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L^p boundedness of commutators of Riesz transforms associated to Schrödinger operator ☆

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Abstract

In this paper we consider L^p boundedness of some commutators of Riesz transforms associated to Schrödinger operator $P = -\Delta + V(x)$ on $\mathbb{R}^n$, $n \geqslant 3$. We assume that $V(x)$ is non-zero, non-negative, and belongs to B_q for some $q \geqslant n/2$. Let $T_1 = (-\Delta + V)^{-1}V$, $T_2 = (-\Delta + V)^{-1/2}V^{1/2}$ and $T_3 = (-\Delta + V)^{-1/2}\nabla$. We obtain that $[b, T_j]$ $(j = 1, 2, 3)$ are bounded operators on $L^p(\mathbb{R}^n)$ when p ranges in a interval, where $b \in \mathbf{BMO}(\mathbb{R}^n)$. Note that the kernel of T_j $(j = 1, 2, 3)$ has no smoothness.

Keywords: Commutator; BMO; Smoothness; Boundedness; Riesz transforms associated to Schrödinger operators

1. Introduction

Let $P = -\Delta + V(x)$ be the Schrödinger differential operator on $\mathbb{R}^n$, $n \geqslant 3$. Throughout the paper we will assume that $V(x)$ is a non-zero, non-negative potential, and belongs to B_q for some $q > n/2$. Let T_j $(j = 1, 2, 3)$ be the Riesz transforms associated to Schrödinger operators, namely, $T_1 = (-\Delta + V)^{-1}V$, $T_2 = (-\Delta + V)^{-1/2}V^{1/2}$ and $T_3 = (-\Delta + V)^{-1/2}\nabla$. L^p boundedness of T_j $(j = 1, 2, 3)$ was widely studied [7,8]. In this paper, we will discuss the L^p boundedness of the commutator operators $[b, T_j] = bT_j - T_j b$ $(j = 1, 2, 3)$, where $b \in \mathbf{BMO}(\mathbb{R}^n)$.

A non-negative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(1 < q < \infty)$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q\,dx\right)^{1/q} \leqslant C\left(\frac{1}{|B|}\int_B V\,dx\right) \tag{1}$$

holds for every ball B in $\mathbb{R}^n$.

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Remark 1. By Hölder inequality we can get that $B_{q_1} \subset B_{q_2}$, for $q_1 \geqslant q_2 > 1$. One remarkable feature about the B_q class is that, if $V \in B_q$ for some $q > 1$, then there exists $\epsilon > 0$, which depends only on n and the constant C in (1), such that $V \in B_{q+\epsilon}$ [2]. It is also well known that, if $V \in B_q$, $q > 1$, then $V(x)\,dx$ is a doubling measure, namely for any $r > 0$, $x \in \mathbb{R}^n$,

$$\int_{B(x,2r)} V(y)\,dy \leqslant C_0 \int_{B(x,r)} V(y)\,dy. \tag{2}$$

It was proved that if $V \in B_n$, then T_3 is a Calderón–Zygmund operator [7]. According to the classical result of R. Coifman, R. Rochberg, and G. Weiss [1], $[b, T_3]$ is bounded on L^p $(1 < p < \infty)$ in this case. So we restrict ourselves to the case that $V \in B_q$ $(n/2 < q < n)$, when considering $[b, T_3]$.

We recall that an operator T taking $C_c^\infty(R^n)$ into $L^1_{\mathrm{loc}}(R^n)$ is called a Calderón–Zygmund operator if

(a) T extends to a bounded linear operator on $L^2(R^n)$,
(b) there exists a kernel K such that for every $f \in L_c^\infty(R^n)$,

$$Tf(x) = \int_{R^n} K(x,y) f(y)\,dy \quad \text{a.e. on } \{\operatorname{supp} f\}^c,$$

(c) the kernel $K(x,y)$ satisfies the Calderón–Zygmund estimate

$$\bigl|K(x,y)\bigr| \leqslant \frac{C}{|x-y|^n}; \tag{3}$$

$$\bigl|K(x+h,y) - K(x,y)\bigr| \leqslant \frac{C|h|^\delta}{|x-y|^{n+\delta}}; \tag{4}$$

$$\bigl|K(x,y+h) - K(x,y)\bigr| \leqslant \frac{C|h|^\delta}{|x-y|^{n+\delta}}; \tag{5}$$

for $x, y \in R^n$, $|h| < \frac{|x-y|}{2}$ and for some $\delta > 0$.

If T is a Calderón–Zygmund operator, $b \in \mathbf{BMO}$, the boundedness on every L^p $(1 < p < \infty)$ of $[b, T]$ was first discovered by Coifman, Rochberg and Weiss [1]. Later, Strömberg [4] gave a simple proof, adopting the idea of relating commutators with the sharp maximal operator of Fefferman and Stein. In both proofs, the smoothness of the kernel (4) plays a key role. However, in our problem the kernel has no smoothness of this kind due to V. This difficulty can be overcome by our basic idea. We discover that the kernels have some other kind of smoothness.

Definition 1. $K(x,y)$ is said to satisfy $H(m)$ for some $m \geqslant 1$, if there exists a constant $C > 0$, such that, $\forall l > 0$, $x, x_0 \in \mathbb{R}^n$ with $|x - x_0| \leqslant l$, then

$$\sum_{k=5}^{\infty} k\bigl(2^k l\bigr)^{\frac{n}{m'}} \Biggl(\int_{2^k l \leqslant |y-x_0| < 2^{k+1} l} \bigl|K(x,y) - K(x_0,y)\bigr|^m\,dy\Biggr)^{1/m} < C, \tag{6}$$

where $1/m' = 1 - 1/m$.

This kind of smoothness was not new. We find that the case $m = 1$ was given by Meyer [5]. It is easily seen that if $K(x,y)$ satisfies (4), then $K(x,y)$ satisfies $H(m)$ for every $m \geqslant 1$. By Hölder inequality we can get that if $K(x,y)$ satisfies $H(m)$ for some $m \geqslant 1$, then $K(x,y)$ satisfies $H(t)$ for $1 \leqslant t \leqslant m$. We now list some results concerning L^p boundedness of T_j $(j = 1,2,3)$, and refer the readers to [7] for further details. We will adopt the notation $1/p' = 1 - 1/p$ for $p \geqslant 1$ throughout the paper.

