem 1. \( \square \)
Proof of Theorem 2. Suppose that $A$ is a sequence satisfying (1) and $R_k(A, n)$ is bounded. Let $T = \{ n \in kA | R_k^{(1)}(A, n) = R_k^{(2)}(A, n) \}$.

Since $R_k^{(1)}(A, n)$ is bounded, we have $x^{-1/k} |A \cap [1, x]|$ is bounded. Note that $R_k^{(2)}(A, n) \geq R_k^{(1)}(A, n)$,

$$\sum_{n \leq x} R_k^{(2)}(A, n) - R_k^{(1)}(A, n) \leq k(k-1)/2 |A \cap [1, x]|^{k-1} \ll x^{1-1/k},$$

and the trivial fact that if the difference is non-zero, then it is at least 1, we obtain $R_k^{(2)}(A, n) = R_k^{(1)}(A, n)$ for almost all $n$. Since (1) implies that $R_k^{(1)}(A, n) > 0$ for all but finitely many $n$, we obtain $R_k^{(2)}(n) = R_k^{(1)}(n) > 0$ for almost all $n$, that is, $T$ is infinite.

By the assumption, if $n \in T$ and $n > n_0$, then

$$R_k(A, n) = k!(d + v(n)), \quad \text{where} \quad 0 \leq v(n) \leq \left\lfloor \frac{2 \sqrt{k!d + 1}}{k!} \right\rfloor. \quad (18)$$

Let $c'$ be a real number such that

$$k!(d + l) - \sqrt{k!(d + l)} < c' < k!d + \sqrt{k!d}, \quad (19)$$

where

$$l = \left\lfloor \frac{2 \sqrt{k!d + 1}}{k!} \right\rfloor.$$ 

It is easy to see that $k!(d + l) - \sqrt{k!(d + l)} < k!d + \sqrt{k!d}$. So $c'$ exists. By (19) we have

$$(k!(d + j) - c')^2 < k!(d + j) \quad \text{for} \quad j = 0, 1, \ldots, l.$$ 

Thus we can choose some $\delta > 0$ such that

$$\frac{(k!(d + j) - c')^2}{k!(d + j)} \leq 1 - \delta \quad \text{for} \quad j = 0, 1, \ldots, l. \quad (20)$$

So by (18) and (20) we have

$$(R_k(A, n) - c')^2 \leq (1 - \delta)R_k(A, n) \quad \text{for} \quad n \in T \quad \text{and} \quad n > n_0. \quad (21)$$

Define $z$ and $F(z)$ as in the proof of Theorem 1, and let

$$J = \int_0^1 \left| F(z) - c' \frac{1}{1-z} \right| \, dz.$$ 

Then similar to the discussion as in the proof of Theorem 1, we have

$$J \geq (1 - o(1))Fk/2(r^2). \quad (22)$$

On the other hand, by (21) and the assumption, we have

$$J \leq \left( \sum_{n=0}^{\infty} (1 - \delta)R_k(A, n)r^{2n} + \sum_{n \geq n_0} \left( (R_k(A, n) - c')^2 - (1 - \delta)R_k(A, n) \right)r^{2n} \right)^{1/2}$$

$$= \left( (1 - \delta)Fk(r^2) + O(1) + O\left( \sum_{n \geq n_0 \cap kA \cap T} r^{2n} \right) \right)^{1/2}.$$

For $m = 1, 2, \ldots, k - 1$, let

$$T_m = \left\{ \sum_{\mu=1}^{m} j_\mu a_{j_\mu} \in kA | \begin{array}{l} 1 \leq j_\mu \leq k, \ a_{j_\mu} \in A, \ \mu = 1, \ldots, m \ \text{and} \\ j_1 + \cdots + j_m = k \\ a_{j_\mu} \neq a_{j_\nu} \quad \text{for} \quad 1 \leq \mu \neq \nu \leq m \end{array} \right\},$$

and let $f(m)$ denote the number of integer solutions of

$$j_1 + \cdots + j_m = k, \quad 1 \leq j_1, \ldots, j_m \leq k.$$
Then for fixed $k \geq 2$, we have
\[ kA \setminus T \subseteq \bigcup_{m=1}^{k-1} T_m \quad \text{and} \quad \sum_{m=1}^{k-1} f(m) = O(1). \]

Note that $r < 1$, we have
\[
\sum_{n > n_0 \atop n \in kA \setminus T} r^{2n} \leq \sum_{n \in kA \setminus T} r^{2n} \leq \sum_{m=1}^{k-1} \sum_{n \in T_m} r^{2m(r_1 + \cdots + r_m a_m)} \\
\leq \sum_{m=1}^{k-1} \sum_{n \in T_m} r^{2(n_1 a_1 + \cdots + n_m a_m)} \\
\leq \sum_{m=1}^{k-1} f(m) \sum_{a_1, \ldots, a_m \in A} r^{2(a_1 + \cdots + a_m)} \\
\ll \left( \sum_{a \in A} r^{2a} \right)^{k-1} = (F(r^2))^{k-1} = o(F^k(r^2)),
\]

so
\[
J \leq (1 - \delta + o(1))^{1/2} F k^{1/2} (r^2).
\]

By (22) and (23), we have $1 - o(1) \leq 1 - \delta$, this cannot hold for large $N$.

This completes the proof of Theorem 2. □

**Proof of Theorem 3.** Suppose that there is a number $n_0$ such that $d \leq R_k^{(2)}(A, n) < d + \frac{2\sqrt{2d} + 1}{k} - \frac{1}{k}$ for $n > n_0$, by $R_k(A, n) \leq k! R_k^{(2)}(A, n)$ we have $R_k(A, n)$ is bounded. Let $T = \{ n \in kA | R_k^{(1)}(A, n) = R_k^{(2)}(A, n) \}$.

Now the remainder of the proof is similar to the proof of Theorem 2. □

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**References**