Theorem A. *(See [7, Theorem 3.1, p. 526].) Suppose $V \in B_q$ and $q \geqslant n/2$. Then, for $q' \leqslant p \leqslant \infty$,*

$$\bigl\|(-\Delta + V)^{-1} V f\bigr\|_p \leqslant C_p \|f\|_p.$$

Theorem B. *(See [7, Theorem 5.10, p. 542].) Suppose $V \in B_q$ and $q \geqslant n/2$. Then, for $(2q)' \leqslant p \leqslant \infty$,*

$$\left\|(-\Delta+V)^{-1/2}V^{1/2}f\right\|_p \leqslant C_p\|f\|_p.$$

Theorem C. *(See [7, Theorem 0.5, p. 514].) Suppose $V \in B_q$ and $\frac{n}{2} \leqslant q < n$. Let $(1/p_0) = (1/q) - (1/n)$. Then, for $p_0' \leqslant p < \infty$,*

$$\left\|(-\Delta+V)^{-1/2}\nabla f\right\|_p \leqslant C_p\|f\|_p.$$

The basic idea in [7] is that, to exploit a pointwise estimate of the kernel and the comparison to the kernel of classical Riesz transform. Generally, it is based on the following two basic facts. If V is large, then one expects the kernel itself has a good decay. On the other hand, if V is small, then it is close to the classical Riesz transform. In this paper, we adopt a different idea. Since we know that the kernel do not satisfy the Calderón–Zygmund estimate (4), we study how close it is. See Section 2.

We will show that the kernels have very good smoothness with respect to the first variable of the following strong type. It is almost (4). There exists a constant $C > 0$ and $\delta > 0$, such that, for some $m > 1$, $\forall l > 0$, $x, x_0 \in \mathbb{R}^n$ with $|x - x_0| \leqslant l$, then

$$\sum_{k=5}^{\infty} 2^{k\delta}\left(2^k l\right)^{\frac{n}{m'}}\left(\int\limits_{2^k l \leqslant |y-x_0| < 2^{k+1} l} \left|K(x,y) - K(x_0,y)\right|^m dy\right)^{1/m} < C. \tag{7}$$

Recall that $T_1 = (-\Delta+V)^{-1}V$, $T_2 = (-\Delta+V)^{-1/2}V^{1/2}$, and $T_3 = (-\Delta+V)^{-1/2}\nabla$. Now we state our main results.

Theorem 1. *Suppose $V \in B_q$ and $q \geqslant n/2$. Let $b \in$ **BMO**. Then, we have*

(i) *If $q' \leqslant p < \infty$,*
$$\left\|[b, T_1]f\right\|_p \leqslant C_p\|b\|_{\mathbf{BMO}}\|f\|_p;$$
(ii) *If $(2q)' \leqslant p < \infty$,*
$$\left\|[b, T_2]f\right\|_p \leqslant C_p\|b\|_{\mathbf{BMO}}\|f\|_p;$$
(iii) *If $p_0' \leqslant p < \infty$, let $(1/p_0) = (1/q) - (1/n)$,*
$$\left\|[b, T_3]f\right\|_p \leqslant C_p\|b\|_{\mathbf{BMO}}\|f\|_p.$$

We know that $T_1^* = V(-\Delta+V)^{-1}$, $T_2^* = V^{1/2}(-\Delta+V)^{-1/2}$, and $T_3^* = -\nabla(-\Delta+V)^{-1/2}$. By duality we can easily get that

$$\left\|\left[b, T_1^*\right]f\right\|_p \leqslant C_p\|b\|_{\mathbf{BMO}}\|f\|_p, \quad 1 < p \leqslant q,$$
$$\left\|\left[b, T_2^*\right]f\right\|_p \leqslant C_p\|b\|_{\mathbf{BMO}}\|f\|_p, \quad 1 < p \leqslant 2q,$$
$$\left\|\left[b, T_3^*\right]f\right\|_p \leqslant C_p\|b\|_{\mathbf{BMO}}\|f\|_p, \quad 1 < p \leqslant p_0.$$

From Theorem 1(i), we can get the result concerning second order Riesz transform. Let $T_4 = (-\Delta+V)^{-1}\nabla^2$, then $T_4^* = \nabla^2(-\Delta+V)^{-1}$. Indeed, $T_4 = (-\Delta+V)^{-1}\nabla^2 = (-\Delta+V)^{-1}\Delta\Delta^{-1}\nabla^2 = (I - (-\Delta+V)V)\frac{\nabla^2}{\Delta} = (I - T_1)\frac{\nabla^2}{\Delta}$. We have

Corollary 1. *Suppose $V \in B_q$ and $q \geqslant n/2$. Then*

$$\left\|[b, T_4]f\right\|_p \leqslant C_p\|b\|_{\mathbf{BMO}}\|f\|_p, \quad q' \leqslant p < \infty,$$

and

$$\left\|\left[b, T_4^*\right]f\right\|_p \leqslant C_p\|b\|_{\mathbf{BMO}}\|f\|_p, \quad 1 < p \leqslant q.$$

For classical Riesz transform, the converse problem was also considered in [1]. This implies a new characterization of **BMO**. In this paper we also discuss the converse problem. Namely, if $[b, T_3]$ is bounded on L^2, do we have $b \in \mathbf{BMO}$? The answer is negative for general $V \in B_q$. It is due to that, for some good V, the kernel of T_3 is better than that of Riesz transform, which makes that the commutator can absorb mild singularity. We give a counterexample for $V \equiv 1$. On the other hand, if imposing some integrability condition on V, we can have the converse.

Throughout this paper, unless otherwise indicated, we will use C and c to denote constants, which are not necessarily the same at each occurrence. By $A \sim B$, we mean that there exist constants $C > 0$ and $c > 0$, such that $c \leqslant A/B \leqslant C$.

The paper is organized as following. In Section 2, we will give the estimates of the kernels K_j $(j = 1, 2, 3)$ of the operators T_j. The proof of Theorem 1 is stated in Section 3. In Section 4, we discuss the converse problem.

2. Estimate of the kernels

This section is devoted to give the estimate of the kernels associated to T_j $(j = 1, 2, 3)$ and denoted by $K_j(x, y)$ $(j = 1, 2, 3)$ respectively. Let $\Gamma(x, y, \tau)$ denote the fundamental solution for the Schrödinger operator $-\Delta + (V(x) + i\tau)$, $\tau \in \mathbb{R}$, and $\Gamma_0(x, y, \tau)$ for the operator $-\Delta + i\tau$, $\tau \in \mathbb{R}$. Clearly, $\Gamma(x, y, \tau) = \Gamma(y, x, -\tau)$.

For $x \in \mathbb{R}^n$, the function $m(x, V)$ is defined by

$$\frac{1}{m(x, V)} = \sup\left\{r > 0\colon \frac{1}{r^{n-2}} \int B(x, r)V(y)\,dy \leqslant 1\right\}.$$

The function $m(x, V)$ reflects the scale of $V(x)$ essentially, but behaves better. It is deeply studied in [7], and will play a crucial role in our proof. We list some properties of $m(x, V)$ here, and their proof can be found in [7].

Lemma A. *(See [7, Lemma 1.4].) Assume $V \in B_q$ for some $q > n/2$, then there exist $C > 0$, $c > 0$, $k_0 > 0$, such that, for any x, y in $\mathbb{R}^n$, and $0 < r < R < \infty$,*

(a) $0 < m(x, V) < \infty$,
(b) *If* $h = \frac{1}{m(x,V)}$, *then* $\frac{1}{h^{n-2}} \int_{B(x,h)} V(y)\,dy = 1$,
(c) $m(x, V) \sim m(y, V)$, *if* $|x - y| \leqslant \frac{C}{m(x,V)}$,
(d) $m(y, V) \leqslant C\{1 + |x - y|m(x, V)\}^{k_0} m(x, V)$,
(e) $m(y, V) \geqslant cm(x, V)\{1 + |x - y|m(x, V)\}^{-k_0/(1+k_0)}$,
(f) $c\{1 + |x - y|m(y, V)\}^{1/(k_0+1)} \leqslant 1 + |x - y|m(x, V) \leqslant C\{1 + |x - y|m(y, V)\}^{k_0+1}$,
(g) $\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\,dy \leqslant C(\frac{R}{r})^{(n/q)-2} \cdot \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)\,dy$.

Estimating the kernels mainly relies on functional calculus and a pointwise estimate of $\Gamma(x, y, \tau)$ that was given in [7].

Theorem D. *(See [7, Theorem 2.7].) Suppose $V \in B_{n/2}$. Then, for any $x, y \in \mathbb{R}^n$, $\tau \in \mathbb{R}$, and integer $k > 0$,*

$$\Gamma(x, y, \tau) \leqslant \frac{C_k}{\{1 + |\tau|^{1/2}|x - y|\}^k\{1 + m(x, V)|x - y|\}^k} \cdot \frac{1}{|x - y|^{n-2}},$$

where C_k is a constant independent of x, y, τ.

The next lemma is used to control the integration of V on a ball.

Lemma 1. *Suppose $V \in B_q$ for some $q > n/2$. Let $N > \log_2 C_0 + 1$, where C_0 is the constant in* (2). *Then for any $x_0 \in \mathbb{R}^n$, $R > 0$,*

$$\frac{1}{\{1 + m(x_0, V)R\}^N} \int_{B(x_0,R)} V(\xi)\,d\xi \leqslant CR^{n-2}.$$

Proof. There exists a integer $j_0 \in \mathbb{Z}$ such that $2^{j_0}R \leqslant \frac{1}{m(x_0,V)} < 2^{j_0+1}R$. We will discuss in following two cases.

Case 1. $j_0 < 0$. By (2), Lemma 1, and (b) of Lemma A, we can get

$$\frac{1}{\{1+m(x_0,V)R\}^N}\int\limits_{B(x_0,R)} V(\xi)\,d\xi \leqslant \frac{1}{(2^{-j_0})^N}\int\limits_{B(x_0,R)} V(\xi)\,d\xi \leqslant \frac{1}{\{2^{-j_0}\}^N}C_0^{-j_0}\big(2^{j_0}R\big)^{n-2} \leqslant R^{n-2} \quad (\text{since } N > \log_2 C_0).$$

Case 2. $j_0 \geqslant 0$. By (b) and (g) of Lemma A, we can get

$$\frac{1}{\{1+m(x_0,V)R\}^N}\int\limits_{B(x_0,R)} V(\xi)\,d\xi \leqslant \int\limits_{B(x_0,R)} V(\xi)\,d\xi \leqslant R^{n-2}\frac{1}{R^{n-2}}\int\limits_{B(x_0,R)} V(\xi)\,d\xi \leqslant R^{n-2}.$$

This completes the proof of Lemma 1. □

Before giving the estimate of the kernels, we still needs one lemma, which is proved in [7].

Lemma B. *(See [7, Lemma 4.6].) Suppose $V \in B_{q_0}$, $q_0 > 1$. Assume that $-\Delta u + (V(x) + i\tau)u = 0$ in $B(x_0, 2R)$ for some $x_0 \in \mathbb{R}^n$, $R > 0$. Then*

(a) *for $x \in B(x_0, R)$,*

$$\big|\nabla u(x)\big| \leqslant C \sup_{B(x_0,2R)}|u| \cdot \int\limits_{B(x_0,2R)} \frac{V(y)}{|x-y|^{n-1}}\,dy + \frac{C}{R^{n+1}}\int\limits_{B(x_0,2R)} \big|u(y)\big|\,dy,$$

(b) *if $(n/2) < q_0 < n$, let $(1/t) = (1/q_0) - (1/n)$, $k_0 > \log_2 C_0 + 1$,*

$$\Bigg(\int\limits_{B(x_0,R)} |\nabla u|^t\,dx\Bigg)^{1/t} \leqslant CR^{(n/q_0)-2}\big\{1 + Rm(x_0,V)\big\}^{k_0}\sup_{B(x_0,2R)}|u|.$$

Now we are ready to give the estimate of the kernels.

Lemma 2. *Suppose $V \in B_q$ for some $q > n/2$. Then, there exists $\delta > 0$ and for any integer $k > 0$, $0 < h < |x-y|/16$,*

$$\big|K_1(x,y)\big| \leqslant \frac{C_k}{\{1+m(x,V)|x-y|\}^k}\cdot\frac{1}{|x-y|^{n-2}}V(y), \tag{8}$$

$$\big|K_1(x+h,y) - K_1(x,y)\big| \leqslant \frac{C_k}{\{1+m(x,V)|x-y|\}^k}\cdot\frac{|h|^\delta}{|x-y|^{n-2+\delta}}V(y). \tag{9}$$

Lemma 3. *Suppose $V \in B_q$ for some $q > n/2$. Then, there exists $\delta > 0$ and for any integer $k > 0$, $0 < h < |x-y|/16$,*

$$\big|K_2(x,y)\big| \leqslant \frac{C_k}{\{1+m(x,V)|x-y|\}^k}\cdot\frac{1}{|x-y|^{n-1}}V(y)^{1/2}, \tag{10}$$

$$\big|K_2(x+h,y) - K_2(x,y)\big| \leqslant \frac{C_k}{\{1+m(y,V)|x-y|\}^k}\cdot\frac{|h|^\delta}{|x-y|^{n-1+\delta}}V(y)^{1/2}. \tag{11}$$

Lemma 4. *Suppose $V \in B_q$ for some $n/2 < q < n$. Then, there exists $\delta > 0$ and for any integer $k > 0$, $0 < h < |x-y|/16$,*

$$\big|K_3(x,y)\big| \leqslant \frac{C_k}{\{1+m(x,V)|x-y|\}^k}\frac{1}{|x-y|^{n-1}}\cdot\left(\int\limits_{B(y,|x-y|)}\frac{V(\xi)}{|y-\xi|^{n-1}}\,d\xi+\frac{1}{|x-y|}\right), \tag{12}$$

$$\begin{aligned}&\big|K_3(x+h,y)-K_3(x,y)\big|\\&\quad\leqslant \frac{C_k}{\{1+m(x,V)|x-y|\}^k}\frac{|h|^\delta}{|x-y|^{n-1+\delta}}\cdot\left(\int\limits_{B(y,|x-y|)}\frac{V(\xi)}{|y-\xi|^{n-1}}\,d\xi+\frac{1}{|x-y|}\right).\end{aligned} \tag{13}$$

Remark 2. If $V \in B_n$, then (4) follows immediately from (13). This can tell us how the kernel behaves when V changes. However, we do not have similar result about the smoothness with respect to the second variable.

Proof of Lemma 2. We easily know that $K_1(x,y)=\Gamma(x,y,0)V(y)$. It immediately follows from Theorem D that, for any $x,y\in\mathbb{R}^n$,

$$\big|K_1(x,y)\big| \leqslant \frac{C_k}{\{1+m(x,V)|x-y|\}^k}\cdot\frac{1}{|x-y|^{n-2}}V(y).$$

For (9), fix $x,y\in\mathbb{R}^n$, and fix $n/2<q_0<\min(n,q)$, then we know $V\in B_{q_0}$. Let $R=\frac{|x-y|}{8}$, $1/t=1/q_0-1/n$, then $\delta=1-n/t>0$ and for any $0<h<\frac{R}{2}$, it follows from the embedding theorem of Morrey (see [3]) and Lemma B that

$$\begin{aligned}\big|K_1(x+h,y)-K_1(x,y)\big| &\leqslant \big|\Gamma(x+h,y,0)-\Gamma(x,y,0)\big|V(y)\\
&\leqslant C|h|^{1-(n/t)}\left(\int\limits_{B(x,R)}\big|\nabla_x\Gamma(z,y,0)\big|^t\,dz\right)^{1/t}V(y)\\
&\leqslant C|h|^{1-(n/t)}R^{(n/q_0)-2}\big\{1+Rm(x,V)\big\}^{k_0}\sup_{z\in B(x,2R)}\big|\Gamma(z,y,0)\big|V(y)\\
&\leqslant C\frac{|h|^\delta}{R^\delta}\big\{1+Rm(x,V)\big\}^{k_0}\sup_{z\in B(x,2R)}\big|\Gamma(z,y,0)\big|V(y)\\
&\leqslant C\frac{|h|^\delta}{R^\delta}\big\{1+Rm(x,V)\big\}^{k_0}\sup_{z\in B(x,2R)}\frac{C_{k_1}}{\{1+m(y,V)|z-y|\}^{k_1}}\cdot\frac{1}{|z-y|^{n-2}}V(y)\\
&\leqslant C_k\frac{|h|^\delta}{|x-y|^\delta}\frac{1}{\{1+m(x,V)|x-y|\}^k}\cdot\frac{1}{|x-y|^{n-2}}V(y)\quad (k_1\text{ large}),\end{aligned}$$

where we used (f) of Lemma A in the last inequality. □

Proof of Lemma 3. By functional calculus, we may write

$$(-\Delta+V)^{-1/2}=-\frac{1}{2\pi}\int\limits_{\mathbb{R}}(-i\tau)^{-1/2}(-\Delta+V+i\tau)^{-1}\,d\tau,$$

then we know that

$$K_2(x,y)=-\frac{1}{2\pi}\int\limits_{\mathbb{R}}(-i\tau)^{-1/2}\Gamma(x,y,\tau)\,d\tau\,V(y)^{1/2}. \tag{14}$$

In order to estimate the integration, we claim that: For $k>2$, then

$$\int\limits_{\mathbb{R}}|\tau|^{-1/2}\big\{1+|\tau|^{1/2}|x-y|\big\}^{-k}\,d\tau\leqslant\frac{C_k}{|x-y|}. \tag{15}$$

In fact, we have

$$\int_{\mathbb{R}} |\tau|^{-1/2}\left\{1+|\tau|^{1/2}|x-y|\right\}^{-k} d\tau = \left(\int_{|\tau|\leqslant|x-y|^{-2}} + \int_{|\tau|\geqslant|x-y|^{-2}}\right)|\tau|^{-1/2}\left\{1+|\tau|^{1/2}|x-y|\right\}^{-k} d\tau$$
$$\leqslant \int_{|\tau|\leqslant|x-y|^{-2}} |\tau|^{-1/2}\,d\tau + \int_{|\tau|\geqslant|x-y|^{-2}} |\tau|^{(-k-1)/2}|x-y|^{-k}\,d\tau$$
$$\leqslant \frac{C_k}{|x-y|}.$$

From Theorem D and the estimate (15), we immediately get (10). For (11), fix $x, y \in \mathbb{R}^n$, and fix $n/2 < q_0 < \min(n,q)$, then we know $V \in B_{q_0}$. Let $R = \frac{|x-y|}{8}$, $1/t = 1/q_0 - 1/n$, then $\delta = 1 - n/t > 0$ and for any $0 < h < \frac{R}{2}$, we have

$$\left|K_2(x+h,y) - K_2(x,y)\right| \leqslant \frac{1}{2\pi}\int_{\mathbb{R}} |\tau|^{-1/2}\left|\Gamma(x+h,y,\tau) - \Gamma(x,y,\tau)\right| d\tau\, V(y)^{1/2}. \tag{16}$$

Similarly, it follows from the embedding theorem of Morrey and Lemma B that

$$\left|\Gamma(x+h,y,\tau) - \Gamma(x,y,\tau)\right| \leqslant C|h|^{1-(n/t)}\left(\int_{B(x,R)} \left|\nabla_x\Gamma(z,y,\tau)\right|^t dz\right)^{1/t}$$
$$\leqslant C|h|^{1-(n/t)}R^{(n/q_0)-2}\left\{1+Rm(x,V)\right\}^{k_0} \sup_{z\in B(x,2R)}\left|\Gamma(z,y,\tau)\right|$$
$$\leqslant C\frac{|h|^\delta}{R^\delta}\left\{1+Rm(x,V)\right\}^{k_0} \sup_{z\in B(x,2R)}\left|\Gamma(z,y,\tau)\right|$$
$$\leqslant C\frac{|h|^\delta}{R^\delta}\left\{1+Rm(x,V)\right\}^{k_0} \sup_{z\in B(x,2R)} \frac{C_k\{1+|\tau|^{1/2}|z-y|\}^{-k}}{\{1+m(y,V)|z-y|\}^k}\cdot\frac{1}{|z-y|^{n-2}}$$
$$\leqslant C_k\frac{|h|^\delta}{|x-y|^\delta}\frac{\{1+|\tau|^{1/2}|x-y|\}^{-k}}{\{1+m(y,V)|x-y|\}^k}\cdot\frac{1}{|x-y|^{n-2}}.$$

Hence, insert this to (16), it follows from the estimate (15) that

$$\left|K_2(x+h,y) - K_2(x,y)\right| \leqslant C_k\frac{|h|^\delta}{|x-y|^{n-1+\delta}}\frac{1}{\{1+m(y,V)|x-y|\}^k}V(y)^{1/2}. \qquad \square$$

Proof of Lemma 4. By partial integral, we know that

$$K_3(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} (-i\tau)^{-1/2}\nabla_y\Gamma(x,y,\tau)\,d\tau. \tag{17}$$

Fix $x, y \in \mathbb{R}^n$, let $R = \frac{|x-y|}{8}$, $1/t = 1/q - 1/n$, $\delta = n/q - 2 > 0$, and for any $0 < h < \frac{R}{2}$, we have

$$\left|K_3(x+h,y) - K_3(x,y)\right| \leqslant \frac{1}{2\pi}\int_{\mathbb{R}} |\tau|^{-1/2}\left|\nabla_y\Gamma(x+h,y,\tau) - \nabla_y\Gamma(x,y,\tau)\right| d\tau. \tag{18}$$

Similarly, it follows from the embedding theorem of Morrey and Lemma B that

$$\left|\nabla_y\Gamma(x+h,y,\tau) - \nabla_y\Gamma(x,y,\tau)\right| \leqslant C|h|^{1-(n/t)}\left(\int_{B(x,R)} \left|\nabla_x\nabla_y\Gamma(z,y,\tau)\right|^t dz\right)^{1/t}$$
$$\leqslant C|h|^{1-(n/t)}R^{(n/q)-2}\left\{1+Rm(x,V)\right\}^{k_0} \sup_{z\in B(x,2R)}\left|\nabla_y\Gamma(z,y,\tau)\right|. \tag{19}$$

Since $\Gamma(z,y,\tau) = \Gamma(y,z,-\tau)$, then $\nabla_y\Gamma(z,y,\tau) = \nabla_x\Gamma(y,z,-\tau)$. It follows from (a) of Lemma B that

$$\sup_{z\in B(x,2R)}\big|\nabla_y\Gamma(z,y,\tau)\big| \leqslant \sup_{z\in B(x,2R)}\big|\nabla_x\Gamma(y,z,-\tau)\big|$$
$$\leqslant \sup_{z\in B(x,2R)}\bigg\{\sup_{\eta\in B(y,|y-z|/4)}\big|\Gamma(\eta,z,-\tau)\big|\cdot\int\limits_{B(y,|z-y|/2)}\frac{V(\xi)}{|y-\xi|^{n-1}}\,d\xi + \frac{C}{|y-z|^{n+1}}\int\limits_{B(y,|z-y|/2)}\Gamma(\xi,z,-\tau)\,d\xi\bigg\}.$$

Using the fact that $|\eta-z|\sim|y-z|$, $|\xi-z|\sim|y-z|$ and $|x-y|\sim|y-z|$, choosing k_1 sufficiently large, it follows from Theorem D and (f) of Lemma A that

$$\sup_{z\in B(x,2R)}\big|\nabla_y\Gamma(z,y,\tau)\big|$$
$$\leqslant \sup_{z\in B(x,2R)}\frac{C_{k_1}}{\{1+|\tau|^{1/2}|y-z|\}^{k_1}\{1+m(z,V)|y-z|\}^{k_1}}\cdot\frac{1}{|y-z|^{n-2}}\int\limits_{B(y,|x-y|)}\frac{V(\xi)}{|y-\xi|^{n-1}}\,d\xi$$
$$+\frac{C_{k_1}}{\{1+|\tau|^{1/2}|y-z|\}^{k_1}\{1+m(z,V)|y-z|\}^{k_1}}\cdot\frac{1}{|y-z|^{n-1}}$$
$$\leqslant \frac{C_k}{\{1+|\tau|^{1/2}|x-y|\}^{k}\{1+m(x,V)|x-y|\}^{k}}\cdot\frac{1}{|x-y|^{n-2}}\int\limits_{B(y,|x-y|)}\frac{V(\xi)}{|y-\xi|^{n-1}}\,d\xi$$
$$+\frac{C_k}{\{1+|\tau|^{1/2}|x-y|\}^{k}\{1+m(x,V)|x-y|\}^{k}}\cdot\frac{1}{|x-y|^{n-1}}. \tag{20}$$

From the estimate (15) and (20), we immediately get (12). Inserting (20) to (19), we get that

$$\big|\nabla_y\Gamma(x+h,y,\tau)-\nabla_y\Gamma(x,y,\tau)\big| \leqslant C_k\frac{|h|^\delta}{|x-y|^\delta}\frac{C_k}{\{1+|\tau|^{1/2}|x-y|\}^{k}\{1+m(x,V)|x-y|\}^{k}}$$
$$\times\bigg(\frac{1}{|x-y|^{n-2}}\int\limits_{B(y,|x-y|)}\frac{V(\xi)}{|y-\xi|^{n-1}}\,d\xi+\frac{1}{|x-y|^{n-1}}\bigg). \tag{21}$$

Inserting (21) to (18), we get from the estimate (15) that

$$\big|K_3(x+h,y)-K_3(x,y)\big|$$
$$\leqslant C_k\frac{|h|^\delta}{|x-y|^\delta}\frac{1}{\{1+m(x,V)|x-y|\}^{k}}\cdot\bigg(\frac{1}{|x-y|^{n-1}}\int\limits_{B(y,|x-y|)}\frac{V(\xi)}{|y-\xi|^{n-1}}\,d\xi+\frac{1}{|x-y|^{n}}\bigg).\qquad\Box$$

3. Proof of main results

We first discuss the problem for general operator $Tf(x)=\int K(x,y)f(y)\,dy$. Later, we will specialize to T_j $(j=1,2,3)$.

Proposition 1. *Let $m>1$, suppose T is bounded on L^p for every $p\in(m',\infty)$, and K satisfies $H(m)$, then $\forall b\in\mathbf{BMO}$, $[b,T]$ is bounded on L^p for every $p\in(m',\infty)$, and*

$$\big\|[b,T]f\big\|_p\leqslant C_p\|b\|_{\mathbf{BMO}}\|f\|_p.$$

We adopt the idea of Strömberg (cf. [6]). Recall that the sharp function of Fefferman–Stein is defined by

$$M^\sharp f(x)=\sup_{x\in B}\frac{1}{|B|}\int\limits_B\big|f(y)-f_B\big|\,dy, \tag{22}$$

where $f_B=\frac{1}{|B|}\int_B f(y)\,dy$, and the supremum is taken on all balls B with $x\in B$.

Recall that **BMO** is defined by

$$\mathbf{BMO}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n):\ \|f\|_{\mathbf{BMO}} = \left\| M^\sharp f \right\|_\infty < \infty \right\}. \tag{23}$$

Two basic facts about **BMO** may be in order. We use $2^k B$ to denote the ball with the same center as B but with 2^k times radius.

$$|f_{2^k B} - f_B| \leqslant C(k+1)\|f\|_{\mathbf{BMO}}, \quad \text{for } k > 0. \tag{24}$$

The second one is due to John–Nirenberg:

$$\|f\|_{\mathbf{BMO}} \sim \sup_B \left(\frac{1}{|B|} \int_B \left| f(y) - f_B \right|^p dy \right)^{1/p}, \quad \text{for any } p > 1. \tag{25}$$

Proposition 1 follows immediately from the following lemma and a theorem of Fefferman–Stein on sharp function.

Lemma 5. *Let T satisfies the same condition in Proposition* 1. *Then $\forall s > m'$, there exists constant $C_s > 0$, such that $\forall f \in L^1_{\mathrm{loc}}$, $b \in$* **BMO**

$$M^\sharp\big([b,T]f\big)(x) \leqslant C_s \|b\|_{\mathbf{BMO}} \left\{ M_s(Tf)(x) + M_s(f)(x) \right\}, \tag{26}$$

where $M_s(f) = M(|f|^s)^{1/s}$ and M is Hardy–Littlewood maximal function.

Proof. Fix $s > m'$, $f \in L^1_{\mathrm{loc}}$, $x \in \mathbb{R}^n$, and fix a ball $I = B(x_0, l)$ with $x \in I$. We only need to control $J = \frac{1}{|I|}\int_I |[b,T]f(y) - ([b,T]f)_I|\,dy$ by the right-hand side of (26). Let $f = f_1 + f_2$, where $f_1 = f\chi_{32I}$, $f_2 = f - f_1$. Then $[b,T]f = [b - b_I, T]f = (b - b_I)Tf - T(b - b_I)f_1 - T(b - b_I)f_2 \triangleq A_1 f + A_2 f + A_3 f$, and we get

$$\begin{aligned} J &\leqslant \frac{1}{|I|}\int_I \left| A_1 f(y) - (A_1 f)_I \right| dy + \frac{1}{|I|}\int_I \left| A_2 f(y) - (A_2 f)_I \right| dy + \frac{1}{|I|}\int_I \left| A_3 f(y) - (A_3 f)_I \right| dy \\ &\triangleq J_1 + J_2 + J_3. \end{aligned}$$

Step 1. First we consider J_1. By Hölder inequality and (25),

$$\begin{aligned} J_1 &\leqslant \frac{2}{|I|}\int_I \left| A_1 f(y) \right| dy = \frac{2}{|I|}\int_I \left| (b - b_I)Tf(y) \right| dy \leqslant 2\left(\frac{1}{|I|}\int_I \left| (b - b_I) \right|^{s'} dy \right)^{1/s'} \left(\frac{1}{|I|}\int_I \left| Tf(y) \right|^s dy \right)^{1/s} \\ &\leqslant 2\|b\|_{\mathbf{BMO}} M_s(Tf)(x). \end{aligned}$$

Step 2. Second we consider J_2. Fix s_1 such that $s > s_1 > m'$, and let $s_2 = \frac{ss_1}{s - s_1}$, then we have

$$\begin{aligned} J_2 &\leqslant 2\frac{1}{|I|}\int_I \left| A_2 f(y) \right| dy \leqslant 2\left(\frac{1}{|I|}\int_I \left| A_2 f(y) \right|^{s_1} dy \right)^{1/s_1} \leqslant 2\left(\frac{1}{|I|}\int_{32I} \left| (b - b_I)f(y) \right|^{s_1} dy \right)^{1/s_1} \\ &\leqslant C\left(\frac{1}{|32I|}\int_{32I} |b - b_I|^{s_2}\,dy \right)^{1/s_2} \left(\frac{1}{|32I|}\int_{32I} \left| f(y) \right|^s dy \right)^{1/s} \leqslant C\|b\|_{\mathbf{BMO}} M_s(f)(x). \end{aligned}$$

Step 3. Last we consider J_3. Set $c_I = \int_{|z - x_0| > 32l} K(x_0, z)(b(z) - b_I)f(z)\,dz$, then we have that

$$\begin{aligned} J_3 &\leqslant \frac{2}{|I|}\int_I \left| A_3 f(y) - c_I \right| dy \leqslant 2\frac{1}{|I|}\int_I \left| \int_{|z - x_0| \geqslant 32l} \left\{ K(y,z) - K(x_0,z) \right\}\big(b(z) - b_I\big) f(z)\,dz \right| dy \\ &\leqslant 2\frac{1}{|I|}\int_I \int_{|z - x_0| > 32l} \left| \left\{ K(y,z) - K(x_0,z) \right\}\big(b(z) - b_I\big) f(z) \right| dz\,dy \\ &= 2\frac{1}{|I|}\int_I \sum_{k=5}^{\infty} \int_{2^k l \leqslant |z - x_0| < 2^{k+1} l} \left| \left\{ K(y,z) - K(x_0,z) \right\}\big(b(z) - b_I\big) f(z) \right| dz\,dy. \end{aligned}$$

From Hölder's inequality, we get

$$\begin{aligned}
J_3 &\leqslant 2\frac{1}{|I|}\int_I \sum_{k=5}^{\infty}\left(\int_{2^k l\leqslant|z-x_0|<2^{k+1}l} \left|K(y,z)-K(x_0,z)\right|^m dz\right)^{1/m} \\
&\quad\times\left(\int_{2^k l\leqslant|z-x_0|<2^{k+1}l} \left|\big(b(z)-b_I\big)f(z)\right|^{m'} dz\right)^{1/m'} dy \\
&\leqslant 2\frac{1}{|I|}\int_I \sum_{k=5}^{\infty}\left(\int_{2^k l\leqslant|z-x_0|<2^{k+1}l} \left|K(y,z)-K(x_0,z)\right|^m dz\right)^{1/m} \big(2^k l\big)^{n/m'} k \\
&\quad\times\frac{1}{(2^k l)^{n/m'}k}\left(\int_{2^k l\leqslant|z-x_0|<2^{k+1}l} \left|\big(b(z)-b_I\big)f(z)\right|^{m'} dz\right)^{1/m'} dy \\
&\leqslant C\sup_{k\geqslant 5}\frac{1}{(2^k l)^{n/m'}k}\left(\int_{2^k l\leqslant|z-x_0|<2^{k+1}l} \left|\big(b(z)-b_I\big)f(z)\right|^{m'} dz\right)^{1/m'} \\
&\leqslant C\sup_{k\geqslant 5}\frac{1}{k}\left(\frac{1}{(2^k l)^n}\int_{|z-x_0|<2^{k+1}l} \left|\big(b(z)-b_{2^{k+1}I}+b_{2^{k+1}I}-b_I\big)f(z)\right|^{m'} dz\right)^{1/m'} \\
&\leqslant C\sup_{k\geqslant 5}\frac{1}{k}(k+2)\|b\|_{\mathbf{BMO}} M_s f(x) \quad \big(\text{by (24)}\big) \\
&\leqslant C\|b\|_{\mathbf{BMO}} M_s f(x).
\end{aligned}$$

This completes the proof of Lemma 5. □

Proof of Theorem 1. Now we begin to prove Theorem 1. Considering Remark 1, we can assume $q > \frac{n}{2}$, $q' < p$. We first prove (i). By Proposition 1 and Theorem A, it suffices to prove that K_1 satisfies $H(q)$ (see (6)). From (9), we have

$$\begin{aligned}
&\left(\int_{2^k l\leqslant|y-x_0|<2^{k+1}l} \left|K_1(x,y)-K_1(x_0,y)\right|^q dy\right)^{1/q} \\
&\quad\leqslant C_N\frac{l^\delta}{(2^k l)^{n-2+\delta}}\frac{1}{\{1+m(x_0,V)2^k l\}^N}\int_{B(x_0,2^{k+3}l)} V(y)^q\, dy^{1/q} \\
&\quad\leqslant C_N\frac{l^\delta}{(2^k l)^{n-2+\delta}}\frac{1}{\{1+m(x_0,V)2^k l\}^N}\big(2^k l\big)^{-n/q'}\int_{B(x_0,2^k l)} V(\xi)\, d\xi \\
&\quad\leqslant C_N\frac{l^\delta}{(2^k l)^{n-2+\delta}}\big(2^k l\big)^{n/q-2} \quad \text{(by Lemma 1)} \\
&\quad\leqslant C\frac{l^\delta}{(2^k l)^{(n/q')+\delta}}.
\end{aligned}$$

Thus, we can get

$$\sum_{k=5}^{\infty} k\big(2^k l\big)^{\frac{n}{q'}}\left(\int_{2^k l\leqslant|y-x_0|<2^{k+1}l} \left|K_1(x,y)-K_1(x_0,y)\right|^q dy\right)^{1/q} \leqslant \sum_{k=5}^{\infty}\frac{Ck}{(2^k)^\delta}\leqslant C.$$

For the proof of (ii) it suffices to prove that K_2 satisfies $H(2q)$. From (11), we have

$$\begin{aligned}
&\Bigg(\int_{2^k l\leqslant|y-x_0|<2^{k+1}l}\big|K_2(x,y)-K_2(x_0,y)\big|^{2q}\,dy\Bigg)^{1/(2q)}\\
&\quad\leqslant C_N\frac{l^\delta}{(2^kl)^{n-1+\delta}}\frac{1}{\{1+m(x_0,V)2^kl\}^N}\int_{B(x_0,2^{k+3}l)}V(\xi)^q\,d\xi^{1/(2q)}\\
&\quad\leqslant C_N\frac{l^\delta}{(2^kl)^{n-1+\delta}}\frac{1}{\{1+m(x_0,V)2^kl\}^N}\big(2^kl\big)^{-n/(2q')}\int_{B(x_0,2^kl)}V(\xi)\,d\xi^{1/2}\\
&\quad\leqslant C_N\frac{l^\delta}{(2^kl)^{n-1+\delta}}\big(2^kl\big)^{-n/(2q')+(n-2)/2}\leqslant C\frac{l^\delta}{(2^kl)^\delta}\big(2^kl\big)^{-n/(2q)'},
\end{aligned}$$

hence,

$$\sum_{k=5}^{\infty}k\big(2^kl\big)^{\frac{n}{(2q)'}}\Bigg(\int_{2^k l\leqslant|y-x_0|<2^{k+1}l}\big|K_2(x,y)-K_2(x_0,y)\big|^{2q}\,dy\Bigg)^{1/(2q)}\leqslant\sum_{k=5}^{\infty}\frac{Ck}{(2^k)^\delta}\leqslant C.$$

At last, we prove (iii). It suffices to prove that K_3 satisfies $H(p_0)$. From (13), we have

$$\begin{aligned}
&\Bigg(\int_{2^k l\leqslant|y-x_0|<2^{k+1}l}\big|K_3(x,y)-K_3(x_0,y)\big|^{p_0}\,dy\Bigg)^{1/p_0}\\
&\quad\leqslant C_N\frac{l^\delta}{(2^kl)^{n-1+\delta}}\frac{1}{\{1+m(x_0,V)2^kl\}^N}\bigg\|\int\frac{V(\xi)\chi_{B(x_0,2^{k+3}l)}}{|y-\xi|^{n-1}}\,d\xi\bigg\|_{L_y^{p_0}}+\frac{l^\delta}{(2^kl)^{(n/p_0')+\delta}}\\
&\quad\leqslant C_N\frac{l^\delta}{(2^kl)^{n-1+\delta}}\frac{1}{\{1+m(x_0,V)2^kl\}^N}\int_{B(x_0,2^{k+3}l)}V(\xi)^q\,d\xi^{1/q}+\frac{l^\delta}{(2^kl)^{(n/p_0')+\delta}}\\
&\quad\leqslant C_N\frac{l^\delta}{(2^kl)^{n-1+\delta}}\frac{1}{\{1+m(x_0,V)2^kl\}^N}\big(2^kl\big)^{-n/q'}\int_{B(x_0,2^kl)}V(\xi)\,d\xi+\frac{l^\delta}{(2^kl)^{(n/p_0')+\delta}}\\
&\quad\leqslant C_N\frac{l^\delta}{(2^kl)^{n-1+\delta}}\big(2^kl\big)^{n/q-2}+\frac{l^\delta}{(2^kl)^{(n/p_0')+\delta}}\leqslant C\frac{l^\delta}{(2^kl)^{(n/p_0')+\delta}},
\end{aligned}$$

therefore,

$$\sum_{k=5}^{\infty}k\big(2^kl\big)^{\frac{n}{p_0'}}\Bigg(\int_{2^k l\leqslant|y-x_0|<2^{k+1}l}\big|K_3(x,y)-K_3(x_0,y)\big|^{p_0}\,dy\Bigg)^{1/p_0}\leqslant\sum_{k=5}^{\infty}\frac{Ck}{(2^k)^\delta}\leqslant C. \qquad\square$$

4. The converse result

This section is devoted to the converse problem. Recall that $T_3=\nabla(-\Delta+V)^{-1/2}$ is the Riesz transform associated to Schrödinger operator. A natural problem is that whether the converse holds. Namely, if $[b,T_3]$ is bounded on L^2, do we have $b\in\mathbf{BMO}$? This is quite subtle. If $V\equiv 0$, it reduces to the classical Riesz transform. However, for general $V\in B_q$, the converse fails. Considering $V\equiv 1$, which is in B_q for every $q>1$, we have the following:

Theorem 2. *There exist a function $b\notin\mathbf{BMO}$, such that $[b,T_3]$ is bounded on L^2.*

Proof. Consider $b = x_j$, we know that $b \notin \mathbf{BMO}$. We have that

$$[b, T_3]f = x_j \nabla(-\Delta+1)^{-1/2} f - \nabla(-\Delta+1)^{-1/2}(x_j f).$$

From Plancherel equality, we can get

$$\big\|[b, T_3]f\big\|_2 = \left\|\partial_j\left(\frac{\xi}{(1+\xi^2)^{1/2}}\hat{f}\right) - \frac{\xi}{(1+\xi^2)^{1/2}}\partial_j \hat{f}\right\|_2 = \left\|\partial_j\left(\frac{\xi}{(1+\xi^2)^{1/2}}\right)\hat{f}\right\|_2 \leqslant \|f\|_2. \qquad \square$$

The converse example in Theorem 2 implies that the assumption $V \in B_q$ is too weak, it cannot guarantee the function $b \in \mathbf{BMO}$. However if we assume V satisfies some additional conditions, for example, if V is L^p integrable, then the converse could be true. Let $T_3' = (-\Delta)^{1/2}(-\Delta+V)^{-1/2}$, then from $T_3' = (-\Delta)^{-1/2}\nabla \cdot T_3$, we know the results above also hold with T_3 replaced by T_3'.

Theorem 3. *If $[b, T_3]$, $[b, T_3']$ and $V^{1/2}(-\Delta)^{-1/2}$ is bounded on L^2, then $b \in \mathbf{BMO}$.*

Proof. From $[b, T_3]$, $[b, T_3']$ is bounded on L^2, and

$$[b, T_3] = \big[b, \nabla(-\Delta)^{-1/2}T_3'\big] = \big[b, \nabla(-\Delta)^{-1/2}\big]T_3' + \nabla(-\Delta)^{-1/2}\big[b, T_3'\big],$$

we have $[b, \nabla(-\Delta)^{-1/2}]T_3'$ is bounded on L^2.

We claim that $[b, \nabla(-\Delta)^{-1/2}]$ is bounded on L^2, which implies the theorem from the well-known theorem of Coifman, Rochberg and Weiss. It suffices to prove that T_3' has a converse bounded on L^2. Note that $T_3'^{-1} = (-\Delta+V)^{1/2}(-\Delta)^{-1/2}$, and

$$\begin{aligned} T_3'^{-1} f &= (-\Delta+V)^{1/2}(-\Delta)^{-1/2} f = (-\Delta+V)^{-1/2}(-\Delta+V)(-\Delta)^{-1/2} f \\ &= (-\Delta+V)^{-1/2}(-\Delta)^{1/2} f + (-\Delta+V)^{-1/2} V^{1/2} V^{1/2}(-\Delta)^{-1/2} f. \end{aligned}$$

Therefore, by using $V^{1/2}(-\Delta)^{-1/2}$ is bounded on L^2, we can easily get the conclusion of Theorem 3. $\square$

Corollary 2. *If $[b, T_3]$, $[b, T_3']$ is bounded on L^2, and $V \in L^{n/2} \bigcap B_q$ for $q > n/2$, then $b \in \mathbf{BMO}$.*

Proof. We only need to prove that $V^{1/2}(-\Delta)^{-1/2}$ is bounded on L^2. This follows directly from Hölder inequality and fractional integration:

$$\big\|V^{1/2}(-\Delta)^{-1/2} f\big\|_2 \leqslant C\big\|V^{1/2}\big\|_n \big\|(-\Delta)^{-1/2} f\big\|_{2n/(n-2)} \leqslant C\|V\|_{n/2}^{1/2}\|f\|_2. \qquad \square$$

References

[1] R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. 103 (1976) 611–635.

[2] F. Gehring, The L^p-integrability of the partial derivatives of a quasi-conformal mapping, Acta Math. 130 (1973) 265–277.

[3] D. Gilberg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Springer-Verlag, 1983.

[4] S. Janson, Mean oscillation and commutators of singular integral operators, Ark. Mat. 16 (1978) 263–270.

[5] Y. Meyer, La minimalité de le espace de Besov $B_1^{0,1}$ et la continuité des opérateurs définis par des intégrales singulières, Monografias de Matematicas, vol. 4, Univ. Autonoma de Madrid, 1985.

[6] A. Torchinsky, Real-Variable Methods in Harmonic Analysis, Academic Press, 1986.

[7] Z. Shen, L^p estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier 45 (2) (1995) 513–546.

[8] J. Zhong, Harmonic analysis for some Schrödinger type operators, PhD thesis, Princeton University, 1993.